**PRESENTATION ON THE TOPIC**

# MULTIVARIATE ARCH AND GARCH MODELS

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## 1. INTRODUCTION:

In the previous chapters, we have learned modelling the conditional mean of the data generation process of a multiple time series. In that context, the variance or covariance matrix of the conditional distribution was assumed to be the time invariant. In fact, the residuals or forecast errors were assumed to be independent white noise.

The conditional mean is the optimal forecast and, hence, changes in volatility are of less importance from a forecasting point of view. This position ignores, however, that the forecast error variances, that is, the variances of the conditional distributions are needed for setting up forecast intervals. Taking into account conditional heteroskedasticity is therefore important also when forecasts of the variables under investigation are desired.

More detailed modelling of the volatility of time series was a natural development which was introduced by Engle's invention in 1982 of ARCH (autoregressive conditional heteroskedasticity) models. ARCH stands for a wide range of models for changing conditional volatility. Therefore, multivariate models for conditional heteroskedasticity are of interest. Multivariate ARCH/GARCH models and dynamic factor models, eventually, in a Bayesian framework are the basic tools used to forecast correlations and covariances.

In the following, a brief review of some facts on univariate ARCH and generalized ARCH (GARCH) models is given and then multivariate extensions will be explained.

## 2. Univariate GARCH Models

### 2.1 Definitions

Consider the univariate serially uncorrelated, zero mean process $u_t$. For instance, $u_t$ may represent the residuals of an autoregressive process. The $u_t$ are said to follow an autoregressive conditionally heteroskedastic process of order q (ARCH(q)) if the conditional distribution of $u_t$, given its past $\Omega_{t-1} = \{u_{t-1}, u_{t-2}, \ldots\}$, has zero mean and the conditional variance is

$$\sigma^2_{t|t-1} := \mathrm{var}(u_t \mid \Omega_{t-1}) = E(u^2_t \mid \Omega_{t-1}) = \gamma_0 + \gamma_1 u^2_{t-1} + \ldots + \gamma_q u^2_{t-q},$$

that is, $u_t|\Omega_{t-1} \sim (0, \sigma^2_{t|t-1})$. Another quite useful way to define an ARCH process is to specify

$$u_t = \sigma_{t|t-1}\varepsilon_t, \qquad \varepsilon_t \sim \text{i.i.d. } (0,1).$$

Here the i.i.d. assumption for $\varepsilon_t$ is slightly more restrictive than the previous definition which makes statements about the first two moments of the conditional distribution only. In the following, aforementioned definition will be used. The $u_t$, generated in this way, will be serially uncorrelated with mean zero.

Originally, Engle (1982), in his seminal paper on ARCH models, assumed the conditional distribution to be normal so that

$$\varepsilon_t \sim \text{i.i.d. } \mathcal{N}(0,1) \quad \text{and} \quad u_t|\Omega_{t-1} \sim \mathcal{N}(0, \sigma^2_{t|t-1})$$

The model is capable of generating series with characteristics similar to those observed time series. In particular, it is capable to generate series with volatility. Bollerslev (1986) and Taylor (1986) proposed to gain greater parsimony by extending the model in a similar manner as the AR model when moving to mixed ARMA models. They suggested the generalized ARCH (GARCH) model with conditional variance given by

$$\sigma^2_{t|t-1} = \gamma_o + \gamma_1 u^2_{t-1} + \ldots + \gamma_q u^2_{t-q} + \beta_1 \sigma^2_{t-1|t-2} + \ldots + \beta_m \sigma^2_{t-m|t-m-1}$$

These models are briefly denoted by GARCH($q$, $m$). They generate processes with existing unconditional variance if and only if the coefficient sum

$$\gamma_1 + \ldots + \gamma_q + \beta_1 + \ldots + \beta_m < 1$$

The similarity of GARCH models and ARMA models for the conditional mean can be seen by defining $v_t := u^2_t - \sigma^2_{t|t-1}$, substituting $u^2_t - v_t$ for $\sigma^2_{t|t-1}$.

## 3. Multivariate GARCH Models

Multivariate extensions of ARCH and GARCH models may be defined in principle similarly to VAR and VARMA models. Multivariate GARCH models have been used to investigate volatility and correlation transaction and spillover effects studies. See Strulz (2003). Early articles on multivariate ARCH and GARCH models are Engle, Granger & Kraft (1986). The simpler ARCH models will be considered first. Some of the best known multivariate GARCH models available include the VECH model of Bollerslev and Wooldridge (1988). It is used for the conditional covariance matrix.

### 3.1 Multivariate ARCH

Suppose that $\mathbf{u}_t = (\mathbf{u}_{1t}, \ldots, \mathbf{u}_{Kt})'$ is a K-dimensional zero mean, serially uncorrelated process which can be represented as

$$\boldsymbol{u}_t = \Sigma^{1/2}_{t|t-1} \boldsymbol{\varepsilon}_t,$$

where $\varepsilon_t$ is a k-dimensional i.i.d. white noise, $\varepsilon_t \sim$ i.i.d. $(0, I_K)$, and $\Sigma_{t|t-1}$ is the conditional covariance matrix of $u_t$, given $\mathbf{u}_{t-1}, \mathbf{u}_{t-2}$. Obviously, the $u_t$'s have a conditional distribution, given $\Omega_{t-1} := \{\mathbf{u}_{t-1}, \mathbf{u}_{t-2}, \ldots\}$, of the form

$$\boldsymbol{u}_t | \Omega_{t-1} \sim (\mathbf{0}, \Sigma_{t|t-1}).$$

They represent a multivariate ARCH($q$) process if

$$\mathbf{vech}(\Sigma_{t|t-1}) = \gamma_0 + \Gamma_1 \,\mathbf{vech}\,(\mathbf{u}_{t-1}\mathbf{u}'_{t-1}) + \ldots + \Gamma_q \,\mathbf{vech}\,(\mathbf{u}_{t-q}\mathbf{u}'_{t-q}),$$

where vech denotes the half-vectorization operator which stacks the columns of a square matrix from the diagonals downwards in a vector, $\gamma_0$ is a ½K(K+1)- dimensional vector of constants and the $\Gamma_j$'s are (½K(K+1) × ½K(K+1) coefficient matrices.

As an example, consider a bivariate (K = 2) ARCH (1) process,

$$\text{vech}\begin{bmatrix} \sigma_{11,t|t-1} & \sigma_{12,t|t-1} \\ \sigma_{12,t|t-1} & \sigma_{22,t|t-1} \end{bmatrix} = \begin{bmatrix} \sigma_{11,t|t-1} \\ \sigma_{12,t|t-1} \\ \sigma_{22,t|t-1} \end{bmatrix} = \begin{bmatrix} \gamma_{10} \\ \gamma_{20} \\ \gamma_{30} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} \begin{bmatrix} u_{1,t-1}^2 \\ u_{1,t-1}u_{2,t-1} \\ u_{2,t-1}^2 \end{bmatrix}$$

This is a simple model for a bivariate series but it has a fair number of parameters.

Baba, Engle, and Kraft & Kroner (1990) investigated the following variant of a multivariate ARCH model.

$$\mathbf{\Sigma_{t|t-1}} = \Gamma_{\mathbf{0}}^{*} + \Gamma_{\mathbf{1}}^{*'} \mathbf{u}_{\mathbf{t-1}}\mathbf{u'}_{\mathbf{t-1}}\Gamma_{\mathbf{1}}^{*} + \ldots + \Gamma_{\mathbf{q}}^{*'} \mathbf{u}_{\mathbf{t-q}}\mathbf{u'}_{\mathbf{t-q}}\Gamma_{\mathbf{q}}^{*}$$

where the $\Gamma_j^{*}$'s are ( $K\times K$ ) matrices. This particular multivariate model has been named as BEKK model .One advantage of this model is that it is relatively parsimonious. For instance, for a bivariate process with $K$=2 and $q$=1, there are only 7 parameters, whereas the full model has 12 coefficients.

### 3.2 MGARCH

MGARCH stands for multivariate GARCH or generalised autoregressive conditional heteroskedasticity. The development of MGARCH models from the orignal univariate specifications represented a major step forward in the modelling of time series. MGARCH models permit time-varying conditional covariances as well as variances, and the former quantity can be of substantial practical use for both modelling and forecasting, especially in finance for example, applications to the calculation of time-varying hedge ratios, value at risk estimation and portfolio construction have been developed.

In principle, multivariate ARCH models may be generalized in the same way as the univariate case. In multivariate GARCH (MGARCH) model for $u_t$, the conditional covariance matrices have the form

$$\mathbf{vech}\ (\Sigma_{t\,|t-1}) = \gamma_0 + \sum_{j=1}^{q} \Gamma_j\ \mathbf{vech}\ (\mathbf{u}_{t-j}\mathbf{u}'_{t-j}) + \sum_{j=1}^{m} \mathbf{G}_j\ \mathbf{vech}(\Sigma_{t-j|\,t-j-1}),$$

where the $G_j$'s are also fixed ( ½K( K + 1 ) × ½K ( K + 1 ) coefficient matrices.

A VARMA representation of an MGARCH process may be obtained similarly to the univariate case. Engle & Kroner (1995) showed that the MGARCH process $u_t$ with conditional covarianes as given above is stationary if and only if all eigenvalues of the matrix

$$\sum_{j=1}^{q} \Gamma_j + \sum_{j=1}^{m} G_j$$

have modulus less than one.

The parameter space of an MGARCH model has a large dimension in general and needs to be restricted to guarantee uniqueness of the representation and to obtain suitable properties of the conditional covariances To reduce the parameter space, Bollerslev et al. (1988) discussed diagonal MGARCH models, where the $\Gamma_j$'s and $G_i$'s as in the above model are diagonal matrices. Alternatively, a BEKK GARCH, Baba-Engle Kraft-Kroner, model of the following form may be useful.

$$\mathbf{\Sigma}_{t|t-1} = \mathbf{C}_0^{*\prime}\ \mathbf{C}_0^{*} + \sum_{N=1}^{N} \sum_{j=1}^{q} \Gamma_{j\,n}^{*\prime}\mathbf{u}'_{t-j}\,\mathbf{u}'_{t-j}\Gamma_{j\,n}^{*} + \sum_{n=1}^{N} \sum_{j=1}^{m} \mathbf{G}_{j\,n}^{*\prime}\,\mathbf{\Sigma}_{t-j|t-j-1}\mathbf{G}_{j\,n}^{*}\,,$$

where again $C_0^*$ is a triangular ( $K\times K$ ) matrix and the coefficient matrices $\Gamma_{j\,n}^*$, $G_{j\,n}^*$ are also ($K\times K$). Given the similarity of MGARCH and VARMA models, it is clear from Chapter 12, Section 12.1, that restrictions have to be imposed on the coefficient matrices to ensure uniqueness of the parameterization.

More general BEKK GARCH (1,1) model:

$$\boldsymbol{\Sigma}_{t\,|\,t-1} = \mathbf{C_o}^{*'}\,\mathbf{C_o}^{*} + \sum_{n=1}^{N} \boldsymbol{\Gamma}_{1\,n}^{*'}\,\mathbf{u}_{t-1}\,\mathbf{u}_{t-1}'\,\boldsymbol{\Gamma}_{1\,n}^{*} + \sum_{n=1}^{N} \mathbf{G}_{1\,n}^{*'}\,\boldsymbol{\Sigma}_{t-1\,|\,t-2}\,\mathbf{G}_{1n}^{*}\,,$$

### 3.3 Other Multivariate ARCH and GARCH Models

These are special BEKK models. For example, Lin (1992 ) specified a **factor GARCH model**, where the $\Gamma_{1\,n}^*$ and $G_1^*n$ ′s in a BEKK GARCH (1,1) model of the form

$$\boldsymbol{\Gamma}_{1\,n}^{*} = \gamma_n\,\eta_n\,\xi_n' \quad \text{and} \quad \mathbf{G}_{1\,n}^{*} = g_n\,\eta_n\,\xi_n', \quad \mathbf{n=1,\ldots.N.}$$

Here $\gamma_n$ and $g_n$ are scalars and $\eta_n$ and $\xi_n$ are ($K\times 1$)-vectors.

A closely related model the so-called **generalized orthogonal GARCH model** was proposed by van der Weide (2002).

In 1990, Bollerslev had obtained **constant conditional correlation ( CCC) MGARCH model.**

Clearly, in this model, the time invariant R is the correlation matrix corresponding to the covariance matrix $\Sigma_{t|t-1}$ for all *t*. Engle (2002) extended the model by allowing for richer dynamics and proposed the so-called **dynamic conditional correlation (DCC) model**.

Many ARCH variants are popular in empirical finance. For example, the **exponential GARCH (EGARCH ) model** by DANIEL NELSON (1991).

$$\mathbf{X_t = \varepsilon_t \exp ( h_t / 2 ).}$$

$$\mathbf{h_t = \gamma_0 + \gamma_1 h_{t-1} + w \in_{t-1} + \lambda ( | \in_{t-1} | - E | \in_{t-1} | )}$$

A range of other models was also proposed and the literature on MGARCH models has grown rapidly over the last years.