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# Scalar Fields

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# Part I.

## Abstract

We will discuss in the following the properties of scalar fields. We will then proceed with the special case of the Klein-Gordon field and show some problems that occur if it is interpreted as a single-particle wave equation. To solve this problem we need to introduce the lagrangian formalism looked at in more detail in the following chapter and then we proceed with an exploration of the canonical quantization of the Klein-Gordon field. We will notice that the problems that occurred earlier vanish if we consider the quantized field. After this the path integral is introduced as another method for quantizing the field. Finally we will introduce some of the applications of quantized fields.

# Part II.

## Conventions and comments

We will use the following signature for the Minkowski metric:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (1)$$

Four vectors are indicated by a latin letter, e.g. the momentum 4-vector is given by  $p$ , while its components are indicated by greek indices  $\Rightarrow p^\mu$ . Consequently the greek indices run from zero to three while the latin indices run from one to three. To avoid confusion we will generally assume  $\phi$  to be a scalar field, while vector fields will be written as  $\Psi$ .

The (pseudo)scalar product of two 4-vectors will be written as

$$pp = p^2 = p^\mu p^\nu g_{\mu\nu} = p_\mu p^\mu \quad (2)$$

We will usually suppress the arguments of fields, such that

$$\phi := \phi(x), \quad \psi := \Psi(x) \quad (3)$$

Additionally we define

$$\partial_\mu \Psi^\mu := \frac{\partial}{\partial x^\mu} \Psi^\mu = \left( \frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla} \right) \Psi^\mu \quad (4)$$

$$\partial^\mu \Psi_\mu := \frac{\partial}{\partial x_\mu} \Psi_\mu = \left( \frac{1}{c} \frac{\partial}{\partial t}, -\vec{\nabla} \right) \Psi_\mu \quad (5)$$

Thereby the D'Alembert-Operator has the following form

$$\square \phi := \partial_\mu \partial^\mu \phi = \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \phi \quad (6)$$

If not stated otherwise the functions operate on  $\mathbb{R}^4$ . The vector  $r^\mu$  is defined as

$$r^\mu = x^\mu - x_0^\mu \quad (7)$$

and the scalar  $r$  is defined as

$$r = \sqrt{r_\mu r^\mu} \quad (8)$$

# Part III.

## Scalar fields

### 1. What is a scalar field?

Mathematically a scalar field is defined as a scalar function operating on a  $n$ -dimensional vector space:

$$\begin{aligned}\phi : \mathbb{R}^n &\rightarrow \mathbb{R} \\ x &\rightarrow \phi(x)\end{aligned}$$

Physically a scalar field can be interpreted as the information about a distribution in space, which is in general assumed to be time dependent. Common examples are the field of the temperature or diverse density fields, e.g. mass-density, charge-density. Especially the interpretation as a probability density field has many applications in modern physics and is one of the key stones in modern quantum mechanics. Later we will discuss the lagrangian density of a scalar field, which can be interpreted as a scalar field itself.

We will now discuss two important quantities that are related to scalar fields. The first one will appear multiple times in the later chapters.

### 2. Properties of the scalar field

#### 2.1. The gradient

In physics we are usually interested in the change of some values and therefore we often need to know how fast things vary. If we consider scalar functions of a single scalar argument this doesn't pose a problem. We simply take a look at the derivative of the function with respect to this argument. In the case of a field we have infinite answers to the question how fast the field varies, namely one for each direction. This means we have to change our question. In generally we end up with two questions that are frequently asked:

1. What is the maximum variation of our field and in which direction occurs this maximum change?
2. How much does the field vary if we move into a chosen direction?

The answer to the first question is given by a theorem of vector analysis, stating that the gradient

$$d\phi(x) := \sum_{i=1}^n \frac{\partial}{\partial x^i} \phi dx_i \tag{9}$$

is a vector in the direction of the maximum increase of  $\phi$  at the point  $x$  and which norm gives the magnitude of the variation. This essentially reduces our problem of testing an

infinite amount of directions to "testing" only  $n$  directions (and as long as we assume our vector space to be finite this solves our problem).

The answer to the second question is actually already given, if we take a look at the first question. Knowing the direction in which the field increases the most, we can simply project the variation in any direction we like, giving the change of the function in towards the chosen direction. This is in fact the reason why we didn't have to ask "In which direction doesn't the field vary?", since in each direction perpendicular to the gradient the field is stable, which again implies that in a case in which the gradient diverges at some point, the system has an instability at this given point.

## 2.2. Equipotential surfaces

Another thing that's often of great value in physics are the so-called equipotential surfaces. In the case, that a vector field is devined as the gradient of a scalar field, the scalar field is usually called the potential of that vector field. The equipotential surfaces are then defined as surfaces for which the potential has a constant value. An important application to this is electrodynamics. Mathmatically they are given by

$$\phi(x) = \text{const.} \tag{10}$$

As already mentioned the gradient has to be perpendicular to any equipotential surface. One should notice that there has neither necessarily to exist a equipotential surface to every value nor has every point in space to lie on some equipotential surface. An example would be the potential of an point charge which is given by

$$\phi(x) \propto q \frac{1}{r}. \tag{11}$$

At  $\vec{0}$  the equipotential surface "reduces" to one point in space and of course there exist cases in which exist equipotential lines instead of equipotential surfaces.

## Part IV.

# Klein-Gordon equation as a single-particle wave equation

The Schrödinger equation can be motivated by taking a look at the classical relation between energy and momentum of a particle. Quantization is done by replacing the physical quantities by operators corresponding to them and a state or wave function on which they operate. This corresponding operators for the energy and momentum are given by

$$\vec{p} \rightarrow -i\hbar\vec{\nabla}, \quad E \rightarrow i\hbar\frac{\partial}{\partial t} \quad (12)$$

Assuming the case of a free particle we get the following relation between momentum and energy:

$$\frac{p^2}{2m} = E \quad (13)$$

Substituting the operators yields

$$i\hbar\frac{\partial}{\partial t}\phi = -\frac{\hbar^2\vec{\nabla}^2}{2m}\phi \quad (14)$$

This is the Schrödinger equation for a free particle. We could now assume that we can obtain an relativistic version of the Schrödinger equation by simply repeating the same procedure with the relativistic correlation between momentum and energy:

$$p^2 = \frac{E^2}{c^2} - \vec{p}^2 = (mc)^2 \quad (15)$$

If we simply substitute the operators we obtain

$$-\hbar^2\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi + \hbar^2\nabla^2\phi = (mc)^2\phi \quad (16)$$

which can be rewritten to

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi + \frac{mc^2}{\hbar}\phi = 0 \Leftrightarrow (\square + \frac{mc^2}{\hbar})\phi = 0 \quad (17)$$

Setting  $\hbar = c = 1$  this leaves us with the simple and beautiful so called Klein-Gordon equation:

$$(\square + m^2)\phi = 0 \quad (18)$$

The Schrödinger equation could therefore be interpreted as the newtonian limit of the Klein-Gordon equation as a single-particle wave equation, and it is said that Schrödinger himself had found this equation earlier, before he had discovered the equation that bears

his name. However, assuming it describes a free particle, we run into problems. The first one is, that the equation for the energy ((15)) allows negativ energy states.

$$E = \pm \sqrt{(mc)^2 + \vec{p}^2} \quad (19)$$

For a non interacting particle we could simple choose the energy to be positive, which would solve the problem by definition. But since we also wish to consider interaction, the problem is still relevant. Nothing prohibits the particle from emitting an infinite amount of energy to the system with which it is interacting. This, of course, is not what we observe in experimets.

Another problem occurs if we look for an interpretation of the Klein-Gordon field. If we interpret it similar to the Schrödinger equation, there should exist a propability density just like the one for the Schrödinger equation

$$\rho = \phi^* \phi \quad (20)$$

and a corresponding probability current

$$\vec{j} = -i \frac{\hbar}{2m} (\phi^* \vec{\nabla} \phi - \phi \vec{\nabla} \phi^*). \quad (21)$$

These two are linked via the continuity equation

$$\frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot \vec{j} = 0 \quad (22)$$

If we search for a probability density for the Klein-Gorden equation, we should demand that it transforms like the time component of a relativistic 4-vector, such that we could rewrite the density and the current as a 4-vector. As an ansatz we choose ((21)) as the spatial part of the vector. This can motivated by assuming that in the Newtonian limit the Klein-Gordon field should obey this equation. Later we will show that the probability density for the Klein-Gordon equation has the form of

$$\rho = i \frac{\hbar}{2m} \left( \phi^* \frac{\partial}{\partial t} \phi - \phi \frac{\partial}{\partial t} \phi^* \right) \quad (23)$$

For both together the following continuity equation

$$\partial_\mu j^\mu = \partial_\mu i \frac{\hbar}{2m} (\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) = 0 \quad (24)$$

can be found.

At first glance this looks promising, but the probability density ((23)) is not positive definite. As we know for a second order ODE there are two arbitrarily conditions necessary to fix it. This allows us to chose  $\phi$  and  $\frac{\partial}{\partial t} \phi$  independently for a given point in time, which allows  $\rho$  to assume negative values. Generally, this wouldn't mean a problem. However, if we want to keep our interpretation as a generalization of the Schrödinger equation, we run into the problem of a negative probability density, which is surely

nothing physical. Altering the equation for the probability density will however lead to problems with identification of the Klein-Gordon equation as the relativistic version of the Schrödinger equation.

We will show later that the Klein-Gordon equation is of use and that the given equations regarding probability density and current are the correct ones. But therefore we need some additional tools which will be derived in the next chapter.

## 1. Application in Physics

We will discuss in chapter (VII) how to quantize the Klein-Gordon field, which means that the field is handled as an operator acting on states. The current we derived will be interpreted as an electric current of the particles described by the quantum field and the density will be the electric charge density of spin 0 particles, which of course can be negative.

# Part V.

## Lagrangian formulation

In classical mechanics the Lagrange formalism with which one can derive the equations of motion is well known. The so called Euler-Lagrange equations can be derived via the principle of the least action, where the latter is defined as

$$S = \int \mathcal{L} dt. \quad (25)$$

$\mathcal{L}$  is called the Lagrange function of the system. We will generalize this concept to derive field equations through a Lagrange density and the principle of extremal action.

### 1. Classical Lagrange formalism

In classical mechanics we wish to derive the trajectory of a particle  $x(t)$ . The principle of the extremal action now states, that, given fixed starting and end points, the trajectory chosen by the particle will be the path for which the action is extremal.

We therefore set the starting point as  $x(t_0)$  and the ending point as  $x(t_1)$ . We will now define a small perturbation of a given trajectory by

$$x(t) \rightarrow x'(t) = x(t) + \lambda a(t), \quad (26)$$

where  $\lambda$  is very small compared to  $x$  and the perturbation  $a(t)$  is zero at the starting and the ending point. If we insert this into the action with the Lagrangian given in the form of

$$\mathcal{L} = T - V, \quad (27)$$

where  $T$  is the kinetic and  $V$  the potential energy, we see that the action is changed to

$$\begin{aligned} S \rightarrow S' &= \int_{t_0}^{t_1} \left( \frac{m}{2} (\dot{x} - \dot{a})^2 - V(x + a) \right) dt \\ &= \int_{t_0}^{t_1} \left[ \frac{1}{2} m \dot{x}^2 + m \dot{x} \dot{a} - (V(x) + aV'(x)) \right] dt + (a^2) \\ &= S + \int_{t_0}^{t_1} [m \dot{x} \dot{a} - aV'(x)] dt \\ &= S + \delta S \end{aligned}$$

by

$$\delta S = \int_{t_0}^{t_1} [m \dot{x} \dot{a} - aV'(x)] dt \quad (28)$$

where  $V$  has been expanded into a Taylor series. We conclude that the action is extremal if and only if  $\delta S = 0$  under a variation of  $x$ . This is clearly the case when the integrand

is equal to zero. If we partially integrate  $a(\dot{t})$  in the first term and since  $a(t)$  is arbitrary we see that the integrand is zero, if and only if

$$m\ddot{x} + V'(x) = 0 \tag{29}$$

which represents the Newtonian equations of motion. Rewritten this becomes the standard form of the Euler Lagrange equations (pulling out one time derivative and integrating with respect to  $\dot{x}$ ):

$$\frac{d}{dt} \frac{\partial}{\partial \dot{x}} \frac{m\dot{x}^2}{2} + \frac{\partial V}{\partial x} = 0 \Leftrightarrow \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0 \tag{30}$$

## 2. Lagrangian formulation of a scalar field

The next step is to generalize the formulation for scalar fields. Since in general a field is dependent on time and space, the formulation should represent this too. Therefore we need to substitute

$$x \rightarrow \phi \qquad t \rightarrow x^\mu \tag{31}$$

This can easily be achieved if we assume a function  $\mathcal{L}$  which is defined as follows

$$\mathcal{L} = \int \mathcal{L}(\phi, \partial_\mu \phi) d^3x. \tag{32}$$

Inserting this into the action yields

$$S = \int \mathcal{L}(\phi, \partial_\mu \phi) d^4x. \tag{33}$$

This is exactly the kind of generalization for the action we were looking for. The function  $\mathcal{L}$  is called the Lagrangian density, which is usually shortened to Lagrangian. However, if we want to keep the principle of extremal action to receive field equations, analogous to the equations of motion, we have to repeat the variation procedure. Consequently there are a few additional generalizations we have to make. Instead of a one dimensional line, like a trajectory, the field spans a 4-dimensional region  $R$  in a 5-dimensional space time. Therefore the starting and ending points of the trajectory are replaced by spacelike hypersurfaces which form parts of the boundary  $\partial R$  of the 4-dimensional region. Intuitively this is clear, because we are no longer varying one dimension, where the boundary equals 2 points, but 4 dimensions, with consequently 2 3-dimensional surfaces as boundary.

Additionally we have to regard not only the variation of the field if the point in space time at which it is evaluated does vary, but additionally the functional variation of the field at each point in space time itself. This can be expressed as

$$\begin{aligned} x^\mu &\rightarrow x'^\mu = x^\mu + \delta x^\mu \\ \phi(x) &\rightarrow \phi'(x) = \phi(x) + \delta\phi(x) \end{aligned}$$

It is appropriate to let  $\mathcal{L}$  depend explicitly on  $\phi, \partial\phi$  and  $x^\mu$ , such that we consider interaction of  $\phi$  with external sources. As already mentioned we have to regard the change of  $\phi$  itself under variation and in addition the variation of  $\phi$  if we let  $x^\mu$  vary. Therefore we can describe the two variations as

$$\begin{aligned}\phi'(x') - \phi(x') &= \delta\phi \\ \phi(x') - \phi(x) &= (\partial_\mu\phi)\delta x^\mu\end{aligned}$$

The total change of  $\phi$  can then be written as

$$\phi'(x') = \phi(x) + \Delta\phi = \phi + \delta\phi + (\partial_\mu\phi)\delta x^\mu \quad (34)$$

Getting back to our problem, we can write the change of the action as

$$\delta S = \int \mathcal{L}(\phi', \partial_\mu\phi', x'^\mu) d^4x' - \int \mathcal{L}(\phi, \partial_\mu\phi, x^\mu) d^4x. \quad (35)$$

where  $d^4x' = J(x'/x)d^4x$  with the Jacobian of the transformation  $x \rightarrow x'$ . Since

$$J\left(\frac{x'}{x}\right) = \det\left(\frac{\partial x'^\mu}{\partial x^\lambda}\right) \quad (36)$$

and is

$$\frac{\partial x'^\mu}{\partial x^\lambda} = \delta_\nu^\mu + \partial_\lambda\delta x^\mu \quad (37)$$

as follows straight from the transformation. The variation of the Lagrangian is given as

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu. \quad (38)$$

Therefore the variation of the action becomes

$$\delta S = \int (\delta\mathcal{L} + \mathcal{L}\partial_\mu\delta x^\mu) d^4x \quad (39)$$

where we disregarded terms with variation in  $\mathcal{L}$  and  $x$ . Taking a look at the transformation  $\phi(x) \rightarrow \phi'(x)$  (and additionally from the fact, that  $\delta$  is a functional operator) it follows, that  $\delta(\partial_\mu\phi) = \partial_\mu\delta\phi$ . This leads to

$$\delta S = \int \left[ \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) + \partial_\mu(\mathcal{L}\delta x^\mu) \right] d^4x. \quad (40)$$

Since the last term is a total divergence we can transform this part into a surface integral via Stokes theorem. Since we demand that the variation of  $\phi$  and  $x$  has to vanish at the boundary, it follows that the term has to vanish. Since

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) = \partial\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) \quad (41)$$

we can replace the second term with a correction term plus a total divergence, which again vanishes by the same argument as before. This leaves us with

$$\delta S = \int_R \left[ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \right] \delta \phi d^4x + \underbrace{\int_{\partial R} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi + \mathcal{L} \delta x^\mu \right) d\sigma_\mu}_0 \quad (42)$$

which again implies that the action is extremal if the integrand vanishes and therefore

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right). \quad (43)$$

This are the Euler-Lagrange equations for the scalar field. It is actually straight forward to show, that they look the same for vector fields.

### 3. The lagrangian of the Klein-Gordon equation

Since we know the form of the Klein-Gordon equation, we can easily derive an associated lagrangian:

$$\mathcal{L} = \frac{1}{2} \left( (\partial_\mu \phi)(\partial^\mu \phi) - m^2 \phi^2 \right) \quad (44)$$

# Part VI.

## Noether's Theorem

### 1. Noether's Theorem in classical mechanics

The main conclusion from Noether's theorem is, that the invariance of the action under a variation of a parameter is equivalent to a conserved quantity. In classical mechanics we can for instance relate the rotation of the coordinates with the conservation of angular momentum, a translation in time with the conservation of energy. One can assume that equivalent statements are true for scalar fields, and we need to be able to recover the classical conserved quantities if we let the scalar field be a projector onto a one dimensional subspace, i.e.

$$\phi(x) = x^0, \dots, x^3 \quad (45)$$

However we can additionally demand the invariance under a functional variation of the field.

### 2. Noether's Theorem in field theory

Since we want to check if the action is invariant under certain transformations of  $x$  and  $\phi$ , it would be senseless to demand, that the variation vanishes on the boundaries of the regarded area of space time. Therefore the divergence terms, we could neglect before, have to be considered now. These terms can be written as

$$\begin{aligned} \delta S_d &= \int_{\partial R} \left( \mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \phi \right) d\sigma_\mu = \\ &= \int_{\partial R} \left[ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} [\delta \phi + (\partial_\nu \phi) \delta x^\nu] - \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta_\nu^\mu \mathcal{L} \right) \delta x^\nu \right] d\sigma_\mu, \end{aligned}$$

where the same term has been added and subtracted. The first two terms contain the total change of  $\phi$  which we called  $\Delta\phi$  in the previous part. The second two terms contain

$$\theta_\nu^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta_\nu^\mu \mathcal{L} \quad (46)$$

the so called energy-momentum tensor  $\theta_\nu^\mu$ . This allows us to rewrite  $\delta S_d$  as

$$\delta S_d = \int_{\partial R} \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - \theta_\nu^\mu \delta x^\nu \right) d\sigma_\mu. \quad (47)$$

We now reparametrise the variations with an infinitesimal parameter  $\delta\omega^\nu$  in such a way, that the infinitesimal transformations can be written as

$$\Delta x^\mu = X_\nu^\mu \delta\omega^\nu, \quad \Delta\phi = \Phi_\nu \delta\omega^\nu \quad (48)$$

where  $\nu$  is considered to be a single index<sup>1</sup>.

If we assume that the transformation is some kind of symmetry than the variation vanishes, which means that

$$\delta S_d = \int_{\partial R} \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi_\nu - \theta_\tau^\mu X_\nu^\tau \right) \delta \omega^\nu d\sigma_\mu = 0. \quad (49)$$

Since  $\delta \omega^\nu$  is arbitrary this means that

$$\int_{\partial R} \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi_\nu - \theta_\tau^\mu X_\nu^\tau \right) d\sigma_\mu = \int_{\partial R} J_\nu^\mu d\sigma_\mu = 0. \quad (50)$$

Considering Stokes theory we can change this back into an space-time integral, which makes it clear to see that the 4-gradient of  $J_\nu^\mu$  has to vanish, since the region of space-time can be chosen arbitrary. This means that

$$\partial_\mu J_\nu^\mu = 0, \quad (51)$$

and that we've found a "conserved" current, conserved in the sense that it fulfils a continuity equation.

We will now derive a conserved charge from it. Therefore we consider integrate the current over a spacelike hypersurface  $\sigma_\mu$ :

$$Q_\nu = \int_\sigma J_\nu^\mu d\sigma_\mu. \quad (52)$$

In the special case  $t = \text{const}$  this yields

$$Q_\nu = \int_V J_\nu^0 d^3x. \quad (53)$$

Using that

$$\partial_\mu J_\nu^\mu = 0 \quad (54)$$

and the requirement for physical fields to become zero at infinity, we see that

$$0 = \int_V \partial_0 J_\nu^0 d^3x + \underbrace{\int_V \partial_i J_\nu^i d^3x}_0 = \int_V \frac{\partial}{\partial t} J_\nu^0 d^3x = \frac{d}{dt} \int_V J_\nu^0 d^3x = \frac{dQ_\nu}{dt} = 0. \quad (55)$$

This is the conserved quantity we we're looking for.

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<sup>1</sup>In general it doesn't have to be restricted to this case, e.g. if one considers Lorentz transformations. The same is true for  $\Phi$  in the case of multiplets of scalar fields.

### 3. Examples

#### 3.1. Energy-momentum conservation

Let us discuss an invariance of the action under space-time translations. This corresponds to changes of  $x^\mu$  and  $\phi$  in the following sense

$$\Delta x^\mu = \xi^\mu, \quad \Delta \phi = 0. \quad (56)$$

Remembering (48) and since  $\phi$  doesn't change at all, the corresponding infinitesimal transformation has to be zero too. The remaining transformation can be achieved if we choose  $\delta\omega^\nu$  to be equal to  $\epsilon^\nu$ , and therefore the two transformations are of the form

$$\Phi_\nu = 0, \quad X_\nu^\mu = \delta_\nu^\mu, \quad (57)$$

where  $\delta_\nu^\mu$  is the Kronecker symbol. This means that our conserved current is just the negative energy-momentum tensor, and the corresponding conserved charge is given by

$$\frac{d}{dt} \int \theta_\nu^0 d^3x = 0, \quad (58)$$

and we therefore have to take another look at the definition of  $\theta_\nu^\mu$  in (46). We will first consider only the zero-eth component of the conserved charge. It is given by

$$\int \theta_0^0 d^3x = \int \left( \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial_0 \phi - \mathcal{L} \right) d^3x = \int \left( \frac{\partial \mathcal{L}}{\partial(\dot{\phi})} \dot{\phi} - \mathcal{L} \right) d^3x \quad (59)$$

where one finds, by remembering classical mechanics, that the second term looks similar to the relation between the lagrangian and the hamiltonian. In fact one can show, that the integrand is the so called Hamilton density, and the integral therefore the hamiltonian of the considered field. Therefore the zero-eth component of the conserved charge is the energy of the field. The property of the conserved charge, to transform under Lorentz transformations like a 4-vector, shows immediately, that the spatial part of the charge corresponds to the momentum conservation.

We can conclude that for any lagrangian that is independent of  $x^\mu$  the energy-momentum tensor is a conserved quantity.

#### 3.2. The conserved current of the Klein-Gordon field

We discussed in a previous chapter, that there exists a conserved current for the Klein-Gordon field, with the following form

$$J^\mu = i \frac{\hbar}{2m} (\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*). \quad (60)$$

Since the current contains the complex conjugated field, we look for a generalisation of the Klein-Gordon lagrangian. It is given by

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi^* \phi. \quad (61)$$

We can interpret this as a lagrangian for two independent fields,  $\phi$  and  $\phi^*$ , yielding two field equations, the Klein-Gordon equation and its complex conjugate. One can immediately see, that the lagrangian is not changed if we multiply  $\phi$  by a complex phase:

$$\phi \rightarrow \exp(-i\alpha)\phi, \quad \phi^* \rightarrow \exp(i\alpha)\phi^* \quad (62)$$

The infinitesimal form of the transformation is then given by

$$\Phi = -i\phi, \quad \Phi^* = i\phi^* \quad (63)$$

and of course  $X = 0$ . Inserting this and the Lagrangian in Noether's theorem we recover the current (60) we we're looking for (except for a constant factor), justifying our assumption for the density we made in the 4<sup>th</sup> chapter.

# Part VII.

## Quantization of a scalar field

### 1. Methods to quantize a field

There exist multiple ways to quantize a field which lead to the same quantum field. We will discuss the following two:

1. The canonical quantization
2. The path integral quantization

The first one is realised by promoting the coefficients of the Fourier decomposition of the field to become operators and demanding certain commutation relations for these operators. The second one is based on functional methods and a so called generating functional.

### 2. The canonical quantization of the Klein Gordon field

#### 2.1. The canonical quantization of conjugate variables

Remembering quantum mechanics we it comes to our mind that one can see quantization as the change from certain Poisson bracket relations to certain commutator bracket relations. The two conjugate variables  $p_i$  and  $q_j$  are handled as operators which will act on states on a Hilbertspace and which have to fulfil the relations

$$[q_i(t), p_j(t)] = i\hbar\delta_{ij} \quad \text{and} \quad [q_i(t), q_j(t)] = [p_i(t), p_j(t)] = 0 \quad (64)$$

The canonical quantisation is in a way trying to repeat this concept, only for fields.

#### 2.2. The general solution to the Klein Gordon equation

We found in Chapter IV that that the Klein Gordon equation is given by

$$(\square + m^2)\phi = 0 \quad (65)$$

We can find the general solution to this equation by using the plain waves ansatz. Inserting a plain wave into the equation gives

$$(\square + m^2)a \exp(\pm ikx) = (m^2 - p^2)a \exp(\pm ipx) = 0, \quad (66)$$

leaving us with the so called mass shell<sup>2</sup> condition

$$p^2 = m^2. \quad (67)$$

The general solution has then to be a linear combination of  $\exp(ipx)$  and  $\exp(-ipx)$  and additionally we have to include all  $p$ 's which fulfil the condition. Thus we can write the general solution as

$$\phi = \int a^\dagger(p) \exp(ikx) + a(p) \exp(-ipx) d\mu(p) \quad (68)$$

where

$$d\mu(p) = \frac{d^3p}{(2\pi)^3 2p^0} \quad (69)$$

and  $p^0$  is chosen such, that the mass shell condition is fulfilled:

$$p^0 = \sqrt{\vec{p}^2 + m^2} \quad (70)$$

What we're now doing by quantizing is to promote the coefficients  $a^\dagger(p)$  and  $a(p)$  to become operators.

### 2.3. Canonical commutator relations

We will now demand analogous commutator relations for the coefficients as we mentioned in (64) above, namely

$$[a(p), a(p')^\dagger] = (2\pi)^3 2p^0 \delta^{(3)}(\vec{p} - \vec{p}') =: \delta(p, p') \quad (71)$$

These commutator relations automatically imply another set of commutator relations, which hold for the field  $\phi$  and its canonical momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \quad (72)$$

which are given by

$$[\phi(t, \vec{x}), \pi(t, \vec{y})] = \delta^{(3)}(\vec{x} - \vec{y}) \quad \text{and} \quad [\phi(t, \vec{x}), \phi(t, \vec{y})] = [\pi(t, \vec{x}), \pi(t, \vec{y})] = 0. \quad (73)$$

If we take a closer look at the commutator relations for  $a$  and  $a^\dagger$  we see that they are similar to those of the harmonic oscillator.

If we think of the interpretation of the harmonic oscillator in quantum mechanics, we remember that there  $a^\dagger$  creates a quanta of energy while  $a$  annihilates it. In Quantum mechanics one defines an operator  $N$  that is called the *particle number operator* and

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<sup>2</sup>The mass shell is a hyperboloid which represents timelike vectors in space-time with the same length in the sense of the Minkowski metric.

that counts the quanta of energy that a given state possesses. We will now create an analogous operator for our two operators,

$$N(p)(2\pi)^3 2\omega_k \delta^3(0) = a^\dagger(p)a(p). \quad (74)$$

Straight forward calculation shows, that this operator satisfies the commutator relations

$$[N(p), N(p')] = 0, \quad [N(p), a^\dagger(p)] = a^\dagger(p), \quad [N(p), a(p)] = a(p). \quad (75)$$

The first one allows us to form a basis out of the eigenstates  $|n(p)\rangle$  of this operator, which have the eigenvalue  $n(p)$ , the second lead to the following relation between  $a^\dagger(p)$ ,  $a(p)$  and  $|n(p)\rangle$ :

$$N(p)a^\dagger(p)|n(p)\rangle = \left(n(p) + 1\right)a^\dagger(p)|n(p)\rangle, \quad N(p)a(p)|n(p)\rangle = \left(n(p) - 1\right)a(p)|n(p)\rangle \quad (76)$$

This justifies that we proceed analogous to the harmonic oscillator and interpret the operators in a similar way, that  $a^\dagger(p)$  creates an quanta of energy, while  $a(p)$  annihilates it, and finally  $N(p)$  gives the amount of quanta that the state possesses. However there is one difference, since the operators are related to the mass term in the Klein-Gordon equation the energy has to be related to it too. Thus we can actually say that they do not create or annihilate a quanta of energy, but create a particle with mass  $m$ .

## 2.4. Solved and new problems

However we did not only gain a new interpretation of the Klein-Gordon equation as an operator which acts on states and creates particles, there were also some problems we had with this equation that miraculously disappeared. First we noted, that the energy of the single particle described by the equation could be negative, but if we now insert the lagrangian into the formula for the energy density of the field we derived earlier, we see the following:

$$H = \int \theta^{00} d^3x = \frac{1}{2} \int \left( (\partial_0 \phi^*)(\partial_0 \phi) - \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi \right) d^3x \quad (77)$$

This is positive definite and therefore the problem of negative energy just disappeared, at least almost, since there is still the possibility, that we could descend down into negative energies by simply using the annihilation operator. However, since

$$\left( a(p)|n(p)\rangle \right)^\dagger \left( a(p)|n(p)\rangle \right) = \langle n(p)|a^\dagger(p)a(p)|n(p)\rangle = n(p)\langle n(p)|n(p)\rangle > 0 \quad (78)$$

is guaranteed by the fact that  $a(p)|n(p)\rangle$  is a state in a Hilbertspace and therefore has to have non negative norm. Since  $a(p)$  reduces  $n(p)$  by one this process has to stop at

one point, which can only be the case if at some point  $n(p)$  becomes zero. We call the state for which this is true the *ground state*  $|0(p)\rangle$  and define it as

$$a(p)|0(p)\rangle = 0 \quad (79)$$

which simply means, that in the ground state (which is also referred to as vacuum) there exists no particle with momentum  $\vec{p}$ . If we insert all our gained relations into the expression for the energy and express it in terms of the annihilation and creation operators, we see that

$$H = \int p_0 \left( N(p) + \frac{1}{2} \right) d^3p \quad (80)$$

and since the integral is taken over the complete momentum space, we see that the last term diverges. Since this term exists always, independent of the state we consider, it is called the vacuum energy, just like the vacuum energy of the harmonic oscillator.

However there exists a way to solve this problem, called renormalization, which we could describe as choosing the ground state to be zero. This can formally be achieved by reordering the operators in such a way, that all creation operators stand on the left, while all annihilation operators stand on the right.

## 2.5. The propagator of the Klein-Gordon field

The propagator, as the name suggests, tells us how a particle propagates through space-time. It's canonical to assume it to be some sort of probability amplitude, given by the inner product of the state of one point in space-time, and that at another point in space-time. We could thus assume, that the propagator  $\Delta(x, y)$  is given as

$$\Delta(x, y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle. \quad (81)$$

However, we already saw, that the field can be interpreted as creating a particle at some point in coordinate space, which gives rise to a problem if we consider the probability of propagating backwards in time. If we assume that  $y^0$  is earlier than  $x^0$ , the particle at  $\vec{x}$  would be created in a state, that is already disturbed by a particle that should not yet been created. Thus we have to consider the so called *time-ordered T* product of the two field operators. Taking this and the translation invariance of the theory into account, we find the following identity

$$\langle 0 | T \phi(x) \phi(y) | 0 \rangle = \langle 0 | T \phi(x - y) \phi(0) | 0 \rangle \quad (82)$$

and therefore we write the propagator as

$$\Delta(x) = i \langle 0 | T \phi(x) \phi(0) | 0 \rangle. \quad (83)$$

The T simply ensures that  $x^0 > y^0$  by reordering and renaming the variables, if it was not the case. There are some important properties of the propagator. First it is symmetric

$$\Delta(x) = \Delta(-x). \quad (84)$$

Second, if we consider that

$$\Delta(x) = i\Theta(x^0) \int \exp(-ipx) d\mu(p) + i\Theta(-x^0) \int \exp(+ipx) d\mu(p) \quad (85)$$

we see that there exist two types of solution, these with positive  $x^0$ , giving the "positive frequency" solutions, and those with negative  $x^0$  giving the "negative frequency" solutions.

Third we see that the propagator is a Green function of the Klein Gordon operator

$$(\square + m^2)\Delta(x) = \delta^{(4)}(x) \quad (86)$$

and that we can write it as the Fourier integral of

$$\Delta(x) = \int \frac{\exp(-ikx)}{m^2 - k^2 - i\epsilon} \frac{d^4k}{(2\pi)^4} \quad (87)$$

where the  $\epsilon$  arises from the so called "Feynman boundary condition" which takes into account a shift from  $m^2 \rightarrow m^2 - i\epsilon$ ,  $\epsilon > 0$ .<sup>3</sup>

### 3. The path integral quantization

We will now reproduce some of the results with another way of quantization, called the path integral quantization. The path integral is not only a tool for quantum field theory, but for quantum mechanics too.

#### 3.1. Motivating the path integral in quantum mechanics

The path integral was introduced by Feynman and it is said that he had the idea already during his time as a student. One can visualize this (like everything else in quantum mechanics) with the double slit experiment. There exists two possible ways for the photon to go. It starts at  $I$ , goes through the slit  $A_1$  and gets detected at the screen  $F$ , or it starts at  $I$ , goes through the slit  $A_2$  and gets detected at the screen  $F$ . The superposition postulate now states that the sum of the two is the actual amplitude for detection, the wave function is a sum of the wave functions describing the two possibilities. However, we could do this again in a triple slit experiment. This means we have the sum of  $\psi(F \rightarrow A_1 \rightarrow F)$ ,  $\psi(F \rightarrow A_2 \rightarrow F)$  and  $\psi(F \rightarrow A_3 \rightarrow F)$ . But this is not where it ends, we can take as many slits as we want, if we have  $N$  slits, we will get

$$\psi = \sum_i^N \psi(I \rightarrow A_i \rightarrow F). \quad (88)$$

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<sup>3</sup>The Green function defined above is not unique since we could add any solution of the homogeneous equation. Therefore we need boundary conditions.

But this is not where Feynman stopped. What happens if we put another screen containing  $M$  slits in? We have to sum over all possible combinations of points that the particle has to pass. We could say, we have to sum over all paths:

$$\psi = \sum_i^N \sum_j^M \psi(I \rightarrow A_i \rightarrow B_j \rightarrow F). \quad (89)$$

Still, there is no limit to this, we can put in as many screens as we like and we can take as many slits as we like. But that's exactly where the physical problem that Feynman was aiming at starts: "If we take an screen with an infinite amount of slits in it, than there is no screen anymore." But quantum mechanics tells us, that we still would have to sum over all the possible paths, that the particle could go, one for each of the infinite number of slits, and we can do this for an infinite number of slits, placed infinitely close together. We now know, that the propagation of the particle should be described by a sum over all possible paths, but how can we calculate with something like that? We start with the normal transition amplitude for a particle in state  $|q_I\rangle$  to a state  $|q_F\rangle$  in time  $T$ , given by

$$\mathcal{A} = \langle q_F | e^{-iHT} | q_I \rangle, \quad (90)$$

where  $H$  is the hamiltonian of the system. Now we divide the time  $T$  into small sections  $\delta t$ , representing the time it needs to go from one screen to another. Looking at  $N$  screens, we get

$$\delta t = T/N, \quad (91)$$

and therefore the amplitude can be written as

$$\mathcal{A} = \langle q_F | e^{-iH\delta t} e^{-iH\delta t} \dots e^{-iH\delta t} | q_I \rangle. \quad (92)$$

Now we use the completeness relation for the states and insert  $\mathbb{1} = \int |q\rangle \langle q| dq$ , which represents one of the screens and the possible ways the particle can take. If we do this we arrive at

$$\langle q_F | e^{-iHT} | q_I \rangle = \int \langle q_F | e^{-iH\delta t} | q_{N-1} \rangle \langle q_{N-1} | e^{-iH\delta t} | q_{N-2} \rangle \dots \langle q_1 | e^{-iH\delta t} | q_I \rangle dq_{N-1} \dots dq_1. \quad (93)$$

Now let's evaluate one of factors  $\langle q_{j+1} | e^{-iH\delta t} | q_j \rangle$ . For simplicity we'll do this for a free particle, and therefore  $H = \frac{\hat{p}^2}{2m}$ . Since there's a momentum operator in the exponent we will insert a complete set of momentum states into the equation, therefore we arrive at

$$\langle q_{j+1} | e^{-i\frac{\hat{p}^2}{2m}\delta t} | q_j \rangle = \int \langle q_{j+1} | e^{-i\frac{\hat{p}^2}{2m}\delta t} | p \rangle \langle p | q_j \rangle \frac{dp}{2\pi} \quad (94)$$

$$= \int e^{-i\frac{p^2}{2m}\delta t} \langle q_{j+1} | p \rangle \langle p | q_j \rangle \frac{dp}{2\pi} \quad (95)$$

$$= \int e^{-i\frac{p^2}{2m}\delta t} e^{iq_{j+1}p} e^{-iq_j p} \frac{dp}{2\pi}. \quad (96)$$

This is a Gaussian integral and evaluation gives

$$\langle q_{j+1} | e^{-i\frac{p^2}{2m}\delta t} | q_j \rangle = \left( \frac{-i2\pi m}{\delta t} \right)^{\frac{1}{2}} \exp \left( i\delta t \frac{m}{2} \left( \frac{q_{j+1} - q_j}{\delta t} \right)^2 \right). \quad (97)$$

Inserting this into (93) yields

$$\langle q_F | e^{-iHT} | q_I \rangle = \left( \frac{-i2\pi m}{\delta t} \right)^{\frac{N}{2}} \int \exp \left( i\delta t \frac{m}{2} \sum_{j=0}^{N-1} \left( \frac{q_{j+1} - q_j}{\delta t} \right)^2 \right) dq_1 \dots dq_{N-1} \quad (98)$$

where we relabelled  $q_I \equiv q_0$  and  $q_F \equiv q_N$ . Now we take the limit  $\delta t \rightarrow 0$ , which means that we put an infinite number of screens in-between the starting and the end point, and thereby the following transformations happen

$$\delta t \sum_{j=0}^{N-1} \rightarrow \int_0^T dt \quad \text{and} \quad \frac{q_{j+1} - q_j}{\delta t} \rightarrow \dot{q}. \quad (99)$$

We therefore have included the idea of an infinite number of screens and we still need to insert the idea of an infinite number of slits. This is simply done by letting  $N \rightarrow \infty$  and we therefore found the integral over paths to be

$$\int [dq(t)] = \lim_{N \rightarrow \infty} \left( \frac{-i2\pi m}{\delta t} \right)^{\frac{N}{2}} \int dq_1 \dots dq_{N-1}. \quad (100)$$

Putting the two together we see that the transition amplitude is given by

$$\langle q_F | e^{-iHT} | q_I \rangle = \int [dq(t)] \exp \left( i \int_0^T \frac{1}{2} m \dot{q}^2 dt \right). \quad (101)$$

Redoing the calculation with a hamiltonian that contains a potential  $V(q)$  yields a very similar result

$$\langle q_F | e^{-iHT} | q_I \rangle = \int [dq(t)] \exp \left( i \int_0^T \left( \frac{1}{2} m \dot{q}^2 - V(q) \right) dt \right) = \int [dq(t)] \exp \left( i \int_0^T \mathcal{L} dt \right). \quad (102)$$

It is important to note that this is an integral over a complex phase and therefore it would not be clear that it converges if it would not follow from its construction. This justifies that we take a closer look at what happens if we integrate. The first thing we see is that the exponent contains the classical action and therefore is zero if the path considered satisfies the Euler Lagrange equations. Furthermore a change of the path represents a variation of the action and thus gives a rapid oscillation of the phase if the path is not very close to a solution of the Euler Lagrange equations, therefore in the mean contributions that don't satisfy these conditions vanish.

This equation can be generalised in two ways. First, if we insert complete sets of states

in such a way, that we can compute the transition from some given initial state  $|I\rangle$  to a given final state  $\langle F|$ :

$$\langle F|\exp(-iHT)|I\rangle = \int \langle F|q_F\rangle \langle q_F|\exp(-iHT)|q_I\rangle \langle q_I|I\rangle dq_F dq_I. \quad (103)$$

The most common case is, to let  $|F\rangle = |I\rangle = |0\rangle$  be the ground state. Second, we can insert conditions of  $q$  at certain time points into the integral and get the following:

$$\langle 0|T Q(t_1)Q(t_2)|0\rangle = \int \exp(i \int \mathcal{L} dt) q(t_1)q(t_2) [dq] \quad (104)$$

This means that we weight the integral with boundary conditions for a given configuration. If we think of the integral as an infinite sum we weight some terms with the positions that these paths have at time  $t_1, t_2$ , which fulfil the Euler-Lagrange conditions.

### 3.2. The path integral in QFT

What we will do now, is just analogous to what we did in the chapter about the Lagrange formalism, we simply replace the lagrangian  $\mathcal{L}$  with the Lagrange density  $\mathcal{L}$ . This way the path integral becomes

$$\int \exp(i \int \mathcal{L} dx) [d\varphi] \quad (105)$$

However in field theory this integral is just representing the vacuum and the vacuum oscillations are not essentially the quantities one is observing and are therefore not of primary interest. Therefore we have to disturb the vacuum by creating particles. This is done by the field operators as we have already discussed in this chapter. In analogy to what we have just discussed we can think of it as configurations of the field we demand at certain points in space time. This means our integral is modified in the following way:

$$\langle 0|T \Phi(x_1)\Phi(x_2)|0\rangle = \int \exp(i \int \mathcal{L} dx) \varphi(x_1)\varphi(x_2) [d\varphi] \quad (106)$$

So what has changed is, that we have gone from paths to fields, and the conditions we set by including  $\varphi(x_1)$  and  $\varphi(x_2)$  are field configurations. However, since we are interpreting the fields as quantized, and therefore consisting of single particles, we have introduced particles into the vacuum. This process is often called "disturbing the vacuum", because the vacuum described by the path integral without the  $\varphi$  is for instance disturbed by a particle created at the space-time point  $x_1$ , travelling to the space time point  $x_2$ . This allows us to describe transitions of particles described by a quantized field and is therefore the basis for handling scattering processes.

## 4. Application

The concept of quantized fields is mainly applied in particle physics, meaning in the standard model. There the fields discussed are the quantized Dirac field in the quantum

electro dynamic and for instance the Yang-Mills field describing the gluons. The calculations represent the experimental data with an astonishing precision. However there is a broad range of other applications. One example is the context of solid state physics where phonons can be understood via a quantized field. In string theory the background fields are assumed to be quantized and quantized fields are therefore at the very basic level of string theory. A modern topic of interest is the discussion of Bose-Einstein condensates, which too are described via quantum fields. Additionally the concept is used in cosmology for describing the so called inflationary universe. The main idea is to explain the tiny inhomogeneties, namely the clusters of masses, are the result of quantum fluctuations that occurred at an early stage of the universe and due to the expansion are now this big, that they are actually visible as our galaxies. A possible application that is currently a hot topic is of course quantum gravity which is suspected to be described by a quantum field too.

# Part VIII.

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