Geometric entanglement witnesses and bound entanglement

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Basics

We consider a Hilbert-Schmidt space $\mathcal{A}_A \otimes \mathcal{A}_B$ of operators on the finite dimensional bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$

Dimension: $d_A \times d_B,\ D := d_A d_B$

States ρ (density matrices) have the properties:

$$\rho^\dagger = \rho,\ \mathrm{Tr}\ \rho = 1 \text{ and } \rho \geq 0$$

A scalar product on $\mathcal{A}_A \otimes \mathcal{A}_B$ is defined by

$$\langle A, B\rangle = \mathrm{Tr}\, A^\dagger B \text{ with } A, B \in \mathcal{A}_A \otimes \mathcal{A}_B$$

Entanglement witnesses

A state ρ is entangled iff there exists a Hermitian operator A such that

$$\begin{aligned} \langle \rho, A\rangle &= \mathrm{Tr}\,\rho A < 0\,, \\ \langle \sigma, A\rangle &= \mathrm{Tr}\,\sigma A \geq 0 \qquad \forall \sigma \in S \end{aligned}$$

Idea: Convexity of the set of separable (not entangled) states

Optimal entanglement witness:

Exists separable state $\tilde{\sigma}$ such that $\mathrm{Tr}\,\tilde{\sigma} A = 0$

→ tangent hyperplanes to the set of separable states

PPT criterion

A separable state stays positive under partial transposition (PPT):

$$\sigma^{T_B} \geq 0$$

$$(\rho_{m\mu,n\nu})^{T_B} := \rho_{m\nu,n\mu}$$

→ States not positive under partial transposition (NPT) are entangled, entangled PPT states are bound entangled (not distillable)

Realignment criterion

Any separable state fulfills

$$\mathrm{Tr}\,\sqrt{\sigma_R^\dagger \sigma_R} \leq 1$$

where

$$(\rho_{m\mu,n\nu})_R := \rho_{mn,\mu\nu}$$

Magic simplex

Set of states that are mixtures of the Bell states P_{nm} (dimension d x d):

$$\mathcal{W} := \left\{ \sum_{n,m=0}^{d-1} q_{nm} P_{nm} \mid q_{nm} \geq 0,\ \sum_{n,m} q_{nm} = 1 \right\}$$

$$P_{nm} := (U_{nm} \otimes \mathbb{1}) |\phi_d^+\rangle\langle\phi_d^+| (U_{nm}^\dagger \otimes \mathbb{1})$$

Using the Weyl operators

$$U_{nm} = \sum_{k=0}^{d-1} e^{\frac{2\pi i}{d} kn} |k\rangle\langle (k+m) \bmod d|$$

Characterizing entanglement

Shift method

Construction of an operator that can detect NPT entangled, PPT (bound) entangled and separable states:

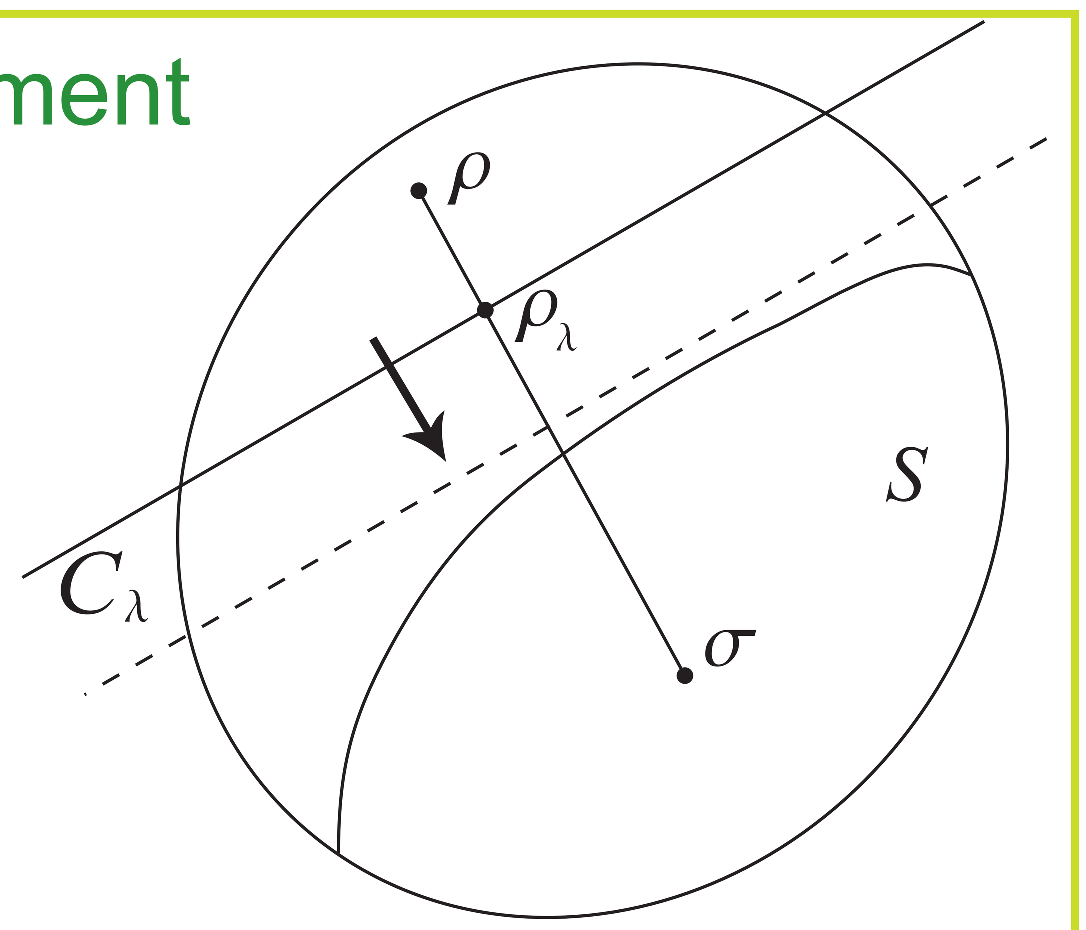
$$C_\lambda = \rho_\lambda - \rho - \langle \rho_\lambda, \rho_\lambda - \rho\rangle \mathbb{1}_{\mathrm{D}}$$

where

$$\rho_\lambda := \lambda\rho + (1-\lambda)\sigma\,, \quad 0 \leq \lambda \leq 1$$

$$\rho \notin S,\ \sigma \in S$$

If C_λ is an entanglement witness, then all states "left" of the hyperplane (including ρ) are entangled.



→ Detection of PPT entangled states (bound entangled states), e.g. if ρ is PPT

→ Constructing the shape of the separable states with C_λ as optimal entanglement witnesses

Application for a family of two-qutrit states:

$$\rho_{\alpha,\beta,\gamma} := \frac{1-\alpha-\beta-\gamma}{9}\mathbb{1} + \alpha P_{00} + \frac{\beta}{2}(P_{10}+P_{20}) + \frac{\gamma}{3}(P_{01}+P_{11}+P_{21})$$

Characterization of the entanglement of this three-parameter family of states (part of the magic simplex):

Difficulty: Proving that C_λ is an entanglement witness in the application of the shift method. For states of the magic simplex, a geometrically constructed operator can be written in terms of Weyl operators:

$$C = a\left((d-1)\,\mathbb{1}_{d^2} + \sum_{n,m=0}^{d-1} c_{nm} U_{nm} \otimes U_{-nm} \right)$$

Then for the EW-inequality we get

$$\langle \sigma, C\rangle = \mathrm{Tr}\sigma^\dagger C = \sum_k p_k \left((d-1)a \left(1 + \sum_{n,m} c_{nm} n_{nm}^{*k} m_{-nm}^{*k} \right)\right)$$

Since the separable states can be written as a Bloch vector of a convex combination of product states:

$$\sigma = \sum_k p_k \frac{1}{d^2}\Big(\mathbb{1}_{\mathrm{d}} \otimes \mathbb{1}_{\mathrm{d}} + \sum_{n,m=0}^{d-1} \sqrt{d-1}\, n_{nm}^k\, U_{nm} \otimes \mathbb{1}_{\mathrm{d}} + \sum_{l,k=0}^{d-1} \sqrt{d-1}\, m_{lk}^k\, \mathbb{1}_{\mathrm{d}} \otimes U_{lk}$$
$$+ \sum_{n,m,l,k=0}^{d_1-1} (d-1)\, n_{nm}^k m_{lk}^k U_{nm} \otimes U_{lk} \Big)\,,$$

$$n_{nm}^k, m_{lk}^k \in \mathbb{C}\,, \quad |\vec{n}^k| \leq 1\,, \quad |\vec{m}^k| \leq 1\,, \quad p_k \geq 0,\ \sum_k p_k = 1\,,$$

PPT and realignment provides all entanglement:

Fig. 1

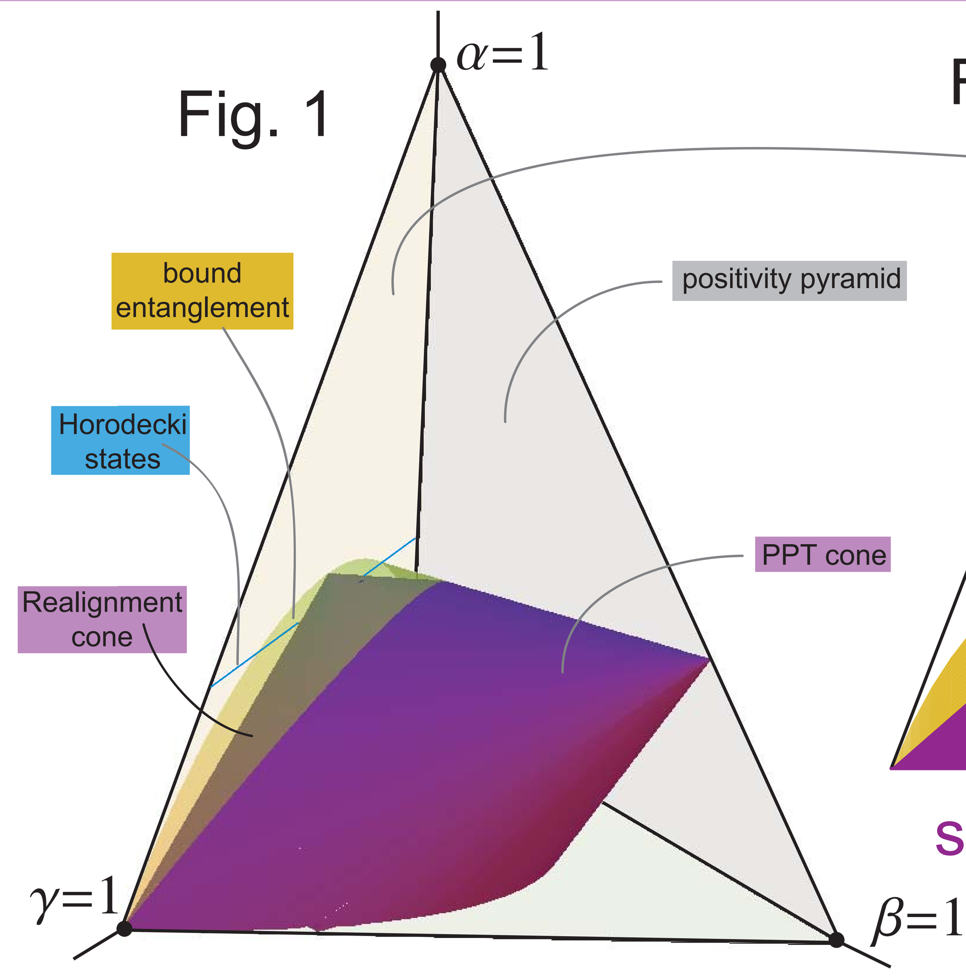
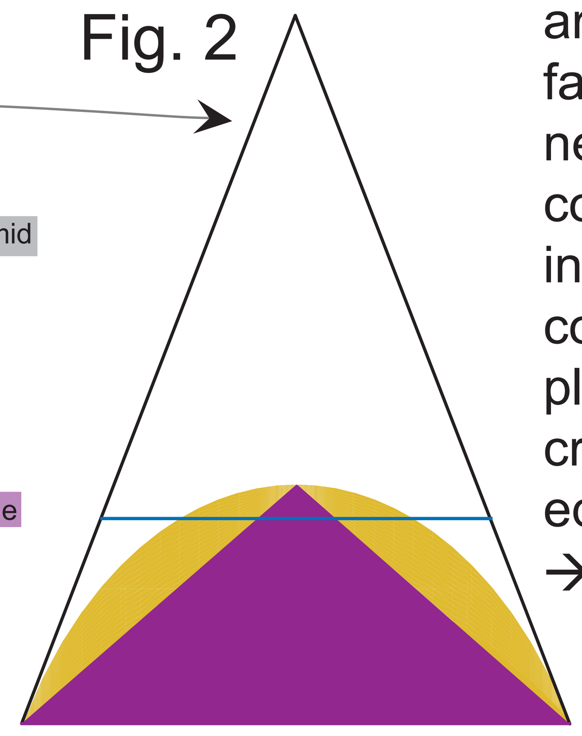


Fig. 2



separable states

→ Incuding Horodecki states $\rho_b = \frac{2}{7}|\phi_+^3\rangle\langle\phi_+^3| + \frac{b}{7}\sigma_+ + \frac{5-b}{7}\sigma_-$

So the EW-inequality is ≥ 0 iff

$$\sum_{n,m} c_{nm} n_{nm} m_{-nm} \geq -1 \quad \text{(prod. state criterion)}$$

for the Bloch vectors $\vec{n}$ and $\vec{m}$ of all *product* states.

Sufficient condition: $|c_{nm}| \leq 1\ \forall n, m$

→ Can detect most of bound entanglement (analytical) using the shift method with $\sigma = 1/9\,\mathbb{1}$. (see Fig. 3)

Verification that purple shape are separable states of the family (Fig. 1): Draw polygon of necessarily separably states connecting edge points (green in Fig. 3) , then shift operators corresponding to boundary planes outside until prod. state criterion is satisfied with equality (numerical)

→ Shifted planes corresponding to optimal entanglement witnesses, draw new polygon with new edge points, repeat shift of boundary planes

Fig. 3