# BACHELOR THESIS

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## Quantum Information and Entropy

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# Contents

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## 2 Shannon and von Neumann Entropy

In 1948 Claude Shannon published his well known paper "A Mathematical Theory of Communication" [1]. The main theme of this paper was to find under what conditions a message sent by a transmitter can be recovered by a receiver. In doing so he defined a quantity called Entropy used as a measure of Information. The problem of recovering a message is similar to the idea in quantum mechanics of measurement: How to gain knowledge of the system by performing a measurement test. In this sense the quantum system represents an analogy to the message to be transmitted in Communication Theory and the measurement outcome to the received message. Shannon derived a formula for the Entropy [1]:

$$H = \sum_i p_i log(p_i) \tag{1}$$

In 1932 John von Neumann extended the idea of Entropy to quantum systems by introducing a density matrix in "Mathematical Foundations of Quantum Mechanics" [2]. Previously a system could be in a superposition of pure states: $|\Psi\rangle = \sum_i |\psi_i\rangle$ But with the help of a density matrix we could create a mixture of pure states by adding projectors of each pure states $|\psi_i\rangle$ with a probability $\lambda_i$ such that $\sum_i \lambda_i = 1$ and $\lambda_i > 0$:

$$\rho = \sum_i \lambda_i |\psi_i\rangle \langle\psi_i| \tag{2}$$

The density operator therefore has a few interesting properties which are easy to prove [3] and are listed here:

| | |
|---|---|
| $\rho^\dagger = \rho$ | Hermitian |
| $\rho^2 = \rho$ | Idempotent if pure |
| $Tr\,[\rho] = 1$ | sum of the eigenvalues have to be 1 |
| $Tr[\hat{A}\rho] = E[A]$ | Procedure for calculating the Expected Value of the Observable $\hat{A}$ |
| $Tr[\rho^2] < 1$ | If it is in a mixed state |
| $\rho \geq 0$ | All eigenvalues (probabilities) are greater or equal to 0 |

Thus since density matrices are the direct analog to the probability Von Neumann defined the Entropy of a quantum system described by the density matrix $\rho$ as:

$$S = -Tr\,[\rho\, log\rho] \tag{3}$$

## 3 Entropy Inequalities for Finite Dimensional Hilbert Spaces

In this section we will restate some of the Lemmas of the paper "Entropy Inequalities" by Araki and Lieb [5] and show some of the proofs.

**Definition 3.1** *If $\rho_{XY}$ is a density matrix in the Hilbert Space $\mathcal{H}_X \otimes \mathcal{H}_Y$. Then the partial trace over the Hilbert Space $\mathcal{H}_Y$ is $\rho_X$*

$$\underset{Y}{Tr}\left[\rho_{XY}\right] = \rho_X \tag{4}$$

We will use throughout the following notation.

**Definition 3.2** *If $\rho_{XY}$ denotes the density matrix on $\mathcal{H}_X \otimes \mathcal{H}_Y$ then we denote the Entropy associated with $\rho_{XY}$ by $S_{XY}$.*

One very important theorem in the Araki-Lieb Paper [5] is the following:

**Theorem 3.3** *let $\rho_{XYZ}$ be a density matrix on $\mathcal{H}_X \otimes \mathcal{H}_Y \otimes \mathcal{H}_Z$. Then the following inequality holds.*

$$S_{XYZ} \leq S_{XY} + S_{YZ} + log\left[\underset{Y}{Tr}\left[\rho_Y^2\right]\right] \leq S_{XY} + S_{YZ} \tag{5}$$

*Furthermore if $[\rho_Y \otimes \mathcal{I}_Z, \rho_{YZ}] = 0$ holds then:*

$$S_{XYZ} + S_Y \leq S_{XY} + S_{YZ} \tag{6}$$

To prove this Theorem we need Klein's, Peierls-Bogolybov and Golden Thompson inequalities, of which the first two we will prove and the last one we will only state.

**Lemma 3.4** ***Klein Inequality*** *[5][10]*
*For convex $f$ and Hermitian $A$ and $B$ the following inequality holds*

$$Tr\left[f(A) - f(B) - (A-B)f'(B)\right] \geq 0 \tag{7}$$

**Proof of Lemma 3.4** Rearrange such that:

$$Tr[f(A) - f(B)] \geq Tr[(A-B)f'(B)] \tag{8}$$

Let $C = A - B$ and $t \in [0,1]$ Then we can define a function

$$\phi(t) = Tr[f(B + tC)] \tag{9}$$

$$\begin{aligned} \phi(1) &= Tr[f(A)] \\ \phi(0) &= Tr[f(B)] \end{aligned} \tag{10}$$

Differentiating $\phi(t)$ wrt. to $t$ yields:

$$\frac{d}{dt}\phi(t)|_{t=0} = \frac{d}{dt}Tr[f(B+tC)]|_{t=0} = Tr[(A-B)f'(B)] \tag{11}$$

Therefore:

$$\phi(1) - \phi(0) \geq \lim_{t\to 0}\frac{\phi(t) - \phi(0)}{t} \to 0 \tag{12}$$

because the RHS is monotonically decreasing with t.

**Lemma 3.5** ***Peierls-Bogolybov inequality*** *[5]*
*If $E$ and $F$ are Hermitian, $Tr[e^E] = 1$ and $f \equiv Tr[Fe^E]$ then*

$$Tr\left[e^{E+F}\right] \geq e^f \tag{13}$$

**Proof of Lemma 3.5**
Using **Lemma 3.4** and setting $f(x) = e^x$ $A = E + F,\ B = E + f\mathbb{1}$.
Rearranging

$$Tr\left[e^{E+F}\right] \geq Tr\left[(F - fI)e^{E+fI} + e^{E+f\mathbb{1}}\right] \tag{14}$$

Noting that

$$[E, \mathbb{1}] = 0 \rightarrow e^{E+f\mathbb{1}} = e^E e^{f\mathbb{1}} = e^E e^f \tag{15}$$

Using the definition $f \equiv Tr[Fe^E]$

$$\begin{aligned} RHS &= Tr[Fe^E e^f] - Tr\left[fe^E e^f\right] + Tr\left[e^{E+fI}\right] \\ &= Tr[Fe^E]e^f - f\,Tr\left[e^E\right]e^f + Tr\left[e^{E+f\mathbb{1}}\right] = e^f \end{aligned} \tag{16}$$

Therefore:

$$Tr\left[e^{E+F}\right] \geq e^f \tag{17}$$

**Lemma 3.6** ***Golden-Thompson Inequality*** *[5]*
*If $A$ and $B$ are Hermitian then*

$$Tr[e^{A+B}] \leq Tr[e^A e^B] \tag{18}$$

**Proof of Theorem 3.3** [5]
Given a positive definite operator $\left(\rho_{XYZ} = e^{R_{XYZ}}\right) \in (\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3)$
We define $\rho_{XY} = e^{R_{XY}}$, $R_{XY} \equiv R_{XY} \otimes \mathcal{I}_Z$. Moreover we will use $Tr$ to denote the Trace over all Hilbert Spaces. Partial Traces will be specially denoted.

$$\Delta = S_{XYZ} - S_{XY} - S_{YZ} = Tr\left[\rho_{XYZ}\left(-R_{XYZ} + R_{XY} + R_{YZ}\right)\right] \tag{19}$$

Choose $F = -R_{XYZ} + R_{XY} + R_{YZ}$, $E = \rho_{XYZ}$ and $\Delta = f$

$$\Delta = S_{XYZ} - S_{XY} - S_{YZ} = Tr\left[e^{R_{XYZ}}(-R_{XYZ} + R_{XY} + R_{YZ})\right] \tag{20}$$

Using Lemma 3.5:

$$e^{\Delta} \leq Tr\left[e^{R_{XYZ} - R_{XYZ} + R_{XY} + R_{YZ}}\right] = Tr\left[e^{R_{XY} + R_{YZ}}\right] \tag{21}$$

And finally using Lemma 3.6:

$$e^{\Delta} \leq Tr\left[e^{R_{XY}} e^{R_{YZ}}\right] = Tr\left[\rho_{XY}\rho_{YZ}\right] \tag{22}$$

Since $\rho_{XY} \equiv \rho_{XY} \otimes \mathcal{I}_Z$ and $\rho_{YZ} \equiv \mathcal{I}_X \otimes \rho_{YZ}$. We can first trace out the $Z$ and then the $X$ and we are then left with

$$e^{\Delta} = \underset{Y}{Tr}\left[\rho_Y^2\right] \leq 1 \quad \text{equal if pure} \tag{23}$$

This shows that:

$$S_{XYZ} - S_{XY} - S_{YZ} \leq log\left(\underset{Y}{Tr}\left[\rho_Y^2\right]\right) \leq 0 \tag{24}$$

or rearranging:

$$S_{XYZ} \leq log\left(\underset{Y}{Tr}\left[\rho_Y^2\right]\right) + S_{XY} + S_{YZ} \leq S_{XY} + S_{YZ} \tag{25}$$

Which is exactly the first part of Theorem 3.3.
For the second part consider that $[\rho_Y, \rho_{YZ}] = 0$ Then we can apply the same procedure as above:

$$\Delta = S_{XYZ} + S_Y - S_{XY} - S_{YZ} \tag{26}$$

$$e^{\Delta} = Tr\left[e^{R_{XY}+R_{YZ}-R_Y}\right] \leq Tr\left[e^{R_{XY}}e^{R_{YZ}-R_Y}\right] = Tr\left[e^{R_{XY}}e^{R_{YZ}}e^{R_Y}\right] \tag{27}$$

$$= Tr\left[\rho_{XY}\rho_{YZ}\rho_Y^{-1}\right] = 1 \tag{28}$$

Which yields the desired result, since we can first Trace out over $\mathcal{H}_Z$ then over $\mathcal{H}_X$ and then we are left with $Tr[\rho_Y^2\rho_Y^{-1}] = 1$:

$$S_{XYZ} + S_Y \leq S_{XY} + S_{YZ} \tag{29}$$

**Corollary 3.7** *If $\rho_{XY}$ is a density matrix on $\mathcal{H}_X \otimes \mathcal{H}_Y$ then*

$$S_{XY} \leq S_X + S_Y \tag{30}$$

**Proof of Corollary 3.7** Using the second part of Theorem 3.3, interchaning $Y \leftrightarrow Z$ and letting $Z$ be one dimensional shows this result.

**Definition 3.8** *A density matrix $\rho$ is pure if it is a projection operator onto a one-dimensional subspace, i.e. $\rho\,|\mu\rangle = \langle\lambda|\mu\rangle\,|\lambda\rangle$ for $|\lambda\rangle$ being fixed and normed.*

**Lemma 3.9** *If $\rho_{XY}$ is a pure state density matrix on $\mathcal{H}_X \otimes \mathcal{H}_Y$ and let $f(.)$ be a real valued function with $f(0) = 0$. Then:*

$$Tr\left[f(\rho_X)\right] = Tr\left[f(\rho_Y)\right] \tag{31}$$

**Proof of Lemma 3.9**
Applying Definition 3.8

$$\rho_{XY}\,|\mu\rangle = \langle\lambda|\mu\rangle\,|\lambda\rangle \; and \, |\lambda\rangle = \sum_i p_i\,|\lambda_{Xi}\rangle \otimes |\lambda_{Yi}\rangle \tag{32}$$

$|\lambda_{Xi}\rangle$ *and* $|\lambda_{Yi}\rangle$ can be chosen as orthogonoal. Now letting $P_{\nu,i} = |\lambda_{\nu,i}\rangle\langle\lambda_{\nu,i}|$ be the Projection onto the one dimensional subspace of $\mathcal{H}_\nu$ then:

$$\rho_\nu = \sum_i p_i^2 P_{\nu,i} = \sum_i p_i^2 |\lambda_{\nu,i}\rangle\langle\lambda_{\nu,i}| \tag{33}$$

Therefore $\rho_X$ and $\rho_Y$ have the same eigenvalues with the same multiplicities. In particular it can be shown then that $S_X = S_Y$.

**Lemma 3.10** *[5] Let $\rho_X$ be a density matrix on $\mathcal{H}_X$. Then there exists a Hilbert Space $\mathcal{H}_Y$ and a pure state density matrix $\rho_{XY}$ on $\mathcal{H}_X \otimes \mathcal{H}_Y$ such that*

$$\rho_X = \underset{Y}{Tr}\left[\rho_{XY}\right] \tag{34}$$

**Proof of Lemma 3.10** Let $\rho_X = \sum_i \lambda_i P_i$ be the spectral decomposition with $\lambda_i \geq 0$ and $P_i|\mu\rangle = |\lambda_i\rangle\langle\lambda_i|\mu\rangle$ where $|\lambda_i\rangle$ form an orthonormal basis. Assume that $dim\left(\mathcal{H}_Y\right) \geq dim\left(\mathcal{H}_X\right)$ and that it has the orthonormal basis $|\delta_i\rangle$. Now let $\rho_{XY}$ be the projection operator on the one dimensional subspace of $\mathcal{H}_X \otimes \mathcal{H}_Y$ containing the vector $\sum_i \lambda^{1/2}|\lambda_i\rangle \otimes |\delta_i\rangle$. Then

$$\underset{Y}{Tr}\left[\rho_{XY}\right] = \rho_X \tag{35}$$

Lemmas 3.9 and 3.10 can be used to prove more Entropy inequalites. However the inequalities derived so far are sufficient for our purpose.

# 4 Problem with the Peierls-Bogylobov-Inequality

In the Peierl's Bogylobov inequality in Lemma 3.5 we make one assumption: Namely that $Tr[e^E] = 1$ where $E$ is Hermitian. If we assume that $E$ is Hermitian then we can do a spectral decomposition.

$$E = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i| \tag{36}$$

If we take the Exponential of this then we can write it as:

$$Tr\left[\sum_i e^{\lambda_i}|\lambda_i\rangle\langle\lambda_i|\right] = 1 \tag{37}$$

because of linearity of the Trace:

$$\sum_i e^{\lambda_i} Tr\left[|\lambda_i\rangle\langle\lambda_i|\right] = 1 \tag{38}$$

Since the trace is independent of basis we can use the Basis of $E$:

$$\sum_i e^{\lambda_i} \sum_j \underbrace{\langle\lambda_j|\lambda_i\rangle}_{\delta_{ij}} \underbrace{\langle\lambda_i|\lambda_j\rangle}_{\delta_{ij}} = 1 \tag{39}$$

$$\sum_i e^{\lambda_i} \sum_j \underbrace{\langle \lambda_j | \lambda_i \rangle}_{\delta_{ij}} \underbrace{\langle \lambda_i | \lambda_j \rangle}_{\delta_{ij}} = 1 \tag{40}$$

$$\sum_i e^{\lambda_i} \underbrace{\sum_j \delta_{ij}\delta_{ij}}_{\delta_{ii}} = \sum_i e^{\lambda_i} \delta_{ii} = \sum_i e^{\lambda_i} = 1 \tag{41}$$

For real $\lambda_i$ this can only be satisfied if there is only one eigenvalue $\lambda = 0$. So $E = 0$. However in the Proof of Theorem 1.3 we chose $E = \rho_{XYZ}$ to prove the inequality given. But then $\rho_{XYZ}$ would be not be a density matrix any more since its trace is not 1. So the only other possibility would be if the eigenvalues could be complex or negative, in which case the density matrix would not really be hermitian or a density matrix respectively.

## 5 Dirac's opinion on the derivative of the logarithm

The problem that the density matrix might have complex eigenvalues could be related to the fact that the derivative of the logarithm does, using a special argument presented by Dirac [13], contain an imaginary part as well. We are used to:

$$\frac{d}{dx}[log(x)] = \frac{1}{x} \Big| x \in \mathbb{R}^+ \backslash \{0\} \tag{42}$$

However Dirac in Principles of Quantum Mechanics [13] states that this equation needs examination in the neighbourhood of the origin. So let us consider the integral:

$$\int_{-a}^{a} \frac{1}{x} dx = \lim_{\epsilon \to 0} \int_{-a}^{-\epsilon} \frac{1}{x} dx + \int_{\epsilon}^{a} \frac{1}{x} dx = 0 \tag{43}$$

Because $1/x$ is odd.
Now let us do the following:

$$\frac{d}{dx} log(x) = \frac{1}{x} \tag{44}$$

To check whether this is true we can integrate this function:

$$\int_{-a}^{a} \frac{d}{dx} log(x) = \int_{-a}^{a} \frac{1}{x} \tag{45}$$

The RHS vanishes as we just showed. So now we need to evaluate the LHS.

$$LHS = \int_{-a}^{a} \frac{d}{dx} log(x) = log(a) - log(-a) = log(-1) \tag{46}$$

However this term is complex.

$$log(-1) \equiv a \to e^a = -1 \to a = i\,(1 + 2n)\,\pi \Big| \text{let us say that}\, n = 0 \tag{47}$$

So we can conclude that we can extend the definition of the derivative of the logarithm:

$$\frac{d}{dx}log(x) = \frac{1}{x} - i\pi\delta(x) \tag{48}$$

We can check this by integrating again:

$$\int_{-a}^{a} \frac{d}{dx}log(x)dx = \int_{-a}^{a} \frac{1}{x} - i\pi\delta(x)dx \tag{49}$$

$$log(-1) = -i\pi\Big|exp(.) \tag{50}$$

$$e^{log(-1)} = e^{-i\pi} \tag{51}$$

$$-1 = e^{-i\pi} \tag{52}$$

Which is correct. Finally we note that an imaginary eigenvalue might have something to do with oscillations. Further reserach for justification is necessary.

# 6 Complex Vector Operations

Since we have shown that complex numbers might play a crucial role in the Entropy and because of the fact that during the mathematical algorithm for finding conditions on the Entropy for 2-qubit states we found that the computer sometimes printed errors that complex comparisons are not allowed. Therefore it might be useful for the future to define complex number comparisons. [14]

## 6.1 Multiplication of Complex Numbers

Let us undertake the multiplication of complex numbers and develop a Matrix-vector formalism for dealing with complex numbers:

$$zw = (a + ib)(c + id) = ac - bd + i(ad + bc) \tag{53}$$

So this can be written in matrix form in two different waves:

$$\begin{aligned} zw &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \times \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac - bd \\ ad + bc \end{bmatrix} \\ wz &= \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \times \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ac - bd \\ ad + bc \end{bmatrix} \end{aligned} \tag{54}$$

So every complex numbers vector representation has a multiplication-matrix representation. And the multiplication of complex numbers can be represented by a matrix multiplication which is real-antisymmetric. Interestingly the determinant of the matrix multiplication representation of a complex numebr is nothing but the modulus of the complex number. How can we define an orthogonal to a complex vector? For general

two dimensional real vectors all we need to do is switch the elements and multiply one by $-1$. We will do exactly the same for complex numbers.

$$z = \begin{bmatrix} a \\ b \end{bmatrix} z_{\perp} = \begin{bmatrix} \pm b \\ \mp a \end{bmatrix} \tag{55}$$

So we can define a matrix which does this operation, call it the complex orthogonalisation matrix $\hat{CO}$.

$$z_{\perp} = \begin{bmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{bmatrix} \times z \tag{56}$$

This is nothing but a multiplication by $i$ or $-i$ and looks very similar to the y-Pauli-matrix but only with real entries. Lets calculate the scalar product of two orthogonal complex numbers

$$z \cdot z_{\perp} = \begin{bmatrix} a & b \end{bmatrix} \cdot \begin{bmatrix} \pm b \\ \mp a \end{bmatrix} = 0 \tag{57}$$

In other words we need not define any metric to calculate the dual of a complex number, but we need only to transpose it.

## 6.2 Comparison of Complex Numbers

Now comes a more interesting question: How would you compare complex numbers? For example:

$$a + ib > c + id \tag{58}$$

How do we compare numbers on the real axis? What we usually do is just see the position of two numbers with respect to the origin and the one which is more to the right is greater than the other one. Actually we can also compare two real numbers with respect to a third number applying the same procedure. We can do a similar thing with complex numbers. The geometric procedure would be as follows:

1. Define a line $\Gamma$ through the complex numbers $z$ and $w$
2. Lay an orthogonal line $\Upsilon$ through $\Gamma$ and the Complex Number $v$ with respect to which $z$ and $w$ should be compared to.
3. Calculate the intersection point $\varsigma$ of $\Upsilon$ with $\Gamma$ then calculate the distance from the intersection $\varsigma$ to $z$ and $w$.
4. The complex number with the larger distance is the greater complex number.

Why did we define it this way? Suppose now if we were to rotate $\Upsilon$ and along with that its orthogonal $\Gamma$ then we could rotate it by an angle $\theta$ such that only a real part or imaginary part is left after the rotation. This would allow simple comparison. This

means that we are not concerned whether a number is to the left or to the right, because of isotropy of space. Let us treat this mathematically: Say we have two complex numbers:

$$w = \begin{bmatrix} a \\ b \end{bmatrix} \; z = \begin{bmatrix} c \\ d \end{bmatrix} \tag{59}$$

Then we can define a line $\Gamma$ as:

$$\Gamma = w + (z - w) \cdot t \tag{60}$$

for some parameter $t$. Now we need an orthogonal line to $\Gamma$. For this purpose we must find an orthogonal $zw_\perp$ to $z - w$. Which we can find using the procedure above:

$$zw_\perp = \begin{bmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{bmatrix} \times (z - w) = \begin{bmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{bmatrix} \times \begin{bmatrix} c - a \\ d - b \end{bmatrix} = \begin{bmatrix} \pm(d - b) \\ \mp(c - a) \end{bmatrix} \tag{61}$$

Now say that our complex with respect to which we want to compare the other two is:

$$v = \begin{bmatrix} e \\ f \end{bmatrix} \tag{62}$$

Then the line $\Upsilon$ is given by a line:

$$\Upsilon = v + s \cdot zw_\perp \tag{63}$$

To find the intersection we must equation equations 60 and 63:

$$\begin{aligned} \Gamma = \Upsilon \rightarrow v + s \cdot zw_\perp &= w + (z - w) \cdot t \\ v - w &= t \cdot (z - w) - s \cdot \hat{CO}(z - w) \\ v - w &= \left(t \cdot \mathbb{1} - s \cdot \hat{CO}\right)(z - w) \end{aligned} \tag{64}$$

This is a linear system of two equations with two variables and therefore has one solution. The signs of the solution depend on what convention we use for $\hat{CO}$. If we write out the full system of equation then we get:

$$\begin{bmatrix} e - a \\ f - b \end{bmatrix} = \begin{bmatrix} t & \pm s \\ \mp s & t \end{bmatrix} \begin{bmatrix} c - a \\ d - b \end{bmatrix} \tag{65}$$

The determinant of this matrix vanishes if:

$$t^2 + s^2 = 0 \rightarrow t = \pm\, i\, s \tag{66}$$

This would mean that the LH complex number is an "eigen-complex-number" of the RH complex number. Which reminds us of rotation matrices. If it is non-vanishing then the system of equations can be solved. And it should be since $s$ is a real parameter. A special case is if all the three complex numbers lie on one line. In this case it turns out

that the orthogonal line is the line itself and reminds of the light-cone in Minkowksi-Space. Moreover the linear system of equations can be solved if $s = 0$, which makes sense since in this case there doesn't really exist a normal complex-vector and you also get two solutions which must as expected correspond to the two complex numbers which are to be compared. As an example consider: $w = \gamma \cdot v$ $z = \delta \cdot v$. This would yield the following system of equation:

$$(1-\gamma)\, v = \left(t \cdot \mathbb{1} - s \cdot \hat{CO}\right)(\delta - \gamma)\, v \tag{67}$$

and we get two equations:

$$\begin{bmatrix}(1-\gamma)v_{Re}\\(1-\gamma)v_{Im}\end{bmatrix} = \begin{bmatrix} t & \pm s \\ \mp s & t \end{bmatrix} \begin{bmatrix}(\delta-\gamma)v_{Re}\\(\delta-\gamma)v_{Im}\end{bmatrix} \tag{68}$$

Due to the antisymmetry of the matrix this can be solved if $s = 0$ in other words: there is no need to define an orthogonal vector since they are all on one line. Using this we get the solution $t = \frac{1-\gamma}{\delta-\gamma}$. This obviously has a singularity if $\delta = \gamma$. Which makes sense since this would mean that the two complex numbers we are trying to compare are equal and therefore there is no need to compare them regardless with respect to which complex number we compare them. Now two special cases are if $\delta = 1$ or $\gamma = 1$. In the first case we get that the point of intersection is $z$ and in the second case $w$. This makes sense. Note that we could have also defined the number which is more towards the right/left of the line through the two complex numbers to be compared the larger one. The choice of left and right is purely a convention.

# 7 Separability - Entanglement - Negentropy - Non-Locality

Let us consider a system of two Quantum objects given by a density matrix $\rho$ in the Hilbert Space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ with $dim(\mathcal{H}_A) = dim(\mathcal{H}_B) = 2 \rightarrow \dim(\mathcal{H}_{AB}) = 2 \times 2$ The scalar product is defined by $\langle A|B\rangle = Tr\left[A^\dagger B\right]$ All Quantum states can be classified as either seperable or entangled.

**Definition 7.1** ***Separable States*** *[7][6]*
*The set of separable states $\mathcal{S}$ are defined as follows*

$$S = \left\{ \rho = \sum_i p_i \rho_{A,i} \otimes \rho_{B,i} \;\middle|\; 0 \leq p_i \leq 1, \sum_i p_i = 1 \right\} \tag{69}$$

A state is therefore entangled if it is not separable. An important distinction is to be made here: Entangled states are not necessarily non-local, hower all non-local states are entangled. Locality is broken iff the CHSH-inequality is violated. For density matrices the CHSH-inequality [6] can be written as:

$$\langle \rho | 2\,\mathbb{1} - B_{CHSH} \rangle \equiv Tr\left[\rho\,(2\,\mathbb{1} - B_{CHSH})\right] \geq 0 \tag{70}$$

Where the Bell operator $B_{CHSH}$ is:

$$B_{CHSH} = \vec{a} \cdot \vec{\sigma}_A \otimes \left(\vec{b} - \vec{b'}\right) \cdot \vec{\sigma}_B + \vec{a'} \cdot \vec{\sigma}_A \otimes \left(\vec{b} + \vec{b'}\right) \cdot \vec{\sigma}_B \tag{71}$$

It turns out that the CHSH Inequality is violated for all **pure entangled states** and that there are mixed entangled local states??

**Theorem 7.2** ***PPT-criterion*** *[7]*
*Given a density matrix $\rho$ in a Hilbert Space $\mathcal{H}$ with $dim(\mathcal{H}) = 2 \times 3$ or $dim(\mathcal{H}) = 2 \times 2$ and denoting $T_X$ as the partial transposition operator in the subspace $\mathcal{H}_X$ Then*

$$(\mathbb{1}_A \otimes T_B)\, \rho \geq 0 \text{ or } (T_A \otimes \mathbb{1}_B)\, \rho \geq 0 \leftrightarrow \rho \in S \tag{72}$$

**Lemma 7.3** ***Werner State Decomposition***
*The Werner States for a Hilbert Space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with $dim[\mathcal{H}] = 2 \times 2$ can be decomposed in the following way.*

$$\rho_{Werner} = \frac{1}{4}\left(\mathbb{1} \otimes \mathbb{1} - \alpha\, \sigma_i \otimes \sigma_i\right) \,\Big|\, \alpha \in [0, 1] \tag{73}$$

*Or in matrix notation using the z Basis:*

$$\rho_{Werner} = \frac{1}{4}\begin{bmatrix} 1-\alpha & 0 & 0 & 0 \\ 0 & 1+\alpha & -2\alpha & 0 \\ 0 & -2\alpha & 1+\alpha & 0 \\ 0 & 0 & 0 & 1-\alpha \end{bmatrix} \tag{74}$$

Using Theorem 7.2 the Werner states turn out to be separable for $\alpha \leq 1/3$ and therefore entangled for $\alpha > 1/3$ [6].

**Theorem 7.4** ***Maximal violation of Bell inequality*** *[6]*
*Given a general density matrix $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with $dim[\mathcal{H}] = 2 \times 2$ in a Hilbert Space in Bloch form*

$$\rho = \frac{1}{4}\left(\mathbb{1} \otimes \mathbb{1} + r_i \sigma_i \otimes \mathbb{1} + u_i \mathbb{1} \otimes \sigma_i \otimes + t_{ij} \sigma_i \otimes \sigma_j\right) \tag{75}$$

*and the Bell operator $B_{CHSH} = \vec{a} \cdot \vec{\sigma}_A \otimes \left(\vec{b} - \vec{b'}\right) \cdot \vec{\sigma}_B + \vec{a'} \cdot \vec{\sigma}_A \otimes \left(\vec{b} + \vec{b'}\right) \cdot \vec{\sigma}_B$ Then the maximal violation of the CHSH inequality is $B^{max} = \sqrt{t_1^2 + t_2^2} > 1$ where $t_1$ and $t_2$ are the two largest eigenvalues of $t_{ij}$*

For the Werner state all eigenvalues $t$ of the matrix $t_{ij}$ are $t = \alpha$.
So $\sqrt{2\alpha^2} > 1 \rightarrow \alpha > 1/\sqrt{2}$. Therefore the Werner state is non-local for $\alpha > \frac{1}{\sqrt{2}}$ [6].

**Hypothesis Do all non local states have Negentropy?**
Given a density matrix $\rho \in (\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B)$ with $dim(\mathcal{H}) = 2 \times 2$ which can be decomposed in Bloch form:

$$\rho = \frac{1}{4}\left(\mathbb{1} \otimes \mathbb{1} + \sum_i \alpha_i\, \sigma_i \otimes \sigma_i\right) \tag{76}$$

Let

$$NL = \left\{\rho \middle| Tr\left[\left(B_{CHSH} - 2\mathbb{1}\right)\rho\right] < 0\right\} \tag{77}$$

i.e the set of all Non Local States and Let

$$NE = \left\{\rho \middle| \left(S(A|B) < 0\right) \cup \left(S(B|A) < 0\right)\right\} \tag{78}$$

i.e the set of all States with negative conditional Entropy.
Then arises the question: Do all Non-Local states have Negentropy?

$$\rho \in NL \rightarrow \rho \in NE \tag{79}$$

$$NL \subset NE \tag{80}$$

In addition to finding this answer we would also like to geometrically locate the Separable, Mixed, Non-Local and Negentropic states. So let us first calculate the set of Non-Local states. For this we can make use of Theorem 7.4. So we just need to find the two largest eigenvalues $t_1$ and $t_2$ of the density matrix $\rho$ and apply the condition:

$$t^2_{max,1} + t^2_{max,2} > 1 \tag{81}$$

Then we have to apply the condition that the Negentropy Entropy is negative.

$$S(AB) - S(A) < 0 \tag{82}$$

For finding the Entropy of the subsystem let us take the partial trace of $\rho$. Since $\rho$ can be decomposed as in equation (76). We must take the partial trace of $\vec{\sigma} \otimes \vec{\sigma}$ and $\mathbb{1}$ on $\mathcal{H}_B$. Using the fact that the Pauli Spin Marices are traceless:

$$\begin{aligned} \langle . \uparrow | \vec{\sigma} \otimes \vec{\sigma} | . \uparrow \rangle + \langle . \downarrow | \vec{\sigma} \otimes \vec{\sigma} | . \downarrow \rangle = 0 \\ \langle . \uparrow | \mathbb{1} \otimes \mathbb{1} | . \uparrow \rangle + \langle . \downarrow | \mathbb{1} \otimes \mathbb{1} | . \downarrow \rangle = 2 \left|\uparrow\right\rangle \left\langle\uparrow\right| + 2 \left|\downarrow\right\rangle \left|\downarrow\right\rangle \end{aligned} \tag{83}$$

Therefore:

$$\underset{B}{Tr}[\rho] = \frac{1}{2} \left|\uparrow\right\rangle \left\langle\uparrow\right| + \frac{1}{2} \left|\downarrow\right\rangle \left|\downarrow\right\rangle \tag{84}$$

With two equal eigenvalues:

$$\lambda_A = \frac{1}{2} \tag{85}$$

Therefore the Entropy of the partial System $A$ is given by:

$$S(A) = -\left(\frac{1}{2} log_2\left(\frac{1}{2}\right) + \frac{1}{2} log_2\left(\frac{1}{2}\right)\right) = log_2(2) = 1 \tag{86}$$

So if the Negentropy Entropy has to be negative then:

$$S(AB) < 1 \tag{87}$$

And for this matter we will now find its eigenvalues. The density matrix $\rho$ can be represented as a matrix using the z-Basis:

$$\rho = \frac{1}{4}\begin{bmatrix} 1+\alpha_3 & 0 & 0 & \alpha_1-\alpha_2 \\ 0 & 1-\alpha_3 & \alpha_1+\alpha_2 & 0 \\ 0 & \alpha_1+\alpha_2 & 1-\alpha_3 & 0 \\ \alpha_1-\alpha_2 & 0 & 0 & 1+\alpha_3 \end{bmatrix} \tag{88}$$

The eigenvalues of $\rho$ turn out to be:

$$\begin{aligned} \lambda_1 &= \frac{1}{4}(1-\alpha_1-\alpha_2-\alpha_3) \\ \lambda_2 &= \frac{1}{4}(1-\alpha_1+\alpha_2+\alpha_3) \\ \lambda_3 &= \frac{1}{4}(1+\alpha_1-\alpha_2+\alpha_3) \\ \lambda_4 &= \frac{1}{4}(1+\alpha_1+\alpha_2-\alpha_3) \end{aligned} \tag{89}$$

In other words if $\lambda_k$ is an eigenvalues of $\rho$ then:

$$\sum_k \lambda_k \, log_2(\lambda_k) < 1 \tag{90}$$

So the three conditions are:

$$\begin{aligned} \alpha_i &\in [-1,1] \\ \lambda_k &\geq 0 \ \forall k \\ NE &\rightarrow \sum_k \lambda_k \, log_2(\lambda_k) < 1 \\ NL &\rightarrow t^2_{max,1} + t^2_{max,2} > 1 \end{aligned}$$

Since the Matrix $t_{i,j}$ defining the decomposition of the density matrix of the qubit state is diagonal. We just need to pick the two largest $\alpha_i$.
To give a broader picture and to demonstrate the geometry of the Qubit state we would also like to also find now a condition for separability and purity. Below Theorem 7.2 we showed the conditions for which the Werner State is separable. We would like to generalize this to all qubit states that can be decomposed as in equation (76). So applying the partial transposition we find:

$$\rho_{TB} = (\mathbb{1} \otimes P_T)\,\rho \geq 0 \ || \ \rho_{TA} = (P_T \otimes \mathbb{1})\,\rho \geq 0 \tag{91}$$

This yields a matrix:

$$\rho_{TB} = \frac{1}{4}\begin{bmatrix} 1+\alpha_3 & 0 & 0 & \alpha_1+\alpha_2 \\ 0 & 1-\alpha_3 & \alpha_1-\alpha_2 & 0 \\ 0 & \alpha_1-\alpha_2 & 1-\alpha_3 & 0 \\ \alpha_1+\alpha_2 & 0 & 0 & 1+\alpha_3 \end{bmatrix} \tag{92}$$

Due to the symmetry of the density matrix positivity of the other partially transposed density matrix is ensured. In this case the sign of the components of $\alpha$ of all the eigenvalues change and invoking the condition for separability through positvity of its eigenvalues yields the following:

$$\begin{aligned}
\lambda_{1,TP} &= \frac{1}{4}\left(1+\alpha_1+\alpha_2+\alpha_3\right) \geq 0 \\
\lambda_{2,TP} &= \frac{1}{4}\left(1+\alpha_1-\alpha_2-\alpha_3\right) \geq 0 \\
\lambda_{3,TP} &= \frac{1}{4}\left(1-\alpha_1+\alpha_2-\alpha_3\right) \geq 0 \\
\lambda_{4,TP} &= \frac{1}{4}\left(1-\alpha_1-\alpha_2+\alpha_3\right) \geq 0
\end{aligned} \tag{93}$$

So the conditions for separability are:

$$\lambda_k \geq 0 \quad \forall k \tag{94}$$

$$\lambda_{i,TP} \geq 0 \ \forall i \tag{95}$$

Now let us find a condition for purity. By definition of the density matrix it is mixed i.e. impure iff

$$Tr[\rho^2] < 1 \tag{96}$$

So let us use the this equation to find a condition on the set of $\alpha_i$:

$$\rho^2 = \left[\frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \sum_i \alpha_i\sigma_i\otimes\sigma_i\right)\right]^2 \tag{97}$$

$$\rho^2 = \frac{1}{16}\left[\mathbb{1}\otimes\mathbb{1} + 2\sum_i \alpha_i\sigma_i\otimes\sigma_i + \sum_{ij}\alpha_i\alpha_j\{\sigma_i,\sigma_j\}\otimes\{\sigma_i,\sigma_j\}\right] \tag{98}$$

Using the anticommutator relation for the Pauli Matrices:

$$\{\sigma_i,\sigma_j\} = 2\delta_{ij}\mathbb{1} \tag{99}$$

This simplifies to:

$$\rho^2 = \frac{1}{16}\left[\mathbb{1}\otimes\mathbb{1} + 2\sum_i \alpha_i\sigma_i\otimes\sigma_i + 4\sum_i \alpha_i^2\mathbb{1}\otimes\mathbb{1}\right] \tag{100}$$

And using the fact that the Pauli Matrices are Traceless so that the middle terms vanish:

$$Tr[\rho^2] = \frac{1}{16}\left[4 + 16\sum_i \alpha_i^2\right] < 1 \tag{101}$$

$$\rightarrow \sum_i \alpha_i^2 < \frac{12}{16} = 0.75 \tag{102}$$

This implies that:

$$\sum_i \alpha_i^2 < 0.75 \tag{103}$$

Let us now summarize the conditions for the various classifications of qubit states:

$$\begin{aligned}
\text{Separability} &\begin{cases} \alpha_i \in [-1,1] \\ \lambda_k \geq 0 \quad \forall k \\ \lambda_{i,TP} \geq 0 \quad \forall i \end{cases} \\
\text{Mixed} &\begin{cases} \alpha_i \in [-1,1] \\ \lambda_k \geq 0 \quad \forall k \\ \sum_i \alpha_i^2 < 0.75 \end{cases} \\
\text{Non-Local} &\begin{cases} \alpha_i \in [-1,1] \\ \lambda_k \geq 0 \quad \forall k \\ \alpha_{max,1}^2 + \alpha_{max,2}^2 > 1 \end{cases} \\
\text{Negative Conditional Entropy} &\begin{cases} \alpha_i \in [-1,1] \\ \lambda_k \geq 0 \quad \forall k \\ \sum_k \lambda_k log_2(\lambda_k) < 1 \end{cases}
\end{aligned} \tag{104}$$

These conditions restrict values of $\alpha_i$ to certain regions in the cube.

## 7.1 Geometry

We would now like to parametrize the states in such a way that it spans an equilateral triangular plane between the state on the top-center (tc) state $\rho_{tc}$, the Bell-state $\rho^- \to |\Psi^-\rangle$ and the other Bell-state $\rho^+ \to |\Psi^+\rangle$. However thanks to symmetry we need not consider the whole plane. We will infact split this equilateral triangular plane into two rectangular triangular planes by considering the following parametrization:

$$\begin{gathered}
\rho_{edge} = (1-\beta)\,\rho^- + \beta\,\rho^+ \\
\rho_{plane} = (1-\gamma)\,\rho_{edge} + \gamma\,\rho_{tc} \\
\beta \in [0, 1/2] \\
\gamma \in [0,1] \\
\gamma \leq 2\beta
\end{gathered} \tag{105}$$

Using the Bloch-decomposition of qubit density matrices.

$$\begin{aligned}
\rho^- &= \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \sum_i \alpha_i^- \ \sigma_i\otimes\sigma_i\right) \\
\rho^+ &= \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \sum_i \alpha_i^+ \ \sigma_i\otimes\sigma_i\right) \\
\rho_{tc} &= \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \sum_i \alpha_i^{tc} \ \sigma_i\otimes\sigma_i\right) \\
\alpha_i^- = [-1,-1,-1] \quad \alpha_i^+ &= [1,1,-1] \quad \alpha_i^{tc} = [0,0,1]
\end{aligned} \tag{106}$$

If we now plug in $\rho_{edge}$ in $\rho_{plane}$ we get the following:

$$\rho_{plane} = (1-\gamma)\left((1-\beta)\,\rho^- + \beta\rho^+\right) + \gamma\rho_{tc} \tag{107}$$

Simplifying this:

$$\rho_{plane} = \frac{1}{4}\mathbb{1}\otimes\mathbb{1} + \frac{1}{4}\sum_i \left[(\beta\gamma-\beta-\gamma)\,\alpha_i^- + (1-\gamma)\,\beta\alpha_i^+ + \gamma\,\alpha_i^{tc}\right]\ \sigma_i\otimes\sigma_i \tag{108}$$

Thus we can define a new $\alpha_i^{tc}$

$$\alpha_i^{plane} = \begin{bmatrix} -2\beta\gamma+2\beta+\gamma \\ -2\beta\gamma+2\beta+\gamma \\ 2\gamma \end{bmatrix} \tag{109}$$

Such that we can describe the states lying on that plane by:

$$\rho_{plane} = \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \sum_i \alpha_i^{plane}\sigma_i\otimes\sigma_i\right) \tag{110}$$

Writing this density matrix in matrix form:

$$\rho_{plane} = \frac{1}{4}\begin{bmatrix} 2\gamma+1 & 0 & 0 & 0 \\ 0 & -2\gamma+1 & -4\beta\gamma+4\beta+2\gamma & 0 \\ 0 & -4\beta\gamma+4\beta+2\gamma & -2\gamma+1 & 0 \\ 0 & 0 & 0 & 2\gamma+1 \end{bmatrix} \tag{111}$$

Thus this density matrix has eigenvalues:

$$\lambda_{\rho_{plane}} = \begin{bmatrix} \gamma/2+1/4 \\ \gamma/2+1/4 \\ 1/4+\beta-\beta\gamma \\ 1/4-\beta-\gamma-\beta\gamma \end{bmatrix} \tag{112}$$

Now we can plot the Geometry in Figure (1) of the states in the plane defined. It is clear from this figure that the negentropic states in red are a subset of the nonlocal states.

Therefore Negentropy is a not necessary condition for Non-Locality, but all Negentropy states are are also non-local. We have included in Figure (2) the Conditional Entropy as it varies along the lower edge.

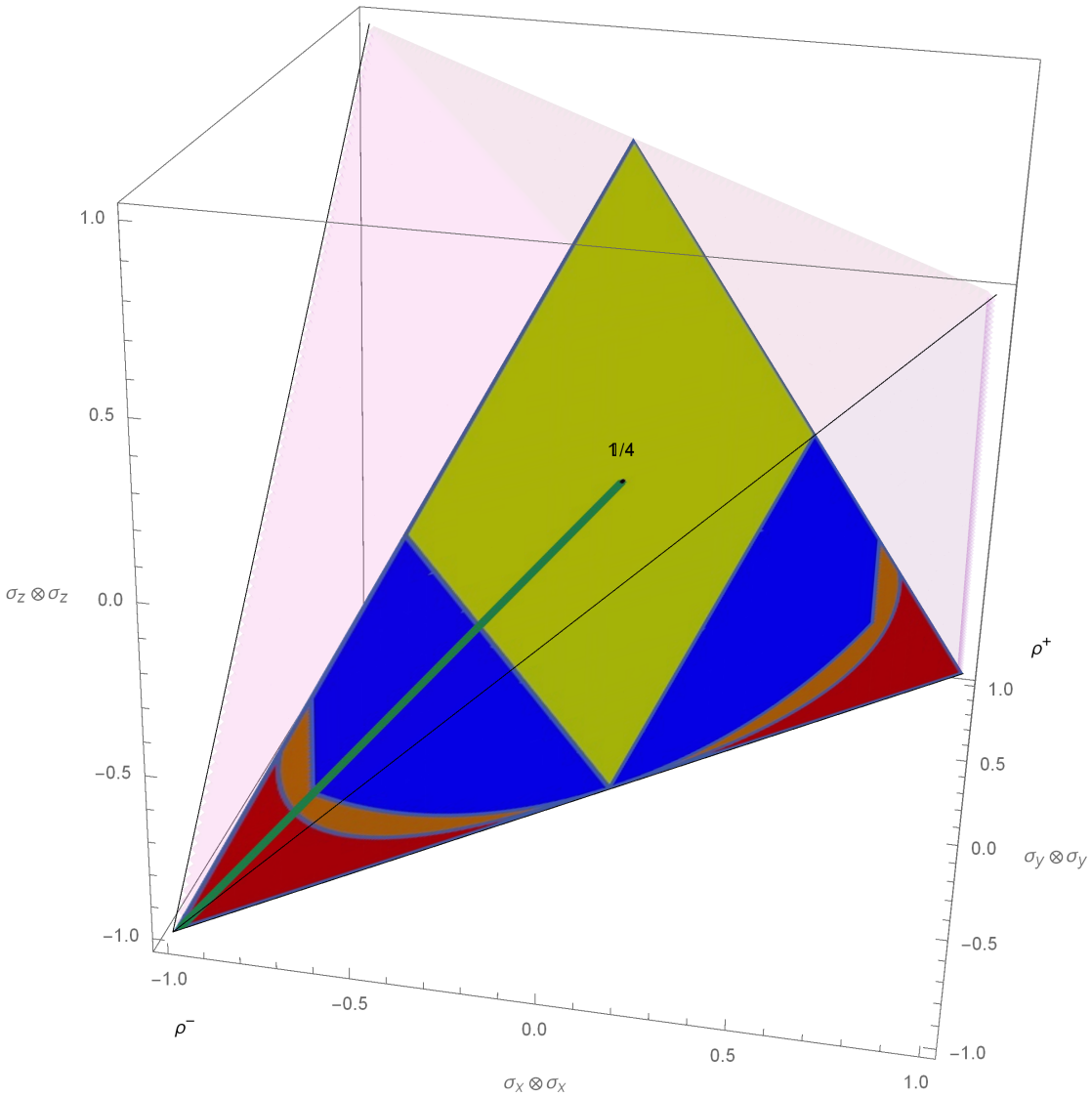


Figure 1: Geometry of states in Bell-Tetrahaedon: The yellow states describe the separable state, blue non-separable, orange non-local and red negentropic

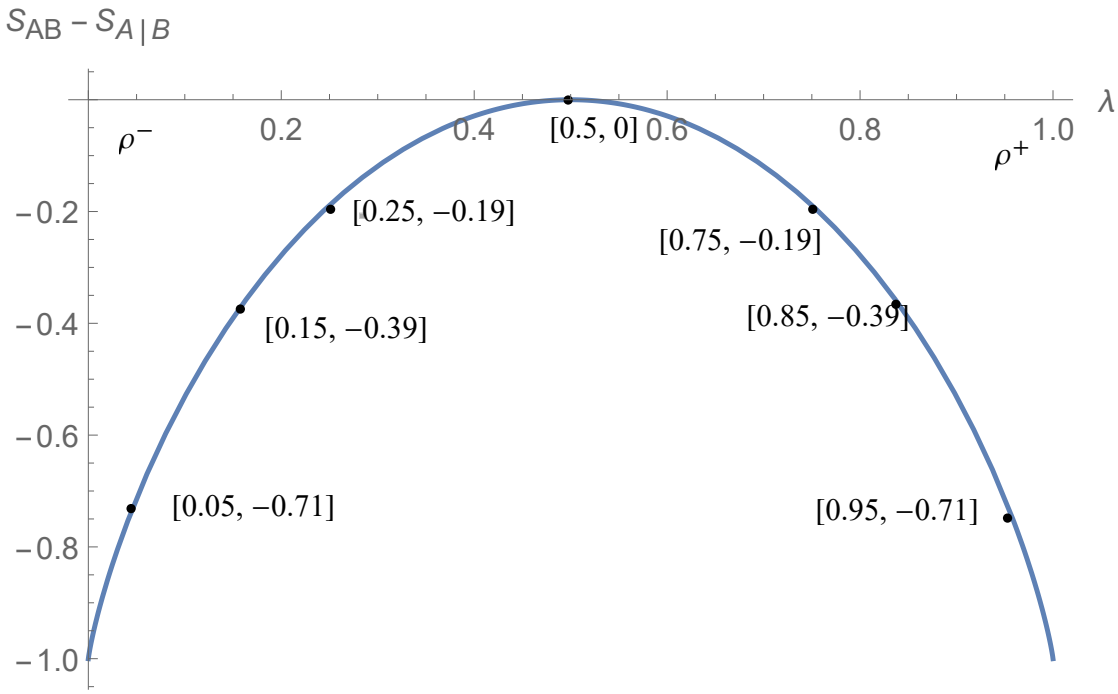


Figure 2: Conditional Entropy as it varies across the lower edge of the triangular plane defined above.

# 8 The Problem of Time

## 8.1 Principle of Least Action in Classical Mechanics

In Classical Mechanics we know that we can desribe any system in terms of its Lagrangian $\mathcal{L}(q, \dot{q}(t))$. The principle of Least Action states that evolution of the system between two configurations $q_a(t_1)$ and $q_a(t_2)$ is such that the action:

$$S := \int_{t_1}^{t_2} L(q_1, ..., q_N, \dot{q}_1, ...\dot{q}_N, t)dt \tag{113}$$

is minimzed over all possible trajectories between $q_a(t_1)$ and $q_a(t_2)$: Infact this condition leads to the famous Euler-Lagrange equations:

$$\frac{\partial L}{\partial q_a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) = 0 \,|a \in \{1, ..., N\} \tag{114}$$

## 8.2 Relativistic Generalization

The generalization of the action principle to a relativistic system faces serious challenges, because absolute time doesn't exist. [8] Consider a system reduced to a single particle $\mathcal{P}$ We also replace the absolute time by a parameter $\lambda$ that increases along $\mathcal{P}$'s worldline.

The Lagrangian is thus defined as a a differentiable function $L : \mathbb{R}^8 \to \mathbb{R}$ such that between any two events $A_1$ and $A_2$ of $\mathcal{P}'s$ worldline the integral:

$$S := \int_{\lambda_1}^{\lambda_2} L\left(x^\alpha(\lambda), \dot{x}^\alpha(\lambda)\right) d\lambda \tag{115}$$

has two properties:

- $S$ has the dimension of $Js$
- $S$ is independent of the parametrization $\lambda$

Requiring that $S$ is independent of the parametrization induces a constaint on the Lagrangian $L$. Considering a second parametrization $\tilde{\lambda}$ of $L$:

$$x^\alpha(\lambda) = \tilde{x}^\alpha(\tilde{\lambda}) \tag{116}$$

Therefore:

$$\dot{\tilde{x}}^\alpha := \frac{d\tilde{x}^\alpha}{d\tilde{\lambda}} = \frac{dx^\alpha}{d\lambda}\frac{d\lambda}{d\tilde{\lambda}} = \dot{x}^\alpha \frac{d\lambda}{d\tilde{\lambda}} \tag{117}$$

This implies that the invariance of the action is equivalent to

$$L\left(x^\alpha(\lambda), \dot{x}^\alpha(\lambda)\right) d\lambda = L\left(\tilde{x}^\alpha(\tilde{\lambda}), \dot{\tilde{x}}^\alpha(\tilde{\lambda})\right) d\tilde{\lambda} \tag{118}$$

Using the last two previous equations:

$$L\left(x^\alpha(\lambda), \dot{x}^\alpha(\lambda)\right) \frac{d\lambda}{d\tilde{\lambda}} = L\left(\tilde{x}^\alpha, \dot{x}^\alpha \frac{d\lambda}{d\tilde{\lambda}}\right) \tag{119}$$

This equation must be fulfilled for any $\frac{d\lambda}{d\tilde{\lambda}}$ thus we can use the Euler Homogeneity theorem:

$$\dot{x}^\alpha \frac{\partial L}{\partial \dot{x}^\alpha} = L \tag{120}$$

Now we would like the use the definition for the canonical conjugate impuls:

$$p^\alpha = \frac{\partial L}{\partial \dot{q}^\alpha} \tag{121}$$

For us $q^\alpha = x^\alpha$ Therefore:

$$\dot{x}^\alpha \frac{\partial L}{\partial \dot{x}^\alpha} = L \tag{122}$$

And now using the definition of the Hamiltonian:

$$H = \dot{x}^\alpha p^\alpha - L = 0 \tag{123}$$

Therefore the Hamiltonian vanishes. In Quantum Mechanics however we know that the Time Evolution is governed by the Unitary Operator:

$$\hat{U} = e^{-i\hat{H}t} \tag{124}$$

This poses a serious problem in unifying Quantum Theory and Relativity and is often referred to as the Problem of Time.

# 9 Clock Time and Entropy

## 9.1 Introduction

In this section we discuss the theoretical framework and recalculate the examples of the paper "Clock Time and Entropy" [11] forward by Don N. Page. He makes a clear distinction between measuring the probability of objects at different times and measuring different records at the same time. The past cannot be known except thought its record in the present, so we can only test the present. So we cannot directly test the conditional probability that the electron has spin up at $t = t_f$ given that it had spin down at $t = t_i < t_f$, but only given that there are records at $t = t_f$ from which we can infer that the electron had spin up at $t = t_i$. Wheeler states this even more strongly: "The past has no existence except as it is recorded in the present". This principle should be extended to say that we cannot directly compare things at different locations either. So that all measurements are localized within the spatial extent of an observers senses and the temporal extent of a single conscious moment. In the formalism of Quantum Mechanics on a spatial hypersurface is given by a density matrix $\rho$, the conditional probability of the result $A$, given a testable condition B, is:

$$p(A|B) = \frac{Tr\left[P_A P_B \rho P_B\right]}{Tr\left[P_B \rho P_B\right]} = \frac{Tr\left[P_B P_A P_B \rho\right]}{Tr\left[P_B \rho\right]} \tag{125}$$

Because of the cyclicity of the trace and the hermitian and idempotent properties of the Projectors.

## 9.2 Inaccessibility of coordinate time

These testable conditional probability as described in 125 seem to be confined to a single "time". However they cannot depend on the value of the coordinate time labelling the hypersurface, which is unobservable. The main arguments for this are:

- For a closed Universe the Wheeler-DeWitt equation, $\hat{H}\left|\Psi\right\rangle = 0$, implies that the wavefunction doesn't depend on time.
- For an asymptotically-flat open universe it turns out that the phases between states of different energy are unmeasureble, so that the coordinate tie dependence of the density matrix $\rho$ is not detectble.
- One doesn't have access to the coordinate time, so one should average over the inaccessible coordinate time dependence of the density matrix $\rho$.

  $$\bar{\rho} = \langle \rho(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \rho(t) dt \tag{126}$$

  So without access to the coordinate time we can only test conditionaly probability of the form:

  $$p(A|B) = \frac{\langle Tr\left[P_B P_A P_B \rho(t)\right]\rangle}{\langle Tr\left[P_B \rho(t)\right]\rangle} = \frac{Tr\left[P_B P_A P_B \bar{\rho}\right]}{Tr\left[P_B \bar{\rho}\right]} \tag{127}$$

Now consider the case where the condition is entirely the reading of a clock subsystem (C) with states $|\Psi(T)\rangle$. Let

$$P_B \to P_T = |\Psi_C(T)\rangle \langle\Psi_C(T)| \otimes I_R \tag{128}$$

Then

$$P(A|T) = \frac{Tr\left[P_T P_A P_T \bar{\rho}\right]}{Tr\left[P_T \bar{\rho}\right]} \tag{129}$$

can vary with clock time.

## 9.3 Examples

### 9.3.1 2-Qubit System

Let us assume we have two coupled spin $\frac{1}{2}$ systems in a vertical magnetic field. One represents the clock and the second represents the rest. Its Hamiltonian is given by:

$$\hat{H} = \frac{1}{4}\vec{\sigma}_C \otimes \vec{\sigma}_R + \frac{1}{2}\sigma_{Cz} \otimes \mathbb{1}_R + \frac{1}{2}\mathbb{1} \otimes \sigma_{Rz} + \frac{3}{4}\mathbb{1}_{CR} \tag{130}$$

Using the z Basis we can write the Pauli Spin Matrices as:

$$\begin{aligned} \sigma_x &= |\uparrow\rangle \langle\downarrow| + |\downarrow\rangle \langle\uparrow| \\ \sigma_y &= i\left(-|\uparrow\rangle \langle\downarrow| + |\downarrow\rangle \langle\uparrow|\right) \\ \sigma_z &= |\uparrow\rangle \langle\uparrow| - |\downarrow\rangle \langle\downarrow| \\ \mathbb{1} &= |\uparrow\rangle \langle\uparrow| + |\downarrow\rangle \langle\downarrow| \end{aligned} \tag{131}$$

Using these:

$$\begin{aligned} \vec{\sigma} \otimes \vec{\sigma} = &+ |\uparrow\uparrow\rangle \langle\downarrow\downarrow| + |\uparrow\downarrow\rangle \langle\downarrow\uparrow| \\ &+ |\downarrow\uparrow\rangle \langle\uparrow\downarrow| + |\downarrow\downarrow\rangle \langle\uparrow\uparrow| \\ &- |\uparrow\uparrow\rangle \langle\downarrow\downarrow| + |\uparrow\downarrow\rangle \langle\downarrow\uparrow| \\ &+ |\downarrow\uparrow\rangle \langle\uparrow\downarrow| - |\downarrow\downarrow\rangle \langle\uparrow\uparrow| \\ &+ |\uparrow\uparrow\rangle \langle\uparrow\uparrow| - |\uparrow\downarrow\rangle \langle\uparrow\downarrow| \\ &- |\downarrow\uparrow\rangle \langle\downarrow\uparrow| + |\downarrow\downarrow\rangle \langle\downarrow\downarrow| \\ = &\, 2\,|\uparrow\downarrow\rangle \langle\downarrow\uparrow| + 2\,|\downarrow\uparrow\rangle \langle\uparrow\downarrow| \\ &+ |\uparrow\uparrow\rangle \langle\uparrow\uparrow| - |\uparrow\downarrow\rangle \langle\uparrow\downarrow| \\ &- |\downarrow\uparrow\rangle \langle\downarrow\uparrow| + |\downarrow\downarrow\rangle \langle\downarrow\downarrow| \end{aligned} \tag{132}$$

$$\begin{aligned} \vec{\sigma} \otimes \mathbb{1} &= |\uparrow\uparrow\rangle \langle\uparrow\uparrow| + |\uparrow\downarrow\rangle \langle\uparrow\downarrow| - |\downarrow\uparrow\rangle \langle\downarrow\uparrow| - |\downarrow\downarrow\rangle \langle\downarrow\downarrow| \\ \mathbb{1} \otimes \vec{\sigma} &= |\uparrow\uparrow\rangle \langle\uparrow\uparrow| + |\downarrow\uparrow\rangle \langle\downarrow\uparrow| - |\uparrow\downarrow\rangle \langle\uparrow\downarrow| - |\downarrow\downarrow\rangle \langle\downarrow\downarrow| \\ \mathbb{1} \otimes \mathbb{1} &= |\uparrow\uparrow\rangle \langle\uparrow\uparrow| + |\uparrow\downarrow\rangle \langle\uparrow\downarrow| + |\downarrow\uparrow\rangle \langle\downarrow\uparrow| + |\downarrow\downarrow\rangle \langle\downarrow\downarrow| \end{aligned} \tag{133}$$

Now with the help of these subcalculationas we can calculate the Hamiltonian as per equation (130):

$$\hat{H} = 2\,|\uparrow\uparrow\rangle \langle\uparrow\uparrow| + \frac{1}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)\left(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|\right) \tag{134}$$

Let us now examine the time-dependant pure state:

$$|\Psi\rangle = \frac{\sqrt{5}}{10}\left[\left(e^{-it}+1\right)|\uparrow\downarrow\rangle + \left(e^{-it}-1\right)|\downarrow\uparrow\rangle\right] \tag{135}$$

So we can calculate its time averaged density matrix and since $\lim\limits_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{\pm it}dt \to 0$

$$\begin{aligned}\bar{\rho} &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\rho dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\Psi\rangle\langle\Psi| \\ &= \frac{1}{10}\left(|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + 2|\uparrow\downarrow\rangle\langle\downarrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow| - 2|\downarrow\uparrow\rangle\langle\downarrow\downarrow| + 2|\downarrow\downarrow\rangle\langle\uparrow\downarrow| - 2|\downarrow\downarrow\rangle\langle\downarrow\uparrow| + 8|\downarrow\downarrow\rangle\langle\downarrow\downarrow|\right)\end{aligned} \tag{136}$$

Using the standard basis for Spin up and Spin down:

$$\begin{aligned}|\uparrow\rangle &= \begin{bmatrix}1\\0\end{bmatrix} \\ |\downarrow\rangle &= \begin{bmatrix}0\\1\end{bmatrix}\end{aligned} \tag{137}$$

We can write the time averaged density matrix in this matrix form:

$$\bar{\rho} = \frac{1}{10}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & -2\\ 0 & 2 & -2 & 8\end{bmatrix} \tag{138}$$

The eigenvalues of this Matrix are:

$$\lambda = \{0, 0.1, 0.9\} \tag{139}$$

And since we defined $0 \cdot log(0) \equiv 0$ its Entropy is given by:

$$S_{CR} = -0.1log(0.1) - 0.9log(0.9) = 0.3251 \tag{140}$$

Now let

$$P_T = \frac{1}{2}\left(|\uparrow\rangle + e^{-iT}|\downarrow\rangle\right)\left(\langle\uparrow| + e^{iT}\langle\downarrow|\right)\otimes \mathbb{1}_R \tag{141}$$

In Matrix form this is:

$$P_T = \frac{1}{2}\begin{bmatrix}1 & 0 & e^{iT} & 0\\ 0 & 1 & 0 & e^{iT}\\ e^{-iT} & 0 & 1 & 0\\ 0 & e^{-iT} & 0 & 1\end{bmatrix} \tag{142}$$

So:

$$P_T\rho P_T = \frac{1}{40}\begin{bmatrix}1 & 0 & e^{iT} & 0\\ 0 & 1 & 0 & e^{iT}\\ e^{-iT} & 0 & 1 & 0\\ 0 & e^{-iT} & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & -2\\ 0 & 2 & -2 & 8\end{bmatrix}\begin{bmatrix}1 & 0 & e^{iT} & 0\\ 0 & 1 & 0 & e^{iT}\\ e^{-iT} & 0 & 1 & 0\\ 0 & e^{-iT} & 0 & 1\end{bmatrix} \tag{143}$$

which turns out to be:

$$P_T \rho P_T = \frac{1}{40} \begin{bmatrix} 1 & -2 & e^{iT} & -2e^{iT} \\ -2 & 9+2e^{-iT}+2e^{iT} & -2e^{iT} & 2+9e^{iT}+2e^{i2T} \\ e^{-iT} & -2e^{-iT} & 1 & -2 \\ -2e^{-iT} & 2+9e^{-iT}+2e^{-i2T} & -2 & 9+2e^{-iT}+2e^{iT} \end{bmatrix} \tag{144}$$

Therefore:

$$Tr\left[P_T \bar{\rho} P_T\right] = \frac{1}{40}\left[20 + 8cos(T)\right] \tag{145}$$

So:

$$\rho_T = \frac{P_T \bar{\rho} P_T}{Tr\left[P_T \bar{\rho} P_T\right]} = \frac{1}{20+8cos(T)} \begin{bmatrix} 1 & -2 & e^{iT} & -2e^{iT} \\ -2 & 9+2e^{-iT}+2e^{iT} & -2e^{iT} & 2+9e^{iT}+2e^{i2T} \\ e^{-iT} & -2e^{-iT} & 1 & -2 \\ -2e^{-iT} & 2+9e^{-iT}+2e^{-i2T} & -2 & 9+2e^{-iT}+2e^{iT} \end{bmatrix} \tag{146}$$

This matrix can infact be split into a tensor product:

$$\rho_T = \rho_{TC} \otimes \rho_{TR} \tag{147}$$

where

$$\rho_{TC} = \frac{1}{2} \begin{bmatrix} 1 & e^{iT} \\ e^{-iT} & 1 \end{bmatrix} \tag{148}$$

$$\rho_{TR} = \frac{1}{4cosT+10} \begin{bmatrix} 1 & -2 \\ -2 & 9+4cos(T) \end{bmatrix} \tag{149}$$

Therefore to find the eigenvalues of $\rho_T$ we have to find the eigenvalues of $\rho_{TC}$ and $\rho_{TR}$ and multiply them in all possible combinations [12].
The eigenvalues of $\rho_{TC}$ and $\rho_{TR}$ are:

$$\begin{gathered} \lambda_{TC} = \{0, 1\} \\ \lambda_{TR} = \{5 + 2cosT \pm \sqrt{20 + 16cosT + 4cos^2T}\} \end{gathered} \tag{150}$$

So the eigenavlues of $\rho_T$ are:

$$\lambda_T = \left\{0, 0, \frac{1}{2} + \frac{2\sqrt{(cosT+2)^2+1}}{10+4cosT}, \frac{1}{2} - \frac{2\sqrt{(cosT+2)^2+1}}{10+4cosT}\right\} = \{0, 0, a+b, a-b\} \tag{151}$$

We can now calculate the Entropy as follows:

$$S = -\left((a+b)log(a+b) + (a-b)log(a-b)\right) = -\left[alog(a^2-b^2) + blog\left(\frac{a+b}{a-b}\right)\right]$$

$$S = log\left(\frac{10+4cosT}{\sqrt{5+4cosT}}\right) - 2\frac{\sqrt{(cosT+2)^2+1}}{5+2cosT} log\left(\frac{5+2cosT+2\sqrt{(cosT+1)^2+1}}{\sqrt{4cosT+5}}\right) \tag{152}$$

So this shows that although $\rho$ is pure and its Entropy vanishes the Entropy of the time averaged density matrix $\bar{\rho}$ is finite however the measured Entropy changes with time.

### 9.3.2 3-Qubit System

Now let us turn our attention to a 3-qubit system. We would like to find the Course Grained Entropy of the following system:
Let the Hamiltonian be given by:

$$\hat{H} = \frac{1}{2}\,\sigma_{Cz}\otimes\mathbb{1}_{R1}\otimes\mathbb{1}_{R2}+\frac{1}{2}\,\mathbb{1}_C\otimes\sigma_{R1z}\otimes\mathbb{1}_{R2}+\frac{1}{2}\,\mathbb{1}_C\otimes\mathbb{1}_{R1}\otimes\sigma_{R2z}+\frac{1}{4}\,\mathbb{1}_C\otimes\vec{\sigma}_{R1}\otimes\vec{\sigma}_{R2}+\frac{1}{4}\,\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} \tag{153}$$

Now we consider the pure time independent state:

$$|\Psi\rangle = \frac{1}{2}\left(|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle + |\downarrow\downarrow\uparrow\rangle\right) \tag{154}$$

Its density matrix can be written in Matrix form as:

$$|\Psi\rangle\langle\Psi| = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & -1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \equiv \frac{1}{4}\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \tag{155}$$

Let us now apply the same Projector as in the first example:

$$P_T = \frac{1}{2}\left(|\uparrow\rangle + e^{-iT}\,|\downarrow\rangle\right)\left(\langle\uparrow| + e^{iT}\,\langle\downarrow|\right)\otimes\mathbb{1}_{R1}\otimes\mathbb{1}_{R2} = \frac{1}{2}\begin{bmatrix} \mathbb{1}_{4\times4} & e^{iT}\,\mathbb{1}_{4\times4} \\ e^{-iT}\,\mathbb{1}_{4\times4} & \mathbb{1}_{4\times4} \end{bmatrix} \tag{156}$$

$P_T\rho P_T =$

$$\frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2+2cosT & -2isinT & 0 & 0 & 1+2e^{iT}+e^{i2T} & 1-e^{i2T} & 0 \\ 0 & 2isinT & 2-2cosT & 0 & 0 & -1+e^{i2T} & -1+2e^{iT}-e^{i2T} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+2e^{-iT}+e^{-i2T} & -1+e^{-i2T} & 0 & 0 & 2+2cosT & -2isinT & 0 \\ 0 & 1+e^{-i2T} & -1+2e^{iT}-e^{i2T} & 0 & 0 & 2isinT & 2-2cosT & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{157}$$

Its Trace:

$$Tr[P_T\rho P_T] = \frac{1}{2} \tag{158}$$

So:

$$\frac{P_T \rho P_T}{Tr[P_T \rho P_T]} =$$

$$\frac{1}{8}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2+2cosT & -2isinT & 0 & 0 & 1+2e^{iT}+e^{i2T} & 1-e^{i2T} & 0 \\ 0 & 2isinT & 2-2cosT & 0 & 0 & -1+e^{i2T} & -1+2e^{iT}-e^{i2T} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+2e^{-iT}+e^{-i2T} & -1+e^{-i2T} & 0 & 0 & 2+2cosT & -2isinT & 0 \\ 0 & 1+e^{-i2T} & -1+2e^{iT}-e^{i2T} & 0 & 0 & 2isinT & 2-2cosT & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{159}$$

This density matrix can be decomposed in the following way:

$$\rho = |\Psi_T\rangle \langle \Psi_T| = \frac{1}{8}\begin{bmatrix} 1 & e^{iT} \\ e^{-iT} & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2+2cosT & -2isinT & 0 \\ 0 & 2isinT & 2-2cosT & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{160}$$

To find the coarse grained Entropy we have to compute the partial trace of $\rho_T$ with respect to all its subparts. Obviously if we Trace over $\mathcal{H}_{R1} \otimes \mathcal{H}_{R2}$ then we get $\rho_{TC}$ because we can form the density matrix as a tensor product as indicated above. Tracing over $\mathcal{H}_C$ we get the following density matrix:

$$\rho_{T,R_1,R_2} = \frac{1}{2}\left[(1+cosT)\,|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + (1-cosT)\,|\downarrow\uparrow\rangle\langle\downarrow\uparrow| - isinT\,|\uparrow\downarrow\rangle\langle\downarrow\uparrow| + isinT\,|\downarrow\uparrow\rangle\langle\uparrow\downarrow|\right] \tag{161}$$

Forming the Trace of this density matrix once over $\mathcal{H}_{R_1}$ and once over $\mathcal{H}_{R_2}$ we get the following two density matrices

$$\rho_{T,R_1} = \frac{1}{2}\begin{bmatrix} 1+cosT & 0 \\ 0 & 1-cosT \end{bmatrix} \quad \rho_{T,R_2} = \frac{1}{2}\begin{bmatrix} 1-cosT & 0 \\ 0 & 1+cosT \end{bmatrix} \tag{162}$$

Therefore the eigenvalues of these matrices are:

$$\lambda_C = \{0,1\}$$
$$\lambda_{R1,2} = \left\{\frac{1}{2}(1+cosT), \frac{1}{2}(1-cosT)\right\} \tag{163}$$

And this the coarse grained Entropy is:

$$S_{T,coarse} = -2\,cos^2(T/2)\,log(cos^2(T/2)) - 2\,sin^2(T/2)\,log(sin^2(T/2)) \tag{164}$$

This clearly varies periodically with time.

# A Proposing an Extension of Einstein's Energy-Mass equivalence

Currently we are not able to unify General Relativity with Thermodynamics. It is however very simple. Imagine that "our" universe is part of a hypothetical larger universe. Then we can treat our universe according to the Grand-Canonical-Ensemble:

$$dE = TdS + pdV + \mu dV \tag{165}$$

Dividing by $dt$ where $x$ we would get:

$$\frac{dE}{dt} = T(x)\frac{dS(x)}{dt} + p(x)\frac{dV(x)}{dt} + \mu(x)\frac{dN(x)}{dt} \tag{166}$$

Now assume that our universe is actually isolated. This means that $dE/dt = 0$ by law of conservation of energy.

$$T(x)\frac{dS(x)}{dt} + p(x)\frac{dV(x)}{dt} + \mu(x)\frac{dN(x)}{dt} = 0 \tag{167}$$

This equation describes some sort of energy flow in time and suggests that Entropy, Space and Mass are interconvertable forms of energy. Note that it might be possible, that the energy can not only flow in time but also in space. Now let us think about a thought experiment and assume that $T, p, c^2 > 0$: We could now interpret the pressure as the Gravitational Cosmological Constant and the chemical potential as $\mu = mc^2$. Relativistic corrections might be necessary. Further research and exact mathematical formulation possibly through tensor fields is probably necessary.

## B Using the Definition of Euler Mascheroni Constant to find $\int_0^\infty 1/x dx$

The Euler Mascheroni Constant is known as [15]:

$$\gamma \equiv \lim_{k\to\infty} \sum_{i=1}^{k} \frac{1}{i} - ln(k) = \int_1^\infty \frac{1}{\lfloor x \rfloor} - \frac{1}{x} dx \tag{168}$$

We want to find:

$$\int_0^\infty 1/x dx \tag{169}$$

For this matter we will take the first equation and change the summation index:

$$\begin{aligned}
\gamma = \lim_{k\to\infty} \sum_{j=0}^{k-1} \frac{1}{j+1} - ln(k) &= \int_0^\infty \frac{1}{\lfloor x+1 \rfloor} - \left(\frac{1}{x} - i\pi\delta(x)\right) dx \\
&= \int_1^\infty \frac{1}{\lfloor x \rfloor} dx - \int_0^\infty \left(\frac{1}{x} - i\pi\delta(x)\right) dx \\
&= \underbrace{\int_1^\infty \frac{1}{\lfloor x \rfloor} - \frac{1}{x} dx}_{\gamma} - \int_0^1 \frac{1}{x} dx + \int_0^\infty i\pi\delta(x) dx \\
&\to \int_0^1 \frac{1}{x} dx = \int_0^\infty i\pi\delta(x) dx
\end{aligned} \tag{170}$$

Since using the substitution $u = 1/x$ it can be shown that

$$\int_0^1 1/x dx = \int_1^\infty 1/x dx \tag{171}$$

We conclude that:

$$\int_0^\infty \frac{1}{x} dx = \int_0^\infty i2\pi\delta(x) dx \tag{172}$$

Now using a sequence of normal distribution to approximate the Dirac-Delta we find, since we are integrating over half of its domain.

$$\int_0^\infty \frac{1}{x}\, dx = i\pi \tag{173}$$

This result can also be shown using complex integration.

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