Course: 390018 (Econometrics of Seasonality)

by

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Periodic Processes

(Chapter 6)

Base on book by

Ghysels/Osborn: The Econometric Analysis of Seasonal Time Series

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1: Introduction

- **Periodic Process**:

Coefficient changes with the season of the year. Deterministic process is the special case of periodic process if intercept changes with seasons.

- Gersovitz and Mackinnon (1978) and Osborn (1988): It arises when modelling seasonal decisions of consumers.
- Hansen and Sargent (1993): It could also arise from seasonal technology.

2: Periodic Process

Simple Example:

First-order autoregressive process with periodically varying coefficients.

$$y_t = \left(\sum_{s=1}^{S} \delta_{st} \emptyset_s{}^{P} \right) y_{t-1} + \varepsilon_t$$

Stability conditions : The absolute value of product of all coefficients should be less than one.

2.1: Periodic ARMA (p,q) process

$$\emptyset_s(B)y_{s\tau} = C_s + \theta_s(B)\varepsilon_{s\tau} \quad (1.1)$$

Where, $s = 1, \dots, S$, $\tau = 1, \dots, T$

and

$$\emptyset_s(B) = 1 - \emptyset_{s1}B - \cdots - \emptyset_{sp}B^p$$

$$\theta_s(B) = 1 - \theta_{s1}B - \cdots - \theta_{sq}B^q$$

are polynomials in the conventional lag operator B and Disturbance $\varepsilon_{s\tau}$ is i.i.d over season and year.

p and q are max AR and MA lags with nonzero coefficients.

$E(\varepsilon_{s\tau}\varepsilon_{kj}) = 0$ unless $s = k$ and $\tau = j$

- Heteroskedasticity over seasons is permitted , so that $E(\varepsilon^2{}_{st}) = \sigma^2{}_s$

- B operates on the season and one period lagged observation is $\mathrm{B}y_{s\tau} = By_{s-1,\tau}$

- ARMA(p,q) must have at least one $\emptyset_{sp} \neq 0$and $\theta_{sq} \neq 0$ over $s = 1, \ldots, S$

- Further, $\emptyset_s(B)$ and $\theta_s(B)$ must have no roots in common in order to identify the parameters of the process.

- Periodic ARMA processes have distinctive stationarity and invertibility properties compared with a conventional ARMA processes.

2.2: Periodic Heteroskedasticity

- Considering a special case of (1.1) Periodic seasonal ARMA process with constant coefficients:

$$\emptyset\ (B) y_{s\tau} = C + \theta\ (B) \varepsilon_{s\tau} \qquad (1.2)$$

With all roots of $\emptyset\ (B)$ and $\theta\ (B)$ out side the unit circle. This is conventional ARMA process except that we assume that

$E(\varepsilon^2{}_{st}) = \sigma^2{}_s$ is not constant over seasons.
$E(y_{s\tau}) = \mu = C/\emptyset\ (1)$ (mean is unaffected)

Zero mean seasonal heteroskedastic AR(1) process in periodic notation:

$$y_{s\tau} = \emptyset y_{s-1,\tau} + \varepsilon_{s\tau} \qquad (1.3)$$

assuming $y_{s\tau}$ corresponds to $s = S$, repeated substitution yields

$$y_{St} = \emptyset^2 y_{S-2,\tau} + \varepsilon_{S\tau} + \emptyset\varepsilon_{S-1,\tau},$$
$$= \emptyset^S y_{S,\tau-1} + \varepsilon_{S\tau} + \emptyset\varepsilon_{S-1,\tau} + \cdots + \emptyset^{S-1}\varepsilon_{1\tau},$$
$$= \emptyset^{\tau S} y_{S0} + \sum_{j=0}^{\tau-1} \emptyset^{Sj}(\varepsilon_{S,\tau-j} + \emptyset\varepsilon_{S-1,\tau-j} + \cdots + \emptyset^{S-1}\varepsilon_{1,\tau-j})$$

- If starting value y_{S0} has the same variance as each sample period $y_{S\tau}$, it follows that

$$Var(y_{S0}) = \gamma_s(0) = \frac{\sigma_s{}^2 + \emptyset^2 \sigma^2{}_{s-1} + \cdots + \emptyset^{2(S-1)} \sigma^2{}_{s-(S-1)}}{1 - \emptyset^{2S}}$$

- $Var(y_{S0})$ is periodically varying since the weighting of each $\sigma_s{}^2 (s = 1, \ldots, S)$ depends on the season s in which $y_{S\tau}$ is observed.

- The above ARMA process with constant coefficient has autocovariance at lag k:

$$\gamma_s(k) = E(y_{s\tau} - \mu)E(y_{s-k,\tau} - \mu)$$

That is also seasonally varying.

- Periodic heteroskedasticity can be easily removed by Standardization i.e.,

$$\frac{E(y_{s\tau} - \mu)}{\sqrt{Var(y_{s\tau})}}$$

That have zero mean, unit variance and autocovariance are independent of S.

2.3: Periodic MA(1) Process

- The Periodic MA(1) process with no deterministic component:

$$y_{s\tau} = \varepsilon_{s\tau} - \theta_s \varepsilon_{s-1,\tau}, \qquad (1.4)$$

- The mean of $y_{s\tau}$ is zero and variance $E(\varepsilon^2{}_{st}) = \sigma^2$ is constant. But variance of $y_{s\tau}$ exhibits periodic heteroskedasticity, since

$$Var(y_{s\tau}) = \gamma_s(0) = E(\varepsilon_{s\tau} - \theta_s \varepsilon_{s-1,\tau})^2$$
$$= (1 + \theta_s{}^2)\sigma^2$$

- Further autocovariance at lag (1) is

$$\gamma_s(1) = E(y_{s\tau} y_{s-1,\tau})$$
$$= E(\varepsilon_{s\tau} - \theta_s \varepsilon_{s-1,\tau})\,(\varepsilon_{s-1,\tau} - \theta_{s-1}\varepsilon_{s-2,\,\tau})$$
$$= -\theta_s \sigma^2$$

For k > 1 $\gamma_s(k)$ = 0. Thus,

- MA(1) exhibits periodic variances and autocovariances, observation 1 year apart, $y_{s\tau}$ and $y_{s,\tau-1}$, are not correlated.

- The characteristic of seasonality in economic variables that the patterns in the observations tend to repeat each year, and hence that $y_{s,\tau-1}$ provides relevant information for the prediction of $y_{s\tau}$, cannot be delivered by a periodic MA (1) process.

2.4: Periodic AR (1) Process

- The simple Periodic AR (1), or PAR (1) process:

$$y_{s\tau} = \emptyset_s y_{s-1,\tau} + \varepsilon_{s\tau}, \qquad (1.5)$$

$s = 1, \ldots, S$

With substitution for lagged y,

$$y_{s\tau} = \emptyset_s \emptyset_{s-1} y_{s-2,\tau} + \varepsilon_{s\tau} + \emptyset_s \varepsilon_{s-1,\tau}$$
$$= \emptyset_S \emptyset_{S-1\ldots} \emptyset_1 y_{s,\tau-1} + \varepsilon_{s\tau} + \emptyset_s \varepsilon_{s-1,\tau} + \emptyset_s \emptyset_{s-1} \varepsilon_{s-2,\tau}$$
$$+ \cdots \emptyset_s \emptyset_{s-1} \ldots \emptyset_{s-(S-1)} \varepsilon_{s-(S-1),\tau}$$

- The coefficient of $y_{s,\tau-1}$ is the product of all S periodic AR(1) coefficients, namely $\psi = \emptyset_1 \emptyset_2 \ldots \emptyset_S$.
- The presence of PMA process implies that $Var(y_{s\tau})$ and its autocovariances vary over s. While starting value $E(y_{s0}) = 0$ and $E(y_{s\tau}) = \mu = 0$

$$\text{Var}(y_{1\tau}) = \gamma_1(0)$$
$$= \left(\frac{1}{1-\psi^2}\right)\left[\sigma_1{}^2 + \emptyset_1{}^2\sigma_4{}^2 + \emptyset_1{}^2\emptyset_4{}^2\sigma_3{}^2 + \emptyset_1{}^2\emptyset_4{}^2\emptyset_3{}^2\sigma_2{}^2\right].$$

- Even with homoskedasticity in the disturbances the periodic AR(1) process $y_{s\tau}$ exhibits periodic heteroskedasticity.
- The autocovariances at lag 1 for the PAR (1)

$$\gamma_s(1) = E\left(y_{s\tau}y_{s-1,\tau}\right) = \emptyset_s\gamma_{s-1}(0),$$

At annual lag S,

$$\gamma_s(S) = E\left(y_{s\tau}y_{s,\tau-1}\right) = \psi_1\gamma_s(0) \qquad (1.6)$$

- The $\gamma_s(1)$ is periodic through both $\emptyset_s$ and $\gamma_{s-1}(0)$,
- While $\gamma_s(S)$ is periodic only through the variance $\gamma_s(0)$.

- Consequently, the autocorrelation of $y_{s\tau}$ at lag S is:

$$\rho_s(S) = \gamma_s(S) / \gamma_s(0) = \psi,$$

Which is constant over $s = 1, \ldots, S$.

- Above $\gamma_s(S)$ in equation (1.6) implies that PAR (1) process gives rise to an annual pattern in the conditional expectations, with,

$$E\left(y_{s\tau} \middle| y_{s,\tau-1}\right) = \psi y_{s,\tau-1} \text{ applies for all s}$$

- In contrast to the PMA(1) process, the PAR(1) process gives rise to a type of seasonal habit persistence whereby an annual pattern in the observations will tend to be repeated when ψ is positive.

3: The VAR Representations

- VAR representation of the PAR (1) process of equation (1.5) becomes,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\emptyset_2 & 1 & 0 & 0 \\ 0 & -\emptyset_3 & 1 & 0 \\ 0 & 0 & -\emptyset_4 & 1 \end{bmatrix} \begin{bmatrix} y_{1\tau} \\ y_{2\tau} \\ y_{3\tau} \\ y_{4\tau} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \emptyset_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{1,\tau-1} \\ y_{2,\tau-1} \\ y_{3,\tau-1} \\ y_{4,\tau-1} \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} + \begin{bmatrix} \varepsilon_{1\tau} \\ \varepsilon_{2\tau} \\ \varepsilon_{3\tau} \\ \varepsilon_{4\tau} \end{bmatrix}$$

- Or more compactly, $\Phi_0 Y_\tau = \Phi_1 Y_{\tau-1} + C + U_\tau$ (1.7)

$$E(U_\tau U'_\tau) = \sum = diag(\sigma_1{}^2, \sigma_2{}^2, \sigma_3{}^2, \sigma_4{}^2)$$

Φ_0 and Φ_1 are $S \times S$ coefficient matrices and C is $S \times 1$ vector of intercept.

- The general Vector representation for a PAR(p) process is the VAR(P):

$$\Phi_0 Y_\tau = \Phi_1 Y_{\tau-1} + \cdots + \Phi_P Y_{\tau-P} + C + U_\tau$$

Using the matrix polynomial lag operator

$$\Phi(B) = \Phi_0 - \Phi_1 B^s - \cdots - \Phi_P B^{Ps}$$

$$\Phi(B) Y_\tau = C + U_\tau$$

- The more usual VAR(P)representation can be obtained by inverting Φ_0,

$$Y_\tau = \Phi_0{}^{-1}\Phi_1 Y_{\tau-1} + \cdots + \Phi_0{}^{-1}\Phi_P Y_{\tau-P} + \Phi_0{}^{-1}C + \Phi_0{}^{-1}U_\tau$$

$$Y_\tau = A_1 Y_{\tau-1} + \cdots + A_P Y_{\tau-P} + \tilde{C} + V_\tau$$

Where,

$A_i = \Phi_0{}^{-1}\Phi_i (i = 1, \ldots, P)$, $\tilde{C} = \Phi_0{}^{-1}C$, and $V_\tau = \Phi_0{}^{-1}U_\tau$

- And $|\Phi_0|$ must be nonsingular then VAR(P) representation is well defined.

4: Integration In PAR

- Stationarity condition for PAR process $|\Phi(B)|$ lie outside the unit circle. In PAR three types of integrated process for first order non stationarity,

- $y_t \sim I(1)$. PAR operator $\emptyset_s(B)$ contains the common factor $\Delta_1 = (1 - B)$ but $\Delta_1 y_{st}$ is a stationary PAR process.

- $y_t \sim SI(1)$. PAR operator $\emptyset_s(B)$ contains the common factor $\Delta_s = (1 - B^s)$ but $\Delta_s y_{st}$ is a stationary PAR process.

- $y_t \sim PI(1)$. $|\Phi(B)|$ contains the factor $(1 - B^s)$ but Δ_s is not common to each polynomial $\emptyset_s(B)(s = 1, \ldots . S)$, with the VAR for $\Delta_s y_{st}$ being stationary.

- $y_t \sim PI(1)$ is the special form of periodic integrated process and the process is non-stationary with $\emptyset_1\emptyset_2\emptyset_3\emptyset_4 = 1$ but not all individual $\emptyset_s = 1$. For stationarity quasi difference is required,

$$D_s y_{st} = y_{st} - \emptyset_s y_{s-1,t}$$

- If $\emptyset_s = 1$ then $I(1)$ process can be viewed as special case of PAR process.

4.1: GHL Test (Ghyzel, Hall and Lee 1996)

- GHL proposed using HEGY test as in chapter 3, who examine the null hypothesis $y_t \sim SI(1)$.

- The test regression of GHZ test by allowing each coefficient varies over the seasons.

$$\Delta_4 y_t = \sum_{s=1}^{4} \pi_{s1}\, \delta_{st} y_{t-1}^{(1)} - \sum_{s=1}^{4} \pi_{s2}\, \delta_{st} y_{t-1}^{(2)} - \sum_{s=1}^{4} \pi_{s3}\, \delta_{st} y_{t-2}^{(3)} - \sum_{s=1}^{4} \pi_{s4}\, \delta_{st} y_{t-1}^{(3)} + \varepsilon_t$$

- δ_{st} is the quarterly dummy variable, $y_{t-1}^{(1)}, y_{t-1}^{(2)}, y_{t-2}^{(3)}$ and $y_{t-1}^{(3)}$ is HEGY transformed variables

- π_{si} = Seasonal varying coefficients where s = 1, … .4 and i = 1, … 4 so there are 16 coefficients

- Seasonal integration null hypothesis implies π_{si} = 0 and adopt a Wald test for estimation.

4.2: Cointegration to test seasonal integration

- In chapter 3 we have studied that y_{st} process can not be cointegrated with each other.

- Franses (1994) adopt this idea and develops a cointegration for testing a null hypothesis of seasonal integration.

- He treats the series separately for seasons S and adopt a VAR(1) representation.

$$\Phi_0 Y_t = \Phi_1 Y_{t-1} + U_t$$

$$\Delta_s Y_t = \mathrm{A} Y_{t-1} + V_t$$

Where A $= -(I_s - \Phi_0^{-1}\Phi_1)$ and $V_t = \Phi_0^{-1} U_t$

- Franses applies Johansen (1988) terminology to this VAR representation.
- Seasonal random walk implies A = 0 and $V_t = U_t$
- Clearly rank of *A* is then zero implies process y_{st} is individually $I(1)$ and not cointegrated.
- When Rank *A* is r then null hypothesis of no cointegration r = 0, also a test that univariate process $y_t \sim SI(1)$.

- The Null hypothesis r = 0 is tested against the alternative r > 0. If null is rejected r = 1 is tested against the alternative r > 1. The process continues until the null cannot be rejected and this specifies Rank of A as r^*.

- Quarterly case: $y_t \sim PI(1)$, r = 3 (Periodic Seasonal Unit Root)

- Matrix *A* is computed from equation 1.7 ,

- $A = -(I_s - \Phi_0^{-1}\Phi_1) = \begin{bmatrix} -1 & 0 & 0 & \phi_1 \\ 0 & -1 & 0 & \phi_1\phi_2 \\ 0 & 0 & -1 & \phi_1\phi_2\phi_3 \\ 0 & 0 & 0 & \phi_1\phi_2\phi_3\phi_4 - 1 \end{bmatrix}$

- We can see that final row contains zero only when $\emptyset_1\emptyset_2\emptyset_3\emptyset_4 = 1$

- The three cointegrating relationships are $y_{2\tau} - \phi_2 y_{1\tau}$, $y_{3\tau} - \phi_3 y_{2\tau}$, and $y_{4\tau} - \phi_4 y_{3\tau}$.

- <u>Another Case</u>: $y_t \sim I(1)$, r = 3 (No seasonal Unit root)

- There is also a cointegration relationship as $y_{2\tau} - y_{1\tau}$, $y_{3\tau} - y_{2\tau}$, and $y_{4\tau} - y_{3\tau}$.

- The coefficients of cointegrating relationships are (1, -1).

4.3: Periodic and nonperiodic integration testing

- Franses Multivariate test unable to consider highly parameterized nature of periodic integration. Because Franses consider null hypothesis that $y_\tau \sim SI(1)$.

- Boswijk and Franses (1996) develop a test for null hypothesis $y_\tau \sim PI(1)$.

- Test Exploits $PI(1)$ implication that matrix A of the VAR representation has a rank of 3 and series y_{1t}, y_{2t}, y_{3t} and y_{4t} posses a single common stochastic trend.

- Test regression proposed by Boswijk and Franses (1996) has the form

$$y_t = \sum_{s=1}^{4} \phi_s \, \delta_{st} y_{t-1} + \varepsilon_t$$

- Under the null hypothesis, Non linear estimation is required in order to impose the periodic integration restriction $\emptyset_1\emptyset_2\emptyset_3\emptyset_4 = 1$, while under the alternative, $\emptyset_1\emptyset_2\emptyset_3\emptyset_4 \neq 1$

- Likelihood ratio test is used to compare the results of these estimations.

- Asymptotically test statistics converges to square of DF t-distribution.

5: Periodic Cointegration

- Periodic cointegration can be defined as stationary linear combination $y_{s\tau} - k_s^P x_{s\tau}$ exist between two integrated series. P represents parameter varying periodically.

- Birchenhall et al. (1989) was the first who consider periodic cointegration and then Franses et al., (1995, 1999),

- Definition:

(Boswijk and Franses, 1995) ,

"Consider $y_{s\tau}$ and $x_{s\tau}$ for $s = 1, \ldots .4$ each of which is $I(1)$. The variables x and y are fully periodically cointegrated of order (1,1) if there exist coefficientsk_s^P such that $y_{s\tau} - k_s^P x_{s\tau}$ is stationary for $s = 1, \ldots .4$ with not all k_s^P equal. Partially cointegrated if k_s^P exist such that $y_{s\tau} - k_s^P x_{s\tau}$ is stationary for only some $s = 1, \ldots .4$ ".

- Acooriding to definition each series y_τ, x_τ contains $I(1)$, $SI(1)$ or $PI(1)$. But Ghysels and Osborn (2001) criticised the definition of Boswijk and Franses (1995) and redefine as it does not permit cointegration vector to be common across all four quarter. Furthermore, cointegrating relationship is periodically varying and that is the genuine definition. Three possibilities to consider,

1. $x_\tau \sim I(1)$. With full periodic cointegration between y_τ and x_τ, then $y_\tau \sim PI(1)$.

$$y_{s\tau} = k_s^P x_{s\tau} + u_{s\tau}$$

where $u_{s\tau}$ is stationary and $x_\tau \sim I(1)$, implies

$$x_{s\tau} = x_{s-1,\tau} + v_{s\tau}$$

$$y_{s\tau} = k_s^P x_{s-1,\tau} + k_s^P v_{s\tau} + u_{s\tau} \quad (1.8)$$

Periodic cointegration also implies

$$y_{s-1,\tau} = k_s^P x_{s-1,\tau} + u_{s-1,\tau} \quad (1.9)$$

- Multiply the equation (1.9) by k_s^P / k_{s-1}^P and subtract from equation (1.8). We obtained

$$y_{s\tau} - \frac{k_s^P}{k_{s-1}^P} y_{s-1,\tau} = k_s^P v_{s\tau} + u_{s\tau} - \frac{k_s^P}{k_{s-1}^P} u_{s-1,\tau}$$

- All right-hand side variable are stationary therefore left-hand side is stationary.

- The coefficient of $y_{s-1,\tau}$ is $\alpha_s^y = \frac{k_s^P}{k_{s-1}^P}$, the $PI(1)$ restriction $\alpha_1^y \alpha_2^y \alpha_3^y \alpha_4^y = 1$ must hold for y_τ.

2. $x_\tau \sim PI(1)$. With full periodic cointegration or non periodic cointegration, then $y_\tau \sim PI(1)$.

- Same logic as in the case 1 but difference in coefficients

$$y_{s\tau} - \frac{\alpha_s^x k_s^P}{k_{s-1}^P} y_{s-1,\tau} = k_s^P v_{s\tau} + u_{s\tau} - \frac{\alpha_s^x k_s^P}{k_{s-1}^P} u_{s-1,\tau}$$

$$\alpha_s^y = \frac{\alpha_s^x k_s^P}{k_{s-1}^P}$$

- $PI(1)$ restriction $\alpha_1^y \alpha_2^y \alpha_3^y \alpha_4^y = 1$ must hold for y_τ

- Non Periodic cointegration then $\alpha_s^y = \alpha_s^x$

3. $x_\tau \sim SI(1)$. Combine with periodic cointegration between y_τ and x_τ.

- The four unit process in $x_{s\tau} (s = 1,2,3,4)$ give rise to four distinct unit root process in y_τ through $y_{s\tau} = k_s^P x_{s\tau} + u_{s\tau}$. Therefore $y_\tau \sim SI(1)$.

6: Comments on Empirical Evidences

- Wells (1997) and Novals and de fruto (1997):

  Periodic models produce less accurate forecasts than nonperiodic models. Novals and de fruto also conclude that important accuracy gains can be made by imposing nonperiodic coefficients across some seasons of periodic model.

- Franses (1995b):

  Seasonal adjustment does not remove all periodic characteristics and hence he recommends the use of periodic approach even for seasonal adjusted data.

- Conclusion:

Periodic processes have some attractive features, great strides have been made in establishing an appropriate toolkit for the statistical analysis of such processes. Overall, however, empirical applications in economics are relatively few to date. While evidence of periodicity has been found, so it is difficult to assume that the majority of important real macroeconomic variables are of this type.

Thank you