Seemingly Unrelated Regressions in Panel Models

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Arnold Zellner
An Efficient Method of Estimating Seemingly Unrelated Regressions and Tests for Aggregation Bias
Journal of the American Statistical Association, 1962

Different regression equations that seem to be unrelated and indivdually satisfy the classical OLS assumption, but are interdependent in the error term.

OLS unbiased and consistent but more efficient estimates by using FGLS to account for interdependence.

$$
\begin{aligned}
y_1 &= X_1\beta_1 + u_1 \\
y_2 &= X_2\beta_2 + u_2 \\
&\vdots \\
y_i &= X_i\beta_i + u_i \\
&\vdots \\
y_m &= X_m\beta_m + u_m
\end{aligned}
$$

- y_i is an $n \times 1$ vector of observations on variable i.
- X_i is an $n \times k_i$ matrix of observations on explanatory variables
- β_i is a $k_i \times 1$ vector of coefficients
- u_i is an $n \times 1$ vector of disturbances

$$
\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X_m \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}
$$

$$
y = X\beta + u
$$

- y is an $nm \times 1$ vector of observations on variable i.
- X is an $n \times \sum_i k_i$ matrix of observations on explanatory variables
- β is a $\sum_i k_i \times 1$ vector of coefficients
- u is an $nm \times 1$ vector of disturbances

$$E[uu\prime] = \begin{bmatrix} E(u_1 u_1\prime) & E(u_1 u_2\prime) & \dots & E(u_1 u_m\prime) \\ E(u_2 u_1\prime) & E(u_2 u_2\prime) & \dots & E(u_2 u_m\prime) \\ \dots & \dots & \ddots & \dots \\ E(u_m u_1\prime) & E(u_m u_2\prime) & \dots & E(u_m u_m\prime) \end{bmatrix}$$

$$E[uu\prime] = \begin{bmatrix} VCV_1 & E(u_1 u_2\prime) & \dots & E(u_1 u_m\prime) \\ E(u_2 u_1\prime) & VCV_2 & \dots & E(u_2 u_m\prime) \\ \dots & \dots & \ddots & \dots \\ E(u_m u_1\prime) & E(u_m u_2\prime) & \dots & VCV_m \end{bmatrix}$$

$$E[uu\prime] = \begin{bmatrix} \sigma_{11} I_N & \sigma_{12} I_N & \dots & \sigma_{1m} I_N \\ \sigma_{21} I_N & \sigma_{22} I_N & \dots & \sigma_{2m} I_N \\ \dots & \dots & \ddots & \dots \\ \sigma_{m1} I_N & \sigma_{m2} I_N & \dots & \sigma_{mm} I_N \end{bmatrix}$$

$$E[u_i u_j \prime] = \Sigma \otimes I_n$$

for a $m \times m$ symmetric matrix with positive entries Σ.

James G. Beierlein, James W. Dunn, James C. McConnon, Jr.
The Demand for Electricity and Natural Gas in the Northeastern United States
The Review of Economics and Statistics, 1981

Simultaneous estimation of the

1. residential gas
2. residential electric
3. commercial gas
4. commercial electric
5. industrial gas
6. industrial electric

demand in logarithmized quantities in the northern US. Both cross-sectional and time dimension for each equation. Estimation as a Two-Way Random Effects SUR model.

The individuals are nine states, the time is yearly from 1967-1977.

Explanatory variables are fuel prices , per capita income and disposable income, value of retail sales and value added by manufacturing.

No qualitative differences between OLS, two-way random effects and two-way random effects SUR, but the latter gives much lower standard errors.

Notation and assumptions

- set of M equations
  $y_i = X_j\beta_j + u_j \ (j = 1, \ldots, M)$
  y_j is $NT \times 1$, X_j is $NT \times k_j$, β_j is $k_j \times 1$
- additive error components
  $u_j = Z_\mu \mu_j + Z_\lambda \lambda_j + \nu_j$ two-way random effects
  with $Z_\mu = I_N \otimes e_t, \quad Z_\lambda = e_N \otimes I_T$
- μ_j, λ_j and ν_j are random vectors with expectation 0 and a variance-covariance matrix given by
  $$E \begin{pmatrix} \mu_j \\ \lambda_j \\ \nu_j \end{pmatrix} \begin{pmatrix} \mu_l & \lambda_l & \nu_l \end{pmatrix} = \begin{pmatrix} \sigma_{\mu_{jl}} I_N & 0 & 0 \\ 0 & \sigma_{\lambda_{jl}} I_T & 0 \\ 0 & 0 & \sigma_{\nu_{jl}} I_{NT} \end{pmatrix}$$

Combining all M equations

- Combining all M equations yields $y = X\beta + u$
- with $E(uu') = \Omega = [\Omega_{jl}]$ and $X = I_M \otimes Xj$

rewriting Ω_{jl}

- $\Omega_{jl} = E(u_j u_l') = \sigma^2_{\mu_{jl}} A + \sigma^2_{\lambda_{jl}} B + \sigma^2_{\nu_{jl}} I_{NT}$ with $A = I_N \otimes e_t e_t'$ and $B = e_N e_N' \otimes I_T$
- this can be rewritten as
  $\Omega_{jl} = \sigma^2_{3jl} \frac{J_{NT}}{NT} + \sigma^2_{1jl}(\frac{A}{T} - \frac{J_{NT}}{NT}) + \sigma^2_{2jl}(\frac{B}{N} - \frac{J_{NT}}{NT}) + \sigma^2_{\nu_{jl}} Q$

with

- $Q = I_{NT} - \frac{A}{T} - \frac{B}{N} + \frac{J_{NT}}{NT}$ and $J_{NT} = e_{NT} e_{NT}'$,
- $\sigma^2_{3jl} = \sigma^2_{\nu_{jl}} + N\sigma^2_{\lambda_{jl}} + T\sigma^2_{\mu_{jl}}$
- $\sigma^2_{1jl} = \sigma^2_{\nu_{jl}} + T\sigma^2_{\mu_{jl}}$
- $\sigma^2_{2jl} = \sigma^2_{\nu_{jl}} + N\sigma^2_{\lambda_{jl}}$

Eigenvalues of Ω

- it can be shown that (Nerlove(1971)) σ^2_{1jl}, σ^2_{2jl}, σ^2_{3jl} and $\sigma^2_{\nu_{jl}}$ form the eigenvalues of Ω_{jl}.
- plugging them in yields the following expression for Ω
  $\Omega = \Omega_3 \otimes \frac{J_{NT}}{NT} + \Omega_1 \otimes (\frac{A}{T} - \frac{J_{NT}}{NT}) + \Omega_2 \otimes (\frac{B}{N} - \frac{J_{NT}}{NT}) + \Omega_\nu Q$
- where $\Omega_3 = [\sigma^2_{3jl}], \Omega_2 = [\sigma^2_{2jl}], \Omega_1 = [\sigma^2_{1jl}]$ and $\Omega_\nu = [\sigma^2_{\nu_{jl}}]$

in search of Ω^{-1}

- the expression for Ω is of the structure $\Omega = \sum_{i=1}^{k} \Omega_i \otimes D_i$ where Ω_i are nonsingular matrizes and D_i are symmetric, sum up to I_n and are mutual orthogonal (i.e. $D_i D_j = 0$ for $i \neq j$ and idempotent.
- it can be shown that for a matrix Ω of this structure the unique inverse of the following form can be obtained: $\Omega^{-1} = \sum_{i=1}^{r} \Omega_i^{-1} \otimes D_i$
- closer inspection of the structure of Ω as derived above shows that indeed Ω is of the form: $\Omega = \sum_{i=1}^{k} \Omega_i \otimes D_i$ with $D_1 = (\frac{A}{T} - \frac{J_{NT}}{NT})$,$D_2 = (\frac{B}{N} - \frac{J_{NT}}{NT})$, $D_3 = \frac{J_{NT}}{NT}$ and $D_4 = Q$

the GLS-Estimator for β

- therefore Ω^{-1} is given by a weighted sum of four matrizes $\Omega^{-1} = \Omega_3^{-1} \otimes \frac{J_{NT}}{NT} + \Omega_1^{-1} \otimes (\frac{A}{T} - \frac{J_{NT}}{NT}) + \Omega_2^{-1} \otimes (\frac{B}{N} - \frac{J_{NT}}{NT}) + \Omega_\nu^{-1} \otimes Q$
- note that $X'(\Omega_3^{-1} \otimes (J_{NT}/NT))X = 0$ as $J_{NT}X_j = 0$
- as usual the GLS estimator is then given by: $\hat{\beta}_{GLS} = (X'\Omega^{-1}X)^{-1})X'\Omega^{-1}Y$
- note also that for $\sigma^2_{\mu jl} = \sigma^2_{\lambda jl} = 0$, $\Omega_1 = \Omega_2 = \Omega_\nu$ which results in $\Omega^{-1} = \Omega_\nu^{-1} \otimes I_{NT} \Rightarrow$ standard SUR estimator without a two-way random effects error specification.
- but unlike the standard SUR estimator, even when all matrizes Xj contain the same information, i.e. $X_1 = X_2 = ...X_M = \hat{X}$ performing GLS on each system seperately is not equivalent to GLS perform on the whole system.

References

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Econometrica, 1977

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Econometrica, 1980

Marc Nerlove
A Note on Error Components Models
Econometrica, 1971

Ingmar R. Prucha
On the Asymptotic Efficiency of Feasible Aitken Estimators for Seemingly Unrelated Regression Models with Error Components
Econometrica, 1984

What estimators of the variance-covariance-matrix are good asymptotically?

Probabilistic limits are with respect to both N and T.

Theorem: Let $\hat{\beta}_{\mathrm{FGLS}}$ be a fesible GLS-estimator satisfying

(i) $\operatorname{plim} \hat{\Omega}_\nu = \Omega_\nu$,

(ii) $\operatorname{plim} \hat{\Omega}_\mu = \Omega_\mu^*$ and

(iii) $\operatorname{plim} \hat{\Omega}_\lambda = \Omega_\lambda^*$,

where Ω_μ^* and Ω_λ^* are *any* finite, positive definite matrices. Then $\operatorname{plim} \sqrt{NT}(\hat{\beta}_{\mathrm{GLS}} - \hat{\beta}_{\mathrm{FGLS}}) = 0$.

A sufficient condition for $\text{plim}\, \sqrt{NT}\big(\hat{\beta}_{\text{FGLS}} - \hat{\beta}_{\text{GLS}}\big) = 0$ is that

(i) $\text{plim}\, X'(\hat{\Omega}^{-1} - \Omega^{-1})X/NT = 0$ and

(ii) $\text{plim}\, X'(\hat{\Omega}^{-1} - \Omega^{-1})u\sqrt{NT} = 0$.

Using Baltagis formula for the inverse of Ω, this can be rewritten as

(i) $\text{plim}(\hat{\sigma}_1^{jl} - \sigma_1^{jl})\frac{X_j'(A/T - J/NT)X_l}{NT} + (\hat{\sigma}_2^{jl} - \sigma_2^{jl})\frac{X_j'(B/N - J/NT)X_l}{NT} + (\hat{\sigma}_\nu^{jl} - \sigma_\nu^{jl})\frac{X_j Q X_l}{\sqrt{NT}} = 0$ for $j, l = 1, \ldots, M$.

(ii) $\text{plim} \sum_l T^{3/4}(\hat{\sigma}_1^{jl} - \sigma_1^{jl})\frac{X_j'(A/T - J/NT)u_l}{\sqrt{NT^{5/2}}} + N^{3/4}(\hat{\sigma}_2^{jl} - \sigma_2^{jl})\frac{X_j'(B/N - J/NT)u_l}{\sqrt{N^{5/2}T}} + (\hat{\sigma}_\nu^{jl} - \sigma_\nu^{jl})\frac{X_j Q u_l}{\sqrt{NT}} = 0$ for $j = 1, \ldots, M$.

By assumption

$$X_j' X_l / NT$$

and

$$X_j' Q X_l / NT$$

converge, so

$$X_j'(A/T - J/NT) X_l / NT$$

and

$$X_j'(B/T - J/NT) X_l / NT$$

converge too.

Write our first condition as

(i) $\text{plim}(\hat{\sigma}_1^{jl} - \sigma_1^{jl})[\ ] + (\hat{\sigma}_2^{jl} - \sigma_2^{jl})[\ ] + (\hat{\sigma}_\nu^{jl} - \sigma_\nu^{jl})[\ ] = 0$ for $j, l = 1, \dots, M$.

Now plim $\frac{X_j'(A/T - J/NT)u_l}{\sqrt{NT^{5/2}}} = \text{plim}\ \frac{X_j'(B/N - J/NT)u_l}{\sqrt{N^{5/2}T}} = 0.$

$X_j' Q u_l / \sqrt{NT} = X_j' Q v_l / \sqrt{NT}$ can be shown to converge in distribution to a random variable by a CLT.

Write our second condition as

(ii) plim $\sum_l T^{3/4}(\hat{\sigma}_1^{jl} - \sigma_1^{jl})0 + N^{3/4}(\hat{\sigma}_2^{jl} - \sigma_2^{jl})0 + (\hat{\sigma}_\nu^{jl} - \sigma_\nu^{jl})[\tilde{]} = 0$ for $j = 1, \ldots, M$.

We only have to show that

(i) $\mathrm{plim}(\hat{\sigma}_1^{jl} - \sigma_1^{jl})[\,] + (\hat{\sigma}_2^{jl} - \sigma_2^{jl})[\,] + (\hat{\sigma}_\nu^{jl} - \sigma_\nu^{jl})[\,] = 0$ for $j, l = 1, \ldots, M$.

(ii) $\mathrm{plim} \sum_l T^{3/4}(\hat{\sigma}_1^{jl} - \sigma_1^{jl})0 + N^{3/4}(\hat{\sigma}_2^{jl} - \sigma_2^{jl})0 + (\hat{\sigma}_\nu^{jl} - \sigma_\nu^{jl})[\tilde{]} = 0$ for $j = 1, \ldots, M$.

That's all Folks!