Periodic VAR Processes and Intervention Models

Based on Chapter 17 of "New Introduction to Multiple Time Series Analysis" from H. Lütkepohl

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Introduction

- Models with potentially time varying first and second moments <u>but</u>
- time invariant coefficients
- Models with time varying coefficients, e.g. time series with seasonal pattern.
- e.g. a model where only the intercept term varies for s seasons

$$y_t = n_{1t}\nu_1 + ... + n_{st}\nu_s + A_1 y_{t-1} + ... + A_p y_{t-p} + u_t$$

where

$$n_{it} = 0 \text{ or } 1 \text{ and } \sum_{i=1}^{s} n_{it} = 1.$$

- A more general model:

$$y_t = \nu_t + A_{1t}y_{t-1} + ... + A_{pt}y_{t-p} + u_t$$

with

$$\begin{aligned} B_t :&= [\nu_t, A_{1t}, ..., A_{pt}] \\ &= n_{1t}[\nu_1, A_{11}, ..., A_{p1}] + ... + n_{st}[\nu_s, A_{1s}, ..., A_{ps}] \\ &= n_{1t}B_1 + ... + n_{st}B_s \end{aligned}$$

and

$$\Sigma_t := E(u_t u_t') = n_{1t}\Sigma_1 + ... + n_{st}\Sigma_s.$$

The VAR(p) Model with Time Varying Coefficients

General form of a K-dimensional VAR(p) model with time varying coefficients:

$$y_t = \nu_t + A_{1t} y_{t-1} + \ldots + A_{pt} y_{t-p} + u_t, \;\; t \in \mathbb{Z} \tag{1}$$

where u_t is a zero mean noise process with covariance matrices $E(u_t u_t') = \Sigma_t$

The VAR(p) model of Equation (1) can be written in VAR(1) form as:

$$Y_t = \boldsymbol{\nu}_t + \mathbf{A}_t Y_{t-1} + U_t$$

where

$$\underset{(Kp\times 1)}{Y_t := \begin{bmatrix} y_t \\ \vdots \\ y_{t-p+1} \end{bmatrix}}, \qquad \boldsymbol{\nu}_t := \underset{(Kp\times 1)}{\begin{bmatrix} \nu_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \qquad U_t := \underset{(Kp\times 1)}{\begin{bmatrix} u_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}},$$

$$\mathbf{A}_t := \underset{(Kp\times Kp)}{\begin{bmatrix} A_{1,t} & \dots & A_{p-1,t} & A_{p,t} \\ I_K & & 0 & 0 \\ & \ddots & \vdots & \vdots \\ 0 & \dots & I_K & 0 \end{bmatrix}}.$$

- By successive substitution we get

$$Y_t = \left(\prod_{j=0}^{h-1} \mathbf{A}_{t-j}\right) Y_{t-h} + \sum_{i=0}^{h-1} \left(\prod_{j=0}^{i-1} \mathbf{A}_{t-j}\right) \boldsymbol{\nu}_{t-i} + \sum_{i=0}^{h-1} \left(\prod_{j=0}^{i-1} \mathbf{A}_{t-j}\right) U_{t-i}.$$

- Using $J := [I_K : 0]$ $((K \times Kp))$ such that $y_t = JY_t$ and premultiplying by this matrix gives

$$y_t = J \left(\prod_{j=0}^{h-1} \mathbf{A}_{t-j}\right) Y_{t-h} + \sum_{i=0}^{h-1} \Phi_{it}\nu_{t-i} + \sum_{i=0}^{h-1} \Phi_{it}u_{t-i}, \qquad (2)$$

- where

$$\Phi_{it} := J \left(\prod_{j=0}^{i-1} \mathbf{A}_{t-j} \right) J'$$

  and it has been used that $J'JU_t = U_t, JU_t = u_t$, and similar results hold for $\boldsymbol{\nu}_t$.

- equation (2) can be represented as

$$y_t = \mu_t + \sum_{i=0}^{\infty} \Phi_{it} u_{t-i} \tag{3}$$

- Equation (3) can be used to derive the autocovariance structure of the process:

$$
\begin{aligned}
E[(y_t - \mu_t)(y_t - \mu_t)'] &= E\left[\left(\sum_{j=0}^{\infty} \Phi_{jt} u_{t-j}\right)\left(\sum_{i=0}^{\infty} \Phi_{it} u_{t-i}\right)'\right] \\
&= E\left[\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \Phi_{jt} u_{t-j} u'_{t-i} \Phi'_{it}\right] \\
&= \sum_{i=0}^{\infty} \Phi_{it} \Sigma_{t-i} \Phi'_{it}.
\end{aligned}
$$

- and for lag 1

$$E[(y_t - \mu_t)(y_{t-1} - \mu_{t-1})'] = E\left[\left(\sum_{j=0}^{\infty} \Phi_{jt} u_{t-j}\right)\left(\sum_{i=0}^{\infty} \Phi_{i,t-1} u_{t-1-i}\right)'\right]$$

$$= E\left[\sum_{j=-1}^{\infty} \sum_{i=0}^{\infty} \Phi_{j+1,t} u_{t-j-1} u'_{t-1-i} \Phi'_{i,t-1}\right]$$

$$= \sum_{i=0}^{\infty} \Phi_{i+1,t} \Sigma_{t-1-i} \Phi'_{i,t-1}.$$

- More generally, for some integer h,

$$E[(y_t - \mu_t)(y_{t-h} - \mu_{t-h})'] = \sum_{i=0}^{\infty} \Phi_{i+h,t} \Sigma_{t-h-i} \Phi'_{i,t-h}.$$

- Optimal forecasts can be obtained recursively from Equation (1) as

$$y_t(h) = \nu_{t+h} + A_{1,t+h} y_t(h-1) + \ldots + A_{p,t+h} y_t(h-p), \quad (4)$$

where $y_t(j) := y_{t+j}$ for $j \leq 0$.

- Alternatively Equation (3) can be used for calculating optimal forecasts:

$$y_t(h) = \mu_{t+h} + \sum_{i=h}^{\infty} \Phi_{i,t+h} u_{t+h-i}$$

- The forecast error is

$$y_{t+h} - y_t(h) = \sum_{i=0}^{h-1} \Phi_{i,t+h} u_{t+h-i}.$$

- The forecast MSE matrices are given by

$$\Sigma_t(h) := MSE[y_t(h)] = \sum_{i=0}^{h-1} \Phi_{i,t+h} \Sigma_{t+h-i} \Phi'_{i,t+h}.$$

- The general model can be written as

$$y_t = B_t Z_{t-1} + u_t,$$

where $B_t := [\nu_t, A_{1t}, ..., A_{pt}], Z_{t-1} := (1, Y'_{t-1})'$, B_t depending on an $(N \times 1)$ vector $\boldsymbol{\gamma}$ of fixed, time invariant parameters and the Σ_t are assumed to depend on an $(M \times 1)$ vector $\boldsymbol{\sigma}$ of fixed parameters.

- Assuming $u_t \sim \mathcal{N}(0, \Sigma_t)$ the log-likelihood function of the general model is

$$ln\ l(\boldsymbol{\gamma}, \boldsymbol{\sigma}) = -\frac{KT}{2} ln\ 2\pi - \frac{1}{2}\sum_{t=1}^{T} ln|\Sigma_t| - \frac{1}{2}\sum_{t=1}^{T} u_t' \Sigma_t^{-1} u_t,$$

where any initial condition terms are ignored.

In periodic VAR or PAR processes the coefficients vary periodically with period s,

$$y_t = \nu_t + A_t Y_{t-1} + u_t, \qquad (5)$$

where

$$\begin{aligned} \nu_t &= n_{1t}\nu_1 + ... + n_{st}\nu_s, & (K \times 1) \\ A_t &= [A_{1t}, ..., A_{pt}] = n_{1t}A_1 + ... + n_{st}A_s, & (K \times Kp) \\ \Sigma_t &= E(u_t u_t') = n_{1t}\Sigma_1 + ... + n_{st}\Sigma_s, & (K \times K) \end{aligned}$$

The n_{it} are seasonal dummy variables which have a value of one if t is associated with the $i - th$ season and zero otherwise.

A VAR Representation with Time Invariant Coefficients

- quarterly process with period $s = 4$ and y_1 belongs to the first quarter.
- define an annual process with vectors

$$\mathfrak{y}_1 := \begin{bmatrix} y_4 \\ y_3 \\ y_2 \\ y_1 \end{bmatrix}, \mathfrak{y}_2 := \begin{bmatrix} y_8 \\ y_7 \\ y_6 \\ y_5 \end{bmatrix}, ..., \mathfrak{y}_\tau := \begin{bmatrix} y_{4\tau} \\ y_{4\tau-1} \\ y_{4\tau-2} \\ y_{4\tau-3} \end{bmatrix}, ...$$

- This process has a representation with time invariant coefficient matrices.

A VAR Representation with Time Invariant Coefficients

- e.g. the process for each quarter is a VAR(1),

$$\begin{aligned} y_t &= \nu_t + A_{1,t} y_{t-1} + u_t \\ &= \nu_i + A_{1,i} y_{t-1} + u_t, \qquad \text{if t belongs to the i-th quarter,} \end{aligned}$$

- the process $\mathfrak{y}_t$ has the representation

$$\begin{bmatrix} I_K & -A_{1,4} & 0 & 0 \\ 0 & I_K & -A_{1,3} & 0 \\ 0 & 0 & I_K & -A_{1,2} \\ 0 & 0 & 0 & I_K \end{bmatrix} \begin{bmatrix} y_{4\tau} \\ y_{4\tau-1} \\ y_{4\tau-2} \\ y_{4\tau-3} \end{bmatrix}$$

$$= \begin{bmatrix} \nu_4 \\ \nu_3 \\ \nu_2 \\ \nu_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{1,1} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{4\tau-4} \\ y_{4\tau-5} \\ y_{4\tau-6} \\ y_{4\tau-7} \end{bmatrix} + \begin{bmatrix} u_{4\tau} \\ u_{4\tau-1} \\ u_{4\tau-2} \\ u_{4\tau-3} \end{bmatrix}.$$

A VAR Representation with Time Invariant Coefficients

- More generally, if there are s different regimes with constant parameters within each regime, we may define the sK-dimensional process

$$\mathfrak{y}_\tau := \underset{(sK\times 1)}{\begin{bmatrix} y_{s\tau} \\ y_{s\tau-1} \\ \vdots \\ y_{s\tau-s+1} \end{bmatrix}}, \quad \tau = 0, \pm 1, \pm 2, ...$$

- This process has the following VAR(P) representation, where P is the smallest integer greater than or equal to p/s:

$$\mathfrak{A}_0 \mathfrak{y}_\tau = \boldsymbol{\nu} + \mathfrak{A}_1 \mathfrak{y}_{\tau-1} + ... + \mathfrak{A}_P \mathfrak{y}_{\tau-P} + \mathfrak{u}_\tau,$$

where

$$\mathfrak{A}_0 := \underset{(sK \times sK)}{\begin{bmatrix} I_K & -A_{1,s} & -A_{2,s} & \dots & -A_{s-1,s} \\ 0 & I_K & -A_{1,s-1} & \dots & -A_{2,s-1} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \dots \\ 0 & 0 & 0 & \dots & I_K \end{bmatrix}}, \quad \boldsymbol{\nu} := \underset{(sK \times 1)}{\begin{bmatrix} \nu_s \\ \nu_{s-1} \\ \vdots \\ \nu_1 \end{bmatrix}},$$

$$\mathfrak{A}_i := \underset{(sK \times sK)}{\begin{bmatrix} A_{is,s} & A_{is+1,s} & \dots & A_{(i+1)s-1,s} \\ A_{is-1,s-1} & A_{is,s-1} & \dots & A_{(i+1)s-2,s-1} \\ \vdots & \vdots & & \vdots \\ A_{is-s+1,1} & A_{is-s+2,1} & \dots & A_{is,1} \end{bmatrix}}, \quad i = 1, ..., P,$$

A VAR Representation with Time Invariant Coefficients

$$\mathfrak{u}_\tau := \underset{(sK\times 1)}{\begin{bmatrix} u_{s\tau} \\ u_{s\tau-1} \\ \vdots \\ u_{s\tau-s+1} \end{bmatrix}}.$$

- All $A_{i,j}$'s with $i > p$ are zero.
- The process $\mathfrak{y}_\tau$ is stationary if the y_t's have bounded first and second moments and the VAR operator is stable, that is,

$$det(\mathfrak{A}_0 - \mathfrak{A}_1 z - ... - \mathfrak{A}_P z^P)$$
$$= det(I_{sK} - \mathfrak{A}_0^{-1}\mathfrak{A}_1 z - ... - \mathfrak{A}_0^{-1}\mathfrak{A}_P z^P) \neq 0 \quad \text{for } |z| \leq 1.$$

A VAR Representation with Time Invariant Coefficients

- Stationarity of $\mathfrak{y}_\tau$ does not imply stationarity of the original process y_t. Even if $\mathfrak{y}_\tau$ has a time invariant mean vector

  $$\boldsymbol{\mu} = \begin{bmatrix} \mu_4 \\ \mu_3 \\ \mu_2 \\ \mu_1 \end{bmatrix},$$

  for example, the mean vectors μ_4 and μ_3 may be different.
- $\mathfrak{y}_\tau$ can also be used to determine an upper bound for the order p of the corresponding periodic process y_t. If $\mathfrak{y}_\tau$ is stationary and order P is selected, then $p < sP$.

A VAR Representation with Time Invariant Coefficients

- Optimal forecasts of a periodic process can be obtaind from the recursions of equation (4)
- Forecast origin t is associated with the last period of the year,

$$
\begin{aligned}
y_t(1) &= \nu_1 + A_{1,1} y_t + ... + A_{p,1} y_{t-p+1} \\
y_t(2) &= \nu_2 + A_{1,2} y_t(1) + ... + A_{p,2} y_{t-p+2} \\
&\vdots \\
y_t(s) &= \nu_s + A_{1,s} y_t(s-1) + ... + A_{p,s} y_t(s-p) \\
y_t(s+1) &= \nu_1 + A_{1,1} y_t(s) + ... + A_{p,1} y_t(s+1-p) \\
&\vdots
\end{aligned}
$$

ML Estimation and Testing for Time Varying Coefficients

Cases of Interest:

1. All coefficients time varying
2. All coefficients time invariant
3. Time invariant white noise
4. Time invariant covariance structure
5. Time varying error covariance matrix only

All Coefficients Time Varying

- a periodic VAR(p) model for which all coefficients are time varying, that is

$$H_1 : B_t = [\nu_t, A_t] = \sum_{i=1}^{s} n_{it} B_i, \quad \Sigma_t = \sum_{i=1}^{s} n_{it} \Sigma_i.$$

- In this case
$\boldsymbol{\gamma} = vec[B_1, ..., B_s]$ and $\boldsymbol{\sigma} = [vech(\Sigma_1)', ..., vech(\Sigma_s)']'$.
- The ML-estimators are obtained to be

$$\tilde{B}_i^{(1)} = \left(\sum_{t=1}^{T} n_{it} y_t Z_{t-1}' \right) \left(\sum_{t=1}^{T} n_{it} Z_{t-1} Z_{t-1}' \right)^{-1}$$

and

$$\tilde{\Sigma}_i^{(1)} = \sum_t n_{it}(y_t - \tilde{B}_i^{(1)} Z_{t-1})(y_t - \tilde{B}_i^{(1)} Z_{t-1})' / T\bar{n}_i,$$

for $i = 1, ..., s$. Here $\bar{n}_i = \sum_{t=1}^{T} n_{it}/T$.

- Except for an additive constant, the corresponding maximum of the log-likelihood function is

$$\lambda_1 := -\frac{1}{2}\sum_t ln\ |\tilde{\Sigma}_t^{(1)}| = -\frac{1}{2}T(\bar{n}_1 ln\ |\tilde{\Sigma}_1^{(1)}| + ... + \bar{n}_s ln\ |\tilde{\Sigma}_s^{(1)}|).$$

All Coefficients Time Invariant

- H_2 is a null hypothesis of interest in the present context:

$$H_2 : B_i = B_1, \quad \Sigma_i = \Sigma_1, \quad i = 2, ..., s.$$

- The ML estimators are

$$\tilde{B}_1^{(2)} = \left(\sum_t y_t Z'_{t-1} \right) \left(\sum_t Z_{t-1} Z'_{t-1} \right)^{-1}$$

$$\tilde{\Sigma}_1^{(2)} = \sum_t (y_t - \tilde{B}_1^{(2)} Z_{t-1})(y_t - \tilde{B}_1^{(2)} Z_{t-1})'/T.$$

- The maximum likelihood is, except for an additive constant,

$$\lambda_2 := -\frac{1}{2} T ln \, |\tilde{\Sigma}_1^{(2)}|.$$

- Just the white noise covariance matrix is time invariant while the other coefficients vary,

$$H_3 : B_t = [\nu_t, A_t] = \sum_{i=1}^{s} n_{it} B_i, \quad \text{and } \Sigma_i = \Sigma_1,\ i = 2, ..., s.$$

- The ML estimators are

$$\tilde{B}_i^{(3)} = \tilde{B}_i^{(1)}, \quad i = 1, ..., s,$$

$$\tilde{\Sigma}_1^{(3)} = \sum_{i=1}^{s} \sum_{t=1}^{T} n_{it} (y_t - \tilde{B}_i^{(1)} Z_{t-1})(y_t - \tilde{B}_i^{(1)} Z_{t-1})' / T.$$

- The maximum likelihood is,

$$\lambda_3 := -\frac{1}{2} T ln\ |\tilde{\Sigma}_1^{(3)}|.$$

- Model with seasonal dummies,

$$H_4 : \nu_t = \sum_{i=1}^{s} n_{it}\nu_i \quad \text{and} \quad A_i = A_1,\ \Sigma_i = \Sigma_1, \quad i = 2, ..., s.$$

- defining

$$W_{t-1} = \begin{bmatrix} n_{1,t} \\ \vdots \\ n_{s,t} \\ Y_{t-1} \end{bmatrix} \quad \text{and} \quad C = [\nu_1, ..., \nu_s, A_1]$$

- the ML estimators are

$$\tilde{C} = \left(\sum_t y_t W_{t-1}' \right) \left(\sum_t W_{t-1} W_{t-1}' \right)^{-1}$$

$$\tilde{\Sigma}_1^{(4)} = \sum_t (y_t - \tilde{C} W_{t-1})(y_t - \tilde{C} W_{t-1})' / T.$$

- The maximum likelihood is (dropping an additive constant)

$$\lambda_4 := -\frac{1}{2} T ln\ |\tilde{\Sigma}_1^{(4)}|. \tag{6}$$

Tests for time varying parameters

Tabelle: LR tests for time varying parameters

null hypothesis	alternative hypothesis	LR statistic λ_{LR}	degrees of freedom
H_2	H_1	$2(\lambda_1 - \lambda_2)$	$(s-1)K[K(p+\frac{1}{2})+\frac{3}{2}]$
H_3	H_1	$2(\lambda_1 - \lambda_3)$	$(s-1)K(K+1)/2$
H_4	H_1	$2(\lambda_1 - \lambda_4)$	$(s-1)K[Kp+(K+1)/2]$
H_2	H_3	$2(\lambda_3 - \lambda_2)$	$(s-1)K(Kp+1)$
H_2	H_4	$2(\lambda_4 - \lambda_2)$	$(s-1)K$

- Time varying error covariance matrix only,

$$H_5 : B_i = B_1, \quad i = 2, ..., s \quad \text{and} \quad \Sigma_t = \sum_{t=1}^{s} n_{it}\Sigma_i.$$

- Testing H_5 against H_1 using a Wald test.
- Testing H_2 against H_5 using a LM-test.

- a particular stationary data generation mechanism until period T_1, another process generates the data after period T_1. For instance,

$$y_t = \nu_1 + A_1 Y_{t-1} + u_t, \quad E(u_t u_t') = \Sigma_1,\ t \leq T_1$$

and

$$y_t = \nu_2 + A_2 Y_{t-1} + u_t, \quad E(u_t u_t') = \Sigma_2,\ t > T_1.$$

- For simplicity, it is assumed that $A_2 = A_1$ and $\Sigma_2 = \Sigma_1$ and that the process is stable.

- Then

$$E(y_t) = \begin{cases} \sum_{i=0}^{\infty} \Phi_i \nu_1, & t \leq T_1 \\ \sum_{i=0}^{t-T_1} \Phi_i \nu_2 + \sum_{i=t-T_1+1}^{\infty} \Phi_i \nu_1, & t > T_1 \end{cases}$$

where the Φ_i's are the coefficient matrices of the moving average representation of the mean-adjusted process, i.e.,

$$\sum_{i=0}^{\infty} \Phi_i z^i = (I_K - A_{11} z - ... - A_{p1} z^p)^{-1}.$$

- The process mean does not reach a fixed new level immediately but only gradually,

$$E(y_t) \xrightarrow{t\to\infty} \sum_{i=0}^{\infty} \Phi_i \nu_2.$$

Interventions in the Intercept Model

- The model setup from the previous section can also be used for intervention models with properly specified n_{it}.
- Same hypotheses, same formulas for test statistics, but
- test statistics do not necessarily have the indicated asymptotic distributions

- e.g. model with all coefficients time varying, model as given before.
- If T_1 is some fixed finite point and $T > T_1$,

$$\tilde{B}_1 = [\tilde{\nu}_1, \tilde{A}_1] = \left(\sum_{t=1}^{T_1} y_t Z'_{t-1} \right) \left(\sum_{t=1}^{T_1} Z_{t-1} Z'_{t-1} \right)^{-1}$$

will not be consistent because the sample information regarding $B_1 := [\nu_1, A_1]$ does not increase when T goes to infinity.

- Then, under common assumptions,

$$plim\tilde{B}_1 = plim\left(\frac{1}{T_1}\sum_{t=1}^{T_1} y_t Z'_{t-1}\right) plim\left(\frac{1}{T_1}\sum_{t=1}^{T_1} Z'_{t-1}\right)^{-1} = B_1$$

- Also asymptotic normality is easy to obtain in this case and the test statistics have the limiting χ^2-distributions obtained before.

A Discete Change in the Mean

- If a one-time jump in the process mean after time T_1 is plausible a model in mean-adjusted form could be considered,

$$y_t - \mu_t = A_1(y_{t-1} - \mu_{t-1}) + ... + A_p(y_{t-p} - \mu_{t-p}) + u_t.$$

- $\mu_t := E(y_t)$ and for simplicity, all other coefficients are considered time invariant and the process as stable.
- u_t is Gaussian white noise with time invariant covariance, $u_t \sim \mathcal{N}(0, \Sigma_u)$.

- The μ_i's may be estimated by

$$\tilde{\mu}_i = \frac{1}{T\bar{n}_i}\sum_{t=1}^{T} n_{it} y_t \quad i = 1, ..., s.$$

- if $T\bar{n}_i = \sum_t n_{it}$ approaches infinity with T, it can be shown under general assumptions that $\tilde{\mu}_i$ is consistent and

$$\sqrt{T\bar{n}i}(\tilde{\mu}_i - \mu_i) \xrightarrow{d} \mathcal{N}(0, \Sigma_{\tilde{\mu}})$$

- Furthermore, the $\tilde{\mu}_i$ are asymptotically independent
- The hypothesis $H_6 : \mu_i = \mu_1 \quad i = 2, ..., s$ can be tested with a Wald test.

Thank you for your attention!