Einstein, Podolsky and Rosen Paradox, Bell Inequalities and the Relation to the de Broglie-Bohm Theory

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Contents

1 Introduction

Is quantum mechanics incomplete? Are hidden variables needed? Does spooky interaction exist? Do we need a new interpretation for quantum mechanics? Is there an equivalent interpretation to the common Copenhagen interpretation? In this work, these questions will be raised and discussed. Quantum mechanics is known to be strange and describe phenomena which do not exist in the world of classical mechanics.

The first main topic is about entanglement, a correlation between two separated systems which cannot interact with each other. Because this contradicts special relativity, Einstein, Podosky and Rosen concluded that quantum mechanics is incomplete and that a hidden variable could solve the problem. At first, this assertion couldn't be verified in any way, until John Bell derived an inequality in 1964, which could answer the question whether a hidden variable was needed.

In 1951, David Bohm presented a new quantum mechanical interpretation, which is still accepted as an equivalent to the Copenhagen interpretation. The second main topic is an introduction to the de Broglie-Bohm theory and a comparison with the Copenhagen interpretation. We will also discuss the fact that the de Broglie-Bohm theory is a hidden variable theory which doesn't satisfy the necessary conditions to be bound by the Bell inequalities.

2 Einstein Podolsky Rosen Paradox

Einstein, Podolsky and Rosen were the first to question the completeness of the mathematical formalism of quantum mechanics in their joint paper [1]. In those days, this question was a purely philosophical one, so they did not provide a proof of incompleteness.

In their paper [1], a theory is required to satisfy the **condition of completeness**. It states that

> *every element of physical reality must have a counterpart in the physical theory.* (Phys. Rev. 47, 1935, p. 777)

The so-called **condition of reality** clarifies what the elements of reality are:

> *If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.* (Phys. Rev. 47, 1935, p. 777)

In quantum mechanics, an element of physical reality is described by an eigenstate ψ_A of a physical quantity represented by the observable A. Hence, if another observable B doesn't commute with the observable A, the wave function ψ_A can't be an eigenstate of the observable B. This result means that the physical quantity represented by B has no physical reality. We are therefore left with two possibilities:

> (1)*[The] quantum mechanical description of reality given by the wave function is not complete.*
>
> (2)*[When] the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality.*
> (Phys. Rev. 47, 1935, p. 777)

Normally, the mathematical description of quantum mechanics is considered to be complete. By taking this assumption as an initial point, Einstein, Podolsky and Rosen derived a contradiction to their condition of reality using a general wave function.
One starts with two systems, denoted by **I** and **II**, which are permitted to interact for a time range between $t = 0$ and $t = T$. After that the systems are separated and can't influence each other. It is assumed that the state of both systems is known before the interaction began ($t < 0$), so the state of the combined system can be calculated for any later time t with the help of Schrödinger's equation, but the states of the individual systems are unknown. A measurement of the first system leads to a collapse of the wave function to an eigenstate. The result of the measurement is then given by the corresponding eigenvalue.
The general wave function Ψ can be expressed by the eigenstates $u_k(x_1)$ of the observable A which corresponds to the measured physical quantity. Then the states $\psi_k(x_2)$ of system **II** can be thought of as coefficients of the eigenstates $u_k(x_1)$.

$$\Psi(x_1, x_2) = \sum_{n=1}^{\infty} \psi_n(x_2) u_n(x_1) \tag{1}$$

After measuring system **I**, the wave function collapses into an eigenstate $u_k(x_1)$ with an eigenvalue a_k. This leaves the second system in the state $\psi_k(x_2)$.
System **I** can also be measured by an observable B corresponding to another physical quantity, which means that the general wave function can be thought of as another composition of different eigenstates $v_r(x_1)$ and coefficients $\phi_r(x_2)$.

$$\Psi(x_1, x_2) = \sum_{s=1}^{\infty} \phi_s(x_2) v_s(x_1) \tag{2}$$

If system **I** is measured, the wave function will collapse into the eigenstate $v_r(x_1)$ with an eigenvalue b_r and System **II** will be in the eigenstate $\phi_r(x_2)$.

Let's assume that $\psi_k(x_2)$ and $\phi_r(x_2)$ are eigenstates of two physical quantities P and Q. Then both functions are real in the sense of the condition of realty. By assumption, there is no interaction between the systems ($t > T$), so the measured system **I** can't interact with system **II**, which implies $\psi_k(x_2)$ and $\phi_r(x_2)$ can be assigned to the same reality.

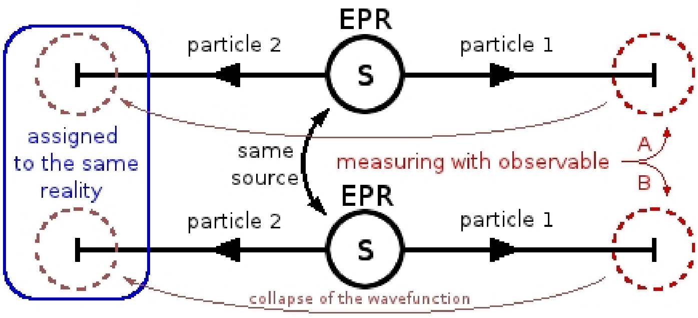

Fig. 1: Two experiments with identical sources. The difference between these experiments is that the first particle is measured with different observables A and B, resulting in a collapse of the wave function. The second particle is then determined by a corresponding wave function. It is possible that this wave function is an eigenstates of an observable, which implies that the corresponding eigenvalue is real. In general the eigenstate in the first experiment is different from the eigenstate in the second experiment. Because the first particle is separated from the second one and the initial states of the experiments where the same both eigenstates can be assigned to the same reality.

However, there is a possibility that P and Q don't commute, in which case one of the states $\psi_k(x_2)$ and $\phi_r(x_2)$ cannot be a real quantity and hence cannot be an eigenstate. This is a contradiction.

Therefore the description of the quantum state by its wave function is incomplete.

2.1 An Example to show Incompleteness

The next calculation is a concrete example to show that the previous discussion is not merely a philosophical one. Let's take the general wave function of two particles

$$\Psi(x_1, x_2) = \int_{-\infty}^{\infty} exp\left(\frac{i}{\hbar}(x_1 - x_2 + x_0)p\right) dp \tag{3}$$

where x_0 is some constant. If the observable A is the momentum of the first particle, then the eigenfunction associated to an eigenvalue p is given by

$$u_p(x_1) = exp\left(\frac{i}{\hbar}px_1\right). \tag{4}$$

After rewriting Ψ we get

$$\Psi(x_1, x_2) = \int_{-\infty}^{\infty} \psi_p(x_2)u_p(x_1)dp \tag{5}$$

and it is clear that

$$\psi_p(x_2) = exp\left(-\frac{i}{\hbar}(x_2 - x_0)p\right) \tag{6}$$

is also an eigenfunction of the momentum operator P of particle two corresponding to the eigenvalue $-p$. This means the momentum of the second particle is real (again in the sense of the condition of reality).

If we measure the coordinates of the first particle using observable B, its eigenfunctions are given by

$$v_x(x_1) = \delta(x_1 - x), \tag{7}$$

where x is the eigenvalue associated to v_x. Rewriting Ψ results in

$$\Psi(x_1, x_2) = \int_{-\infty}^{\infty} \phi_x(x_2)v_x(x_1)dx \tag{8}$$

with

$$\phi_x(x_2) = \int_{-\infty}^{\infty} exp\left(\frac{i}{\hbar}(x - x_2 + x_0)p\right) dp = 2\pi\hbar\delta(x - x_2 + x_0). \tag{9}$$

These $\phi_x(x_2)$ are eigenfunctions of the position-operator Q with eigenvalues $x+x_0$. That also means that the spacial coordinate of particle 2 is real.
It is known that the momentum-operator P and the position-operator Q do not commute. As mentioned above, if $\phi_x(x_2)$ is an eigenstate, $\psi_p(x_2)$ can't be one simultaneously, which also is true the other way round. But the gedanken experiment above showed that both are eigenstates at the same time. This stands in contradiction with quantum mechanics and shows that quantum mechanics should be incomplete.
At this point it's good to recall that it is not necessary to measure system **I** with A and B simultaneously. The reason is that the two systems cannot interact with

each other, which results in the conclusion that both collapsed wave functions can be assigned to the same reality.

Niels Bohr, who responded to Einstein's, Podolsky's and Rosen's paper with his own paper of the same title [2], claimed that their arguments did not justify their conclusion that quantum mechanics is not complete. He reasoned that it is impossible to determine whether the eigenvalue in the second system corresponding to observable B is a real quantity or not after one has measured system **I** with observable A, provided A and B do not commute. According to his idea, this is because there is no way to measure the other physical quantity exactly after interacting with the measuring apparatus, so it is not justified to conclude that quantum mechanics is incomplete.

In his paper, Bohr neither wrote about the experiment nor did he try to explain how the interaction between the separated systems is possible. Einstein, Podolsky and Rosen, however, made a suggestion to create a hidden variable to describe the missing part of the incomplete theory of quantum mechanics. This hidden variable should encode the information required to explain why the not-measured system collapses into the right state even if it cannot possibly determine whether the other system is measured with A or B for times $t > T$. On the other hand, quantum theory without hidden variables would imply that the two systems could interact with each other instantaneously, which is a contradiction to one of the main points of special relativity, namely that there is no interaction faster than speed of light. Therefore, Einstein called this *spooky action at a distance.*

2.2 Contributions from D. Bohm and Y. Aharonov

In 1957 D. Bohm and Y. Aharonov published a paper [3], which presented a more intuitive experimental setup to show what is meant by spooky action at a distance. They considered a system of two spin-1/2-particles with a total spin of 0. Its wave function is

$$\Psi = \frac{1}{\sqrt{2}} \left[\psi_+(1)\psi_-(2) - \psi_-(1)\psi_+(2) \right], \tag{10}$$

where $\psi_\pm(1)$ is the wave function of the first particle (1) with spin $\pm\hbar/2$ and $\psi_\pm(1)$ is the analog for the other wave functions. Without any disturbances the two particles get separated. When the particles are no longer able interact with each other, a spin component of the first particle is measured in an arbitrary direction. Due to the total spin of 0, the second particle has the exact opposite component. If this were a classical system, this experiment could easily be explained, because the components of the spin are well defined at any given time. Since the spin components of both particles are correlated from the beginning, the value of the

second particle in any direction will always be in the opposite direction of the first one.
However, this is a quantum mechanical system and the three orthogonal spin components do not commute with each other. This means that the measured component is determined but the orthogonal components are uncertain. In Bohm's and Aharonov's paper these uncertain components are regarded as some kind of random fluctuations, and can be interpreted as disturbances from the measuring apparatus. After the measurement of the first particle, the collapsed wave function is $\psi_+(1)\psi_-(2)$ or $\psi_-(1)\psi_+(2)$. Also, the correlation between the particles is gone. It is crucial to see that the direction used for the measurement can be chosen arbitrarily and that the axis of the spin is determined by this choice. The measurement will determine both particles and due to the collapsed wave function, the spins of both particles point in opposite directions.
The problem that occurs here is that there is no plausible reason why the second particle collapses into this state and why the orthogonal components are fluctuating. Moreover, if this phenomenon is tried to be explained with a hidden interaction, this interaction would have to be instantaneous because the measuring apparatus can be changed at any moment. In particular, a hidden variable must be created to preserve locality.

Locality A theory is called **local** if the measurement of a particle is not able to interfere with another particle, provided the latter is not causally linked with the measuring apparatus or the measured particle.

Such a hidden variable will also maintain the reality criterion.

Reality A theory is called **real** if you can assure that a measured quantity existed even before the particle was measured.

3 Bell's Inequality

John S. Bell [4] was the first physicist who found a way to experimentally verify some of the long unanswered questions posed in the first chapter. He derived an inequality which establishes a boundary for **local & real** theories based on Bohm's and Aharanov's experiment.
For further calculations the following notation will be used.

$\vec{\sigma_1}$	spin of the first particle
$\vec{\sigma_2}$	spin of the second particle
$\vec{a}$	normal vector for the spin measurement of the first particle
$\vec{b}$	normal vector for the spin measurement of the second particle
λ	hidden variable or set of hidden variables or even a set of functions
$\rho(\lambda)$	normalized probability function of the hidden variable

The properties of the experiment are as follows. If $\vec{\sigma_1}\vec{a} = 1$, the value of the measured spin component in the direction of $\vec{a}$ is 1. Similarly, $\vec{\sigma_2}\vec{a} = -1$ means that the measured spin component of the second particle in the same direction is -1. Assuming perfect anti-correlation σ_1 and σ_2 will always lead to results of opposite sign when measured in the same direction.
In a hidden variable theory (HVT), the measurement results $A = \vec{\sigma_1}\vec{a}$ and $B = \vec{\sigma_2}\vec{b}$ will depend on the hidden variable λ and the directions $\vec{a}$ and $\vec{b}$.

$$A(\vec{a}, \lambda) = \pm 1 \qquad B(\vec{b}, \lambda) = \pm 1 \tag{11}$$

Locality, in this case, means that A is independent of $\vec{b}$ and B independent of $\vec{a}$. The expectation value for the local behavior of the particles is expressed by

$$P(\vec{a}, \vec{b}) = \int \rho(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda) d\lambda. \tag{12}$$

The maximum anti-correlation is given when $\vec{a}$ equals $\vec{b}$:

$$A(\vec{a}, \lambda) = -B(\vec{a}, \lambda) \tag{13}$$

Therefore, the expectation value can be rewritten in the following form.

$$P(\vec{a}, \vec{b}) = -\int \rho(\lambda) A(\vec{a}, \lambda) A(\vec{b}, \lambda) d\lambda \tag{14}$$

If a new normalized measuring direction $\vec{c}$ is introduced, the difference in expectation values of two setups $P(\vec{a}, \vec{b})$ and $P(\vec{a}, \vec{c})$ has a boundary value depending on a new measuring setup $P(\vec{b}, \vec{c})$.

$$\begin{aligned} P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c}) &= -\int \rho(\lambda) \left[A(\vec{a}, \lambda) A(\vec{b}, \lambda) - A(\vec{a}, \lambda) A(\vec{c}, \lambda) \right] d\lambda \\ &= \int \rho(\lambda) A(\vec{a}, \lambda) A(\vec{b}, \lambda) \left[\frac{1}{A(\vec{b}, \lambda)} A(\vec{c}, \lambda) - 1 \right] d\lambda \end{aligned} \tag{15}$$

From this, Bell's inequality can be derived by using $A(\vec{a}, \lambda) = \pm 1 = \frac{1}{A(\vec{a}, \lambda)}$.

$$\left| P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c}) \right| \leqslant \int \rho(\lambda) \left[1 - A(\vec{b}, \lambda) A(\vec{c}, \lambda) \right] d\lambda$$

$$\boxed{\boxed{\left| P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c}) \right| \leqslant 1 + P(\vec{b}, \vec{c})}} \tag{16}$$

With this inequality local & real theories can be experimentally verified by measuring three different expectation values.
Quantum mechanics violates this inequality due to the following calculations. The state that is used in this experiment is the Bell state $|\Psi^-\rangle$ (10), which is convenient because the correlation between the particles is the same in every basis. The resulting expectation value is then given by

$$\langle \Psi^- | \ \vec{\sigma_1} \vec{a} \ \vec{\sigma_2} \vec{b} \ | \Psi^- \rangle = -\vec{a}\vec{b}. \tag{17}$$

Because $\vec{a}$ and $\vec{b}$ are normalized, the result can be described using the angle α between the vectors: $-\vec{a}\vec{b} = -cos(\alpha)$. Denoting the angle between $\vec{b}$ and $\vec{c}$ by β, the angle between $\vec{a}$ and $\vec{c}$ is given by $\alpha + \beta$. Then the Bell inequality with quantum mechanical expectation values is of the following form.

$$|-cos(\alpha) + cos(\alpha + \beta)| + cos(\beta) - 1 \leqslant 0 \tag{18}$$

The next figures show graphs of expression 18 depending on α and β. In a following figure (Fig.3) it can be clearly seen that the Bell inequality is violated. Maximal violation is reached with the angles $\alpha = \beta = \frac{\pi}{3}$. Thus quantum mechanics is not a **local & real** theory.

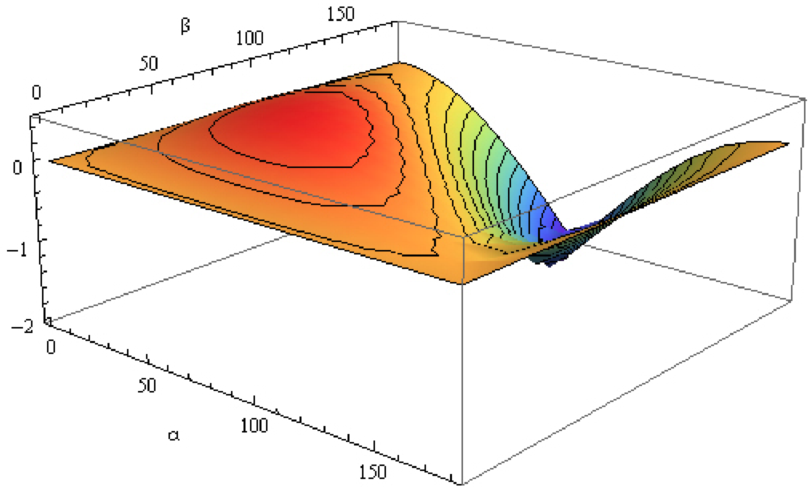

Fig. 2: Bell's inequality with the quantum mechanical expectation value $< \vec{\sigma}_1\vec{a}\ \vec{\sigma}_2\vec{b} >_{\Psi^-}$ depending on the two angles α and β between the three measuring directions $\vec{a}$,$\vec{b}$,$\vec{c}$.

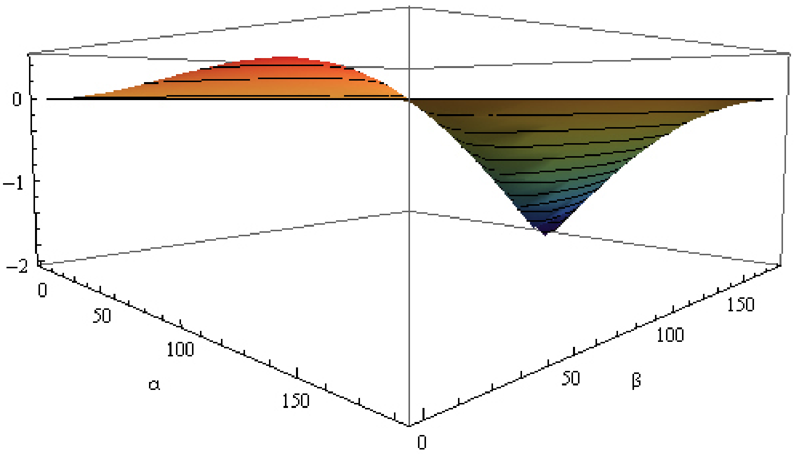

Fig. 3: The same picture as in Fig. 2, adjusted for a better view of violation of Bell's inequality.

3.1 Experimental Verification

Other people derived similar inequalities. One of them is the CHSH inequality, named after Clauser, Horne, Shimony, Holt [5]. This expanded version of Bell's inequality has an additional measuring direction $\vec{d}$ and therefore a third angle γ.

$$S = \left|P(\vec{a},\vec{b}) - P(\vec{a},\vec{c})\right| - \left|P(\vec{d},\vec{b}) + P(\vec{d},\vec{c})\right| \leqslant 2 \tag{19}$$

This inequality reaches its maximal violation at $\alpha = \beta = \gamma = \frac{\pi}{4}$.

$$S^{QM}_{max} = 2\sqrt{2} \leqslant 2 \tag{20}$$

As an experimental example, Gregor Weihs and a team lead by Anton Zeilinger used the CHSH inequality to determine the outcome of an experiment with two photons described by the wave function Ψ^-. Their result was presented in [6] and showed that the inequality was indeed violated.

$$S_{experimental} = 2,73 \pm 0.02 \leqslant 2 \tag{21}$$

The first time that the violation was verified was in 1972 by Freedman and Clauser [7] and a decade later by the group of Aspect [8]. Up until now this phenomenon, called entanglement, was checked in many different ways and with many different particles (for an overview, see e.g.[9]).

4 A new Interpretation

4.1 Copenhagen Interpretation

Nowadays, there is more then one interpretation of quantum mechanics. The first and most common viewpoint is the Copenhagen interpretation, which states that the mathematical description of the wave function and it's propagation governed by the Schrödinger equation has no physical meaning. Only the modulus squared of the wave function represents a physical part, namely the probability density, which represents the probability to find the quantum in a certain eigenvalue of an observable, if it is measured. The Copenhagen interpretation is based on two assumptions:

> (1) *The wave function with its probability interpretation determines the most complete possible specification of the state of an individual system.*

(2) *The process of transfer of a single quantum from observed system to measuring apparatus is inherently unpredictable, uncontrollable, and unanalyzable.*

(Phys. Rev. 85, 1952, p. 166) [10]

These assumptions are based on Heisenberg's uncertainty principle by interpreting the two different derivations of the uncertainty principle. The first is based on the wave character of a quantum particle. In general, the inequality $\Delta x \Delta k \leqslant 1$ holds for every kind of wave, where Δx denotes the range of positions and Δk the range of the wave numbers. With the next three quantum mechanical interpretations, the first assumption of the Copenhagen interpretation (1) follows.

(i) *The de Broglie equation $p = \hbar k$ creates a relationship between wave numbers and momentum, which is not present in classical waves. A classical electromagnetic wave with a given wave number k, for example, can have arbitrary amplitude and, therefore, arbitrary momentum.*

(ii) *Whenever either the momentum or the position of electron is measured, the result is always some definite number. Because of the de Broglie relation, a definite momentum implies a definite wave number k. On the other hand, a classical wave packet always covers a range of positions and a range of wave numbers.*

(iii) *The wave function $\psi(x)$ determines only the probability of a given position, whereas the Fourier component $\phi(k)$ determines only the probability of a given momentum. This means that it is impossible to predict or control the exact location of the electron within the region Δx in which $|\psi(x)|$ is appreciable; and that it is impossible to predict or control the exact momentum of the electron within the region Δk, in which $|\phi(k)|$ is appreciable. Thus Δx is a measure of the minimum uncertainty, or lack of complete determinism of, the position that can be ascribed to the electron. Δk is, similarly, a measure of the minimum uncertainty, or lack of complete determinism of the momentum that can be ascribed to it.*

(D. Bohm, Quantum Theory (Prentice-Hall, Inc., New York, 1951), p.100) [11]

The second derivation is based on mathematical formalism. For an observable A, we define the deviation operator $\bar{A}$ by

$$\bar{A} := A - \langle A \rangle \tag{22}$$

and its uncertainty ΔA by

$$\Delta A := \sqrt{\langle A^2\rangle_\psi - \langle A\rangle_\psi^2}. \tag{23}$$

Given another observable B with deviation operator $\bar{B}$ and uncertainty ΔB, we consider the non-hermitian operator Z

$$Z := \frac{\bar{A}}{\Delta A} + i\frac{\bar{B}}{\Delta B}. \tag{24}$$

Using the scalar product we obtain

$$\langle Z\psi|Z\psi\rangle = \langle\psi|Z^\dagger Z|\psi\rangle = \langle\psi|(\frac{\bar{A}}{\Delta A} - i\frac{\bar{B}}{\Delta B})(\frac{\bar{A}}{\Delta A} + i\frac{\bar{B}}{\Delta B})|\psi\rangle \geqslant 0. \tag{25}$$

Further calculations yield

$$\langle\psi|\frac{\bar{A}^2}{(\Delta A)^2}|\psi\rangle + \langle\psi|i\frac{\bar{A}\bar{B} - \bar{B}\bar{A}}{\Delta A\Delta B}|\psi\rangle + \langle\psi|\frac{\bar{B}^2}{(\Delta B)^2}|\psi\rangle \geqslant 0 \Leftrightarrow \tag{26}$$

$$\Leftrightarrow 2 \geqslant -i\frac{\langle\psi|[\bar{A},\bar{B}]|\psi\rangle}{\Delta A\Delta B} = -\,i\frac{\langle\psi|[A-\langle A\rangle, B-\langle B\rangle]|\psi\rangle}{\Delta A\Delta B} = -i\frac{\langle\psi|[A,B]|\psi\rangle}{\Delta A\Delta B}, \tag{27}$$

which is already the uncertainty relation for observables.

$$\Delta A\Delta B \geqslant \frac{1}{2}|\langle[A,B]\rangle| \tag{28}$$

By solving this inequality for the position and momentum operator we get the known Heisenberg uncertainty relation

$$\Delta X\Delta P \geqslant \frac{\hbar}{2} \sim \hbar. \tag{29}$$

In this derivation we used the expectation value of an operator, which corresponds to the measurement of a quantum particle. The outcome states that there is a non vanishing disturbance for measurements associated with the non-commuting observables[1], but we can still try to predict and control the disturbances to get better results. To get a consistent interpretation the second assumption (2) is needed. This imposes an inherent and unavoidable limitation on the precision of all measurements[2].

[1]That doesn't necessary mean that the measurements where taken simultaneous. More on this topic can be found in chapter 4.4

[2]One may ask whether this limitation is caused by the measurement or is inherent to the particle. Are we looking at a blurry picture or an image of fog? Bohm stated in his book "Quantum Theory" that if both assumptions (1) & (2) are made then this property must already be inherent to the particle.

Because nothing can be said about the particle before a measurement is performed, the Copenhagen interpretation must be nonlocal. The particle is described by a wave function, which in turn is used to describe the probability in every point in space. At the instant the measurement is performed, the wave function collapses. This means that the probability changes in every point in space, which is a case of nonlocal behavior.
The assumption (2) is essentially the description of nonreality. One cannot assign a definite state to the particle before the measurement because it is inherently unpredictable, uncontrollable and unanalyzable and therefore cannot be described by hidden variables, which implies that the Copenhagen interpretation is not real.

4.2 de Broglie-Bohm Theory

The Copenhagen interpretation is consistent in itself and the most symmetric interpretation of quantum theory we know. But for some reasons, which will be discussed later, Bohm wrote a paper [10] in which he presented a new interpretation of quantum mechanics. Back in 1951 it was the first attempt to look at quantum mechanics from an different perspective. He presented an interpretation based on an idea of de Broglie and which uses hidden variables to describe a particle guided by a wave function. It is still accepted as a valid interpretation but is nonlocal and therefore not bounded by a Bell inequality.
The first question we want to answer is why Bohm thought there was a need for another interpretation. We shall first try to answer how an interpretation can be verified.
In principle, a chosen interpretation determines which mathematical formalisms are applicable. If there is only one mathematical formalism it is possible, to test the combination of the interpretation and the mathematical formalism through experiment. In case the results of an experiment are inconsistent, a change of the mathematical formalism is required. Normally, the interpretation changes when new postulates are needed to make the mathematical formalism consistent with the experiment, but if there is more then one mathematical formalism that are consistent with the interpretation, the verification by experiment is impossible.
The Copenhagen interpretation is based on the postulates (1) & (2), but does not impose sufficient restrictions on the mathematical formalism to permit experimental verification according to Bohm. In his paper [10], Bohm argues as follows:

> *..., one can contemplate particular arbitrary changes on the Hamiltonian operator, including, for example, the postulation of an unlimited range of new meson fields each having any conceivable rest mass, charge, spin, magnetic moment, etc. And if such postulates should*

prove to be inadequate, it is conceivable that we may have to introduce nonlocal operators, nonlinear fields, S-matrices, etc.
(Phys. Rev. 85, 1952, p.166)

Regardless of the aforementioned changes, we can still stick to the Copenhagen interpretation. Even if the Copenhagen interpretation is wrong, we won't be able to conclude that from an experiment.
Bohm's idea was to build a hidden variable theory (HVT) instead of using the postulates (1) & (2) and then find an experiment which depends on the hidden variables to verify their existence[3].
Bohm himself published a HVT in the same paper and proposed an alternative interpretation. As mentioned before, the de Broglie-Bohm theory is not bounded by a Bell inequality. To give a more detailed answer, we are going to explain the basics of Bohmian mechanics.

4.3 Introduction to the de Broglie-Bohm Theory

The first idea of Bohm's interpretation can be understood by looking at the Schrödinger equation applied to a wave function in spherical coordinates.

$$\Psi(x,t) = R(x,t)e^{\frac{i}{\hbar}S(x,t)} \tag{30}$$

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V(x)\Psi \tag{31}$$

With further calculation (see appendix A) one can derive the following two equations.

$$\frac{\partial R}{\partial t} = -\frac{1}{-2m}(R\nabla^2 S + 2\nabla R\nabla S) \tag{32}$$

$$\frac{\partial S}{\partial t} = -\left(\frac{(\nabla S)^2}{2m} + V(x) - \frac{\hbar^2\nabla^2 R}{2mR}\right) \tag{33}$$

By introducing the probability density $P(x) = R^2(x)$ we can rewrite this as follows.

$$\frac{\partial P}{\partial t} + \nabla\left(P\frac{\nabla S}{m}\right) = 0 \tag{34}$$

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V(x) = \frac{\hbar^2}{4m}\left[\frac{\nabla^2 P}{P} - \frac{1}{2}\frac{(\nabla P)^2}{P^2}\right] \tag{35}$$

[3]Even a failure to prove the existence of these hidden variables does not vitiate the Copenhagen interpretation.

In the classical case ($\hbar \to 0$), S(x,t) is the solution of the Hamilton-Jacobi equation

$$H + \frac{\partial S}{\partial t} = 0, \tag{36}$$

where H is the Hamiltonian.
The Hamilton-Jacobi formalism is another formulation of classical mechanics and is equivalent to Newton Law's of motion, Lagrangian mechanics and Hamiltonian mechanics.
As quoted in Bohm's paper [10], there exists a theorem in classical mechanics stating that for an ensemble of particle trajectories, which are solutions of the equation of motion and normal to some particular level set[4] of S, then the trajectories are normal to all other level sets of S as well. Moreover, the velocity vector $\vec{v}(x)$ is proportional to the gradient of S at x.

$$\vec{v}(x) = \frac{\nabla S(x)}{m} \tag{37}$$

Looking again at equation (34), we can rewrite it to a conservation law:

$$(34) \Rightarrow \frac{\partial P}{\partial t} + \nabla(P\vec{v}) = 0 \tag{38}$$

By interpreting P as a probability density we can describe $P\vec{v}$ as the mean current of the particles in an ensemble. Then (34) represents the conservation of probability.

In the quantum mechanical case ($\hbar \neq 0$) the term

$$U(x) = -\frac{\hbar^2}{4m}\left[\frac{\nabla^2 P}{P} - \frac{1}{2}\frac{(\nabla P)^2}{P^2}\right] \tag{39}$$

appears. If we interpret it as a new potential which acts on a particle, we can use the same interpretation as in the classical case ($\hbar \to 0$). Usually U(x) is called the quantum potential.
The description of path and momentum of a particle still remains in quantum mechanics. Since the particle has a given location and momentum at the same time, the question if this violates the uncertainty relation arises quite naturally.

[4] A level set for S is a set of the form $\{(x,t) \mid S(x,t) = c\}$ for some constant c.

4.4 The de Broglie-Bohm Theory and the Uncertainty Relation

One can obtain another perspective of the uncertainty principle by thinking about measurement. For this it is necessary to determine the magnitude of the average and the mean square value of the position measurements of an ensemble of particles with the same initial wave function $\Psi_0(x)$.

$$\langle x\rangle_n = \frac{1}{n}\sum_{i=1}^{n} x_i(t) \tag{40}$$

$$(\Delta x)_n^2 = \frac{1}{n}\sum_{i=1}^{n} \left[x_i(t) - \langle x\rangle_n\right]^2 \tag{41}$$

This can also be done with momentum measurements in the x-direction for the same initial wave function.

$$\langle p_x\rangle_n = \frac{1}{n}\sum_{i=1}^{n} p_{xi}(t) \tag{42}$$

$$(\Delta p_x)_n^2 = \frac{1}{n}\sum_{i=1}^{n} \left[p_{xi}(t) - \langle p_x\rangle_n\right]^2 \tag{43}$$

Furthermore we define the following functions.

$$\Delta x = \lim_{n\to\infty} (\Delta x)_n \tag{44}$$

$$\Delta p_x = \lim_{n\to\infty} (\Delta p_x)_n \tag{45}$$

The main point of this experimental view is the absence of any statement of simultaneous measurement of position and momentum on a single particle. Moreover, the assumption that a position measurement causes an unpredictable and uncontrollable distortion in the momentum of a particle is not valid in this context.
In the de Broglie-Bohm theory, the wave function in momentum space, obtained by the Fourier transformation of the wave function in position space, doesn't carry any information regarding the actual momentum. It can only be used to describe the measured momentum. However, the actual momentum - and therefore the velocity - is still governed by

$$\vec{p} = \nabla S. \tag{46}$$

Note that the position representation for the wave function is the only representation which inherits the information about the actual position.

With this view of the uncertainty relation in mind and with some further calculations (see [12], chapter 8.5), one arrives at a description of the uncertainty relation, which states that it represents the information that can be extracted from a particle through measurements.

The following passage from the paper "Understanding Bohmian mechanics: A dialogue"[13] sheds some light on what is meant by actual momentum and measured momentum.

Alice: *OK, I'll give a different example. Suppose a particle is confined between two impermeable walls. Its wave function is a multiple of* $e^{ikx} + e^{-ikx}$, *where* k *is chosen so that the wave function vanishes at the walls. Again, the Bohmian particle stands still.*

Bob: *Yes.*

Alice: *But, quantum mechanics says the momentum is, up to small corrections, either* $\hbar k$ *or* $-\hbar k$, *so the particle can't be at rest.*

Bob: *The word "momentum" doesn't have a meaning.*

Alice: *But, we can measure the momentum.*

Bob: *Tell me how you measure the momentum.*

Alice: *Take away the walls and let the particle move freely for an amount of time. Then, detect its position. If the amount of time was large enough and the distance between the walls small enough, we know quite precisely how far the particle traveled. Now, divide by time and multiply by mass.*

Bob: *The result of this experiment is perfectly predicted by Bohmian mechanics. The trajectory of the Bohmian particle in your experiment looks like this: it is a smooth curve* $t \mapsto X(t)$ *which is constant,* $X(t) = x_0$, *before the walls are removed and which is asymptotic to the line* $X(t) \approx (\hbar k/m)t$ *+ constant if* x_0 *lies right of the center, and asymptotic to the line* $X(t) \approx -(\hbar k/m)t$ *+ constant if* x_0 *lies left of the center. Each of these two cases occurs with probability 1/2.*

Alice: *So the particle slowly accelerates until it reaches the velocity* $\pm\hbar k/m$*?*

Bob: *Yes.*

Alice: *But, I always imagined the particle going back and forth between the walls, having velocity either* $\hbar k/m$ *or* $-\hbar k/m$, *each half of the time.*

Bob: *That's Newtonian mechanics, and Newtonian mechanics is refuted by experiment.*

Alice: *But, Newtonian mechanics for our experiment makes the true prediction that the particle will, with a certain fixed velocity, move either in the x or in the $2x$ direction after the walls have been removed. So, why should we give up Newtonian mechanics in this case?*

Bob: *Because it can't cope with other experiments, such as the double-slit.*

4.5 Reality and Nonlocality in the de Broglie-Bohm Theory

In the de Broglie-Bohm theory, quantum mechanics is described by the wave function and a particle[5] with a specific path and a momentum determined by S.

$$\dot{x} = \frac{\nabla S}{m}\bigg|_{x=x_0} \tag{47}$$

The only variable which can be freely chosen is the position at a specific time. Fixing this position results in a uniquely determined path from the initial point up to the moment of observation. In this case the position is therefore a hidden variable to describe the physical state of the particle before measurement. In particular, the de Broglie-Bohm theory is a real interpretation in the sense of the ERP paper [1].
As a consequence of these observations, the de Broglie-Bohm theory can be regarded as a HVT which can usually be bounded by Bell inequalities. However it is nonlocal and thus the Bell inequalities are not applicable to it. To see that the de Broglie-Bohm theory is nonlocal one has to consider a many-body situation. The necessary equations can be obtained through the Schrödinger equation like in the single body case.

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[\sum_{i=n}^{n}\frac{-\hbar^2}{2m_i}\nabla_i^2 + V(x_1, ..., x_n, t)\right]\Psi \tag{48}$$

Here m_i denotes the mass of the i-th particle and V is the classical potential energy, which includes interparticle as well as external potentials. The value of Ψ is uniquely determined once we specify its value at $t = 0$.
By using $\Psi = Re^{iS/\hbar}$ and separating the resulting expression into its real and

[5] Remember that in the Copenhagen interpretation no statement is given whether the transfered quantum is a particle or ware. Only a wave function is given, which evolutes according to the Schrödinger equation.

imaginary part one obtains

$$\frac{\partial S}{\partial t}+\sum_{i=1}^{n}\frac{(\nabla_i S)^2}{2m_i}+Q+V=0 \tag{49}$$

$$\frac{\partial R^2}{\partial t}+\sum_{i=1}^{n}\nabla_i\left(R^2\frac{\nabla_i S}{m_i}\right), \tag{50}$$

$$\text{where}\qquad Q=\sum_{i=1}^{n}-\hbar^2\frac{\nabla_i^2 R}{2m_i R} \tag{51}$$

The function Q is called the many-body quantum potential energy.
As a generalization of the one body theory we now treat (49) as a generalized Hamilton-Jakobi equation for a system with n particles. Their momentum is given by the following equations.

$$p_i(x_q,...,a_n,t)=\nabla_i S(x_1,...,x_n,t)\quad i=1,...,n \tag{52}$$

To solve these equations for a particular trajectory $x_i(t)$, we have to specify the initial positions x_{i0} for all the particles.
The expression for the quantum potential Q is the cause of the nonlocal connection between the particles for a specific time. Like in classical physics, all the particles have an instantaneous change in their path when the position of one particle is altered. What is different from the classical potential is that its quantum mechanical counterpart doesn't have to loose strength with a bigger separation of two particles. It is even possible for it to get stronger. In particular, particles cannot be neglected regardless of their distance.

5 Criticism of the de Broglie-Bohm Theory

In this chapter we are going to examine some points mentioned in the paper "Why isn't every physicist a Bohmian?"[14]. For structural reasons it is nice to separate the arguments against the de Broglie-Bohm theory into metaphysical and theory immanent arguments.

5.1 Metaphysical Debate

5.1.1 Ockham's Razor

Ockham's razor states that if two theories are equivalent the one which uses fewer premises should be preferred.

In case of de Broglie-Bohm theory, which predicts the same outcomes as the Copenhagen interpretation, many physicist use Ockham's razor because they find the complications caused by the additional particle path needless and unappealing.
On the other hand it is questionable whether Ockham's razor can be applied, because the de Broglie-Bohm theory eliminates the postulations related to the measurement process (e.g. the collapse of the can be calculated). It also provides a completely new interpretation of quantum phenomena in which probability plays no fundamental role.

5.1.2 Asymmetry in de Broglie-Bohm Theory

One can dislike the fact that the de Broglie-Bohm theory is asymmetric in respect to position and momentum, like Heisenberg and Pauli did. This asymmetry refers to the fact that the real position is reflected by the modulus squared of the wave function in the position representation. This isn't true for the momentum or other observables in their representation.
Furthermore, even though the wave function causes the non-classical behavior of the particle, there is no influence the other way round. This asymmetry is due to the homogeneity of the Schrödinger equation. Bohm already mentioned this in his paper [10] and suggested to change the simple equivalence between ∇S and p. Such a modification should only influence the wave function in a range covered by the uncertainty relation ($< 10^{-13} cm$) and should also be negligible outside this region. This implicates that the wave function can be absorbed and emitted by particles, like an electromagnetic field. It is even possible to test it in the neighborhood of the domain of the uncertainty relation.
Dürr suggested to interpret the wave function like the Hamiltonian in classical mechanics and not as an element of reality. He also motivated the quantum potential energy using symmetries [15] and did not consider it as the crucial key element between classical mechanics and quantum theory. This gave the wave function a central role in the de Broglie-Bohm theory. The quantum potential energy is only a byproduct to see the relation to classical mechanics. His interpretation of the de Broglie-Bohm theory is usually called "Bohmian Mechanics".

5.2 Theory Immanent Debate

5.2.1 The "Surreal Trajectory" Objection

An objection to the de Broglie-Bohm theory was brought up by Englert, Scully, Süssmann and Walther (ESSW) [16]. By altering the delayed-choice double slit experiment they claimed that the trajectories are "surreal".

We will therefore review the delayed-choice double slit experiment invented by Wheeler [18]. This experiment arranges two detectors, each of which is pointed to a slit in order to detect from which the particle originated. We know that the interference pattern cannot be obtained in such a setup. Now Wheeler's suggestion is to switch the interference-detector (screen) and the slit-detectors randomly after the particle has passed the slit. This modification leads us to the question if the delay of the choice of the observation method has an effect on the particle when it passes the double slit. Furthermore, how can the particle know the random future and pass through the slit as either particle or wave.
In the de Broglie-Bohm theory the calculated paths for the double slit experiment are symmetric, as can be seen in the following images.

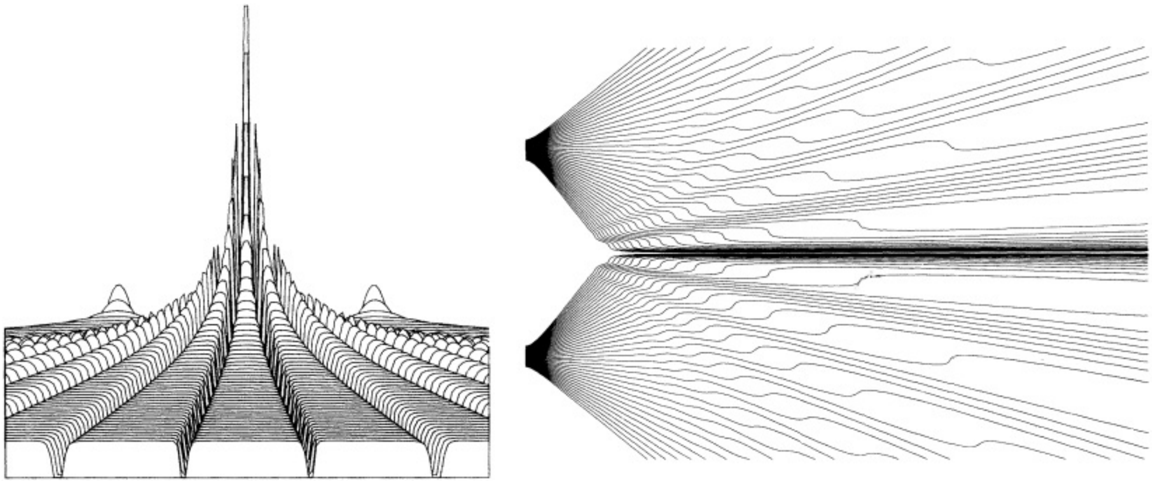

Fig. 4: left: quantum potential energy after the double slit — right: trajectories of particles after the double slit (picture from: Holland - The Quantum Theory Of Motion [12])

All particles going through the right slit stay on the right side and the other way round. Adding the slit-detectors breaks the symmetry ([19] p.111 for more details), which allow the paths to cross the mid plane.
ESSW swapped of slit-detectors for quantum optical devices called "which-way detectors". These detectors respond to the transition of a single particle without affecting the translational part of the wave function. Similarly to the previous setup, the interference pattern should vanish because of the possibility to determine where the particle was when it has passed the double slit. But the symmetry for the path of the particles shouldn't change since the wave function doesn't change. Thus a paradoxic situation can emerge where the left "which-way detector" fires even though the particle passed on the right hand side. Therefore ESSW

proposed to deny the realistic property of the path and called it “surrealistic”. This gedanken experiment ignited a big debate [17] which won’t be reviewed here. Some physicists argued that the “which-way detector” is not a measurement device in the de Broglie-Bohm theory because the wave function isn’t changed in the measurement process. In fact, with the nonlocal properties of the de Broglie-Bohm theory it can be shown that a “which-way detector” can fire even when no path passes through it. [20],[21]
The details to resolve the “surreal trajectory problem” in the de Broglie-Bohm theory can be found in [22].

6 Conclusion

The work at hand tries to present some reasons why Bell’s derivation of his inequalities was a profound step in physics. By deriving a boundary condition for real & local theories, Einstein’s arguments become physically testable. To demonstrate that the Bell inequalities don’t bound every HVT, the real but nonlocal properties of the de Broglie-Bohm theory were described by analyzing the path of a particle and the quantum potential. It is real because we can calculate the state of the particle before it was measured with the deduced formula. Nonlocality is obtained by deriving the quantum potential which has a stronger nonlocal behavior then a classical potential. Because locality is a condition for Bell inequalities, the de Broglie-Bohm theory is not bound by it.
To provide further insight into the de Broglie-Bohm theory, the additional path and momentum for a particle between the beginning and the end of a quantum mechanical experiment as well as the asymmetric treatment of the observables was introduced. Moreover, a chapter about the Copenhagen interpretation was inserted to make the differences between those two interpretations as clear as possible.
Furthermore, the work should make clear that an equivalence between the de Broglie-Bohm theory and the Copenhagen interpretation exists, because their predictions are exactly the same for all experiments. This seems natural because both predictions are based on the Schrödinger equation.

7 Appendix

7.1 Appendix A

$$i\hbar\frac{\partial R}{\partial t}e^{\frac{i}{\hbar}S} - Re^{\frac{i}{\hbar}S}\frac{\partial S}{\partial t} = -\frac{\hbar^2}{2m}\nabla\left(\nabla Re^{\frac{i}{\hbar}S} + \frac{i}{\hbar}Re^{\frac{i}{\hbar}S}\nabla S\right) + V(x)Re^{\frac{i}{\hbar}S} =$$
$$= -\frac{\hbar^2}{2m}\left(\nabla^2 Re^{\frac{i}{\hbar}S} + 2\frac{i}{\hbar}e^{\frac{i}{\hbar}S}\nabla R\nabla S - \frac{1}{\hbar^2}Re^{\frac{i}{\hbar}S}(\nabla S)^2 + \frac{i}{\hbar}Re^{\frac{i}{\hbar}S}\nabla^2 S\right) + V(x)Re^{\frac{i}{\hbar}S}$$

If we split this equation into its real and imaginary part, we find:

$$\frac{\partial R}{\partial t} = -\frac{1}{-2m}(R\nabla^2 S + 2\nabla R\nabla S) \tag{53}$$
$$\frac{\partial S}{\partial t} = -\left(\frac{(\nabla S)^2}{2m} + V(x) - \frac{\hbar^2\nabla^2 R}{2mR}\right) \tag{54}$$

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