A Class of Nonlinear Stochastic Volatility Models

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Introduction

- Stochastic volatility (SV) models have become important tools for modelling the volatility of financial time series
- Prices of options based on SV models are shown to be more accurate than those based on the Black-Scholes model
- SV models offer an alterative to GARCH-type models to explain time varying volatility
- Empirical successes of lognormal SV model relative to GARCH-type models

→ Lognormal SV model is the most widely used SV model
(pricing stock and currency options)

- However, there is a lack of procedure for selecting an appropriate functional form of stochastic volatility → Possibility of Mispricing options and varying marginal distribution of volatility

- Nonlinear SV model:
  - It includes as special cases many SV models, hence, it is easy to test different specifications on SV
  - In fact, specification test is based on a single parameter
  - Furthermore, this way of modelling SV induces marginal normality of volatility
  - Smoother volatility series
  - Most importantly, an application of nonlinear SV model to option pricing shows that lognormal SV model overprices currency options

A Class of Nonlinear SV Models

- The lognormal SV model:

$$X_t = \sigma_t e_t,$$

$$\ln \sigma_t^2 = \mu + \phi(\ln \sigma_{t-1}^2 - \mu) + \sigma v_t$$

  - X(t) – continuously compounded return
  - e(t), v(t) are two sequences of i.i.d N(0,1) random variables with $corr(e_t, v_{t+1}) = \rho$

- Equivalent representation of this model:

$$X_t = \exp(\frac{1}{2}h_t)e_t,$$

$$h_t = \mu + \phi(h_{t-1} - \mu) + \sigma v_t \qquad \text{where } h_t = \ln \sigma_t^2$$

- The lognormal SV model specifies that the logarithmic volatility follows an AR(1) process → However, the data do not always support this relationship
- A natural generalization to this problem is to allow a general (nonlinear) smooth function of volatility to follow an AR(1) process →

$$
\begin{aligned}
X_t &= \sigma_t e_t, \\
h(\sigma_t^2, \delta) &= \mu + \phi[h(\sigma_{t-1}^2, \delta) - \mu] + \sigma v_t
\end{aligned}
$$

  - e(t), v(t) with same properties as on the previous slide
  - h(. , δ) is a smooth function indexed by a parameter δ and is a general nonlinear function (Box-Cox power function):

$$
h(t, \delta) = \begin{cases} (t^\delta - 1)/\delta, & \text{if } \delta \neq 0, \\ \ln t, & \text{if } \delta = 0. \end{cases}
$$

- Some features of this nonlinear SV model (N-SV):
  - Includes other SV models
  - Adds great flexibility to the functional form
  - Allows a simple test for the lognormal SV specification (e.g. H_0: δ = 0 or tests on other SV specifications)

- Let $h_t = h(\sigma_t^2, \delta)$ and re-write the N-SV model:

$$
\begin{aligned}
X_t &= [g(h_t, \delta)]^{1/2} e_t, \\
h_t &= \mu + \phi(h_{t-1} - \mu) + \sigma v_t
\end{aligned}
$$

where

$$
g(h_t, \delta) = \begin{cases} (1 + \delta h_t)^{1/\delta}, & \text{if } \delta \neq 0, \\ \exp(h_t), & \text{if } \delta = 0. \end{cases}
$$

The N-SV model includes the following SV models

	Models	δ	μ	ϕ
Wiggins (1987) Scott (1987) Chesney and Scott (1989) Taylor (1994) Jacquier, Polson and Rossi (1994) Harvey, Ruiz and Shephard (1994) Kim, Shephard and Chib (1998)	$\ln \sigma_t^2 = \mu + \phi(\ln \sigma_{t-1}^2 - \mu) + \sigma v_t$	0		
Scott (1987) Stein and Stein (1991) Andersen (1994)	$\sigma_t = \mu + \phi(\sigma_{t-1} - \mu) + \sigma v_t$	0.5		
Heston (1993)	$\sigma_t = \phi \sigma_{t-1} + \sigma v_t$	0.5	0	
Hull and White (1987) Johnson and Shanno (1987)	$\ln \sigma_t^2 = \mu + \ln \sigma_{t-1}^2 + \sigma v_t$	0		1
Andersen (1994)	$\sigma_t^2 = \mu + \phi(\sigma_{t-1}^2 - \mu) + \sigma v_t$	1		
Clark (1973)	$\ln \sigma_t^2 = \mu + \sigma v_t$	0		0
Nonlinear SV	$\frac{(\sigma_t^2)^\delta - 1}{\delta} = \mu + \phi[\frac{(\sigma_{t-1}^2)^\delta - 1}{\delta} - \mu] + \sigma v_t$			

- Some basic properties of the N-SV model:
  - h(t) is stationary and ergodic if $\phi < 1$ and that if so

$$\mu_h \equiv E(h_t) = \mu,\ \sigma_h^2 \equiv Var(h_t) = \frac{\sigma^2}{1-\phi},\ \text{ and }\ \rho(\ell) \equiv Corr(h_t, h_{t-\ell}) = \phi^\ell$$

  - Some properties make GMM and QML difficult to implement for N-SV
    - Almost impossible to obtain analytical form for the moments of the model
    - Non-linearizable mean equation

- Interpretation of δ:
  - Define m = 1/δ and re-write the inverse Box-Cox transformation yields

$$\sigma_t^2 = (1 + \frac{h_t}{m})^m = \prod_{i=1}^{m} (1 + h_{it})$$

  - Let {h(it)} a sequence of intra-day volatility movement → it is primarily caused by the arrival of new information → m is the average daily information arrival and h(t) represents the average impact of the information on volatility

Implications on Option Pricing

- Pricing of European call options by approximation via MC simulations
- Based on Black-Scholes price integrated over the distribution of the mean volatility:

$$C = e^{-\tau r_d} \int_0^\infty \mathrm{BS}(w_\tau) pdf(w_\tau | h_0) dw_\tau$$

$$w_\tau^2 = \int_0^\tau g(h_s, \delta) ds$$

$$\mathrm{BS}(w_\tau) = F_0 \mathrm{N}(d_1) - X\mathrm{N}(d_2)$$

$$F_0 = S_0 e^{(r_d - r_f)\tau}$$

$$d_1 = \frac{\ln(F_0/X) + w_\tau^2}{w_\tau}$$

$$d_2 = d_1 - w_\tau$$

- Data set:
  - daily Dollar/Pound, Dollar/CD, Dollar/FF, Dollar/GM, Dollar/JY exchange rates
  - Sample period: from January 1, 1986 to December 31, 1998

Table 3: Empirical Results for dollar/pound Exchange Rate

	N-SV					Lognormal SV			
	Mean	SD	90% CI	MC SE	IACT	Mean	SD	MC SE	IACT
ϕ	.9595	.0101	(.9417, .9745)	.00017	138.3	.9676	..0091	.00026	408.2
σ	.2066	.0269	(.1672, .2543)	.00050	174.9	.1873	..0268	.00090	568.1
μ	-.2244	.1044	(-.3913, -.0495)	.00087	35.0	-.2579	..1095	.00103	44.1
δ	.1716	.1203	(.0039, .3684)	.00214	189.0				
Loglik	-4369.792					-4371.606			
LR Stat	3.628								
p-Val	0.0568								

Table 4: Comparison of Call Option Prices on Currency Based on Lognormal SV and N-SV Models; Option Parameters: $\tau = 126$ Days, $S_0 = 1.5$, $r_d = 0$, $r_f = 0$, $\sigma_0 = 0.006349$ Per Day

	Lognormal SV	N- SV	Percentage
S_0/X	Option Price	Option Price	Difference
0.75	2.401e-5	1.172e-5	-104.86
0.8	1.511e-4	1.032e-4	-46.41
0.85	8.645e-4	7.231e-4	-19.55
0.9	0.00415	0.00386	-7.513
0.95	0.01548	0.01507	-2.721
1	0.04257	0.04213	-1.044
1.05	0.08701	0.08661	-0.462
1.1	0.1413	0.1410	-0.213
1.15	0.1971	0.1969	-0.102
1.2	0.2504	0.2503	-0.040
1.25	0.3001	0.3001	0.000

Note: In all cases, the parameter estimates in Table 3 are used.

Table 5: Empirical Results for Other Exchange Rates

		CD	FF	GM	JY
Log-Normal SV	ϕ	.9575 (.0094) [.9372, .9741]	.9618 (.0099) [.939, .9781]	.9617 (.0099) [.9395, .9733]	.8654 (.0278) [.8158, .9069]
	σ	.233 (.0248) [.1864, .284]	.1841 (.0244) [.1417, .2361]	.1772 (.0231) [.1383, .2285]	.4284 (.0515) [.3473, .5169]
	μ	-.3024 (.1027) [-.5016, -.0946]	-.2293 (.0918) [-.4096, -.0476]	-.2219 (.0883) [-.3956, -.0464]	-.378 (0.0654) [-.4842, -.2695]
	Loglik	-4316.61	-4434.54	-4440.67	-4330.89
N-SV	ϕ	.9579 (.0092) [.9414, .9716]	.9538 (.0106) [.9346, .9691]	.9585 (.0105) [.9395, .9733]	.8601 (.0285) [.8078, .9005]
	σ	.2363 (.0248) [.1989, .28]	.2053 (.0234) [.1715, .2472]	.1869 (.0235) [.1519, .2284]	.4380 (.0508) [.3647, .5293]
	μ	-.3308 (.1120) [-.516, -.1489]	-.2223 (.0934) [-.3769, -.0708]	-.2364 (.0945) [-.3924, -.0829]	-.3880 (0.0730) [-.5105, -.2704]
	δ	-.0486 (.0940) [-.2108, .1003]	.0441 (.1534) [-.2104, .2975]	-.0786 (.1668) [-.3514, .1988]	-.0158 (.0902) [-.1662, .1296]
	Loglik	-4316.44	-4434.52	-4440.37	-4330.66
LR Stat		0.3418	0.0524	0.6174	0.4654
p-Value		0.5588	0.8189	0.4320	0.4951

Note: The number in parentheses is the standard deviation while the number in brackets represents the 90% Bayesian confidence interval.

Conclusions and Extensions

- Dollar/Pound:
  - One has to reject all SV models → evidence of N-SV
  - Lognormal SV model tends to overprice the options
  - N-SV model tends to generate smoother volatility series
- Dollar/ . :
  - Indicating suitability of lognormal SV model
  - Marginal distribution of volatility well approximated by lognormal distribution
  - Empirical results are reliable

- Possible extensions:
  - Incorporate jumps and long memory volatility into the model