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The Appearance of a Classical World in Quantum Theory

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1 Introduction

The "paradox" is only a conflict between reality and your feeling of what reality "ought to be".

-Richard P. Feynman

The main features of quantum theory, *quantum entanglement* and *quantum superposition*, are at the same time responsible for most of the confusion. They are the reason why quantum theory seems to be in conflict with our experience of a macroscopic (classical) world, since any attempt of extrapolating the formalism to our macroscopic world immediately produces paradoxical situations. Perhaps the most famous one is Schrödinger's 'quite absurd' thought experiment, known as the Schrödinger's cat paradox[1].

In fact, our macroscopic world is built up by microscopic objects for which quantum theory has shown to be true in various experiments. However, the classical macroscopic world does not show any quantum features. The important and interesting question arises: How does the classical world emerge out of quantum physics? This fundamental question still remains unanswered. However, there exist several approaches to the quantum-to-classical transition. The aim of this work is to discuss some of these approaches, namely the *decoherence program*, a specific *collapse theory* and the quite new approach of *coarse-grained measurements*.

In chapter 2, we will start with explaining the measurement problem. Since every measurement apparatus belongs to the macroscopic realm of our every-day life, the question of how to describe a measurement in quantum theory is strongly connected to the problem of the quantum-to-classical transition. We will explain in some detail the problems arising when treating a measurement process entirely quantum-mechanical.

In chapter 3, we will explain the main ideas of the decoherence program and show what decoherence can account for the quantum-to-classical transition. We will see that a system constantly interacting with some environment will not be able to show its quantum nature anymore.

In chapter 4, we will introduce the idea of collapse theories. It will be shown that the modification of the Schrödinger equation can lead to a fundamental breakdown of quantum superpositions on the macroscopic scale.

In chapter 5, we will introduce the notion of macroscopic realism and discuss the quite new approach of coarse-grained measurements. We will see that imprecise measurements are limiting the observability of quantum phenomena and the quantum nature of certain systems becomes unobservable in the macroscopic (classical) limit.

2 The Measurement Problem

Since every realistic measurement apparatus belongs to the macroscopic realm, the problem of a quantum-to-classical transition is strongly connected with the problem of how to describe a measurement in quantum theory, i.e. the *measurement problem*. Following [2], the measurement problem can be composed by three important parts:

1. *The problem of the preferred basis.* Which observables are accessible?
2. *The problem of nonobservability of interference.* Why is it so difficult to observe interference effects, especially on a macroscopic level?
3. *The problem of definite outcomes.* Why do measurements have outcomes at all and what selects a particular outcome?

In literature, it is often the case that only the third problem is referred to as "the" measurement problem. However, talking about outcomes will not make any sense if the set of possible outcomes is not clearly defined.

At this point, it is already worth mentioning that the first and the second problem are resolved by the *decoherence program* and several *collapse models*, respectively. Collapse models also provide a resolution for the third problem by modifying the Schrödinger equation, whereas in the decoherence program the problem is left to the different interpretations of quantum theory. We will discuss the two approaches in more detail in sections 3 and 4.

2.1 Von Neumann Measurement

To give a more detailed explanation of the problems mentioned above, it is reasonable to introduce the *von Neumann measurement* as an ideal quantum measurement. Von Neumann's idea [3] was to describe the measurement process in entirely quantum-mechanical terms. This means that not only the measured system but also the apparatus should be treated as a quantum-mechanical object. Note that this approach is in sharp contrast to the Copenhagen interpretation, where the measurement apparatuses are excluded from the quantum-mechanical description and are postulated as intrinsically classical.

As a result of describing the system and the apparatus in quantum-mechanical terms, they can be represented by vectors in a Hilbert space, respectively. Let us denote the Hilbert space of the system \mathcal{S} by $\mathcal{H}_{\mathcal{S}}$ with basis vectors $\{|s_i\rangle\}$ and the Hilbert space of the apparatus \mathcal{A} by $\mathcal{H}_{\mathcal{A}}$ with basis vectors $\{|a_i\rangle\}$. Since we want our apparatus to measure the state of the

system \mathcal{S} , it is reasonable to think of the states $\{|a_i\rangle\}$ as "pointer" states. This means each $|a_i\rangle$, representing a pointer position "i" of the apparatus, corresponds to a state $|s_i\rangle$ of the System \mathcal{H}_S . Assuming the apparatus starts in an initial "ready" state $|a_r\rangle$ before the measurement takes place, the evolution of the system \mathcal{SA} , described by the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_A$, will be of the form

$$|s_i\rangle |a_r\rangle \longrightarrow |s_i\rangle |a_i\rangle \quad \forall i. \quad (2.1)$$

We see from (2.1) that the measurement interaction does not change the state of the system. Hence, a one-to-one correspondence between the state of the system and the pointer state has established. Due to this assumption, the von Neumann measurement scheme (2.1) is often called *ideal*.

Now, in the more general case, the system starts in a state of the form

$$|\psi\rangle = \sum_i c_i |s_i\rangle. \quad (2.2)$$

In this case, due to the linearity of the Schrödinger equation and (2.1), the combined system \mathcal{SA} will evolve according to

$$|\psi\rangle |a_r\rangle = \left(\sum_i c_i |s_i\rangle \right) |a_r\rangle \longrightarrow |\phi\rangle = \sum_i c_i |s_i\rangle |a_i\rangle. \quad (2.3)$$

It can easily be seen, that the right-hand side of (2.3) represents an entangled state for at least two non-vanishing c_i . For this reason, it is no longer possible to describe neither the system nor the apparatus individually. The superposition, initially contained by the system only, is now carried by the composite system \mathcal{SA} . Hence, the apparatus has become involved in the superposition. Therefore, it is not clear at all in how far this scheme can be regarded as a measurement in the usual sense and is often called *premeasurement* for this reason.

2.2 The Problem of the Preferred Basis

Considering the von Neumann scheme stated above, one can easily see, that the expansion of the composite state on the right-hand side of (2.3) is not uniquely defined. And so the observable is not. To see this, let us introduce the *Schmidt decomposition*[4] for a pure bipartite state.

Theorem. *Schmidt Decomposition Theorem*

Consider two systems \mathcal{A} and \mathcal{B} with Hilbert spaces $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$, respectively. An arbitrary pure state $|\psi\rangle$ of the composite system \mathcal{AB} with Hilbert space $\mathcal{H}_{\mathcal{AB}}$ can always be written in the form

$$|\psi\rangle = \sum_i \sqrt{\lambda_i} |a_i\rangle |b_i\rangle. \quad (2.4)$$

The *Schmidt states* $|a_i\rangle$ and $|b_i\rangle$ form local (orthonormal) bases of $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$, respectively. The *Schmidt coefficients* λ_i fulfill $\sum_i \lambda_i = 1$ and they are uniquely defined by $|\psi\rangle$, whereas $|a_i\rangle$ and $|b_i\rangle$ are not.

Proof. Given two arbitrary local bases $\{|\varphi_i\rangle\}$ and $\{|\phi_i\rangle\}$ in the spaces $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$, respectively. Then, any pure state in $\mathcal{H}_{\mathcal{AB}}$ can be written as

$$|\psi\rangle = \sum_{ij} d_{ij} |\varphi_i\rangle |\phi_j\rangle. \quad (2.5)$$

The *singular value decomposition* provides that there exist two unitary matrices U and V such that $(d_{ij}) = U\Lambda V^\dagger$, where Λ is a rectangular diagonal matrix with non-negative real numbers $\sqrt{\lambda_k}$ on the diagonal. Now, it is possible to rewrite (2.5):

$$|\psi\rangle = \sum_{ijk} U_{ik} \underbrace{\Lambda_{kk}}_{\sqrt{\lambda_k}} V_{jk}^* |\varphi_i\rangle |\phi_j\rangle \quad (2.6)$$

Note that $|a_k\rangle \equiv \sum_i U_{ik} |\varphi_i\rangle$ and $|b_k\rangle \equiv \sum_j V_{jk}^* |\phi_j\rangle$, the Schmidt decomposition (2.4) follows at once.

□

Let us now turn back to the problem of the preferred basis. It is reasonable to demand from the apparatus states $|a_i\rangle$ to be mutually orthogonal. This ensures that they represent distinguishable outcomes of the measurement. Furthermore, we want the states of the system $|s_i\rangle$ to form an orthonormal basis. Since, in general, we assume the system observables to be hermitian, this should be possible. In this case, the right-hand side of (2.3) can be identified with the Schmidt representation. It can be shown that the Schmidt decomposition is unique, if and only if, all the coefficients c_i are different from one another. In general, this is not the case. Thus, the set of possible outcomes is not uniquely defined.

As an easy example one can take two spin- $\frac{1}{2}$ particles in an maximally entangled Bell state

$$|\psi^-\rangle = \frac{1}{\sqrt{2}} (|0\rangle_{\mathcal{S}} |1\rangle_{\mathcal{A}} - |1\rangle_{\mathcal{S}} |0\rangle_{\mathcal{A}}), \quad (2.7)$$

where $|0\rangle$ and $|1\rangle$ represent the spin eigenstates in the z -direction. Let us assume that the first particle represents the system \mathcal{S} and the second one the apparatus \mathcal{A} . Then, (2.7) can be considered to be the final state of a von Neumann measurement (2.3). It is well known that the state (2.7) looks the same in the spin- $\frac{1}{2}$ basis of the x, y and z direction, respectively. For example, in the x -direction (2.7) reads

$$|\psi^-\rangle = \frac{1}{\sqrt{2}} (|-\rangle_{\mathcal{S}} |+\rangle_{\mathcal{A}} - |+\rangle_{\mathcal{S}} |-\rangle_{\mathcal{A}}) \quad (2.8)$$

with $|\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)$. If one interprets the apparatus \mathcal{A} as a measuring device for the system \mathcal{S} , one will have to conclude from (2.7) and (2.8) that the apparatus is capable of measuring both, the spin of \mathcal{S} in the z - and the x -direction. However, the rules of quantum mechanics do not allow to measure simultaneously two non-commuting observables (in this case σ_x and σ_z). Furthermore, it is a desired feature of measuring devices to measure only particular quantities. Unfortunately, such a "preferred observable" is not given by the von Neumann measurement.

This example illustrates the *problem of the preferred basis*. It is at least as important as the *problem of outcomes* for any quantum measurement.

2.3 The Problem of Outcomes

The final state of the von Neumann scheme (2.3) represents a superposition of system-apparatus product states. We have to point out, that this situation is fundamentally different from a classical ensemble, in which the system-apparatus *is* in one of this product states $|s_i\rangle |a_i\rangle$ but we simply do not know in which. This difference is one of the main resources of confusion when dealing with quantum theory. However, our experience tells us that every measurement leads to a definite outcome, i.e. a definite pointer state $|a_i\rangle$. The confusion arises from the difficulty to combine this two situations. Why do we have outcomes at all, rather than a superposition of pointer states? And what selects a *specific* outcome in each run of the experiment? These two questions can be considered as *the problem of outcomes*. The problem has its origin in the question of how to realize a particular result in a probabilistic

theory. In contrast to classical physics the probabilistic aspect is an intrinsic feature of quantum mechanics. As a result, the problem of outcomes can be seen as intrinsic to quantum mechanics itself. For this reason standard quantum mechanics is not able to provide any resolution and the problem has to be shifted to an interpretative level.

3 Decoherence

The *decoherence theory* is the most commonly used approach when studying the quantum-to-classical transition. Let us therefore start with giving a brief introduction of the decoherence program. Thereby we will limit ourselves to the general concepts that allow us to see the major consequences of this approach. A more detailed view on decoherence theory, including some explicit models, can be found in [6, 2] for example.

The underlying idea of *decoherence* is that in general we cannot ensure a system to be perfectly isolated. In classical mechanics, the interaction with the environment is considered to be merely some kind of "disturbance". However, in quantum mechanics, entanglement makes things more complicated. Since the interaction with the environment can make a big difference due to entanglement, it is in most cases not possible to speak of an isolated or *closed* system. The idea is that in general the environment (which is considered to have a very large number of degrees of freedom, inaccessible for all practical purposes) constantly monitors a macroscopic system and therefore locally suppresses interference of macroscopically distinct states. Thus, decoherence is a theory of *open quantum systems*. Furthermore, it is remarkable that the idea of decoherence and the resulting formalism does not require any modification or interpretation of quantum mechanics. It is a direct consequence of it.

3.1 General Formalism

Given the main ideas of decoherence, let us now show how the statement given above can be described in terms of quantum mechanics. Assume that we have a system of interest \mathcal{S} interacting with its environment \mathcal{E} . Furthermore, we assume the interaction to be of the von Neumann type (see chapter 2), i.e. the environment does not change the state of the system. Consider the system-environment state to be initially in the product form

$$|\psi\rangle|E_0\rangle = \left(\sum_i c_i |s_i\rangle \right) |E_0\rangle, \quad (3.1)$$

its evolution can then be described by

$$\left(\sum_i c_i |s_i\rangle \right) |E_0\rangle \longrightarrow \sum_i c_i |s_i\rangle |E_i\rangle \quad (3.2)$$

with system states $|s_i\rangle$ and states of the environment $|E_i\rangle$. Note, the evolution (3.2) is similar to evolution in the von Neumann scheme. In this sense one can say "the environment measures the system". As mentioned before, this means that the superposition, initially carried by state of the system, is now carried by the system-environment state. Unfortunately, caused by the huge number of degrees of freedom carried by the environment, our observation is restricted to the system only. For this reason let us introduce the idea of *subsystems* and the *partial trace*¹.

Definition. *Reduced Density Matrix and Partial Trace.*

Given a composite state $|\psi\rangle_{\mathcal{AB}}$ in the Hilbert space $\mathcal{H}_{\mathcal{AB}} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ with its corresponding density matrix $\rho_{\mathcal{AB}}$, we can define the subsystems \mathcal{A} and \mathcal{B} , represented by the *reduced density matrices*

$$\rho_{\mathcal{A}(\mathcal{B})} \equiv Tr_{\mathcal{B}(\mathcal{A})} \rho_{\mathcal{AB}}, \quad (3.3)$$

where " $Tr_{\mathcal{B}(\mathcal{A})}$ " is called the *partial trace* over $\mathcal{B}(\mathcal{A})$. It contains all the information that can be extracted by an observer of system $\mathcal{A}(\mathcal{B})$.

Equipped with this definition, the state of the system after the interaction with the environment is represented by the reduced density matrix

$$Tr_{\mathcal{E}} \rho_{\mathcal{SE}} = \rho_{\mathcal{S}} = \sum_{i,j} c_j^* c_i \langle E_j | E_i \rangle |s_i\rangle \langle s_j|. \quad (3.4)$$

We can rewrite (3.4) to obtain:

$$\rho_{\mathcal{S}} = \sum_{i=j} |c_i|^2 |s_i\rangle \langle s_i| + \underbrace{\sum_{i \neq j} c_j^* c_i \langle E_j | E_i \rangle |s_i\rangle \langle s_j|}_{\text{interference terms}} \quad (3.5)$$

It can easily be seen from (3.5) that the interference terms depend on the overlap $\langle E_j | E_i \rangle$. Typically for macroscopically distinct states, the system-environment interaction will lead to distinguishable environmental states, i.e.

¹A more detailed view can be found in [2].

$\langle E_j | E_i \rangle \approx 0$. Therefore, we will have a loss of local phase coherence and the system can be described by the density matrix

$$\rho_S \approx \sum_i |c_i|^2 |s_i\rangle \langle s_i| \quad (3.6)$$

which is formally identical to a classical mixture. However, it is an important fact that the superposition has been destroyed only locally but is still carried by the system-environment state. Thus, one must not conclude that this process provides a resolution of the measurement problem. Since, however, "macroscopic" objects have to be treated as open quantum systems interacting with their environment in the way presented above, decoherence can at least explain why it is so difficult to observe interference on a macroscopic scale.

It is worth mentioning that the description of decoherence given above is seen to be true for the majority of macroscopic distinct states. However, (3.5) shows that the destruction of local interference depends only on the interaction with the environment. The formalism makes no *a priori* difference between the system of interest being either macroscopic or microscopic. Thus, coherence between macroscopically distinct states is not excluded *per se*.

In reality the composite system evolves according to the Schrödinger equation

$$|\psi\rangle |E_0\rangle \xrightarrow{t} e^{-\frac{i}{\hbar} H_{tot} t} |\psi\rangle |E_0\rangle \quad (3.7)$$

with the total Hamiltonian $H_{tot} = H_S + H_E + H_{int}$. The evolution (3.7) shows that the overlap $\langle E_j | E_i \rangle$ depends on the specific Hamiltonian. It is an important implication that the local loss of coherence is not a necessary consequence in a particular case.

In fact, under certain assumptions², one can derive a general equation for the time evolution of the system. The so called *master equation* can be written in the *Lindblad form*

$$\partial_t \rho_S = \mathcal{L} \rho_S = -i[H, \rho_S] + \sum_k^N \gamma_k \left(L_k \rho_S L_k^\dagger - \{L_k^\dagger L_k, \rho_S\} \right), \quad (3.8)$$

where $[,]$ and $\{ , \}$ denote the commutator and the anticommutator, respectively. The operators L_k are called *Lindblad operators* and γ_k are cor-

²No environmental memory and the existence of a semigroup.

responding positive rates. The *Liouville super operator* \mathcal{L} is the generator of a *quantum dynamical semigroup* $\{\mathcal{W}_t\}$, describing the time evolution of ρ_S :

$$\mathcal{W}_t : \rho_S(0) \longmapsto \rho_S(t). \quad (3.9)$$

It can be shown [7] that for hermitian operators L , a damping of the off-diagonal elements of ρ_S in the eigenbasis of L will occur. For a detailed discussion of master equations we refer to [7, 2].

3.2 Environment-Induced Superselection Rules

We have shown in the previous section that the decoherence mechanism follows the von Neumann scheme. As a result, we also have to deal with the problem of the preferred basis. Put it more explicitly, we have to deal with the question for which system-states $|s_i\rangle$ the interaction with the environment causes a local damping of interference and which remain stable under this interaction. The observable states or the *preferred states* will emerge dynamically as the ones that are most robust concerning the system-environment interaction, i.e. become least entangled with the environment. Since the other states decohere quite fast, this *stability criterion* leads to *effective superselection rules*. The term "superselection" has its origin in historically *postulated* superselection. It was used to exclude never observed states, e.g. the superposition of charges [8] or the spatial superposition of macroscopic objects. Some of this postulated "exact" superselection rules can be replaced by environment induced superselection rules.

Let us consider a simple model introduced by Zurek [9] to show more explicitly how decoherence can lead to effective superselection rules. Assume a system \mathcal{S} with Hilbert space \mathcal{H}_S is interacting with its environment \mathcal{E} with Hilbert space \mathcal{H}_E . The total Hamiltonian H_{tot} generating the evolution of the combined system \mathcal{SE} with Hilbert space $\mathcal{H}_{SE} = \mathcal{H}_S \otimes \mathcal{H}_E$ can be considered to be of the form

$$H_{tot} = H_S + H_E + H_{int} \quad (3.10)$$

with the interaction Hamiltonian H_{int} and the self-Hamiltonians of the system and the environment, respectively:

$$H_S = \sum_i \delta_i |s_i\rangle \langle s_i|, \quad (3.11)$$

$$H_E = \sum_j \epsilon_j |e_j\rangle \langle e_j|, \quad (3.12)$$

$$H_{int} = \sum_{ij} \gamma_{ij} |s_i\rangle \langle s_i| \otimes |e_j\rangle \langle e_j|. \quad (3.13)$$

Assuming the state vector of the combined system to be initially in a product state

$$\begin{aligned} |\varphi(t=0)\rangle &= |\phi_S\rangle \otimes |\psi_E\rangle \\ &= \left(\sum_i \alpha_i |s_i\rangle \right) \otimes \left(\sum_j \beta_j |e_j\rangle \right) \end{aligned} \quad (3.14)$$

and using units in which $\hbar = 1$, the time evolution is given by

$$\begin{aligned} |\varphi(t)\rangle &= \sum_{ij} \alpha_i \beta_j \exp[-it(\delta_i + \epsilon_j + \gamma_{ij})] |s_i\rangle \otimes |e_j\rangle \\ &= \sum_i \left[\alpha_i |s_i\rangle \otimes \left(\sum_j \beta_j \exp[-it(\delta_i + \epsilon_j + \gamma_{ij})] |e_j\rangle \right) \right] \\ &\equiv \sum_i \alpha_i |s_i\rangle \otimes |\psi_{Ei}(t)\rangle. \end{aligned} \quad (3.15)$$

Like in the previous section, our observation is restricted to the system. For this reason we calculate the partial trace

$$\rho_S(t) = Tr_E |\varphi(t)\rangle \langle \varphi(t)| \quad (3.16)$$

and we obtain

$$\rho_S(t) = \sum_{ij} \rho_{ij}(t) |s_i\rangle \langle s_j| \quad (3.17)$$

with

$$\rho_{ii}(t) = |\alpha_i|^2 \quad (3.18)$$

$$\begin{aligned} \rho_{ij}(t) &= \alpha_i \alpha_j^* \langle \psi_{Ej}(t) | \psi_{Ei}(t) \rangle \\ &= \alpha_i \alpha_j^* \exp(-it(\delta_i - \delta_j)) \cdot \sum_k |\beta_k|^2 \exp(-it(\gamma_{ik} - \gamma_{jk})). \end{aligned} \quad (3.19)$$

We can see from (3.18) that the diagonal elements are time-independent. The off-diagonal elements (responsible for interference), however, evolve in time according to (3.19) where the time-dependence is carried by the overlap

$$\begin{aligned}\langle \psi_{\mathcal{E}j}(t) | \psi_{\mathcal{E}i}(t) \rangle &= \exp(-it(\delta_i - \delta_j)) \cdot \sum_k |\beta_k|^2 \exp(-it(\gamma_{ik} - \gamma_{jk})) \\ &\equiv \exp(-it(\delta_i - \delta_j)) \cdot z_{ij}(t)\end{aligned}\quad (3.20)$$

with z_{ij} being the *correlation amplitude*. The properties of z_{ij} have been analyzed with great care in [9]. For our purpose it is important that $|z_{ij}| \ll 1$ for large³ t unless

$$\gamma_{ik} - \gamma_{jk} = 0 \quad \forall k. \quad (3.21)$$

Thus, interference between different eigenstates $|s_i\rangle$ will be locally suppressed and any pure state starting out as a superposition of different $|s_i\rangle$ will decay rapidly into a classical mixture. However, in general, it could be the case that several eigenvalues γ_{nk} are degenerated. Then, the corresponding eigenvectors form a subspace \mathcal{H}_n of the Hilbert space \mathcal{H}_S . As a result any state $|\varphi_n\rangle \in \mathcal{H}_n$ interacting with the environment will evolve according to

$$|\varphi_n\rangle \otimes |\psi_{\mathcal{E}}\rangle \longrightarrow |\varphi_n\rangle \otimes |\psi_{\mathcal{E}\varphi}(t)\rangle \quad (3.22)$$

and therefore remain stable. That means that the Hilbert space of the system can be decomposed into a direct sum

$$\mathcal{H}_S = \bigoplus_n \mathcal{H}_n \quad (3.23)$$

and all *available* pure states lie within

$$\bigcup_n \mathcal{H}_n. \quad (3.24)$$

As a consequence, any *available* observable \mathcal{O} can be written as

$$\mathcal{O} = \sum_n o_n P_n \quad (3.25)$$

³For large environments, consisting of N non-interacting systems $\mathcal{H}_{\mathcal{E}} = \bigotimes_{\nu}^N \mathcal{H}_{\mathcal{E}}^{\nu}$ the correlation amplitude can be shown to be a product of correlation amplitudes: $z_{ij} = \prod_{\nu=1}^N z_{ij}^{\nu}$. Since each $|z_{ij}^{\nu}| < 1$ the damping appears very fast.

with P_n being the projection operators on the different subspaces \mathcal{H}_n , respectively. This is equivalent to the statement that all available observables have to fulfill

$$[\mathcal{O}, H_{int}] = 0 \quad (3.26)$$

which is sometimes called *commutativity criterion*.

This model is an demonstrative example for how the interaction with the environment superselects the most robust and therefore *quasiclassical* states. It can easily be extended to the case where a combined system-apparatus is interacting with its environment in the following way:

$$|s_i\rangle |a_i\rangle |E_0\rangle \longrightarrow |s_i\rangle |a_i\rangle |E_i\rangle \quad \forall i \quad (3.27)$$

In this case the interaction with the environment superselects the pointer states and therefore implies the selection of the system states that can be measured by the apparatus[2, 9].

The model presented above makes use of a specific Hamiltonian. In general, however, we have to deal with an arbitrary total Hamiltonian, describing the evolution of the system-environment state. Again, the stability criterion will lead to preferred system states. We can consider three different cases of how the preferred states emerge through the interaction with the environment[2, 10]:

1. *The quantum-measurement limit.* In this case the evolution of the system is dominated by the interaction Hamiltonian H_{int} ($H_{tot} \approx H_{int}$). The model shown above falls into this category and it is therefore easy to see that the preferred system states will be the eigenstates of the interaction Hamiltonian. In many cases, especially with macroscopic objects, the interaction with the environment depends on force-laws which depend on the distance. Thus, leading to position eigenstates (see for example the scattering model of Joos and Zeh [7]).
2. *The quantum limit of decoherence.* This is the case when the evolution of the system is dominated by the system's self-Hamiltonian in the sense that the highest energy available in the environment is smaller than the separation between the system's energy eigenstates - a situation often appearing in the microscopic regime. In this case the environment can only "monitor" constants of motion, i.e., the energy. Thus, the interaction with the environment, although it is weak in the above sense, will lead to the energy eigenstates of H_S being the preferred states[10].

3. *The intermediary regime.* In this case the interaction Hamiltonian and the self-Hamiltonian contribute roughly equally strong. Thus, the resulting preferred states, will represent a compromise between the first two cases. In the study of quantum Brownian motion, for example, the preferred states turn out to be localized in phase-space[6, 11].

In this section we showed the general idea of how superselection rules can emerge dynamically through the interaction with an environment. This approach proposes an resolution of the preferred basis problem. In addition, it provides an explanation for the existence of physical properties that appear determinate to us, since it excludes the existence of certain superpositions. For example, Giulini, Kiefer and Zeh [12] showed that a superposition of different charges never appears due to the interaction of the charge with its own Coulomb field. Therefore, they provided a physical explanation of the formerly postulated superselection rule.

3.3 Quantum-to-Classical Transition in the Decoherence Theory

Given the main results of the decoherence theory, we are now able to discuss its consequences for the problem of the quantum-to-classical transition. We have shown in the previous sections that for a system interacting with its environment, coherence gets delocalized into the system-environment correlation. Therefore, interference is destroyed locally and a measurement performed on the system only cannot reveal its quantum nature anymore. In great accordance with empirical data, many decoherence models can explain the absence of interference, especially on the macroscopic scale, where it seems to be impossible to avoid the influence of the environment⁴. As a result the decoherence theory provides a resolution of the problem of nonobservability of interference. Furthermore, we have discussed how the environment-induced superselection leads to an observerfree selection of the basis. As a result, we have argued that the existence of determinate quantities can be explained by environment-selected, quasiclassical states. Thus, the decoherence theory also provides an explanation for the problem of the preferred basis. Furthermore, as already mentioned, the formalism leading to the results presented above, does neither require any modification nor interpretation of quantum mechanics. It is therefore remarkable, that decoherence can explain the emergence of a classical world from purely quantum mechanical principles.

Despite this great success, decoherence cannot explain the fundamental problem of outcomes. Baccialgaluppi([13], sect. 2.2) puts it like this:

⁴However, macroscopic superpositions are not excluded *per se*.

Intuitively, if the environment is carrying out, without our intervention, lots of approximate position measurements, then the measurement problem ought to apply more widely, also to these spontaneously occurring measurements.(...) The state of the object and the environment could be a superposition of zillions of very well localized terms, each with slightly different positions, and which are collectively spread over a macroscopic distance, even in the case of everyday objects. (...) If everything is in interaction with everything else, everything is entangled with everything else, and that is a worse problem than the entanglement of measuring apparatuses with the measured probes.

The superposition remains existent, at least globally, until some sort of "collapse" has taken place. Therefore, the problem is shifted to the level of the total system-environment system and decoherence cannot provide any resolution. Any attempt to solve this problem requires an interpretation or an extension of standard quantum mechanics. We will discuss one of the latter approaches in the next chapter.

Finally, we want to give an important remark. The entire concept of decoherence is based on a crucial assumption, namely that it is possible to divide the universe into subsystems. The state vector of the universe⁵ $|\psi\rangle$, evolves deterministically due to the Schrödinger equation. This poses no difficulties, since our experience of classicality doesn't apply to the observation of the entire universe from the outside. The problems arise from the division into subsystems, which are the only subjects accessible for our observations. The universe is defined as a closed system. It is therefore, impossible to apply the theory of decoherence to the entire universe. Thus, we cannot define quasiclassical properties for the universe itself by simply using the decoherence theory. One could therefore argue, that classicality is merely the result of our subjective perception of only parts of the universe and the observed properties are determined by the correlations between those parts. However, in general, there exists no criterion of how it is possible to divide the total Hilbert space into subspaces. As a conclusion, the crucial assumption, that it is possible to divide the total Hilbert space into subspaces, is definitely nontrivial.

⁵If it is even possible to postulate such vector.

4 Collapse Theories

In this chapter we want to introduce another important approach to the quantum-to-classical transition. In contrast to the decoherence theory, *Collapse theories* not only deal with the problem of the preferred basis and the problem of nonobservability of interference but also with the problem of outcomes. The basic idea is to modify the Schrödinger time evolution such that an objective physical collapse can be achieved for each state vector. Therefore, the state vector obtains a "realistic" status in collapse theories. The aim is to give a unified evolution for microscopic and macroscopic objects. Thus, the collapse process has to be very effective for macroscopic objects (such as measurement devices) on the one hand and negligible for microscopic objects on the other hand. We will show, that this can be achieved by postulating a fundamental stochastic process contributing to the time evolution of the state vector. Thereby, we restrict our self to the most studied models, the *spontaneous localization model* (GRW) introduced by Ghirardi, Rimini and Weber[14] and the *continuous spontaneous localization model* (CSL) introduced by Pearle [15].

4.1 Spontaneous Localization

In the original model of Ghirardi, Rimini and Weber, they assume that each particle is subject to a *localization process* at random times. For that purpose they consider a non-Hamiltonian evolution equation for the statistical operator of the form

$$\frac{d}{dt}\rho = -i[H, \rho] - \lambda(\rho - T[\rho]) \quad (4.1)$$

with the localization process

$$T[\rho] = \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-(\alpha/2)(\hat{q}-x)^2} \rho e^{-(\alpha/2)(\hat{q}-x)^2}. \quad (4.2)$$

They further assume the localization process to take place at random times, according to a Poisson process with mean frequency λ . It is therefore commonly referred to as a hitting process. In fact, it is shown in [14] that (4.1) leads to a damping of the off-diagonal elements of ρ in the position basis

$$\langle q' | \rho | q'' \rangle \simeq e^{-\lambda t} \langle q' | \rho_s | q'' \rangle; \quad q' - q'' > 2\sqrt{\pi/\alpha}, \quad (4.3)$$

where we denote with ρ_s the pure Schrödinger evolution. It is further shown that the diagonal elements evolve according to

$$\langle q | \rho | q \rangle \simeq \langle q | \rho_s | q \rangle. \quad (4.4)$$

Let us illustrate this by writing down the localization process according to Bell [16] who reformulates the model to explicitly describe the effect on a state vector.

Assume a system of n distinguishable particles, each labeled with index i , can be represented by the state vector $|\psi\rangle$. Then, each particle experiences with mean frequency λ a sudden spontaneous localization around some point x described by

$$|\psi\rangle \rightarrow |\psi_x^i\rangle = L_x^i |\psi\rangle, \quad (4.5)$$

where L_x^i is a positive, selfadjoint, linear operator representing the localization of particle i around some point x . In [14] L_x^i was chosen to be of the form

$$L_x^i = \left(\frac{\alpha}{\pi}\right)^{1/2} \exp(-(\alpha/2)(\hat{q}_i - x)^2). \quad (4.6)$$

Since (4.6) does not conserve the norm, one has to renormalize it. We can rewrite the localization process in the nonlinear norm-conserving form

$$|\phi\rangle \rightarrow |\phi_x^i\rangle = |\psi_x^i\rangle / \|\psi_x^i\|, \quad (4.7)$$

$$|\psi_x^i\rangle = L_x^i |\phi\rangle, \quad (4.8)$$

and tactically assume that the probability for the occurrence of a hit at x is

$$p(x) = \|\psi_x^i\|^2. \quad (4.9)$$

In fact, assumption (4.9) ensures that the probability for a collapse to take place around some point x , coincide with the probability of an particular outcome in standard quantum mechanics. Let us consider a simple example. Consider a N -particle system represented by the superposition $|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle$, where $|\psi_1\rangle$ and $|\psi_2\rangle$ correspond to macroscopically different positions x_1 and x_2 (separated by more than $1/\sqrt{\alpha}$), respectively. Now, if only one particle is subject to a spontaneous localization process, say at x_1 , the state vector $|\psi\rangle$ will transform according to (4.7), causing a collapse around x_1 , since $L_{x_1}^i |\psi_2\rangle$ vanishes. Furthermore, the frequency of the localization process is given by $N\lambda$, since each particle is subject to a hit with mean frequency λ . One can easily see that for an adequate choice of the parameters λ and $1/\sqrt{\alpha}$, a macroscopic object will be subject to a hit very fast, whereas a system consisting of a small number of particles will remain unaffected for a long time. In [14] the parameters are proposed to be

$$\frac{1}{\sqrt{\alpha}} \simeq 10^{-5} \text{ cm},$$

$$\lambda \simeq 10^{-16} \text{ sec}^{-1}.$$

This means that a microscopic system will undergo a localization process once every 10^8 years, while a macroscopic system is subject to a hit $10^{23} \times 10^{-16} = 10^7$ times per second. However, the process will only be effective for systems separated by more than 10^{-5} cm, as it happens in the macroscopic case.

Let us note that the spontaneous localization process provides a model for the measurement process. Suppose a system interacting with an (macroscopic) apparatus, it is easy to see that the process described above, leads quite fast to a definite pointer position. Hence, achieving a real collapse independent of any observer.

4.2 Continuous Spontaneous Localization

Although the process described above has an intuitive physical meaning it is lacking a noise term, describing a continuous evolution of the state vector. However, a continuous evolution would be a desired property of a theory aiming at the unification of the dynamics for microscopic and macroscopic systems. In fact, it can be shown [15, 17] that one can design a continuous reduction model, providing all the features of GRW, by using a norm-conserving, non-linear stochastic evolution equation of the Itô form ⁶

$$d|\phi_{\mathbf{B}}\rangle = \left\{ \left[-iH - \frac{1}{2}\gamma(\mathbf{A} - \mathbf{R})^2 \right] dt + (\mathbf{A} - \mathbf{R})d\mathbf{B} \right\} |\phi_{\mathbf{B}}\rangle,$$

$$\mathbf{R} = \langle \phi | \mathbf{A} | \phi \rangle, \quad (4.10)$$

where $\mathbf{A} \equiv \{A_i\}$ is a set of self-adjoint operators, and $\mathbf{B} \equiv \{B_i\}$ is a real Wiener process ("noise"):

$$\overline{dB_i} = 0,$$

$$\overline{dB_i dB_j} = \delta_{ij} \gamma dt. \quad (4.11)$$

Note that the products between two operators in (4.10) have the obvious meaning of, e.g. $\mathbf{A} \cdot d\mathbf{B} = \sum_i A_i dB_i$. Furthermore, starting with a state vector $|\phi(0)\rangle$, together with a particular realization of the random function $\mathbf{B}(t)$,

⁶In the original work, they started from a more general linear stochastic equation, arriving at the equation presented here by demanding normconservation, and some other details. For a more detailed view we refer to [17, 6].

(4.10) gives rise to a certain state $|\phi_{\mathbf{B}}(t)\rangle$. Therefore, (4.10) generates an ensemble of state vectors. Given a set of commuting operators $\{A_i\}$, following [17], we will show that each $|\phi_{\mathbf{B}}(t)\rangle$ will converge towards an eigenvector of the operators \mathbf{A} , corresponding to a reduction of $|\phi(0)\rangle$ onto the common eigenspaces of the operators \mathbf{A} .

To simplify notation we will drop the dependence of $|\phi\rangle$ on \mathbf{B} and t . Furthermore, for simplicity, we consider only the non-Hamiltonian terms of (4.10):

$$\begin{aligned} d|\phi\rangle &= \left[-\frac{1}{2}\gamma(\mathbf{A} - \mathbf{R})^2 dt + (\mathbf{A} - \mathbf{R})d\mathbf{B}\right] |\phi\rangle. \\ \mathbf{R} &= \langle\phi| \mathbf{A} |\phi\rangle, \end{aligned} \quad (4.12)$$

Let us start by writing

$$\mathbf{A} = \sum_{\alpha} \mathbf{a}_{\alpha} P_{\alpha}, \quad \mathbf{a}_{\alpha} \neq \mathbf{a}_{\sigma}, \quad (4.13)$$

$$\mathbf{R} = \sum_{\alpha} \mathbf{a}_{\alpha} z_{\alpha}, \quad (4.14)$$

$$z_{\alpha} = \langle\phi| P_{\alpha} |\phi\rangle, \quad (4.15)$$

where we denote with P_{α} the orthogonal projections. It follows that

$$\sum_{\alpha} z_{\alpha} = 1 \quad (4.16)$$

Then, (4.12) implies:

$$dP_{\alpha} |\phi\rangle = \left[-\frac{1}{2}\gamma(\mathbf{a}_{\alpha} - \mathbf{R})^2 dt + (\mathbf{a}_{\alpha} - \mathbf{R})d\mathbf{B}\right] P_{\alpha} |\phi\rangle. \quad (4.17)$$

Using the Itô calculus (Appendix A), one obtains the differential equations

$$dz_{\alpha} = 2z_{\alpha}(\mathbf{a}_{\alpha} - \mathbf{R})d\mathbf{B}. \quad (4.18)$$

Again using the Itô calculus, one finds

$$dz_{\alpha}^2 = 2z_{\alpha}dz_{\alpha} + [2z_{\alpha}(\mathbf{a}_{\alpha} - \mathbf{R})]^2 \gamma dt. \quad (4.19)$$

Taking the average over all \mathbf{B} and using the property of the Wiener process (4.11), we have

$$\overline{dz_{\alpha}^2} = \overline{dz_{\alpha}^2} = [2z_{\alpha}(\mathbf{a}_{\alpha} - \mathbf{R})]^2 \gamma dt. \quad (4.20)$$

It obviously follows that

$$\frac{d}{dt} \overline{z_\alpha^2} \geq 0. \quad (4.21)$$

Together with the upper bound,

$$\overline{z_\alpha^2} \leq 1 \quad (4.22)$$

induced by (4.16), we have for $t \rightarrow \infty$

$$\frac{d}{dt} \overline{z_\alpha^2} \rightarrow 0. \quad (4.23)$$

We can see from (4.20) that this is equivalent to the statement

$$z_\alpha(\mathbf{a}_\alpha - \mathbf{R}) \rightarrow 0. \quad (4.24)$$

One can show (Appendix B) that the only solutions of (4.24) are such that one z is 1 and all others vanish. We see from (4.17) that (4.12) leaves the orientation of the vectors $P_\alpha |\phi\rangle$ unchanged within the Hilbertspace. Thus, $|\phi\rangle$ converges towards one of its initial components $P_\alpha |\phi(0)\rangle$ (times a normalization factor), lying in one of the common eigenspaces of the operators A_i .

Finally, we are interested in the probabilities for the occurrence of a particular state vector corresponding to a particular $z_\alpha = 1$. Again, the properties of the Wiener process (4.11) imply that $d\overline{z_\alpha} = d\overline{z_\alpha} = 0$ and therefore

$$\overline{z_\alpha} = z_\alpha(0). \quad (4.25)$$

On the other hand we have

$$\overline{z_\alpha} \rightarrow p(z_\alpha(\infty) = 1), \quad (4.26)$$

so that the probability for $|\phi(0)\rangle$ ending up in one of the common eigenspaces of \mathbf{A} is given by

$$p(z_\alpha(\infty) = 1) = z_\alpha(0) \quad (4.27)$$

in accordance with the probability for a particular outcome in standard quantum mechanics.

We can conclude that (4.10) produces an observer-free, physical collapse for each particular set of random functions \mathbf{B} providing the same probabilities as standard quantum mechanics for the ensemble of $|\phi_{\mathbf{B}}\rangle$. The timescale of the effect is further controlled by the parameter γ . It has to be chosen such that the collapse appears very fast for macroscopic objects and remains

practically unobservable in the microscopic case. Note that the argument presented above is valid for any set of commuting, self-adjoint operators A_i .

Before we specialize the process to a localization process, let us write down the evolution equation for the statistical operator corresponding to the general process described above. From the definition of the statistical operator, it is obvious to write

$$\rho = \overline{|\phi\rangle\langle\phi|} \quad (4.28)$$

which gives, by using the Itô calculus,

$$\frac{d\rho}{dt} = -i[H, \rho] + \gamma \mathbf{A} \rho \mathbf{A}^\dagger - \frac{1}{2} \gamma \{ \mathbf{A}^\dagger \mathbf{A}, \rho \}, \quad (4.29)$$

where $[,]$ and $\{ , \}$ denote the commutator and the anticommutator, respectively. This is exactly the Lindblad form (3.8) for the generator of a quantum dynamical semigroup, presented in chapter 3. It is remarkable that the Lindblad form of the master equation, describing the time evolution of a system coupled to the environment, can be obtained from a stochastic process in Hilbert space. However, we will postpone the comparison of the two approaches till the end of this work.

Given the general formalism of a stochastic process in Hilbert space, we can now specify it to a localization process. For this reason, a particular choice of \mathbf{A} is given in [17] by a locally averaged density operator⁷

$$A_x = N(x) \equiv \sum_s (\alpha/2\pi)^{1/2} \int dy \exp(-(\alpha/2)(y-x)^2) a^\dagger(y, s) a(y, s), \quad (4.30)$$

where a and a^\dagger are the creation and annihilation operators, respectively. In this case, (4.29) reads

$$\frac{d\rho}{dt} = -i[H, \rho] + \gamma \int dx N(x) \rho N(x) - \frac{1}{2} \gamma \left\{ \int dx N^2(x), \rho \right\}. \quad (4.31)$$

According to [17], in the representation of the eigenvectors $|q, s\rangle$ of N (4.31) reads

⁷A generalization to 3 dimensions is straightforward.

$$\begin{aligned}
\frac{d}{dt} \langle q', s' | \rho | q'', s'' \rangle = & - i \langle q', s' | [H, \rho] | q'', s'' \rangle \\
& + \gamma \sum_{i,j} [G(q'_i - q''_j) - \frac{1}{2}G(q'_i - q'_j) \\
& - \frac{1}{2}G(q''_i - q''_j)] \times \langle q', s' | \rho | q'', s'' \rangle, \quad (4.32)
\end{aligned}$$

where

$$G(q'_i - q''_j) = \exp\left(-\frac{1}{4}\alpha(q'_i - q''_j)^2\right) \quad (4.33)$$

and the indices i and j label the particles. Then (4.32) describes the localization process for systems of particles separated by more than $1/\sqrt{\alpha}$.

Let us follow an easy example presented in [18] to illustrate how the localization mechanism works. For this reason, we will concentrate on the collapse dynamics alone and set $H = 0$ in (4.32). Now, consider a system of particles in a superposition $a|1\rangle + b|2\rangle$, where $|1\rangle$ and $|2\rangle$ describe N particles localized around some region 1 and some region 2, respectively. Further assume that region 1 and region 2 are separated by much more than $1/\sqrt{\alpha}$. It is easy to see that (4.32) then yields

$$\begin{aligned}
\frac{d}{dt} \langle 1 | \rho | 2 \rangle &= \gamma \sum_{i,j} (0 - \frac{1}{2} \cdot 1 - \frac{1}{2} \cdot 1) \times \langle 1 | \rho | 2 \rangle \\
&= -\gamma N^2 \langle 1 | \rho | 2 \rangle, \quad (4.34)
\end{aligned}$$

corresponding to a decay of the off-diagonal elements of ρ at the rate γN^2 . A similar result was obtained in the GRW model. In fact, it is shown in [17] that the hitting process of the GRW model and the continuous process of the CSL model are equivalent, taking the infinite frequency limit of the GRW process

$$\lambda \rightarrow \infty, \quad \alpha \rightarrow 0, \quad \frac{1}{2}\lambda\alpha = \gamma. \quad (4.35)$$

4.3 Quantum-to-Classical Transition in the Collapse Theory

Based on the short introduction into spontaneous localization models given above, we are now able to discuss their consequences for the quantum-to-classical transition. We have seen in the previous section that adding a stochastic term to the Schrödinger equation leads to a real physical collapse. In both, the GRW model as well as in the CSL model, an initial

superposed state turns into a statistical mixture of its constituents. The underlying mechanism is acting universally on every state vector, independent of any interaction with any other systems. Motivated by our experiences in the everyday world, the process is chosen to take place in the position basis. However, we have shown in the previous section, that in principle the mechanism works in an arbitrary basis, depending on the choice of the self-adjoint, commuting operators. Furthermore, for an adequate choice of the model's parameters, the collapse mechanism is very effective for macroscopic objects, whereas it leaves the Schrödinger evolution unchanged in the microscopic case. Thus, providing an unified description for the occurrence of superpositions in the microscopic regime and the absence of the latter on the macroscopic scale. The true merit of the theory lies in the fact, that it provides a resolution for the problem of outcomes. Since each individual state vector is subject to a reduction process, the collapse model itself provides a description of the measurement process, leading to real outcomes. The reduction postulate gets replaced by a physical reduction mechanism.

Despite this success, there are some remaining questions. The whole theory is based on a modification of the Schrödinger equation. However, this modification is postulated (e.g. the "noise" field in the CSL model) and is lacking a physical motivation. Furthermore, the reduction process is never fully completed for a finite time. This is sometimes called the *tails problem*, which will become problematic if one wants to ascribe some kind of "reality" to the reduced state, which is aspired in physical collapse theories.

Finally, we want to give an important remark. The modification of the Schrödinger equation should in principle produce some predictions, different from them made by standard quantum mechanics. Thus, physical collapse theories have to be seen as rival theories. We will discuss this in more detail in the last chapter.

5 Macroscopic Realism and Coarse-Grained Measurements

Beside decoherence and the collapse theories, presented in the previous chapters, there exists another interesting approach concerning the quantum to classical transition. In a recent work of Brukner and Kofler [24], it is argued that classicality arises from the restriction to *coarse-grained measurements*, i.e. the imprecision of a real measurement apparatus. Before we will show how this can work, we have to introduce the notion of *macroscopic realism* (macrorealism) first defined by Leggett [22]. The idea is to explicitly write

down what we demand from a theory describing the macroscopic classical world we observe in our every-day life.

5.1 Legget-Garg Inequality

Macrorealism can be defined by the conjunction of the following three postulates[19]:

- (1) *Macrorealism per se.* A macroscopic object which has available to it two or more macroscopically distinct states is at any given time⁸ in a definite one of those states.
- (2) *Non-invasive measurability.* It is possible in principle to determine which of these states the system is in without any effect on the state itself or on the subsequent system dynamics.
- (3) *Induction* The properties of ensembles are determined exclusively by initial conditions (and in particular not by final conditions).

It is quite obvious that classical physics belongs to this class of macrorealistic theories.

Starting from these assumptions one can derive an inequality quite similar to the inequality introduced by Bell [20] or the Clauser-Horne-Shimony-Holt (CHSH) inequality [21]. Consider a macroscopic system and a corresponding quantity A , which whenever measured takes the values ± 1 . Further consider a series of measurements performed on the system starting from identical initial conditions. In the first measurement A is measured at time t_1 and t_2 , at t_2 and t_3 on the second, at t_3 and t_4 on the third and at t_1 and t_4 on the fourth. Let us further denote with A_i the corresponding values of a particular outcome at time t_i . Similar to the CHSH case one can write:

$$A_1(A_2 - A_4) + A_3(A_2 + A_4) = \pm 2 \quad (5.1)$$

Macrorealism per se is reflected by the existence of definite values of A_i at all times, and *non-invasive measurability* combined with *induction* is reflected by the fact that the A_i 's are independent of the combination in which they occur. Given a series of such measurements, we can introduce temporal correlation functions

$$C_{ij} = \overline{A_i A_j} \quad (5.2)$$

⁸Except for small transit times

Taking the average over (5.1) it follows that any theory based on the assumptions of macrorealism has to satisfy the *Leggett-Garg inequality* [22]

$$K \equiv C_{12} + C_{23} + C_{34} - C_{14} \leq 2. \quad (5.3)$$

This equality can be used to test whether or not a system's time evolution can be understood classical. Since standard quantum mechanics makes no *a priori* difference between microscopic and macroscopic systems, an isolated quantum system violates the Leggett-Garg inequality (5.3). In fact, it can be explicitly shown [23] that every non-trivial Hamiltonian leads to a violation of the inequality. However, in many cases (although not in all) decoherence (or physical collapse) is sufficient to restore macrorealism, since it turns superpositions into classical mixtures.

5.2 Coarse-Grained Measurement

Equipped with the definition of macrorealism, let us now illustrate the idea of coarse-grained measurement. Following [24], we will use a single spin coherent state. *Spin- j coherent* states are the eigenstates with maximal eigenvalue j of a spin operator pointing in the $\Omega \equiv (\vartheta, \varphi)$ direction:

$$\mathbf{J}_{\vartheta, \varphi} |\Omega\rangle = j |\Omega\rangle \quad (5.4)$$

Further consider the Hamiltonian

$$H = \frac{\mathbf{J}^2}{2I} + \omega J_x, \quad (5.5)$$

where \mathbf{J} is the rotor's total spin vector operator, I the moment of inertia and ω the angular precision frequency. Since \mathbf{J}^2 commutes with the individual spin components the time evolution operator can be written:

$$U_t \equiv e^{-i\omega t J_x}. \quad (5.6)$$

Given a general spin coherent state, its representation in terms of J_z eigenstates $|m\rangle$, with corresponding possible eigenvalues $m = -j, -j+1, \dots, j$, is given by:

$$|\Omega_t\rangle = \sum_m \binom{2j}{j+m}^{1/2} \cos^{j+m}(\frac{\vartheta_t}{2}) \sin^{j-m}(\frac{\vartheta_t}{2}) e^{-im\varphi_t} |m\rangle. \quad (5.7)$$

The probability of finding the particular outcome m in a J_z measurement at time t is given by the *binomial distribution* (Appendix C):

$$p(m, t) = |\langle m | \Omega_t \rangle|^2. \quad (5.8)$$

In the macroscopic limit⁹, i.e. $j \gg 1$, (5.8) can be written as a Gaussian distribution

$$p(m, t) \approx \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(m-\mu)^2}{2\sigma^2}}, \quad (5.9)$$

with standard deviation $\sigma \equiv \sqrt{j/2} \sin(\vartheta_t)$ and mean value $\mu \equiv j \cos(\vartheta_t)$. Now assume that the resolution of the measurement apparatus is restricted to Δm (which is a reasonable assumption in every realistic experiment). Thus, the $2j + 1$ possible outcomes become reduced to $\frac{2j+1}{\Delta m}$ "slots" \bar{m} of size Δm . In the case where the slot size is much larger than the standard derivation¹⁰

$$\Delta m \gg \sigma \sim \sqrt{j} \quad (5.10)$$

the Gaussian (5.9) can not be distinguished from the discrete Kronecker delta

$$\Delta m \gg \sqrt{j} : p(m, t) \rightarrow \delta_{\bar{m}, \bar{\mu}} \quad (5.11)$$

where \bar{m} is numbering the slots and $\bar{\mu}$ number of the slot where the Gaussian is centered. If in addition $j \rightarrow \infty$ (classical limit), the Kronecker delta will become the Dirac delta function

$$p(m, t) \rightarrow \delta(\bar{m} - \bar{\mu}). \quad (5.12)$$

As a result, it is obvious that the action of a coarse grained measurement, represented by a projection operator

$$P_{\bar{m}} \equiv \sum_{m \in \{\bar{m}\}} |m\rangle \langle m|, \quad (5.13)$$

can be written as

$$P_{\bar{m}} |\Omega\rangle \approx \begin{cases} |\Omega\rangle & \text{for } \bar{\mu} \text{ inside } \bar{m} \\ 0 & \text{for } \bar{\mu} \text{ outside } \bar{m}. \end{cases} \quad (5.14)$$

We can conclude that under the restriction of coarse-grained measurement we are not able to resolve the superposition anymore and a the state seems to have a definite value (the value assigned to the slot in which $\bar{\mu}$ lies) for a J_z measurement at any time. This is *macrorealism per se*. Furthermore,

⁹The term "macroscopic" used in [24] belongs to a system with a high dimensionality rather than to a low-dimensional system with large parameters like mass or size.

¹⁰This is reasonable for large j since the countable number of distinguishable outcomes is limited by the apparatus $\frac{2j+1}{\Delta m} \leq \text{constant}$ and j grows faster than \sqrt{j}

(5.14) shows that a measurement can be seen in good approximation as non-invasive (together with induction). Hence, no violation of the Leggett-Garg inequality is possible anymore.

It is shown in [24] that the argumentation given above even holds for an arbitrary spin- j quantum state represented by a density matrix in the *overcomplete* basis of the coherent states

$$\rho = \int \int f(\Omega) |\Omega\rangle \langle \Omega| d^2\Omega \quad (5.15)$$

where $f(\Omega)$ is a *quasi-probability distribution* with $\int \int f(\Omega) d^2\Omega = 1$.

The argumentation given above uses an explicit Hamiltonian (5.5). In the general case, it is shown in [23] that under the restriction of coarse-grained measurements an arbitrary spin- j quantum state can be described by a classical mixture at any time. Thus, establishing *macrorealism per se*. However, *non-invasive measurability* follows not automatically. In [23] a sufficient condition for macrorealism is given:

$$P_{\bar{m}} U_t |\Omega\rangle \approx \begin{cases} U_t |\Omega\rangle & \text{for one } \bar{m} \\ 0 & \text{for all the others.} \end{cases} \quad (5.16)$$

Therefore, the time evolution operator must produce superpositions of macroscopically distinct states (states belonging to different slots) to violate macrorealism. However, Brukner and Kofler argue that Hamiltonians leading to a violation of macrorealism are unlikely to appear in nature, due to their high complexity [23].

Let us emphasize that the approach of coarse-grained measurement relies only on standard quantum mechanics and the (reasonable) assumption of insufficient precision of our measurement apparatuses. Furthermore, it is not at variance with the decoherence program, it only differs conceptually. However, the formalism presented above deals with spin systems only. A generalization to other systems has not been done yet but would be an interesting task.

6 Conclusion

We have seen various approaches to the question of the quantum-to-classical transition. We now want to summarize what they can offer to answer this question and point out what they have in common or where they differ.

Let us start with the most commonly used approach, the decoherence program. We have seen that the interaction with an uncontrollable environment

not only explains quite well, why we do not see superpositions of macroscopic objects, but also shows that the interaction even selects a preferred (pointer) basis. However, macroscopic superpositions are not *per se* excluded, since one could always (at least in principle) reduce the environment's influence. Several experiments, e.g. interference of large molecules, have shown that we are able to control the environment better and better, maybe arriving at a point where we are faced with a real cat paradox. In addition, the decoherence program is inherently quantum mechanical and therefore not able to resolve the problem of outcomes. A further interpretation is needed to explain why measurements have definite outcomes.

Now what can the collapse theories, presented in this work, tell us about the quantum-to-classical transition? Although they are based on the modification of the Schrödinger equation, they lead to predictions similar to decoherence. This is not surprising, since in both cases a macroscopic superposition evolves into a classical mixture. In fact, the time evolution of a system in both the decoherence and collapse theory can be described by a master equation of the Lindblad form. However, the two approaches differ on a fundamental level. In the case of the decoherence program, coherence becomes destroyed only locally (and is still existing in the larger system-environment), whereas the collapse theory achieves a "real" loss of coherence on a fundamental level, i.e. independent of any interaction with some environment. Contrary to the decoherence program, the real physical collapse of each state vector explains why we have definite outcomes in every measurement. Furthermore, the collapse theory provides a real border for the observability of superpositions, i.e. superpositions are *a priori* excluded at some level. However, the mechanism provided by the collapse theories suffers from the preferred basis problem. This means that the choice of the operator in the additional term of the Schrödinger equation determines in which basis the collapse can be achieved. This seems to be unsatisfying. Furthermore the modification lacks a physical motivation in general.

Anyhow, we want to emphasize that it is not a matter of taste, which approach one prefers. We point out again that, due to the modification of the Schrödinger equation, the collapse theory is a real rival theory, concerning standard quantum mechanics. However, since decoherence effects are found to be much stronger in the present experimental setups, it is still impossible to test collapse models against quantum theory. Hopefully, future experiments, e.g. with huge molecules, will become sensitive enough, to confirm or exclude collapse theories.

Since the approach of coarse-grained measurements only differs conceptually from decoherence on the one hand, and is still demanding a generalization on the other hand, we do not want to say much about it. However, in the

authors view it is an interesting idea. Especially due to the fact that the realistic assumption of imprecise measurement apparatuses can lead to the emergence of classicality.

Finally, we have seen that there exist a lot of explanations for the appearance of a classical world in quantum theory. However, there still exist an important issue which we have not talked about. Even if it was possible to explain the nonobservability of macroscopic superpositions, to answer the question of the quantum-to-classical transition we still would have to explain explicitly the emergence of classical physics, i.e. to show that it is possible to derive Newton's laws from quantum theory.

A Itô's Lemma

Assume X is an Itô process satisfying the Itô stochastic differential equation

$$dX = A(X)dt + C(X)dB \quad (\text{A.1})$$

with a Wiener process B

$$\overline{dB} = 0, \quad \overline{dB^2} = \gamma dt. \quad (\text{A.2})$$

For a twice differentiable function $Y = f(X)$ the change of variable is given by

$$A'(Y) = A(X)\partial_X f(X) + \frac{\gamma}{2}C(X)^2\partial_X^2 f(X) \quad (\text{A.3})$$

$$C'(Y) = C(X)\partial_X f(X) \quad (\text{A.4})$$

and

$$dY = A'(Y)dt + C'(Y)dB. \quad (\text{A.5})$$

Furthermore, it can be shown that the following explicit formula holds for $|\psi\rangle$ satisfying an Itô stochastic differential equation:

$$d\langle\psi|\psi\rangle = (d\langle\psi|)|\psi\rangle + \langle\psi|(d|\psi\rangle) + \overline{(d\langle\psi|)(d|\psi\rangle)} \quad (\text{A.6})$$

B State Vector Reduction via CSL

Following [17] we want to calculate the solutions of

$$0 = z_\alpha(\mathbf{a}_\alpha - \mathbf{R}) \quad (\text{B.1})$$

$$= z_\alpha \left(\mathbf{a}_\alpha - \left(\sum_\sigma \mathbf{a}_\sigma z_\sigma \right) \right) \quad (\text{B.2})$$

for non-negative, real variables z_α with the condition

$$\sum_\alpha z_\alpha = 1. \quad (\text{B.3})$$

One obvious solution is $z_\alpha = 1$ and $z_\sigma = 0$ for $\alpha \neq \sigma$. One can easily show that these are the only ones, provided $\mathbf{a}_\alpha \neq \mathbf{a}_\sigma$ for $\alpha \neq \sigma$.

Suppose that $z_1 \neq 0$ and $z_2 \neq 0$. It follows from (B.2) that

$$\mathbf{a}_1 - \mathbf{R} = 0, \quad \mathbf{a}_2 - \mathbf{R} = 0. \quad (\text{B.4})$$

By subtraction one gets

$$\mathbf{a}_1 - \mathbf{a}_2 = 0. \quad (\text{B.5})$$

This is a contradiction to the hypothesis $\mathbf{a}_\alpha \neq \mathbf{a}_\sigma$.

□

C Outcome Probabilities of a Spin Coherent State

A spin coherent state in the z -basis is given by:

$$|\Omega_t\rangle = \sum_m \binom{2j}{j+m}^{1/2} \cos^{j+m}\left(\frac{\vartheta_t}{2}\right) \sin^{j-m}\left(\frac{\vartheta_t}{2}\right) e^{-im\varphi_t} |m\rangle \quad (\text{C.1})$$

The probability that an outcome m occurs in a J_z measurement is given by:

$$p(m, t) = |\langle m | \Omega_t \rangle|^2 = \binom{2j}{j+m} \cos^{2(j+m)}\left(\frac{\vartheta_t}{2}\right) \sin^{2(j-m)}\left(\frac{\vartheta_t}{2}\right) \quad (\text{C.2})$$

Denoting $2j \equiv n$, $j + m \equiv k$ and $\cos^2\left(\frac{\vartheta_t}{2}\right) \equiv w$ the binomial distribution follows at once:

$$p = \binom{n}{k} w^k (1-w)^{n-k} \quad (\text{C.3})$$

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