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Decoherence and open quantum sytems

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Chapter 1

Introduction

In quantum information theory entanglement is used as a source for many different applications, for instance in quantum cryptography, teleportation schemes or quantum computation. To keep this entanglement in the system as long as possible we have to protect it against different environmental influences. For this reason, the study of diverse decoherence models of open quantum systems is essential to develop a better understanding of the occuring phenomena.

In practice, any realistic physical system has to be regarded as a system coupled to an uncontrollable environment which influences it in a non-negligible way. Therefore the theory of open quantum systems plays an important role in many different applications in the field of quantum mechanics, where it is often not possible to isolate the systems or to control the environmental degrees of freedom. Most of these systems are too complicated to describe them completely with the underlying microscopic laws of physics. Nevertheless such a detailed description is often not necessary since experience shows that the evolution of large physical systems can be described quite accurate by just a small number of relevant variables. The theory of open quantum systems provides us with a set of effective equations of motion describing the coupling to the environment just by a small set of parameters.

Another more fundamental reason for the study of open quantum systems is that the measurement process could be interpreted as a kind of open system dynamics. The collapse of the wavefunction is generated by the destruction of the superpositions between the states of the system. This relation could explain the origin of the classical world out of quantum theory.

The present work is split into two main sections. The first part introduces the underlying theoretical background whereas in the second part different examples for open quantum systems are discussed.

We start with the formulation of quantum states by density operators which will be our objects of interest throughout this work. After the definition of entanglement and the discussion of some special states, we focus on the geometry of different quantum mechanical systems. These geometrical pictures will help us a lot to get a good intuition for all the possible evolutions of the considered systems. The theoretical part ends with an introduction to the theory of decoherence and open quantum systems. This section also provides us with the relevant equations of motion which will be solved for different quantum mechanical systems in the second part. We start our calculations with the simplest possible quantum system where entanglement already occurs - the two qubit system. Different initial states are considered and their behaviours under certain decoherence effects are illustrated. Finally we are going to raise the degrees of freedom and consider bipartite systems with abritrary dimensions but, however, the focus mainly stays on the evolution of systems consisting of two qutrits.

Part I

Basic Concepts

Chapter 2

Basic concepts of quantum mechanics

2.1 Quantum mechanics with pure and mixed states

We briefly sketch the most important aspects of pure and mixed states and moreover their formulation with the help of density operators. Besides the discussion of the properties of such density matrices we introduce a measure of purity which allows us a simple determination of pure and mixed states.

The time evolution of quantum mechanical systems is skipped here but a short introduction can be found in Chapter 4 during the discussion of closed and open system dynamics. If a more detailed description is desired, the books of Ballentine [1], Nielsen and Chuang [40] and Peres [41] provide a comprehensive discussion.

2.1.1 Pure states

Pure quantum mechanical states are described by a state vector $|\Psi\rangle$ living in the Hilbert space $\mathcal{H}$. If a basis $|k_i\rangle$ is chosen for the Hilbert space, any state vector can be expanded as a linear combination of those basis elements like

$$|\Psi\rangle = \sum_i^d c_i |k_i\rangle, \tag{2.1}$$

where the coefficients c_i are complex and $d = dim\,\mathcal{H}$. The general state $|\Psi\rangle$ can be normalized by claiming the condition

$$\langle\Psi|\Psi\rangle = 1 \qquad \Longleftrightarrow \qquad \sum_i^d |c_i|^2 = 1. \tag{2.2}$$

If we ask for the expectation value of an observable A, it can be calculated by

$$\langle A\rangle = \langle A\rangle_\Psi = \langle\Psi|A|\Psi\rangle. \tag{2.3}$$

2.1.2 Mixed states

If the considered quantum mechanical system has a particular probability p_i to be in one state out of a whole set of states $|\Psi_i\rangle$, we talk of a mixed state. It is not possible anymore to describe this statistical ensemble of pure states just by a single state vector, therefore a new formalism is needed. We define a density operator as a convex combination of pure state projectors weighted with a corresponding probability like

$$\rho = \sum_i p_i|\Psi_i\rangle\langle\Psi_i| = \sum_i p_i\rho_i \qquad \text{with} \qquad \sum_i p_i = 1 \text{ and } p_i \geq 0. \tag{2.4}$$

If, for instance, the state is pure with $p_1 = 1$ then the density matrix is reduced to

$$\rho = |\Psi\rangle\langle\Psi|, \tag{2.5}$$

whereas for a given general density matrix describing a pure state it is often not so trivial to find exactly this decomposition. Therefore it is convenient to introduce a measure of **purity** given by $P = Tr\rho^2$ with the properties

$$\begin{aligned} Tr\rho^2 &= 1 \qquad \text{for pure states} \\ Tr\rho^2 &< 1 \qquad \text{for mixed states} \end{aligned} \tag{2.6}$$

The lowest possible amount of purity for any physical state is $P = \frac{1}{d}$ with $d = dim(\mathcal{H})$ again.
To guarantee that ρ really describes a physical state it has to satisfy the following properties:

- $\rho^\dagger = \rho$ hermiticity
- $\rho \geq 0$ positivity
- $Tr\rho = 1$ normalization

- positive and real eigenvalues

where Tr means the trace. The expectation value for a given observable A can be calculated for a mixed state by

$$\langle A\rangle = \langle A\rangle_\rho = \sum_i p_i \langle\Psi_i|A|\Psi_i\rangle = Tr\rho A \tag{2.7}$$

Since it is indistinguishable in which ensemble $\{p_i, \rho_i\}$ the mixed state is decomposed, the whole physical information of the state is expressed by the density operator ρ.
In all the following discussions of quantum mechanical systems we will mere use the density matrix formalism because more states can be considered.

2.2 Entanglement

If we consider systems that consist of two or more parties, a new quantum mechanical feature arises called entanglement. This property caused Einstein and many other physicist to dislike quantum machanics since it predicts highly counterintuitive processes like the EPR Paradoxon [22]. Einstein called entanglement a "spukhafte Fernwirkung" or "spooky action at distance", where Schrödinger [45] stated that this is *the* essence of quantum mechanics.

Here, we just want to focus on bipartite systems whose Hilbert space is constructed by the tensor product of the Hilbert space of the two subsystems $\mathcal{H} = \mathcal{H}^{(\mathcal{A})} \otimes \mathcal{H}^{(\mathcal{B})}$, where we call the first system Alice and the second Bob. Surprisingly in general the density operators can not be written in such a product form like the Hilbert space. If this is actually possible we call the state separated and can formulate the following definition

Separability:
If a state ρ can be written as the convex combination of product states like

$$\rho = \sum_i p_i \, \rho_i^{(\mathcal{A})} \otimes \rho_i^{(\mathcal{B})} \tag{2.8}$$

then ρ is called separable. Here $\rho^{(\mathcal{A})}$ and $\rho^{(\mathcal{B})}$ are density matrices of the subsystems and the weights satisfy $p_i \geq 0$ and $\sum_i p_i = 1$.

The set of all separable states is convex and in the following discussion denoted by $\mathcal{S}$. But note that nevertheless *any* state can be expressed by

$$\rho = \sum_{i,j} p_{ij}\, \rho_i^{(\mathcal{A})} \otimes \rho_j^{(\mathcal{B})}. \tag{2.9}$$

Now we can define entanglement due to Werner [48] as

Entanglement:
If a state ρ is not separable then it is called entangled.

The set of all entangled states should be denoted by $\mathcal{S}^{\mathcal{C}}$. Werner called the separable states classically correlated states since they can be produced "classically" by local preparations of the subsystems, whereas entanglement is a global feature of the whole Hilbert space.
Under local unitary operations any entangled state stays entangled like

$$\rho \longrightarrow \rho' = (U_{\mathcal{A}} \otimes U_{\mathcal{B}})\rho(U_{\mathcal{A}} \otimes U_{\mathcal{B}})^{\dagger}, \tag{2.10}$$

where both unitary operators U satisfy $U^{\dagger}U = UU^{\dagger} = \mathbb{1}$. The two entangled density operators ρ and ρ' are called equivalent.

2.2.1 Positive partial transposition - PPT

Although the separability given by Eq.(2.9) is mathematically well defined, in practise it is difficult to check whether a given density matrix describes a separable or an entangled state. The problem is that there exist infinitely many possibilities of decomposing a density operator and to be sure that the state can indeed be expressed by the form of Eq.(2.9), one has to check all of the possibilities.
That is why different operational criteria for testing separability have been proposed. Since all of them use another approach, they are different powerful. Besides the Peres-Horodecki criterion which will be discussed here, the following criteria have been found: Schmidt rank criterion [42, 43] (only valid for pure states), reduction criterion [29], complete positive maps [30] and entanglement witnesses [30].

Here we will just focus on one criterion which was found by Peres [43] and the Horodeckis [30] and is therefore called the Peres-Horodecki or positive partial transposition (PPT) criterion.
If we choose the basis $\{|m_i\rangle\}$ for Alice's system and $\{|n_i\rangle\}$ for Bob's we can

write the density operator as

$$\rho = \sum_{i,j,k,l} \rho_{ij,kl} |m_i\rangle \otimes |n_k\rangle\langle m_j| \otimes \langle n_l|. \tag{2.11}$$

The two possible partial transpositions according to the respective subsystem are defined as

$$(\rho^{T_\mathcal{A}})_{ij,kl} = \rho_{ji,kl} \tag{2.12}$$
$$(\rho^{T_\mathcal{B}})_{ij,kl} = \rho_{ij,lk} \tag{2.13}$$

Although the partial transposition must be performed in a certain basis, the eigenvalues of the gained matrix are independent of the particular choice of basis. Therefore we can define a socalled ppt-state (positive partial transpose) which has only positive or zero eigenvalues $\rho^{T_{\mathcal{A}/\mathcal{B}}} \geq 0$ and a npt-state (negative partial transpose) with at least one negative eigenvalue. The separablity theorem can then be formulated as

PPT-criterion:

The partial transposition of a separable state with respect to any subsystem is always positive. (All separable states are ppt-states)

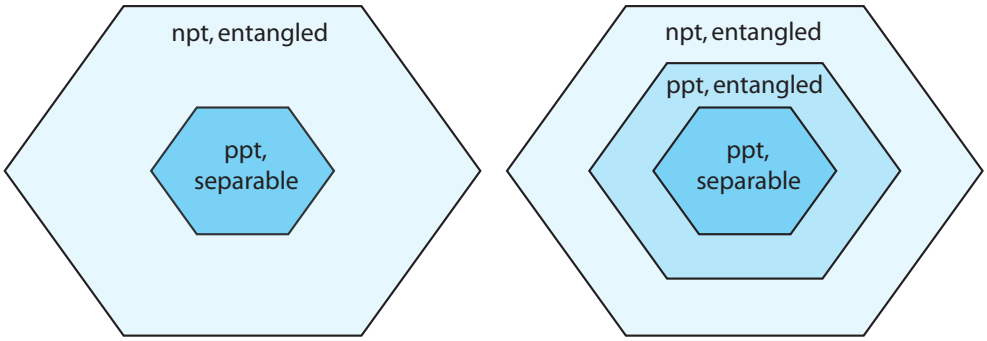


Figure 2.1: A schematic illustration of ppt- and npt-states for the Hilbert spaces $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^3$ on the left side and for higher dimensions on the right side, where bound entangled (ppt and entangled) states can be found.

This criterion holds for bipartite systems with abritrary dimensions but ppt is only a necessary and sufficient condition for the Hilbert spaces $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^3$. In higher dimensions there also exist states that are ppt *and* entangled. These states are called bound entangled states [32]

because they can not be distilled into a maximally entangled state. A region of bound entangled states in a two qutrit system is discussed in Sect.3.4. The Fig.2.1 shows a schematical picture of the PPT-criterion and its propositions for different dimensions.

2.2.2 Concurrence

Although von Neumann found an appropriate measure of entanglement for all pure states with the definition of the von Neumann entropy, a quantification of the amount of entanglement for general states has not yet been found. By now the search of this measure is like the search for the Holy Grail in quantum information theory. Although there still exist many different kinds of measures like the concurrence, the entanglement of formation, the entanglement of distillation or the relative entropy of entanglement, none of them satisfy all properties a good measure should have.

Here we want to introduce the concurrence which is a proper measure of entanglement but only valid for two qubit systems. It was found 1996 by Bennett, DiVincenzo, Smolin and Wootters [7] for pure states and then generalized by Wootters and Hill [27, 49, 50] also for mixed.
If we consider a **pure state** $|\Psi\rangle$, we can express it in the following complete basis constructed with maximally entangled states called the Bell states (see Sect.2.3.2)

$$\begin{aligned} |e_1\rangle &= |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ |e_2\rangle &= i|\Phi^-\rangle = \frac{i}{\sqrt{2}}(|00\rangle - |11\rangle) \\ |e_3\rangle &= i|\Psi^+\rangle = \frac{i}{\sqrt{2}}(|01\rangle + |10\rangle) \\ |e_4\rangle &= |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{aligned} \tag{2.14}$$

Thus we can write

$$|\Psi\rangle = \sum_{j=1}^{4} \alpha_j |e_j\rangle. \tag{2.15}$$

Then the concurrence C is defined as

$$C(|\Psi\rangle) = |\sum_j \alpha_j^2|, \tag{2.16}$$

which is zero for a separable and 1 for a maximally entangled state.
Considering a **mixed state** represented by the denisity operator ρ we can perform a spin-flip operation like

$$\tilde{\rho} = (\sigma_2 \otimes \sigma_2)\rho^*(\sigma_2 \otimes \sigma_2), \tag{2.17}$$

where the complex conjugate of the density matrix is taken in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and σ_2 is a Pauli matrix (see Sect.3.1). Now we can define $R^2 := \rho \cdot \tilde{\rho}$ and λ_i which are the square roots of the eigenvalues of the matrix R^2. The concurrence of the mixed state is then given by

$$C(\rho) = max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{2.18}$$

where $\lambda_i \geq 0$ and λ_1 is the largest eigenvalue. If ρ is pure then the matrix R has just one eigenvalue which is directly the amount of concurrence.
We want to give the concurrence explicitly for all the density matrices of the form

$$\rho(t) = \begin{pmatrix} a(t) & 0 & 0 & f(t) \\ 0 & b(t) & 0 & 0 \\ 0 & 0 & c(t) & 0 \\ f*(t) & 0 & 0 & d(t) \end{pmatrix}. \tag{2.19}$$

The concurrence is then

$$C(\rho(t)) = 2\{max\Big(0, |f(t)| - \sqrt{b(t) \cdot c(t)}\Big)\}. \tag{2.20}$$

2.3 Special bipartite states

We want to discuss different sets of states which are special concerning their amount of entanglement or become relavant later in the discussion of various decoherence modes (see Chapter 5 and 6). From now on we want to consider Hilbert spaces of the form $\mathbb{C}^d \otimes \mathbb{C}^d$.

2.3.1 Maximally mixed state

The maximally mixed state is given by the identity parametrised so that $Tr\rho = 1$ is satisfied. In general the state is

$$\rho = \frac{1}{d}\mathbb{1}_d \otimes \frac{1}{d}\mathbb{1}_d = \frac{1}{d^2}\mathbb{1}_{d^2}, \tag{2.21}$$

where the purity is $P = 1/d^2$. No states exist with a purity smaller than that.

2.3.2 Bell states

All pure and maximally entangled states are called Bell states. For example in a 2 qubit system, where the Hilbert space is of the form $\mathbb{C}^2 \otimes \mathbb{C}^2$, a set of four Bell states build a basis for all states. These Bell states are given in the so-called computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ by

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \tag{2.22}$$

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \tag{2.23}$$

These states have concurrence and purity 1. The corresponding density matrices are in the computational basis

$$|\Psi^{\pm}\rangle\langle\Psi^{\pm}| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm 1 & 0 \\ 0 & \pm 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.24}$$

and

$$|\Phi^{\pm}\rangle\langle\Phi^{\pm}| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{pmatrix}. \tag{2.25}$$

If the degrees of freedom of the correlated systems are higher than 2, it is not so easy to give a set of Bell states. Only one state is for sure a Bell state in the general Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^d$, it is given by

$$|\Phi^d\rangle = \frac{1}{\sqrt{d}} \sum_{s=0}^{d-1} |s\rangle \otimes |s\rangle. \tag{2.26}$$

Based on this state it is possible to construct so-called Bell-type states (see Sect.3.4) but the problems are that these states do not form a complete basis of the system and that they are not necessarily maximally entangled.

2.3.3 Werner and isotropic state

These states are defined as a mixture of the maximally mixed state and a Bell state. For dimension d of the subspaces the isotropic state is therefore given by

$$\rho_{iso} = \frac{1-\alpha}{d^2}\ \mathbb{1}_{d^2} + \alpha|\Phi^d\rangle\langle\Phi^d|, \tag{2.27}$$

where $|\Phi^d\rangle$ is a general Bell state and defined in the previous section. Typically the state is only called isotropic if the dimension of the subsystems is higher than 2 and for $d = 2$ we speak of Werner states which are of the form

$$\begin{aligned}\rho_W &= \frac{1-\alpha}{4}\,\mathbb{1}_4 + \alpha|\Psi^-\rangle\langle\Psi^-| = \\ &= \frac{1}{4}\begin{pmatrix} 1-\alpha & 0 & 0 & 0 \\ 0 & 1+\alpha & -2\alpha & 0 \\ 0 & -2\alpha & 1+\alpha & 0 \\ 0 & 0 & 0 & 1-\alpha \end{pmatrix},\end{aligned} \tag{2.28}$$

where $0 \leq \alpha \leq 1$. Here the purity of the state is given by $P_W(\alpha) = \frac{1+3\alpha^2}{4}$. The Werner state gets separable for $\alpha \leq \frac{1}{3}$ and if α is bigger than $\frac{1}{3}$ the concurrence of the state is

$$C_W(\alpha) = \frac{1}{2}(3\alpha - 1). \tag{2.29}$$

2.3.4 Maximally entangled mixed states - MEMS

Ishizaka and Hiroshima [34] and Munro et al.[39] solved the question what the highest possible amount of entanglement is for a given purity of a given two qubit system. They found that the following states are indeed the maximally entangled mixed states or short MEMS. They must be splitted in to different parametrisations. The first holds for $\alpha \leq \frac{2}{3}$ and is given by

$$\begin{aligned}\rho_{MEMS\,1} &= \alpha|\Phi_+\rangle\langle\Phi_+| + \frac{1}{3}|01\rangle\langle 01| + (\frac{1}{3} - \frac{\alpha}{2})(|00\rangle\langle 00| + |11\rangle\langle 11|) \\ &= \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{\alpha}{2} \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\alpha}{2} & 0 & 0 & \frac{1}{3} \end{pmatrix} \qquad \text{for } \alpha \leq \frac{2}{3},\end{aligned} \tag{2.30}$$

with the properties

$$P_{MEMS\,1} = \alpha^2 + (1-\alpha)^2, \tag{2.31}$$

$$C_{MEMS\,1} = \alpha, \tag{2.32}$$

$$C(P) = \frac{1}{2} + \sqrt{\frac{1}{2}(P - \frac{1}{2})} \qquad \text{for } \frac{1}{4} \leq P \leq \frac{5}{9}. \tag{2.33}$$

The second part of the maximally entangled mixed states is valid for $\alpha \geq \frac{2}{3}$ and of the form

$$\begin{aligned} \rho_{MEMS\,2} &= \alpha|\Phi_+\rangle\langle\Phi_+| + (1-\alpha)|01\rangle\langle 01| = \\ &= \frac{1}{2}\begin{pmatrix} \alpha & 0 & 0 & \alpha \\ 0 & 2-2\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & \alpha \end{pmatrix} \qquad \text{for } \alpha \geq \frac{2}{3}, \end{aligned} \tag{2.34}$$

with

$$P_{MEMS\,2} = \frac{1}{3} + \frac{\alpha^2}{2}, \tag{2.35}$$

$$C_{MEMS\,2} = \alpha, \tag{2.36}$$

$$C(P) = \sqrt{2P - \frac{2}{3}} \qquad \text{for } \frac{5}{9} \leq P \leq 1. \tag{2.37}$$

If we draw a diagram (see Fig.2.3.4) where the concurrence is plotted against the purity, we can easily see that the maximally entangled mixed states form the boundary for all physical states. Additionally the Werner state is depicted in the figure whose function is given by

$$C_W(P) = \frac{1}{2}\left[3\sqrt{\frac{4P-1}{3}} - 1\right]. \tag{2.38}$$

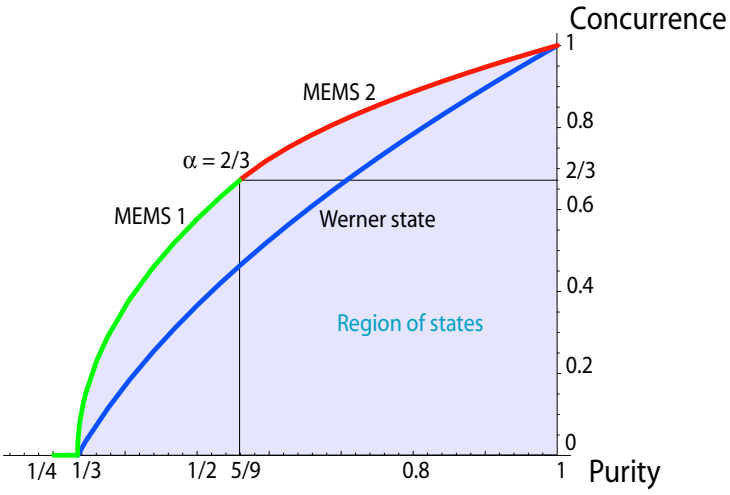


Figure 2.2: CP-diagram showing the MEMS and Werner state for 2 qubits

Note that only states with a purity higher than $\frac{1}{3}$ can be entangled.

Chapter 3

The geometry of quantum mechanical systems

Although the most geometric pictures describing physical systems are not able to consider the whole space, they nevertheless give often further inside into some unsolved problems and develop our intuition. The challenge is to lower the degrees of freedom of a system and in addition conserve important physical properties.
We are introducing geometric pictures for single as well as bipartite qubit and qutrit systems which will be used for the discussion of open quantum system effects in Chapter 5 and 6.

3.1 Single Qubit

We start with the simplest possible case where the quantum machanical system should just have two degrees of freedom, this could be realized for example by a spin-$\frac{1}{2}$ particle or the polarization of a photon. The so-called qubit state lives in a two dimensional complex Hilbert space $\mathcal{H} \equiv \mathbb{C}^2$.
Any **pure** state of the qubit can be described in the orthonormal or computational basis of the Hilbert space constructed with the vectors $|0\rangle$ and $|1\rangle$. Then the qubit is given by

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{3.1}$$

where, due to the normalization of the state, the complex amplitudes have to satisfy $|\alpha|^2 + |\beta|^2 = 1$. Using this condition we can reparameterize the qubit state vector to

$$|\Psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle, \tag{3.2}$$

where the parameters θ and ϕ define a point on the unit sphere in 3 dimensional space. Any pure qubit state lies on the so-called Bloch sphere.

By including **mixed** states we can decompose any abritrary state, expressed by a density matrix, into terms of the well known Pauli matrices[1].

$$\rho = \frac{1}{2}\left(\mathbb{1}_2 + n_i\sigma^i\right), \tag{3.3}$$

where $n_i \in \mathbb{R}^3$ is called the Bloch vector. Due to the positivity of the density matrix the Bloch vector n_i has to satisfy the condition $|\vec{n}|^2 \leq 1$. If $|\vec{n}|^2 = 1$, the state ρ is a pure state and sits on the surface of the 3 dimensional ball (Bloch sphere, see Fig.(3.1)). Otherwise if $|\vec{n}|^2 < 1$, ρ is mixed and located inside the sphere (Bloch ball).

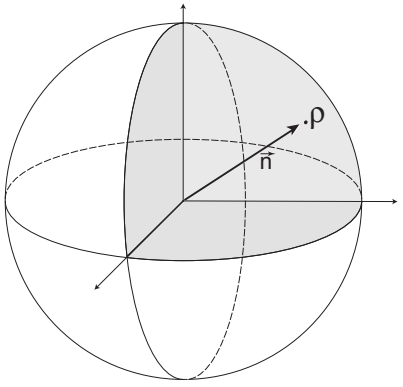


Figure 3.1: Bloch ball

The Bloch ball is a very powerful geometric picture for qubits because it nearly[2] covers all possible quantum states and it is rather easy to distinguish between pure and mixed states.

3.2 Bipartite Qubits

The so-called spin geometry picture was introduced by the Horodeckis [31, 33] and discussed by Vollbrecht and Werner [47] and also by Bertlmann,

[1] The 3 Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

[2] The phase freedom of the state is neglected.

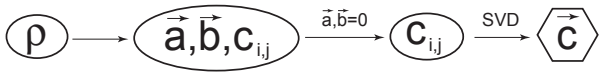
Narnhofer and Thirring [17].
Since the density matrix of 2 qubits lives in the Hilbert space $\mathcal{H} \equiv \mathbb{C}^2 \otimes \mathbb{C}^2$, we need to construct the basis with 16 complex 4×4 matrices. These basis operators can be chosen as tensor products of the single qubit basis $\{\mathbb{1}_2 \otimes \mathbb{1}_2, \mathbb{1}_2 \otimes \sigma^i, \sigma^i \otimes \mathbb{1}_2, \sigma^i \otimes \sigma^j\}$. Any 2 qubit density matrix is then given by

$$\rho = \frac{1}{4}\left(\mathbb{1}_2 \otimes \mathbb{1}_2 + a_i \mathbb{1}_2 \otimes \sigma^i + b_i \sigma^i \otimes \mathbb{1}_2 + c_{ij}\sigma^i \otimes \sigma^j\right), \tag{3.4}$$

where the local parameters a_i , $b_i \in \mathbb{R}^3$ determine the statistic of the reduced density matrices, i.e. of Alice's and Bob's system. The elements c_{ij} are real and responsible for the nonlocal correlations between the subsystems.
Since our main goal is the construction of a 3 dimensional geometric picture out of 16 independent real elements, we have to reduce the number of parameters. In the following we consider only states where the local parameters are equal zero ($\vec{a} = \vec{b} = \vec{0}$). In other words we take the set of all locally maximally mixed states with $Tr_A(\rho) = Tr_B(\rho) = \frac{1}{2}\mathbb{1}_2$, where Tr_A means the partial trace over Alice's system.
At next we try to reduce the remaining 9 parameters of c_{ij} to 3. As the property of separability is invariant under local unitary transformation and classical communication (LOCC) we can perform a unitary transformation of the form $U_1 \otimes U_2$ on the density matrix ρ and diagonalize the matrix c_{ij}. The new subclass of physical states can now be written as

$$\rho = \frac{1}{4}(\mathbb{1}_2 \otimes \mathbb{1}_2 + c_i \sigma^i \otimes \sigma^i). \tag{3.5}$$

This diagonalization mechanism is called singular value decompostion (SVD) and has the advantage that the obtained diagonal elements are all real compared to the eigenvalue decompostion (EVD). An overview of the different stages from the density matrix to the geometric picture is summerized below.
The 3 real parameters form the socalled correlation vector $\vec{c}$ which draw a 3



dimensional picture in Euclidean space for bipartite qubits. For example the related correlation vectors of the four Bell states and the identity are given by

$$\rho = \frac{1}{4}\mathbb{1}_4 \qquad \vec{c} = (0,0,0) \tag{3.6}$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad \vec{c} = (1,-1,1) \tag{3.7}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \qquad \vec{c} = (-1,1,1) \tag{3.8}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \qquad \vec{c} = (1,1,-1) \tag{3.9}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \qquad \vec{c} = (-1,-1,-1) \tag{3.10}$$

The vector $\vec{c}$ creates a cube with the length of 2 but all possible physical states lie inside a tedrahedron in the cube (see Fig 3.2) which is gained by the positivity condition of the density matrix. If we express the positivity in terms of the correlation vectorelements we get 4 inequalities which are of the form

$$1 - c_1 - c_2 - c_3 \geq 0 \qquad 1 - c_1 + c_2 + c_3 \geq 0 \tag{3.11}$$

$$1 + c_1 - c_2 + c_3 \geq 0 \qquad 1 + c_1 + c_2 - c_3 \geq 0 \tag{3.12}$$

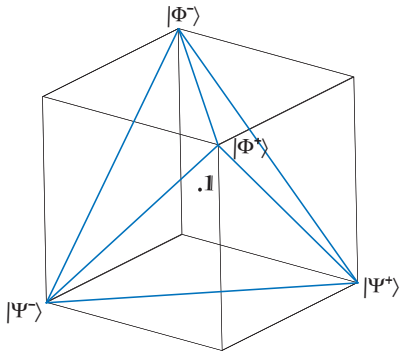


Figure 3.2: All possible physical states with the 4 Bell states in the corners of the positivity border (the blue tedrahedron) and the identity in the center.

Now we want to focus on the region of separable states. A method of checking if a state is entangled is the partial transposition, see Sect.2.2.1. If the

state has at least one negative eigenvalue under partial transposition then the state is entangled. Since the inversion of this argument holds for 2 qubit systems we can construct the region of separable states. It shows that all states with positve eigenvalues under partial transposition (PPT) lie inside a reflection of the positivity tetrahedron, illustrated in Fig 3.3, and the intersection of both gives us the set of all separable states, an octahedron or a doublepyramid.

On the right side of Fig. 3.3 it is shown that any locally mixed 2 qubit

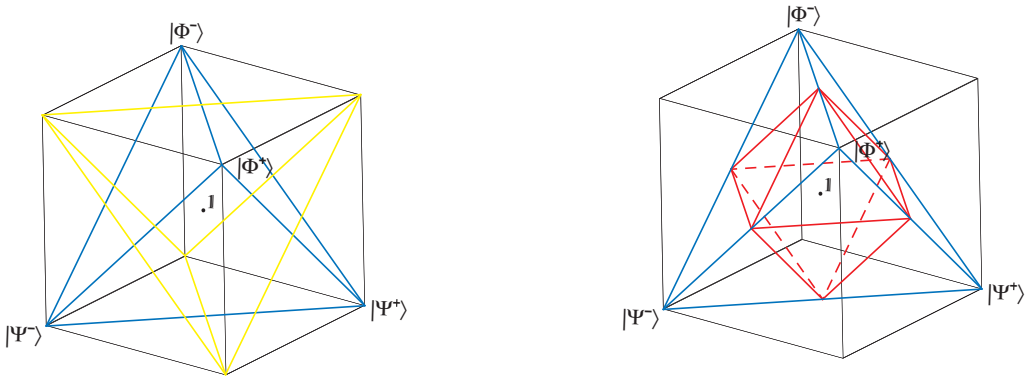


Figure 3.3: Positivity borders and the region of separable states

state lies inside the positivity borders (blue lines) and is entangled if located outside the red octahedron and separated inside this doublepyramid.

The big advantages of the spin geometry picture are that many interesting states, like the Werner state or the Bell states, are included and that it is very easy to calculate the purity and concurrence of the states (see next section). The downside is that no states with local parameters have been considered.

Purity and Concurrence

An important advantage of this three dimensional picture is that properties like concurrence and purity can be depicted in a beautiful way. For the

calculation of these properties we take the density matrix

$$\rho = \frac{1}{4}\left(\mathbb{1}_2 \otimes \mathbb{1}_2 + c_i\sigma^i \otimes \sigma^i\right) \tag{3.13}$$

$$= \frac{1}{4}\begin{pmatrix} 1+c_3 & 0 & 0 & c_1-c_2 \\ 0 & 1-c_3 & c_1+c_2 & 0 \\ 0 & c_1+c_2 & 1-c_3 & 0 \\ c_1-c_2 & 0 & 0 & 1+c_3 \end{pmatrix}. \tag{3.14}$$

The purity is then

$$P = Tr\rho^2 = \frac{1}{4}(1 + c_1^2 + c_2^2 + c_3^2) = \frac{1}{4}(1 + |\vec{c}|^2). \tag{3.15}$$

Note that the purity is dependent of the distance between the state and the unity at the center. This means that all states with the same amount of purity lie on a spherical shell and that the only pure states in the picture are the four Bell states where $|\vec{c}|^2 = 3$ and therefore $P = 1$. The maximally mixed state, the unity at the center, has purity of $\frac{1}{4}$, the lowest amount for a 2 qubit state.

If we want to determine the concurrence of this set of states we have to choose the signs of the considered correlation vector elements. The reason for that lies in the calculation where one has to subtract from the largest absolute value of all eigenvalues and this procedure depends on the signs (see Sect.2.2.2). We take the states where all elements are negative - this is the octant which include the state $|\Psi^-\rangle$. For all states in this area we get

$$C = max\{0, \frac{1}{2}(-1 - c_1 - c_2 - c_3)\}. \tag{3.16}$$

All states outside the doublepyramid with the same amount of concurrence are located on a plane parallel to the surface of the separability border. They have all the same nearest distance to the octahedron. If ρ is inside, the Concurrence is zero.

In Fig.3.4 the concurrence and purity are illustrated for a specific state ρ with negative vector elements. Furthermore it is shown where the isotropic state or Werner state (ρ_W is located in the picture.

3.3 Single Qutrit

Analogical to the single qubit it is possible to describe a qutrit with the help of a generalization of the Pauli matrices for three dimensions - the 8

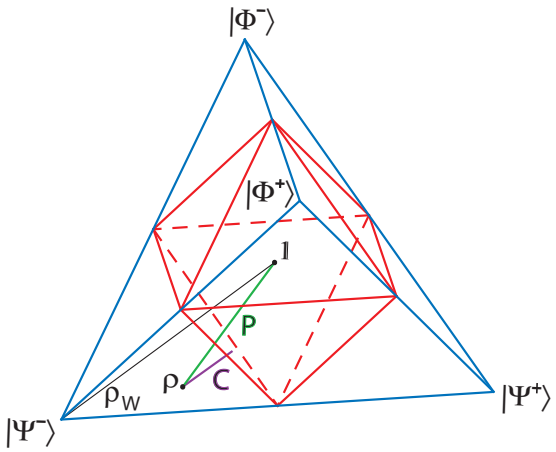


Figure 3.4: The concurrence C and purity P are shown in the geometric picture for the state ρ. Further the Werner state ρ_W is illustrated.

Gell-Mann matrices [3]. Then a qutrit state which lives in $\mathcal{H} \equiv \mathbb{C}^3$ can be expressed by

$$\rho = \frac{1}{3}(\mathbb{1}_3 + \sqrt{3} n_i \lambda^i), \tag{3.17}$$

where $n_i \in \mathbb{R}$ and $|\vec{n}|^2 = \sum_i n_i^2 \leq 1$. The Gell-Mann matrices satisfy $Tr\lambda^i = 0$, $Tr\lambda^i\lambda^j = 2\delta^{ij}$. The problem is that not all vectors n_i describe necessarily a positive density matrix in contrast to the qubit where any Bloch vector represents a physical state. For example the matrix with $n_8 = 1$ and all other components zero is not positive definite.

[3] The 8 Gell-Mann matrices are defined as

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

3.4 Bipartite Qutrits

Weyl matrices

A geometric picture for bipartite qutrits could be constructed analog to the two qubit case where again only density matrices with vanishing local parameters are chosen. The problem is that not all such states can be decomposed into maximally entangled states called Bell type states for qutrits. Therefore the set of states has to be reduced further. This was done by Baumgartner, Hiesmayr and Narnhofer [4] with the help of Weyl matrices which are a kind of gerneralization of the Pauli matrices and defined by

$$W_{kl}|s\rangle = w^{k(s-l)}|s-l\rangle \qquad \text{with } w = e^{2\pi i/3}, \tag{3.18}$$

where the letters k, l, s used in this context stand for $0, 1, 2$ and all calculations are understood as modulo 3, e.g. $(-1 \equiv 2)$. These operators obey the following Weyl relations

$$W_{jl}W_{km} = w^{kl}W_{(j+k)(l+m)}, \tag{3.19}$$

$$W_{kl}^{\dagger} = W_{kl}^{-1} = w^{kl}W_{(-k)(-l)}, \tag{3.20}$$

$$W_{00} = \mathbb{1} \tag{3.21}$$

and are explicitly given in the computational basis by

$$W_{00} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad W_{01} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad W_{02} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$W_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w^* \end{pmatrix}, \quad W_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & w \\ w^* & 0 & 0 \end{pmatrix}, \quad W_{12} = \begin{pmatrix} 0 & 0 & 1 \\ w & 0 & 0 \\ 0 & w^* & 0 \end{pmatrix},$$
$$W_{20} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & w^* & 0 \\ 0 & 0 & w \end{pmatrix}, \quad W_{21} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & w^* \\ w & 0 & 0 \end{pmatrix}, \quad W_{22} = \begin{pmatrix} 0 & 0 & 1 \\ w^* & 0 & 0 \\ 0 & w & 0 \end{pmatrix}. \tag{3.22}$$

In contradiction to the Pauli matrices the Weyl operators are not Hermitian but still unitary. Any qutrit state w can then be written as

$$\rho = \sum_{k,l=0}^{3} c_{kl} W_{kl}|\Phi\rangle\langle\Phi|W_{kl}^{\dagger}\ , \tag{3.23}$$

where the parameters c_{kl} have to satisfy

$$c_{kl} = c_{kl}^* \qquad \text{and} \qquad \sum_{k,l} c_{kl} = 1. \tag{3.24}$$

$|\Phi\rangle$ is an abritrary initial state which could be chosen to be the state $|0\rangle$ for example.
For the construction of bipartite qutrits we have to take the tensor product of the Weyl operators $W_{kl} \otimes W_{k'l'}$ and to be sure that only locally maximally mixed states are produced the operators can be reduced to $W_{kl} \otimes W_{(-k)l}$. It is shown in [4] that it is sufficient to operate just on one subspace, e.g. with $W_{kl} \otimes \mathbb{1}$. In the following section the socalled magic simplex $\mathcal{W}$ is constructed and studied with the help of this operator.

The magic simplex $\mathcal{W}$

Starting with the two qutrit Bell type state which is a pure and maximally entangled

$$\Omega_{00} = \frac{1}{\sqrt{3}} \sum_{s=0}^{2} |s\rangle \otimes |s\rangle. \tag{3.25}$$

Now the other eight Bell states can be constructed with the help of the Weyl operators acting just on one subspace of the state. The other one is left unmodified. The states are then given by

$$\Omega_{kl} = W_{kl}\Omega_{00}. \tag{3.26}$$

These obtained nine Bell type states can be formulated as projectors like

$$P_{kl} = |\Omega_{kl}\rangle\langle\Omega_{kl}|, \tag{3.27}$$

with the property

$$W_{nm} P_{kl} W_{nm}^\dagger = P_{n+k\ m+l}. \tag{3.28}$$

The explicit form of the Bell type states is given in the appendix A Then the magic simplex $\mathcal{W}$ is constructed by a linear combination of these states

$$\mathcal{W} = \{\sum c_{kl} P_{kl} | c_{kl} \geq 0, \sum c_{kl} = 1\} \tag{3.29}$$

The coefficients build an eight dimensional picture which can be discussed concerning the region of seperability and the regions which are PPT but not seperable - the bound entangled states. The focus of the chapter is the presentation of the most important results which are worked out explicitly by

Baumgartner, Hiesmayr and Narnhofer in [5].

Since it is impossible to draw the full 8 dimensional simplex we are going to discuss two representative cases. The first one imply all density matrices which are mixtures of the total mixed state and two Bell type states. The entanglement witness constructed with PT is not a linear one and furthermore regions of bound entangled states occur inside the witness. In the second case density matrices are studied which are mixtures of the total mixed state, a Bell type state and an equal mixture of two other Bell type states. Here the PT witness is linear and no bound entanglement can be found. Due to the high symmetry of the simplex these two set of states are nearly enough for a complete description of the geometry. The left mixture of states is skipped here but discussed in [5].

The first set of states (Fig.3.5)

The first set contains all density matrices which can be constructed by

$$\rho = (1-\alpha-\beta)\omega + \alpha P_{00} + \frac{\beta}{2}(P_{10}+P_{20}) \tag{3.30}$$

where

$$\omega = \frac{1}{9}(\sum_{k,l=0}^{2} P_{kl}) = \frac{1}{9}\mathbb{1}. \tag{3.31}$$

These density matrices (Fig.(3.5)) are a linear combination of the identity matrix and the three Bell type states P_{k0}, where two of them are mixed equally. According to the symmetries of W_{kl} (see Eqs.(3.19) and (3.20)) the same picture can be drawn for the set P_{k1} and P_{k2}. All physical states lie inside the blue triangle which shows the border of the positivity condition. This is analog to the tedrahedron in the bipartite qubit picture. The region of all positive matrices under PT is illustrated by the grey area inside the red triangle. Since all these states are separable as well, no bound entanglement exists.
The vertex points of the separable region are now explicitly mentioned because it will be easier for the discussion of certain decoherence modes later

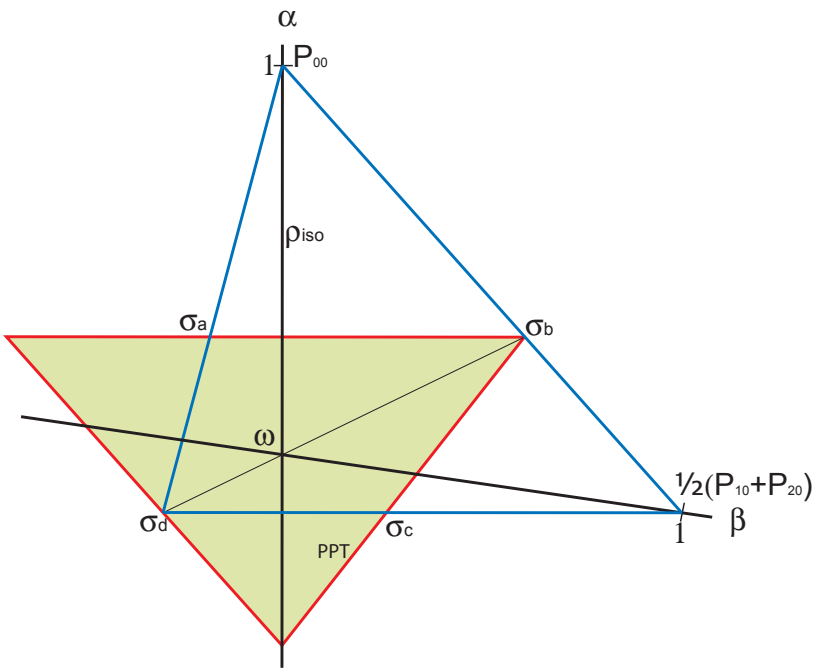


Figure 3.5: The picture for all density matrices with $\rho = (1-\alpha-\beta)\omega + \alpha P_{00} + \frac{\beta}{2}(P_{10}+P_{20})$. All physical states (positive) are inside the blue triangle and all PPT states inside the red one. The intersection of both gives the set of all separable states; no bound entanglement occurs.

in this work (see Chapter 6)

$$\sigma_a = \omega + \frac{2}{9}P_{00} - \frac{1}{9}P_{10} - \frac{1}{9}P_{20} \tag{3.32}$$

$$\sigma_b = \frac{1}{3}(P_{00} + P_{10} + P_{20}) \tag{3.33}$$

$$\sigma_c = \frac{3}{4}(\omega - \frac{1}{9}P_{00} + \frac{2}{9}P_{10} + \frac{2}{9}P_{20}) \tag{3.34}$$

$$\sigma_d = \frac{3}{2}(w - \sigma_b) \tag{3.35}$$

The point σ_b, the equal mixture of all P_{k0}, is separable and analog to the lower peak of the doublepyramid in the spin geometry picture, the equal mixture of $|\Psi^-\rangle$ and $|\Psi^+\rangle$.
Another interesting family of states is labeled in the picture - the isotropic states, which are here given by

$$\rho_{iso} = (1-\alpha)\omega + \alpha P_{00}. \tag{3.36}$$

The second set of states (Fig.3.6)

The focus is on the states which are a mixture of the identity and two Bell type states

$$\rho = (1 - \alpha - \beta)\omega + \alpha P_{10} + \beta P_{20} \tag{3.37}$$

where ω is again given by Eq.(3.31).

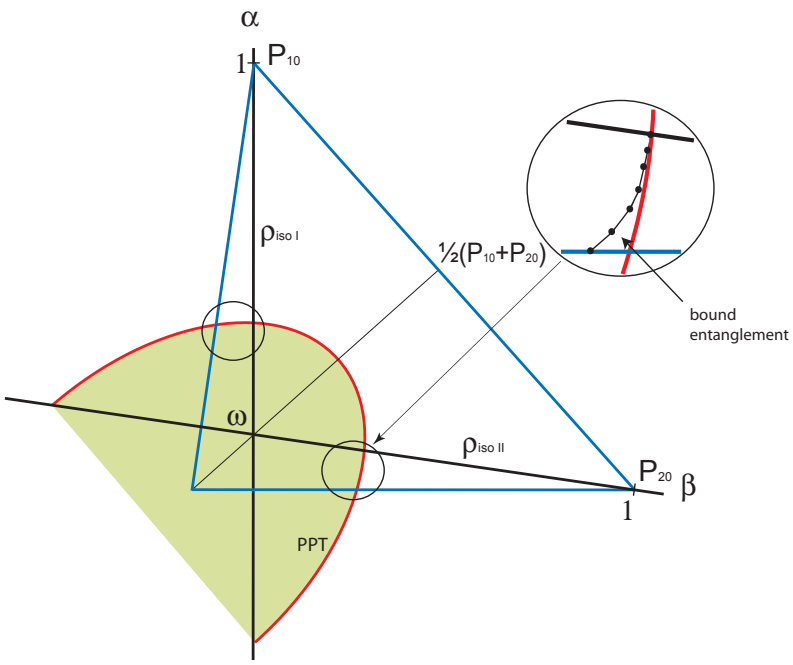


Figure 3.6: Mixtures of the Identity and two Bell type states with the region of bound entangled states in the enlarged picture.

In Fig.(3.6) all physical states are located inside the blue triangle which gives again the positivity borders. The grey region shows all PPT states. But the most remarkable fact is that not all of these states are separable, one finds bound entangled states on the borders for $\alpha < 0$ or $\beta < 0$. The states with α *and* β positive or negative are all separabel. The enlarged picture in Fig.(3.6) shows some numerical found states which are entangled and inside the PPT space. Since it is known that the region of separability is a convex set, the states between the PPT border and the numerical state are bound entangled. In [8] it is shown that a pretty large family of states is bound entangled.
The states of Fig.(3.5) lie on the line where α is equal to β ending on the

positivity border with

$$\rho = \frac{1}{2}(P_{10} + P_{20}) \qquad \text{where} \qquad \alpha = \beta = \frac{1}{2} \tag{3.38}$$

This geometric picture can also be generalized for a two qudit system, shown in [6].

Chapter 4

The Theory of decoherence and open quantum systems

Since experiments on quantum mechanical systems get more and more complex, it is no longer sufficient enough to describe this systems as closed. In practise any realistic system has an uncontrollable coupling to the environment which influences the time evolution of the system. Therefore the theory of open quantum systems describes their behaviour by considering different assumptions about the coupling or the environment since a complete description of the environmental degrees of freedom is not feasible. The dynamical evolution of the open system is then described with an effective equation of motion - the master equation.
Another motivation for the study of open quantum systems is of more fundamental origin. It is the question: How does the classical world emerge from the underlying quantum mechanical framework? This question is relatively young since the mainstream attitude in the 1970's and 1980's was that the boundary between the quantum and the classical world was a purely philosophical problem, out of physic's scope. A change of this opinion was initiated by the work of Zeh [35] and Zurek [55] and by the understanding that there is no reason why one should deny quantum mechanics even for macroscopic objects.
This chapter is understood as a short introduction in the subject of open quantum systems and the associated decoherence. We start with a repetition on dynamical evolutions of closed systems followed by the definition of open systems, (Sect.4.1). The focus of Sect.4.2 is on dynamical maps that guarantee us to stay in a physical describtion. At next these maps are expressed by Kraus operators derived in Sect.4.3. In Sect.4.4 two different ways of a proper derivation of a Markovian master equation are presented. The first approach is based on dynamical maps satisfying the semigroup property,

whereas the second derivation is performed on a microscopic regime. Since the terminology of open system effects is not well defined for the different physical communities, we want to explain some concepts due to the definition in quantum information theory (Sect.4.5).

4.1 From closed to open quantum systems

Closed systems

First of all we want to discuss how it is possible to describe the time evolution of closed quantum mechanical systems. If we consider a pure states $|\Psi(t)\rangle$, the time evolution of this state is given by the Schrödinger equation

$$i\frac{d}{dt}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle, \tag{4.1}$$

where $\hbar$ is set to 1 and $H(t)$ is the Hamiltonian of the system. Since we are describing closed systems, the evolution of the state can be expressed by an unitary time-evolution operator $U(t, t_0)$ which transforms the initial state $|\Psi(t_0)\rangle$ to the state $|\Psi(t)\rangle$,

$$|\Psi(t)\rangle = U(t, t_0)|\Psi(t_0)\rangle \qquad \text{with} \qquad U(t_0, t_0) = \mathbb{1}. \tag{4.2}$$

By substituting this state in Eq.(4.1) we get an operator equation for $U(t, t_0)$

$$i\frac{\partial}{\partial t}U(t, t_0) = H(t)U(t, t_0). \tag{4.3}$$

If we are considering isolated systems which means that they are not driven by an external force, e.g. an external electromagnetic field, the Hamiltonian H is time independent. Hence the unitary operator can be expressed by the well known equation

$$U(t, t_0) = exp\,[-iH(t - t_0)]. \tag{4.4}$$

If the system under consideration is a quantum statistical ensemble, then this ensemble can't be described just by a state vector. The system may be characterized by its density matrix ρ. To get the equation of motion for this formalism, we start with the initial state $\rho(t_0)$ at the time t_0 analog to the definition of the density matrix,

$$\rho(t_0) = \sum_\alpha w_\alpha |\Psi_\alpha(t_0)\rangle\langle\Psi_\alpha(t_0)|. \tag{4.5}$$

Since the state vectors $|\Psi_\alpha(t)\rangle$ are time-dependent according to the unitary evolution given in Eq.(4.2), the density matrix of the system for the time t is then

$$\rho(t) = \sum_\alpha w_\alpha U(t,t_0)\ |\Psi_\alpha(t_0)\rangle\langle\Psi_\alpha(t_0)|\ U^\dagger(t,t_0) \tag{4.6}$$

$$= U(t,t_0)\ \rho(t_0)\ U^\dagger(t,t_0) \tag{4.7}$$

After differentiating the equation with respect to time, the obtained equation of motion is

$$\frac{d}{dt}\rho(t) = -i[H(t),\rho(t)], \tag{4.8}$$

which is often called **Liouville - von Neumann equation**.

Open systems

After we have briefly sketched the most important equations describing the dynamics of closed quantum systems, we want to focus now on open systems. Fig.(4.1) shows a schematic picture of what is meant by an open quantum system.

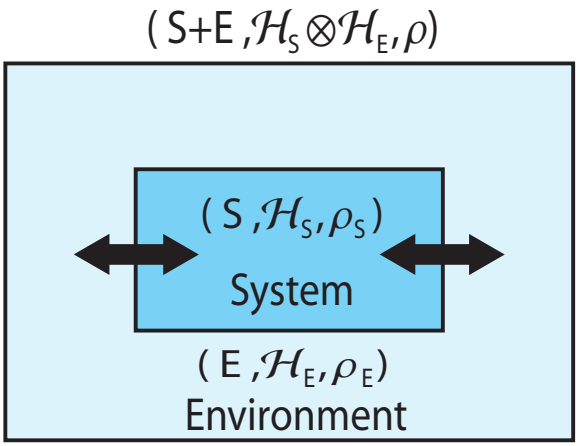


Figure 4.1: Schematic picture of an open quantum system

The system of interest S is coupled to another quantum mechanical system E called the environment. Although the combined system is closed and follows the discussed Hamiltonian dynamics, the time evolution of the 'open' subsystem S is not, in general, unitary. The interaction between the system and the environment generates a certain correlation between them which is

responsible for the non-unitary character of the open system. Hence the reduced system has to be described by a quantum master equation, which will be derived in the next sections.

The Hilbert space of the total system $S+E$ is given by the tensor product $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$, where the Hilbert space of the system is denoted by $\mathcal{H}_S$ and the space of the environment by $\mathcal{H}_E$. Then the total Hamiltonian $H(t)$ can be taken of the form

$$H(t) = H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_E + H_I(t), \tag{4.9}$$

where H_S and H_E are the free, time-independent Hamiltonians of the system and the environment. The interaction is described by the Hamiltonian $H_I(t)$ which operates in the total Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$.
In most of the mathematical models describing open system dynamics the environment is represented by a reservoir or a heat bath consisting of an infinite number of degrees of freedom. This generally leads to an irreversible behaviour of the open system and to the problem that one has to solve an infinite number of coupled equations of motion. To avoid this problem one tries to stay in the reduced system space described by a restricted set of physically relevant parameters.
We are just interested in observables which are related to the open system S and therefore of the form $A \otimes \mathbb{1}_E$, where the operator A is acting in $\mathcal{H}_S$ and $\mathbb{1}_E$ stands for the identity matrix of $\mathcal{H}_E$. At next we want to define the density matrices representing the states of the systems. The density matrix ρ without any index should be the state of the total system whereas the indices S and E stand for the particular subsystems. The reduced density matrix ρ_S can be gained by taking the partial trace over the degrees of freedom of the environment on the combined system like

$$\rho_S = Tr_E \rho. \tag{4.10}$$

Since the total density matrix evolves unitarily in time we can substitute ρ from Eq.(4.7) and get

$$\rho_S(t) = Tr_E\{U(t,t_0)\rho(t_0)U^\dagger(t,t_0)\}, \tag{4.11}$$

where $U(t,t_0)$ is here the time-evolution operator of the total system. To get an equation of motion for the reduced system we can take the partial trace over the environment on both sides of the Liouville - von Neumann equation (4.8) for the total system,

$$\frac{d}{dt} Tr_E \rho(t) = \frac{d}{dt}\rho_S(t) = -iTr_E[H(t), \rho(t)]. \tag{4.12}$$

All these equations are exact but in practise they are not very applicable because the state of the total system ρ is still included. This state is typically unknown due to the lack of knowledge of the environmental modes. In the following sections some approximations are presented that lead us to equations of motion which are only consisting of the reduced density state ρ_S, the object of primary interest.

4.2 Dynamical maps

We want to sketch briefly the concept of dynamical maps and their relation to the theory of open quantum systems. We suppose that the initial state of the total system is a product state like

$$\rho(0) = \rho_S(0) \otimes \rho_E(0), \tag{4.13}$$

which means that no correlation between the system and the environment takes place at $t = 0$. The state evolution of the reduced system ρ_S can then be expressed with the help of a quantum dynamical map V_t in the way that

$$\begin{aligned} V_t : \qquad \mathcal{B}(\mathcal{H}_S) &\longrightarrow \mathcal{B}(\mathcal{H}_S) \\ \rho_S(0) &\longmapsto \rho_S(t) = V_t \rho_S(0) \qquad \text{for } t \in \mathbb{R}^+. \end{aligned} \tag{4.14}$$

Here, $\mathcal{B}(\mathcal{H})$ denotes the Algebra of bounded operators on $\mathcal{H}$. The map V_t is a one-parameter family of **trace-preserving**, **convex linear** and **complete positive** maps. With these properties this map is the most general description of a time evolution which maps an abritrary initial state $\rho(0)$ to valid physical states $\rho(t)$. The commutative diagram in Fig.4.2 illustrates the action of this map.

$$\begin{array}{ccc} \rho(0) = \rho_S(0) \otimes \rho_E & \xrightarrow{\text{unitary evolution}} & \rho(t) = U(t,0)[\rho_S(0) \otimes \rho_E]U^\dagger(t,0) \\ \Big\downarrow {\scriptstyle Tr_E} & & \Big\downarrow {\scriptstyle Tr_E} \\ \rho_S(0) & \xrightarrow{\text{dynamical map}} & \rho_S(t) = V_t \rho_S(0) \end{array}$$

Figure 4.2: Commutative diagram of the dynamical map V_t

Trace-preserving means that the state always stays normalized and the convex linearity is given by

$$V_t(\lambda\rho + (1-\lambda)\rho') \leq \lambda V_t(\rho) + (1-\lambda)V_t(\rho') \tag{4.15}$$

which has to hold for all $0 \leq \lambda \leq 1$. The criteria of complete positivity is much stronger than mere positivity which maps positive states into positive. For complete positivity all the tensor product extensions of V_t to spaces of higher dimension are positive which means that

$$V_t \otimes \mathbb{1}_{ext} > 0 \tag{4.16}$$

and that any state remains positive although it is the reduced part of a state evolving in a higher dimensional space.

Now, we have defined the dynamical map V_t with all its properties such that it is guaranteed to get physical states but how can we construct such maps. We will answer this question in the next section with the help of Kraus operators.

4.3 Kraus representation

Any dynamical map can be expressed by operators that just act in the Hilbert space $\mathcal{H}_S$ of the open system, the socalled **Kraus operators**. To obtain this relation we need the spectral decomposition of the environment which is given by

$$\rho_E = \sum_\alpha \lambda_\alpha |\Phi_\alpha\rangle\langle\Phi_\alpha|, \tag{4.17}$$

where $\{|\Phi_\alpha\rangle\}$ form a orthonormal basis in $\mathcal{H}_E$ and λ_α are non-negative real coefficients satisfying $\sum_\alpha \lambda_\alpha = 1$. Using Eqs.(4.11), (4.13), (4.14) and (4.17) we immediatly get

$$\rho_S(t) = V_t \rho_S(0) = \sum_{\alpha,\beta} W_{\alpha\beta}(t) \rho_S(0) W^\dagger_{\alpha\beta}(t), \tag{4.18}$$

where the Kraus operators $W_{\alpha\beta}(t)$ are

$$W_{\alpha\beta}(t) = \sqrt{\lambda_\beta}\, \langle\Phi_\alpha|U(t,0)|\Phi_\beta\rangle \tag{4.19}$$

and satisfy the completeness relation[1]

$$\sum_{\alpha,\beta} W^\dagger_{\alpha\beta}(t) W_{\alpha\beta}(t) = \mathbb{1}_S. \tag{4.20}$$

There exists $(dim(\mathcal{H}_S))^2$ different Kraus operators but their choice is not necessarily unique.

[1] If the map is trace-decreasing instead of trace-preserving, the relation (4.20) is replaced by $\sum_{\alpha,\beta} W^\dagger_{\alpha\beta}(t) W_{\alpha\beta}(t) < \mathbb{1}_S$.

4.4 Markovian dynamics

To derive a quantum master equation which only depends on variables of the reduced system, we have to make assumptions. One of these assumptions is that we neglect memory effects. This means the environment 'forgets' the past interactions with the system due to the dispersion of correlations into the many environmental degrees of freedom. This is a rather strong assumption but on a coarse-grained time scale quite reasonable. We will see in the microscopic derivation where this Markovian approximation actually comes in. For dynamical maps this statement can be formulated by the semigroup property

$$V_{t_1} V_{t_2} = V_{t_1+t_2} \qquad \text{for } t_1, t_2 \geq 0. \tag{4.21}$$

In the following sections we are going to derive the Lindblad master equation first of all by a dynamical semigroup approach and later by an underlying microscopic theory where we will exactly see which approximations come into play.

4.4.1 The Lindblad master equation

It is a pretty old question to formulate a semigroup with general operators and this question was already answered in 1976 by Lindblad [38] and independently by Gorini, Kossakowski and Sudarshan [25]. We are also going to derive the Lindblad master equation to get a feeling for the emerging Lindblad operators.
Under rather weak mathematical conditions there exists a generator $\mathcal{L}$ for a given dynamical semigroup that satisfies

$$V_t = exp(\mathcal{L}t). \tag{4.22}$$

With this linear map $\mathcal{L}$ we can construct a first-order differential equation for the reduced density matrix of the open system, the **Markovian quantum master equation**,

$$\frac{d}{dt}\rho_S(t) = \mathcal{L}\rho_S(t). \tag{4.23}$$

Now we want to derive the most general form of this generator. For simplicity we consider a finite dimensional Hilbert space with $dim(\mathcal{H}_S) = d \leq \infty$. The corresponding operators then form a d^2-dimensional complex vector space which can be represented by a complete basis of orthonormal operators $\{E_j, 1 \leq j \leq d^2\}$ satisfying

$$(E_i, E_j) := Tr(E_i^\dagger E_j) = \delta_{ij} \tag{4.24}$$

For simplicity and without loss or generality one of the basis operators can be chosen proportional to the identity operator,

$$E_d^2 = \frac{1}{\sqrt{d}}\mathbb{1} \tag{4.25}$$

whereas all other basis operators are traceless,

$$Tr(E_j) = \begin{cases} 0 & \text{for } j = 1, \ldots, d^2 - 1 \\ \sqrt{d} & \text{for } j = d^2. \end{cases} \tag{4.26}$$

Any operator in this space can then be expanded as

$$W_k = \sum_{j=1}^{d^2} (E_j, W_k) E_j \tag{4.27}$$

and if this operators are the Kraus operators we get for the dynamical map

$$V_t \rho_S = \sum_{i,j=1}^{d^2} c_{ij}(t) E_i \rho E_j \tag{4.28}$$

with the positive, hermitian and time-dependent coefficient matrix

$$c_{ij}(t) = \sum_{k=1}^{d^2} (E_i, W_k(t))(E_j, W_k^\dagger(t)). \tag{4.29}$$

Now it is possible to formulate the semigroup generator in terms of the differential quotient created by the map given in (4.28). We have

$$\begin{aligned}
\mathcal{L}\rho_S &= \lim_{\tau \to 0} \frac{V_\tau \rho_S - \rho_S}{\tau} \\
&= \underbrace{\lim_{\tau \to 0} \frac{\frac{1}{d} c_{d^2 d^2}(\tau) - 1}{\tau}}_{c_0} \rho_S + \underbrace{\lim_{\tau \to 0} \sum_{j=1}^{d^2-1} \frac{c_{j d^2}(\tau)}{\sqrt{d}\tau} E_j}_{B} \rho_S \\
&\quad + \rho_S \underbrace{\lim_{\tau \to 0} \sum_{j=1}^{d^2-1} \frac{c_{d^2 j}(\tau)}{\sqrt{d}\tau} E_j^\dagger}_{B^\dagger} + \sum_{i,j=1}^{d^2-1} \underbrace{\lim_{\tau \to 0} \frac{c_{ij}(\tau)}{\tau}}_{\alpha_{ij}} E_i \rho_S E_j^\dagger \qquad (4.30) \\
&= c_0 \rho_S + B \rho_S + \rho_S B^\dagger + \sum_{i,j=1}^{d^2-1} \alpha_{ij} E_i \rho_S E_j^\dagger \qquad (4.31) \\
&= -i[H_U, \rho_S] + \{G, \rho_S\} + \sum_{i,j=1}^{d^2-1} \alpha_{ij} E_i \rho_S E_j^\dagger. \qquad (4.32)
\end{aligned}$$

Two new hermitian operators were introduced,

$$G = \frac{1}{2}(B + B^\dagger + c_0 \mathbb{1}) \tag{4.33}$$

$$H_U = -\frac{i}{2}(B - B^\dagger) \tag{4.34}$$

Since the semigroup conserves the trace of the open system and the trace is invariant under cyclic rotations of its arguments, we can write

$$0 = Tr(\mathcal{L}\rho_S) = 0 + Tr\left[\left(2G + \sum_{i,j=1}^{d^2-1} \alpha_{ij} E_j^\dagger E_i\right)\rho_S\right], \tag{4.35}$$

which gives us a relation between the operator G and the matrix α_{ij},

$$G = -\frac{1}{2}\sum_{i,j=1}^{d^2-1} \alpha_{ij} E_j^\dagger E_i. \tag{4.36}$$

Substituting this expression immediately leads to the **first standard form** of the generator

$$\mathcal{L}\rho_S = -i[H_U, \rho_S] + \sum_{i,j=1}^{d^2-1} \alpha_{ij}\left(E_i \rho_S E_j^\dagger - \frac{1}{2}E_j^\dagger E_i \rho_S - \frac{1}{2}\rho_S E_j^\dagger E_i\right). \tag{4.37}$$

Because of the positivity of the correlation matrix $\alpha = (\alpha_{ij})$, it can be diagonalized with the help of a unitary transformation u that satisfies

$$u\alpha u^\dagger = diag(\gamma_1, \ldots, \gamma_d^2 - 1) \tag{4.38}$$

$$A_k := \sum_{j=1}^{d^2-1} E_j u_{jk}^\dagger \tag{4.39}$$

With this unitary transformation and the new defined **Lindblad operators** $\mathbf{A_k}$ we get the second standard form or the so-called Lindblad form

$$\mathcal{L}\rho_S = \frac{d}{dt}\rho_S = -i[H_U, \rho_S] - \frac{1}{2}\sum_{k=1}^{d^2-1} \gamma_k \left(A_k^\dagger A_k \rho_S + \rho_S A_k^\dagger A_k - 2A_k \rho_S A_k^\dagger\right) \tag{4.40}$$

We see that the most general form of the generator of a dynamical semigroup is constructed by an unitary part with the hermitian operator H_U, which in general is not equal to the Hamiltonian of the open system H_S, and by a set of $d^2 - 1$ abritrary operators A_k weighted with the so-called

decoherence parameters γ_k. The non-unitary part of eq.(4.40) is often called the **dissipator**

$$\mathcal{D}(\rho_S) := \frac{1}{2}\sum_{k=1}^{d^2-1}\gamma_k\left(A_k^\dagger A_k\rho_S + \rho_S A_k^\dagger A_k - 2A_k\rho_S A_k^\dagger\right) \tag{4.41}$$

and the corresponding **Lindblad master equation** is then

$$\frac{d}{dt}\rho_S = -i[H_U, \rho_S] - \mathcal{D}(\rho_S). \tag{4.42}$$

For a given generator the operators A_k and H_U are not unique and the reason is that the Lindblad equation is invariant under the following transformations:

- Unitary transformation

$$A_k \to A_k' = \sum_j U_{kj}A_j \tag{4.43}$$

  with the unitary matrix U_{kj}.

- Inhomogenous transformation

$$A_k \to A_k' = A_k + a_k \tag{4.44}$$

$$H_U \to H_U' = H_U + \frac{1}{2i}\sum_j \gamma_j\left(a_j^* A_j - a_j A_j^\dagger\right) + b \tag{4.45}$$

  where $a_k \in \mathbb{C}$ and $b \in \mathbb{R}$. This transformation allows us to choose only traceless Lindblad operators.

4.4.2 Microscopic derivation in the weak coupling limit

The focus of this section is the derivation of the Markovian master equation from an underlying Hamiltonian dynamics of the total system. There exist different possible approaches and approximation schemes which all lead to the same result. Beside for example the 'Low-density limit', the 'Singular-coupling limit' or the 'Relaxation to equilibrium' we are going to discuss the microscopic derivation in the so-called weak-coupling limit.
The Hamiltonian of the total system is assumed to be of the form

$$H(t) = H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_E + H_I(t), \tag{4.46}$$

where H_I is the interaction part of the Hamiltonian acting in $\mathcal{H}_S \otimes \mathcal{H}_E$ and considered to be weak so that a pertubative treatment of the interaction is permissible. Here, the first assumption appears, called the **Born approximation** which states that the total system can be factorized for $t > 0$ in the 2^{nd} order of H_I,

$$\rho(t) \cong \rho_S(t) \otimes \rho_E \qquad \text{for } t > 0 \text{ in } O(H_I^2). \tag{4.47}$$

The initial state is again given by $t = 0$. Since the microscopic derivation of the Markovian master equation is most easily performed in the interaction picture, we start the discussion with the Liouville-von Neumann equation given in this picture by

$$\frac{d}{dt}\rho(t) = -i[H_I(t), \rho(t)] \tag{4.48}$$

The formal integral solution is then

$$\rho(t) = \rho(0) - i\int_0^t ds[H_I(s), \rho(s)] \tag{4.49}$$

Substituting this recursive solution of $\rho(t)$ in eq.(4.48) and taking the partial trace over the environment we get

$$\frac{d}{dt}\rho_S(t) = -\int_0^t ds Tr_E[H_I(t), [H_I(s), \rho(s)]], \tag{4.50}$$

where we have disregarded the term $Tr_E[H_I(t), \rho(0)]$ because it can vanish if a proper reformulation of the model Hamiltonian H_E is performed. Using the Born approximation (4.47) we obtain a closed integro-differential equation for the reduced density matrix $\rho_S(t)$

$$\frac{d}{dt}\rho_S(t) = -\int_0^t ds Tr_E[H_I(t), [H_I(s), \rho_S(s) \otimes \rho_E]]. \tag{4.51}$$

Now this equation can be simplified by the **Markov approximation** which replaces the densitiy matrix given in the retarded time $\rho_S(s)$ with $\rho_S(t)$ at the present time. The gained master equation which is now local in time is called the **Redfield equation**,

$$\frac{d}{dt}\rho_S(t) = -\int_0^t ds Tr_E[H_I(t), [H_I(s), \rho_S(t) \otimes \rho_E]]. \tag{4.52}$$

This master equation is still not Markovian since it is dependent on the choice of the initial preparation at $t = 0$. To satisfy the semigroup property we have

to substitute the time s by $t-s$ and let the upper limit of the integral go to infinity. Thus, we finally get the Markovian master equation

$$\frac{d}{dt}\rho_S(t) = -\int_0^\infty ds Tr_E[H_I(t),[H_I(t-s),\rho_S(t)\otimes\rho_E]]. \tag{4.53}$$

Now, we want to recall again that the evolution is described on a coarse-grained time scale and that the dynamical behaviour over times of the order of magnitude of the correlation time between the system and the environment is not resolved.
All these approximations do not guarantee that only positive definite density matrices are allowed. Therefore we have to use another assumption described by the **rotating wave approximation**. For this purpose we first of all write the interaction Hamiltonian H_I of the Schrödinger picture in the most general form

$$H_I = \sum_k A_k \otimes B_k \tag{4.54}$$

If we suppose the spectrum of H_S to be discrete, the system operators A_k can then be decomposed in the system energy eigenbasis like

$$A_k(\omega) = \sum_{e-e'=\omega} |e\rangle\langle e|A_k|e'\rangle\langle e'| \tag{4.55}$$

with the properties $[H_S, A_k(\omega)] = -\omega A_k(\omega)$ and $A_k^\dagger(\omega) = A_k(-\omega)$, where $A_k(\omega)$ and $A_k^\dagger(\omega)$ can be seen as the eigenoperators of H_S with their corresponding frequency eigenvalues $\mp\omega$. Then the interaction Hamiltonian in the Schrödinger picture can be expressed by

$$H_I = \sum_{k,\omega} A_k(\omega)\otimes B_k. \tag{4.56}$$

Transforming into the interaction picture we have

$$H_I(t) = \sum_{k,\omega} e^{-i\omega t} A_k(\omega)\otimes B_k(t), \tag{4.57}$$

where $B_k(t) = e^{iH_E t}B_k e^{-iH_E t}$ are the interaction picture operators of the environment. If we insert the interaction Hamiltonian given in eq.(4.56) in the Markovian master equation (4.53) and neglect the terms with different frequencies ω, we obtain after some algebra

$$\begin{aligned}\frac{d}{dt}\rho_S(t) &= \sum_{\omega,k,l}\Big[\Gamma_{kl}(\omega)\Big(A_l(\omega)\rho_S(t)A_k^\dagger(\omega) - A_k^\dagger(\omega)A_l(\omega)\rho_S(t)\Big)\\ &\quad +\Gamma_{kl}^*(\omega)\Big(A_l(\omega)\rho_S(t)A_k^\dagger(\omega) - \rho_S(t)A_k^\dagger(\omega)A_l(\omega)\Big)\Big],\end{aligned} \tag{4.58}$$

where the function Γ is given by the fourier transformation of the reservoir correlation functions

$$\Gamma_{kl}(\omega) = \int_0^\infty ds\, e^{i\omega s} \left\langle B_k^\dagger(t) B_l(t-s) \right\rangle . \tag{4.59}$$

It is useful to decompose Γ as follows

$$\Gamma_{kl}(\omega) = \frac{1}{2}\gamma_{kl}(\omega) + iS_{kl}(\omega). \tag{4.60}$$

This leads to the more familiar expression

$$\frac{d}{dt}\rho_S(t) = -i[H_{LS}, \rho_S(t)] - \mathcal{D}(\rho_S(t)) \tag{4.61}$$

with

$$H_{LS} = \sum_{\omega,k,l} S_{k,l}(\omega) A_k^\dagger(\omega) A_l(\omega) \tag{4.62}$$

$$\mathcal{D}(\rho_S(t)) = \frac{1}{2}\sum_{\omega,k,l} \gamma_{kl}(\omega) \Big(A_k^\dagger(\omega) A_l(\omega) \rho_S(t) + \rho_S(t) A_k^\dagger(\omega) A_l(\omega) - 2A_l(\omega)\rho_S(t) A_k^\dagger(\omega) \Big). \tag{4.63}$$

The Hermitian operator H_{LS} generates the unitary Hamiltonian dynamics of the system and is often called the Lamb shift Hamiltonian since it describes a renormalization of the unperturbed energy levels due to the coupling with the environment. To reach the form of the dissipator given in eq.(4.41) the positive matrix $\gamma_{kl}(\omega)$ must be diagonalized by an unitary transformation.

4.4.3 Representations of the dissipator

The most important result we gained in the last sections is the Lindblad master equation,

$$\frac{d}{dt}\rho = -i[H, \rho] - \mathcal{D}(\rho), \tag{4.64}$$

describing the time evolution of open quantum systems under certain approximations. Then the general form of the dissipator is given by

$$\mathcal{D}(\rho) = \frac{1}{2}\sum_{k=1}^{d^2-1} \gamma_k \left(A_k^\dagger A_k \rho + \rho A_k^\dagger A_k - 2A_k \rho A_k^\dagger \right). \tag{4.65}$$

The structure of $\mathcal{D}(\rho)$ is determined by a complete positive map

$$V_t : \rho(0) \longmapsto \rho(t) = V_t \rho(0) \tag{4.66}$$

satisfying the semigroup property $V_{t_1}V_{t_2} = V_{t_1+t_2}$.
Now we try to find other representations of the dissipator, where we perform further simplifications. Although these assumptions do not have a deeper physical motivation or reason, we can nevertheless draw conclusions on how the system behaves.
Assuming hermitian Lindblad operators $A_k^\dagger = A_k$ the von Neumann entropy of the system $S(\rho) = -Tr(\rho \ln \rho)$ does not decrease due to a theorem by Benatti and Narnhofer [2]. If the Lindblad operators are commuting with the Hamiltonian of the system, $[A_k, H] = 0$, the energy is conserved and the "dissipator"[2] $\mathcal{D}(\rho)$ can be written in the form

$$\mathcal{D}(\rho) = \frac{1}{2}\sum_k^{d^2-1} \gamma_k [A_k, [A_k, \rho]]. \tag{4.67}$$

The next simplification is rather strong but its big advantage is that we can calculate analytically all possible decoherence modes, at least for 2 qubits. We replace the Lindblad operators by projection operators like $\sqrt{\gamma_k}A_k = \sqrt{\lambda_k}P_k$ and get for the dissipator

$$\mathcal{D}(\rho) = \frac{1}{2}\sum_k^{d} \lambda_k (P_k\rho + \rho P_k - 2P_k\rho P_k), \tag{4.68}$$

where the projection properties $P_k^\dagger = P_k$ and $P_k^2 = P_k$ have been used. The positive and real parameters λ_k are called decoherence parameters. If we further assume that all these decoherence weights have the same value λ and use the completeness condition of the projection operators $\sum_k P_k = \mathbb{1}$, we can write

$$\mathcal{D}(\rho) = \lambda(\rho - \sum_k^{d} P_k\rho P_k). \tag{4.69}$$

In the next two chapters we are trying to solve the master equation with this decoherence representation for a 2 qubit and a 2 qutrit system.

4.5 Nomenclature of some open system effects

Since all kind of different decoherence phenomena occur in the various subcommunities of physics, the terminology is often not consistent. Furthermore

[2]To understand why the dissipator is between quotation marks see the definition of the dissipator in Sect.4.5.

the theory of open quantum systems is pretty young, whereas the effects caused by the theory are studied in the meantime many decades. Therefore we want to present an attempt of defining some concepts according to Hornberger [28]:

- **decoherence:**
  Originally decoherence described an environmental quantum effect manipulating macroscopically distinct states. By now the term is also applied to microscopic states if they are affected quantum mechanically by the environmental degrees of freedom, which are in practice unobservable.
  Often decoherence is used for a process that reduces the purity of a microstate.

- **dephasing:**
  Dephasing is the process where the coherence, given by the off-diagonal elements of the density matrix, get reduced in the energy eigenbasis of the system. This effect could also be reversible but under decoherence it is always irreversible.
  Especially this term has to be used with great care because in many communities they are understood differently. In nonlinear optics or in molecular physics for example dephasing is a synonym to 'dispersion', whereas in condensed matter physics it is understood as 'decoherence'.

- **phase averaging:**
  This is a classical noise phenomenon caused by fluctuating external parameters under the unitary evolution of the system.
  Examples are the vibration of an interferometer grating or the technical noise of an electron interferometer produced by the flactuations of a classical magnetical field. In practice it is often difficult to distinguish between a phase averaging or a decoherence effect.

- **dispersion:**
  Dispersion is the coherent broadening of wave packets under unitary evolution. Reasons are for example the velocity dependence of the group velocity or non-harmonic energy spacings. This leads often to a reduction of signal oscillations.

- **dissipation:**
  The energy exchange between the system and the environment is called dissipation. It is usually combined with decoherence but not necessarily.

We see that the name of the dissipator $\mathcal{D}(\rho)$ is a little bit misleading since it actually describes three different physical effects - decoherence, dephasing and dissipation. Nevertheless we will call it further dissipator due to the literature and the common convention.

Part II

Applications

Chapter 5

Decoherence of a two qubit system

In this chapter we want to study how decoherence can influence a system which consists of two qubits described by $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$. We will use two different theoretical models. First we want to allow that our environment has a certain memory, which is described by a non-Markovian master equation. But the focus of this discussion is the Markovian case where the dissipator is given in the projector form. The calculations presented here are based on the work of Durstberger [20].
For both models we will use the concurrence as our measure of entanglement. Further we illustrate the evolution of the state under Markovian decoherence in the spin geometry picture and in the CP-diagram. It is shown that if we initially start with a Bell state we could reach any point in the geometric picture just by tuning the decoherence. If we plot the region of all possible decoherence states in the CP-diagram, we can give upper and lower bounds for the evolution of the 2 qubit system.

5.1 NON-Markovian model

The focus of this section is not on different theoretical models describing non-Markovian dynamics but rather on the results of a particular model discussed by Glendinning, Jakob and Noelle in [24]. We will see that if the considered system is coupled to an environment where certain memory effects are allowed, then the entanglement of the system can oszillate in time. It can be stored in the environment and is then given back. The system and the surrounding reservoir are in a correlated state.
At first we want to explain the model. The considered system consists of

2 qubits represented by 2 two level atoms denoted as 1 and 2 in Fig.5.1. They are each coupled to another two level atom 3 and 4 which, in turn, are damped by a thermal reservoir at a specific temperature that can be characterized with the bosonic excitation number $\bar{n}$. The interaction with the reservoir is described Markovian. The non-Markovian behaviour of the system comes from the "near-environmental" coupling to the atoms 3 and 4.

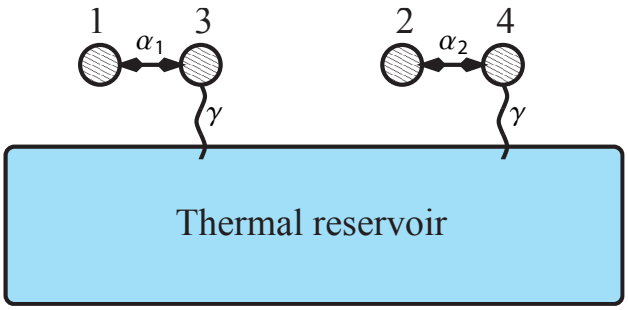


Figure 5.1: Non Markovian decoherence model for the system atom 1 and 2. Each of these atoms is coupled to atom 3 or 4 which simulate the memory of the thermal reservoir.

The atoms 1 and 2 should initially be in a Bell state and uncorrelated with 3 and 4. We can write

$$\rho_{1234}(0) = \rho_{12}(0) \otimes \bar{\rho}_3 \otimes \bar{\rho}_4, \tag{5.1}$$

where atoms 3 and 4 are in a thermal state

$$\bar{\rho}_3 = \bar{\rho}_4 = \frac{1}{2\bar{n}+1} \begin{pmatrix} \bar{n} & 0 \\ 0 & \bar{n}+1 \end{pmatrix}. \tag{5.2}$$

The whole dynamics of the system can then be written as

$$\frac{d}{dt}\rho(t) = -i\left[\sum_{n=1}^{4} H_n + H_{int}^{(13)} + H_{int}^{(24)}, \rho(t)\right] - \mathcal{D}_3(\rho(t)) - \mathcal{D}_4(\rho(t)), \tag{5.3}$$

where we have

$$H_n = \omega_n |1\rangle_{nn}\langle 1|, \tag{5.4}$$

$$H_{int}^{(13)} = \alpha_1(\sigma_1^+\sigma_3^- + \sigma_1^-\sigma_3^+), \tag{5.5}$$

$$H_{int}^{(24)} = \alpha_2(\sigma_2^+\sigma_4^- + \sigma_2^-\sigma_4^+), \tag{5.6}$$

and the Markovian dissipators for atom 3 and 4 are

$$\begin{aligned}\mathcal{D}_3(\rho) &= \gamma(\bar{n}+1)(\sigma_3^+\sigma_3^-\rho + \rho\sigma_3^+\sigma_3^- - 2\sigma_3^-\rho\sigma_3^+) \\ &\quad +\gamma\bar{n}(\sigma_3^-\sigma_3^+\rho + \rho\sigma_3^-\sigma_3^+ - 2\sigma_3^+\rho\sigma_3^-), \qquad (5.7)\\ \mathcal{D}_4(\rho) &= \gamma(\bar{n}+1)(\sigma_4^+\sigma_4^-\rho + \rho\sigma_4^+\sigma_4^- - 2\sigma_4^-\rho\sigma_4^+) \\ &\quad +\gamma\bar{n}(\sigma_4^-\sigma_4^+\rho + \rho\sigma_4^-\sigma_4^+ - 2\sigma_4^+\rho\sigma_4^-). \qquad (5.8)\end{aligned}$$

Here, ω_n represents the energy of the excited state $|1\rangle_n$ for the atom n, the coupling strengths between atoms $1-3$ and $2-4$ are given by the dimensionless parameters α_1 and α_2. The thermal bosonic excitation number is denoted by $\bar{n}$ and γ gives the thermalization rate for atoms 3 and 4. The interactions with the "mini-reservoir" can be tuned by the coupling strengths α_1 and α_2 and by the energy differences $\Delta_k = \omega_k - \omega_{2k}$ with $k = 1, 2$
We will skip the whole calculation where the non-Markovian differential equations (5.3) are solved for the density matrix ρ_{12}. We will rather directly present the evolution of the concurrence for different coupling strengths and energy levels in Fig.5.2.

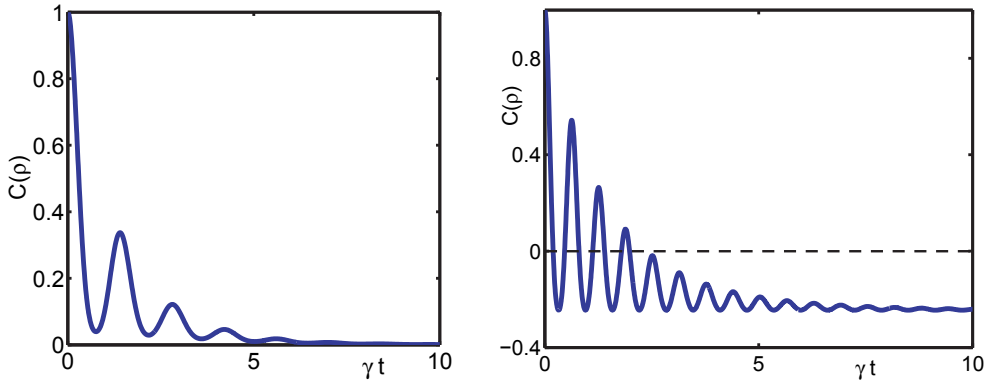


Figure 5.2: The evolution of the concurrence is illustrated for a non-Markovian model with a initially maximally entangled Bell state. Due to the memory effects of the environment the concurrence shows an oszillating behaviour. The entanglement is lost to the reservoir but can be partially restored by the memory effects. On the left side the concurrence is displayed with the parameters $\Delta_1 = \Delta_2 = 2$, $\alpha_1 = \alpha_2 = 2$,$\gamma = 0.5$ and $\bar{n} = 0$. Here the state gets separable for $t \to \infty$. The right side shows also the "negative" concurrence of the state for $\Delta_1 = \Delta_2 = 0$, $\alpha_1 = \alpha_2 = 5$,$\gamma = \frac{1}{3}$ and $\bar{n} = 0.2$. We see that the concurrence can completely vanish for a short time but is partially recovered again till the amount of entanglement is too small and lost.

5.2 Markovian model

In this section we study a certain Markovian decoherence model where the Lindblad generators are given as projection operators P_k, see Sect.4.4.3. Additionally we allow just one decoherence parameter λ which characterizes the strenght of the interaction between the Markovian environment and the considered 2 qubit system. Under these simplifications the dissipator can be written in the form

$$\mathcal{D}(\rho) = \lambda(\rho - \sum_{k=1}^{4} P_k \rho P_k). \tag{5.9}$$

Further we consider that the dynamical evolution of the system should just depend on the decoherence generated by the environment and not on the unitary evolution described by the Hamiltonian H. Therefore we can write the Lindblad master equation in the reduced form

$$\frac{d}{dt}\rho(t) = -\mathcal{D}(\rho(t)). \tag{5.10}$$

Since the arising coupled differential equations has already been solved in [20], we want to focus on the whole set of states that can be reached through different decoherence modes starting from several initial states.

5.2.1 Notation

For characterising how decoherence can act on the system we have to manipulate the projection operators in a certain basis. We take the notation where the eigenstates $|e_k\rangle$ of the undisturbed Hamiltonian form this basis and are defined by

$$H|e_k\rangle = E_k|e_k\rangle. \tag{5.11}$$

Then we can split these eigenstates of the whole system in eigenstates of the subspace of the first qubit $\{|a_1\rangle, |a_2\rangle\}$ and of the second one $\{|b_1\rangle, |b_2\rangle\}$ in the following way

$$\begin{aligned} |e_{1,3}\rangle &= |a_{1,2}\rangle|b_1\rangle \\ |e_{2,4}\rangle &= |a_{1,2}\rangle|b_2\rangle. \end{aligned} \tag{5.12}$$

Further we rotate in one subspace in such way that the new states are given by

$$\begin{aligned} |\alpha_1\rangle &= \cos\frac{\theta}{2}|a_1\rangle + \sin\frac{\theta}{2}|a_2\rangle \\ |\alpha_2\rangle &= -\sin\frac{\theta}{2}|a_1\rangle + \cos\frac{\theta}{2}|a_2\rangle. \end{aligned} \tag{5.13}$$

We call this basis GR for general rotated, a rotation of $\theta = \frac{\pi}{2}$ is denoted by R and the eigenbasis by E. If we want to be accurate, we have to put a phase factor in the rotation but the general solution including the factor shows that physical properties like concurrence and purity are independent of the phase. Although other states can be reached by tuning this factor, we nevertheless want to neglect it by now because we are just interested in these two properties. In order to get an impression how these basis could be constructed, here in the computational basis, we give a short example.

Example

Let the eigenstates of the two subsystems be

$$\begin{aligned} |a_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle \qquad |b_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle \\ |a_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle \qquad |b_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle, \end{aligned} \tag{5.14}$$

where $|a_i\rangle$ denotes states in Alice's space and $|b_i\rangle$ states in Bob's space. For the composite system the basis can be constructed like

$$\begin{aligned} |e_1\rangle = |a_1\rangle \otimes |b_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle \qquad |e_2\rangle = |a_1\rangle \otimes |b_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle \\ |e_3\rangle = |a_2\rangle \otimes |b_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle \qquad |e_4\rangle = |a_2\rangle \otimes |b_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle. \end{aligned}$$

If we take for example the GR basis of the first subspace, we would get

$$\begin{pmatrix} |\hat{e}_{1,2}\rangle \\ |\hat{e}_{3,4}\rangle \end{pmatrix} = \begin{pmatrix} |\alpha_1\rangle|b_{1,2}\rangle \\ |\alpha_2\rangle|b_{1,2}\rangle \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \cdot \begin{pmatrix} |e_{1,2}\rangle \\ |e_{3,4}\rangle \end{pmatrix}. \tag{5.15}$$

Important to note is that if we just rotate the basis in one subsystem, the new basisvectors are linear combinations of just two eigenvectors. This feature gives the equations of motion a certain structure which will be of great importance in the discussion of two qutrits decoherence in the next chapter.

5.2.2 Decoherence modes

We differ between three possible decoherence modes called mode A, B and C depending on the particular choice of basisvectors for the subsystems. Mode A labeled by $E \otimes E$ should be the simplest case where we take the eigenstates of the Hamiltonian in both subsystems. For mode B ($GR \otimes E$) we rotate the eigenbasis in one subsystem and for mode C ($GR \otimes GR$) we rotate in both.

Mode A ($E \otimes E$)

For this decoherence mode the projection operators of the dissipator are given by $P_k = |e_k\rangle\langle e_k|$ and project onto the eigenstates of the Hamiltonian. The solution of this mode is rather simple because the differential equations decouple. It is given by

$$\begin{aligned} \rho_{ij}(t) = &\quad e^{-\lambda_A t}\rho_{ij}(0) \quad \text{for } i \neq j \\ \rho_{ii}(t) = &\quad \rho_{ii}(0), \end{aligned} \tag{5.16}$$

where λ_A is the decoherence parameter of mode A. One can see that only the off-diagonal elements of the density matrix are suppressed with the factor $e^{-\lambda_A t}$ and that the diagonal elements are robust under this kind of decoherence. This implies that an initially entangled system gets separated only for $t \to \infty$ and not in finite times.

Mode B ($GR \otimes E$)

This decoherence mode describes the case where we take the general rotated basis in the first subsystem and the eigenbasis in the second subsystem. The new basisvectors are of the form

$$\begin{aligned} |\hat{e_1}\rangle = |\alpha_1\rangle|b1\rangle = &\quad \cos\tfrac{\theta}{2}|e_1\rangle + \sin\tfrac{\theta}{2}|e_3\rangle \\ |\hat{e_2}\rangle = |\alpha_1\rangle|b2\rangle = &\quad \cos\tfrac{\theta}{2}|e_2\rangle + \sin\tfrac{\theta}{2}|e_4\rangle \\ |\hat{e_3}\rangle = |\alpha_2\rangle|b1\rangle = &\quad -\sin\tfrac{\theta}{2}|e_1\rangle + \cos\tfrac{\theta}{2}|e_3\rangle \\ |\hat{e_4}\rangle = |\alpha_2\rangle|b2\rangle = &\quad -\sin\tfrac{\theta}{2}|e_2\rangle + \cos\tfrac{\theta}{2}|e_4\rangle \end{aligned} \tag{5.17}$$

and the projection operators are given by

$$P_k = |\hat{e_k}\rangle\langle\hat{e_k}|. \tag{5.18}$$

We do not want to present the general solution of the differential equations which are explicitly worked out in [20]. We will rather focus on the special

case where $\theta = \frac{\pi}{2}$ and denote this mode with $R \otimes E$. The resulting dissipator (see Eq.5.9) is

$$\mathcal{D}(\rho) = \lambda_B \begin{pmatrix} \frac{1}{2}(\rho_{11} - \rho_{33}) & \rho_{12} & \frac{1}{2}(\rho_{13} - \rho_{31}) & \rho_{14} \\ \rho_{21} & \frac{1}{2}(\rho_{22} - \rho_{44}) & \rho_{23} & \frac{1}{2}(\rho_{24} - \rho_{42}) \\ \frac{1}{2}(\rho_{31} - \rho_{13}) & \rho_{32} & \frac{1}{2}(\rho_{33} - \rho_{11}) & \rho_{34} \\ \rho_{41} & \frac{1}{2}(\rho_{42} - \rho_{24}) & \rho_{43} & \frac{1}{2}(\rho_{44} - \rho_{22}) \end{pmatrix}. \tag{5.19}$$

One gets a set of pairwise coupled differential equations which are solved by

$$\begin{aligned} \rho_{11}(t) &= \frac{1}{2}[(1 + e^{-\lambda_B t})\rho_{11}(0) + (1 - e^{-\lambda_B t})\rho_{33}(0)] \\ \rho_{33}(t) &= \frac{1}{2}[(1 + e^{-\lambda_B t})\rho_{33}(0) + (1 - e^{-\lambda_B t})\rho_{11}(0)]. \end{aligned}$$

The same solution is given for the sets $\{\rho_{22}, \rho_{44}\}$,$\{\rho_{13}, \rho_{31}\}$,$\{\rho_{24}, \rho_{42}\}$. All other elements of the density matrix behave like mode A with an exponential suppression of $e^{-\lambda_B t}$.

Mode C $(GR \otimes GR)$

The third possibility is a rotation of the basis vectors in both subspaces. In fact one can perform an unitary transformation in such way that the mode C $(GR \otimes GR)$ acts like a rotation in just one subsystem $(GR \otimes E)$. Since the relevant properties of our investigations like concurrence and purity stay untouched under unitary transformations, it is sufficient to focus on mode A and B for a complete characterisation of the Markovian decoherence described by projection operators.
Note that mode C can actually reach different states than mode A and B, for instance see the geometrical illustration given in Sect.5.2.4.

5.2.3 Different initial states

In order to get a sense how these modes are acting on a certain physical state we use different initial states and calculate the evolution of their concurrence and purity. Most of the used states have been already introduced in Sect.2.3. Furthermore we illustrate the evolution of these properties in a CP-diagram (concurrence vs. purity) where we can present an allowed region of all reachable states.

Bell singlet state

The first initial state we focus on is the Bell singlet state

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \tag{5.20}$$

whose density matrix is given by

$$\rho_{Bell} = |\Psi^-\rangle\langle\Psi^-| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.21}$$

Of course, if any other Bell state would be used, we would get analog results.

If the decoherence **mode A** acts on the Bell state one gets

$$\rho_{Bell,mode\,A}(t) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -e^{-\lambda_A t} & 0 \\ 0 & -e^{-\lambda_A t} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.22}$$

As mentioned already this mode affects just the two off-diagonal elements which are responsible for the entanglement of the state. The purity and the concurrence are given by

$$P_A(t) = \frac{1}{2}(1 + e^{-2\lambda_A t}) \tag{5.23}$$

$$C_A(t) = e^{-\lambda_A t}. \tag{5.24}$$

We note that the system initially starts with a pure and maximally entangled state ($P = 1$ and $C = 1$) and reaches a mixed but not maximally mixed seperable state ($P = \frac{1}{2}$ and $C = 0$) for $t \to \infty$.
To plot the result in the CP-diagram we have to substitute the exponential function and get the relation

$$C_A(P) = \sqrt{2P - 1}. \tag{5.25}$$

If **mode B** is acting on the Bell singlet state with a rotation of $\theta = \frac{\pi}{2}$, the state changes like

$$\rho_{Bell,mode\,B}(t) = \frac{1}{4}\begin{pmatrix} 1 - e^{-\lambda_B t} & 0 & 0 & 0 \\ 0 & 1 + e^{-\lambda_B t} & -2e^{-\lambda_B t} & 0 \\ 0 & -2e^{-\lambda_B t} & 1 + e^{-\lambda_B t} & 0 \\ 0 & 0 & 0 & 1 - e^{-\lambda_B t} \end{pmatrix}, \tag{5.26}$$

where the concurrence and the purity are given by

$$P_B(t) = \frac{1}{4}(1 + 3e^{-2\lambda_B t}) \tag{5.27}$$

$$C_B(t) = \max\{0, \frac{1}{2}(3e^{-\lambda_B t} - 1)\}. \tag{5.28}$$

The system starts in a pure state with $P = 1$ and finally ends in the maximally mixed state with $(P \overset{t\to\infty}{\to} \frac{1}{4})$ given by the identity matrix. Interesting to note is the decrease of the entanglement. It goes from the Bell state with $C = 1$ to a seperable one at a finite time $t = \frac{\ln 3}{\lambda_B}$ where the purity has a value of $P = \frac{1}{3}$. If we substitute $\alpha = e^{-\lambda_B t}$ we gain the **Werner state**, which means that the evolution of the Bell state directly goes to the identity.
To get the relation between C and P we substitute again and get

$$C_B(P) = \frac{1}{2}\left[3\sqrt{\frac{4P-1}{3}} - 1\right]. \tag{5.29}$$

Fig.5.2.3 shows the relation of the concurrence and the purity for the decoherence modes A $(E \otimes E)$ and B $(R \otimes E)$. The boundary of all possible physical states is given by the MEMS, see Sect.2.3.4. The lower curve represents mode A where $\theta = 0$ and the upper one represents mode B with $\theta = \frac{\pi}{2}$. Since all decoherence paths with other possible choices of θ lie in the region between these two curves, we have found a region of all achievable states under Markovian decoherence starting with a Bell state.

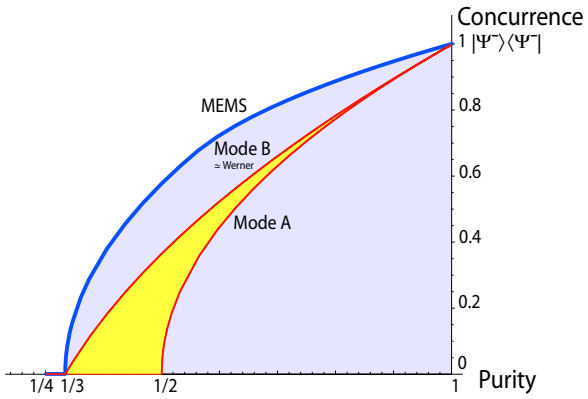


Figure 5.3: CP-Graph of a 2 qubit system under Markovian decoherence starting with a Bell state. Mode B with $\theta = \frac{\pi}{2}$, which is analog to the Werner state, gives a upper border of all achievable states(the yellow region). Mode A is the lower border and all other choices of θ are inside the region.

Pure states

We start with a set of pure states which are parameterised by the angle φ with the condition $0 \leq \varphi \leq \frac{\pi}{2}$. The state should be given by

$$|\Psi\rangle = \sin\frac{\varphi}{2}|01\rangle - \cos\frac{\varphi}{2}|10\rangle \tag{5.30}$$

$$\rho_{pure} = |\Psi\rangle\langle\Psi| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sin^2\frac{\varphi}{2} & -\frac{1}{2}\sin\varphi & 0 \\ 0 & -\frac{1}{2}\sin\varphi & \cos^2\frac{\varphi}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.31}$$

If $\varphi = \frac{\pi}{2}$ we get the Bell singlet state and for $0 \leq \varphi \leq \frac{\pi}{2}$ we can tune the amount of entanglement of the initial state. The concurrence of this set of pure states is simply

$$C(\varphi) = \sin\varphi. \tag{5.32}$$

After we let **mode A** act on the set we get

$$\rho_{pure,mode\,A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sin^2\frac{\varphi}{2} & -\frac{1}{2}\sin\varphi\, e^{-\lambda_A t} & 0 \\ 0 & -\frac{1}{2}\sin\varphi\, e^{-\lambda_A t} & \cos^2\frac{\varphi}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.33}$$

and for the purity and the concurrence

$$P_A(t,\varphi) = \frac{1}{2}\left(2 - \sin^2\varphi + \sin^2\varphi\, e^{-2\lambda_A t}\right) \tag{5.34}$$

$$C_A(t,\varphi) = \sin\varphi\, e^{-\lambda_A t} \tag{5.35}$$

$$C_A(P,\varphi) = \sqrt{2P - 2 + \sin^2\varphi}. \tag{5.36}$$

At next we let **mode B** again with the rotation angle of $\theta = \frac{\pi}{2}$ act on the initial states and get for the evolution of the system

$$\rho_{pure,mode\,B}(t) = \begin{pmatrix} (1-e^{-\lambda_B t})(1-\xi^2) & 0 & 0 & 0 \\ 0 & (1+e^{-\lambda_B t})\xi^2 & -\sin\varphi\, e^{-\lambda_B t} & 0 \\ 0 & -\sin\varphi\, e^{-\lambda_B t} & (1+e^{-\lambda_B t})(1-\xi^2) & 0 \\ 0 & 0 & 0 & (1-e^{-\lambda_B t})\xi^2 \end{pmatrix} \tag{5.37}$$

where $\xi = \sin\frac{\varphi}{2}$. For the physical properties concurrence und purity depending on the angle φ we get the evolution

$$P_B(t,\varphi) = \frac{1}{2}\left(1 - \frac{1}{2}\sin^2\varphi + (1 + \frac{1}{2}\sin^2\varphi)\, e^{-2\lambda_B t}\right) \tag{5.38}$$

$$C_B(t,\varphi) = \frac{1}{2}\sin\varphi\, \left(3e^{-\lambda_B t} - 1\right) \tag{5.39}$$

$$C_B(P,\varphi) = \frac{1}{2}\sin\varphi \left[3\sqrt{\frac{2P - 1 + \frac{1}{2}\sin^2\varphi}{1 + \frac{1}{2}\sin^2\varphi}} - 1\right]. \tag{5.40}$$

In Fig.5.4 on the next page both decoherence modes are illustrated in a CP-diagram for different initial pure states. We note that all of these states except the state with $\varphi = 0$(initially $C = 0$) become separable under the influence of mode B after the same time $t = \frac{\ln 3}{\lambda_B}$ (see Eq.(5.39)). This means that it doesn't matter with which amount of entanglement we start, we always end in a separable state at the same time.

Maximum entangled mixed states (MEMS)

The MEMS, see also Sect.2.3.4, have the property that their entanglement is maximized for a given purity. Its not possible to describe the whole set of these states with only one parameter so we have to split it in two groups of states. In our discussion we want to start again at the Bell state and then go along the MEMS with the initial state. That is why we begin with the "second" set of states which is valid for $\alpha \in [\frac{2}{3}, 1]$ and given by

$$\rho_{MEMS2} = \alpha|\Phi_+\rangle\langle\Phi_+| + (1-\alpha)|01\rangle\langle 01| = \tag{5.41}$$

$$\rho_{MEMS2} = \frac{1}{2}\begin{pmatrix} \alpha & 0 & 0 & \alpha \\ 0 & 2-2\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & \alpha \end{pmatrix} \tag{5.42}$$

$$P(\alpha) = \alpha^2 + (1-\alpha)^2 \tag{5.43}$$

$$C(\alpha) = \alpha \tag{5.44}$$

$$C(P) = \frac{1}{2} + \sqrt{\frac{1}{2}(P - \frac{1}{2})}, \tag{5.45}$$

After calculating **mode A** we get the typical exponential suppression of

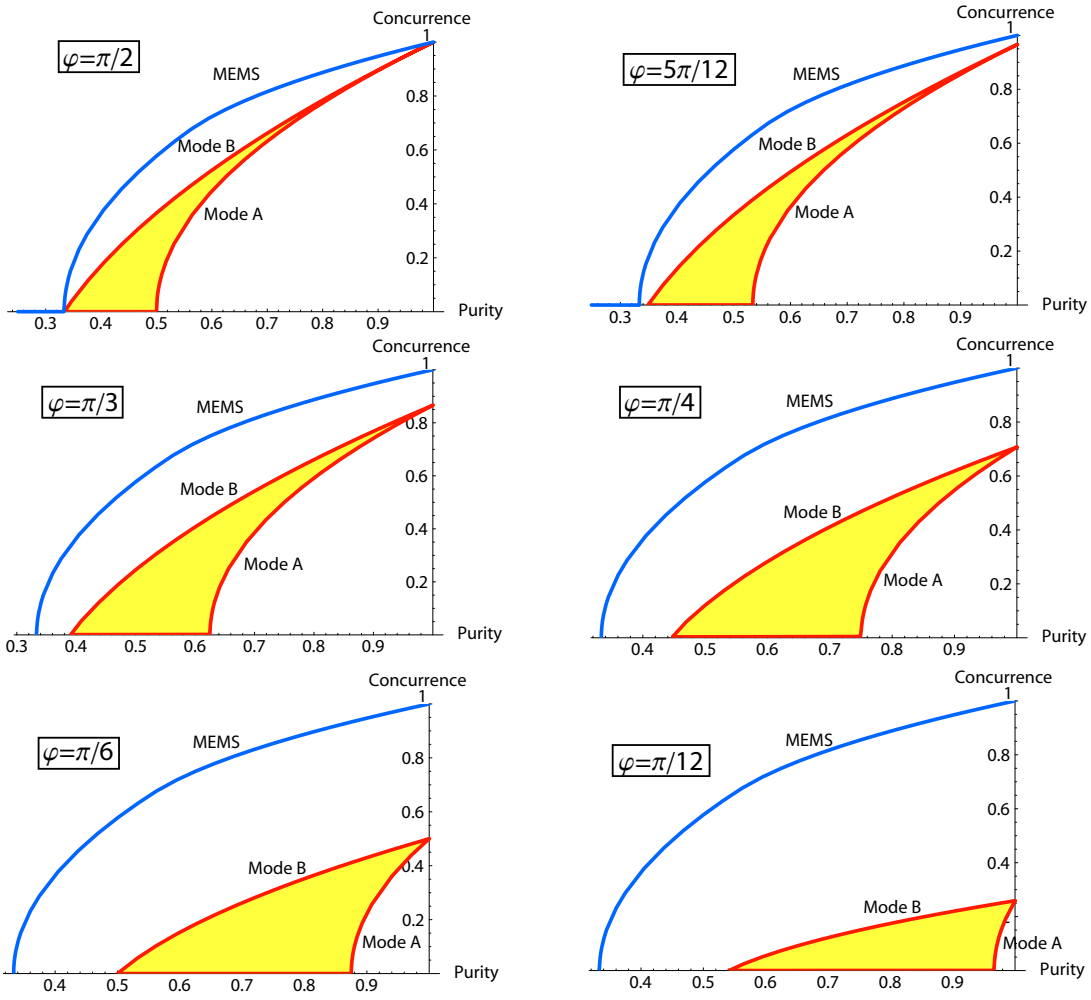


Figure 5.4: CP-diagrams showing the possible achievable states for a Markov decoherence with projectors for different initial pure states: $|\Psi\rangle = \sin\frac{\varphi}{2}|01\rangle - \cos\frac{\varphi}{2}|10\rangle$. Mode B with a rotation angle of $\theta = \frac{\pi}{2}$ is always the upper border of the evolution region whereas mode A gives us the lower border.

the off-diagonal elements with

$$\rho_{(MEMS\,2,mode\,A)} = \frac{1}{2}\begin{pmatrix} \alpha & 0 & 0 & \alpha\, e^{-\lambda_A t} \\ 0 & 2-2\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha\, e^{-\lambda_A t} & 0 & 0 & \alpha \end{pmatrix}. \tag{5.46}$$

The purity and concurrence of this state is given by

$$\begin{aligned} P(t,\alpha) &= \frac{\alpha^2}{2}(1+e^{-2\lambda_A t}) + (1-\alpha)^2 && (5.47)\\ C(t,\alpha) &= \alpha e^{-\lambda_A t} && (5.48)\\ C(P,\alpha) &= \sqrt{2(P-(1-\alpha)^2)-\alpha^2}. && (5.49) \end{aligned}$$

We are not going to discuss the evolution of the concurrence and the purity in detail at the moment because we can better see their behaviour in the CP diagram of Fig.5.5.

Now we want to calculate how the density matrix of the state changes under **mode B**

$$\rho_{(MEMS\,2,mode\,B)} = \tfrac{1}{4}\cdot \begin{pmatrix} (1+e^{-\lambda_B t})\alpha & 0 & 0 & 2\alpha\, e^{-\lambda_B t} \\ 0 & 2-\alpha+(2-3\alpha)e^{-\lambda_B t} & 0 & 0 \\ 0 & 0 & (1-e^{-\lambda_B t})\alpha & 0 \\ 2\alpha\, e^{-\lambda_B t} & 0 & 0 & 2-\alpha+(3\alpha-2)e^{-\lambda_B t} \end{pmatrix}. \tag{5.50}$$

with

$$\begin{aligned} P(t,\alpha) &= \frac{1}{4}(2-2\alpha+\alpha^2+(2-6\alpha+7\alpha^2)e^{-2\lambda_B t}) && (5.51)\\ C(t,\alpha) &= \frac{1}{2}\left(2\alpha e^{-\lambda_B t} - \sqrt{\alpha(1-e^{-\lambda_B t})(2-\alpha+(2-3\alpha)e^{-\lambda_B t})}\right) && (5.52) \end{aligned}$$

We do not want to present the function $C(P,\alpha)$ explicitly because the analytical expression is not very handsome. The characteristics of this function will be shown and discussed in Fig.5.5.

Finally we take the "first" parametrisation of the MEMS as the initial

states and have

$$\begin{aligned}
\rho_{MEMS\,1} &= \alpha|\Phi_+\rangle\langle\Phi_+| + \frac{1}{3}|01\rangle\langle 01| + \\
&\quad +(\frac{1}{3} - \frac{\alpha}{2})(|00\rangle\langle 00| + |11\rangle\langle 11|) && (5.53)\\
\rho_{MEMS\,1} &= \frac{1}{2}\begin{pmatrix} \frac{2}{3} & 0 & 0 & \alpha \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & \frac{2}{3} \end{pmatrix} && (5.54)\\
P(\alpha) &= \frac{1}{3} + \frac{\alpha^2}{2} && (5.55)\\
C(\alpha) &= \alpha && (5.56)\\
C(P) &= \sqrt{2P - \frac{2}{3}}, && (5.57)
\end{aligned}$$

where $\alpha \in [0, \frac{2}{3}]$.

If the decoherence acts with **mode A** we get

$$\begin{aligned}
\rho_{(MEMS\,1,\,mode\,A)} &= \frac{1}{2}\begin{pmatrix} \frac{2}{3} & 0 & 0 & \alpha\, e^{-\lambda_A t} \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha\, e^{-\lambda_A t} & 0 & 0 & \frac{2}{3} \end{pmatrix} && (5.58)\\
P(t,\alpha) &= \frac{1}{3} + \frac{\alpha^2}{2}\, e^{-2\lambda_A t} && (5.59)\\
C(t,\alpha) &= \alpha\, e^{-\lambda_A t} && (5.60)\\
C(P) &= \sqrt{2P - \frac{2}{3}}, && (5.61)
\end{aligned}$$

Interesting to note is that the function $C(P)$ is independent of the initial state which was characterized by α. We can see in Fig.5.5 that starting with $\alpha = \frac{2}{3}$ this decoherence mode exactly produces the MEMS with $\alpha \leq \frac{2}{3}$.

By calculating the decoherence mode B for all MEMS 1 states the time

evolution of the density matrix is given by

$$\rho_{(MEMS\,1,mode\,B)} = \begin{pmatrix} \frac{1}{6}(1+e^{-\lambda_B t}) & 0 & 0 & \frac{\alpha}{2}\,e^{-\lambda_B t} \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{6}(1-e^{-\lambda_B t}) & 0 \\ \frac{\alpha}{2}\,e^{-\lambda_B t} & 0 & 0 & \frac{1}{3} \end{pmatrix} \quad (5.62)$$

$$P(t,\alpha) = \frac{1}{18}(5+(1+9\alpha^2)\,e^{-2\lambda_B t}) \quad (5.63)$$

$$C(t,\alpha) = \frac{1}{3}(3\alpha\,e^{-\lambda_B t} - \sqrt{2-2e^{-\lambda_B t}}). \quad (5.64)$$

The function $C(P,\alpha)$ is again shown in the picture.

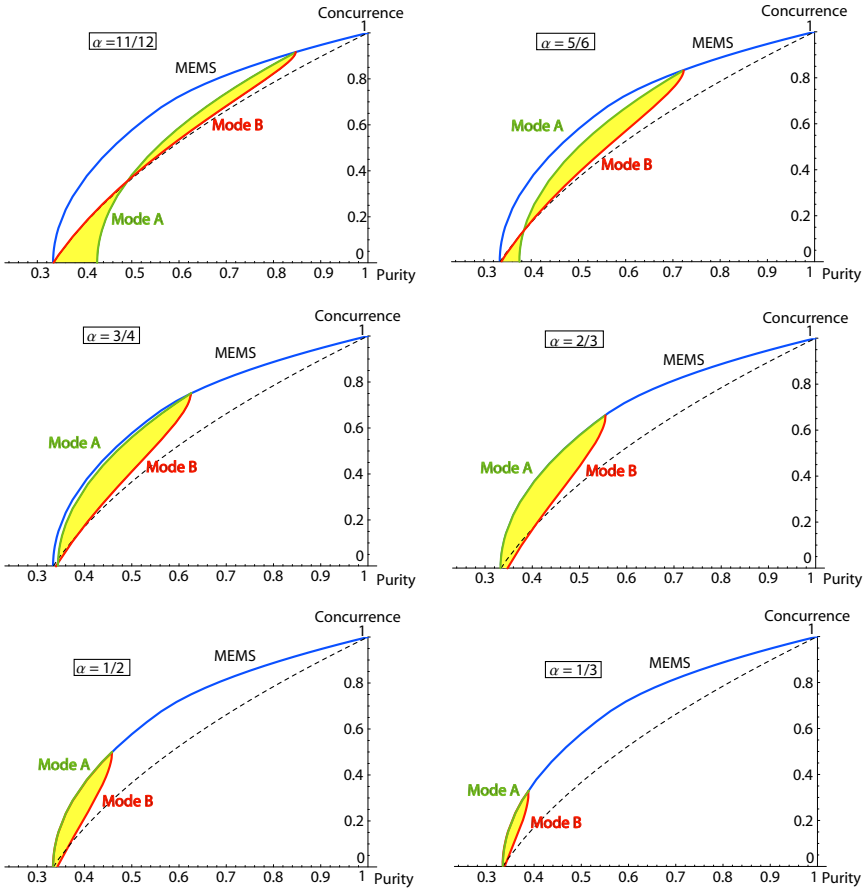


Figure 5.5: CP-diagram showing the possible achievable states (yellow region) under Markov decoherence with projectors for different initial MEMS parametrised with α. Mode A and mode B (with $\theta = \frac{\pi}{2}$) are still borders for all possible states but they cross close to the Werner state(dashed line).

Mode B is now for the second parametrisation, where α is between $[0, \frac{2}{3}]$, the lower border of the decoherence region. In case of α larger than $\frac{2}{3}$ the two extremal modes are actually crossing, which means that if we start with such a maximally entangled mixed state and the decoherence can be described with projection operators that the state will reach a point in the CP-diagram independent of the particular mode. It seems that mode B always tries to "follow" the Werner state or to mix the elements of the density matrix so that all the diagonal elements have nearly the same value, which in fact is only possible for the initial Bell state.

5.2.4 Geometric picture

We have introduced a very useful geometric picture for a 2 qubit system in Sect.3.2, which can be used now for a disussion of the decoherence modes. Unfortunately only the Bell state as the initial state can be considered because all other pure states and the MEMS have local parameters and are therefore not contained in the picture.
We skip most of the calculation, which was done by Durstberger in her PhD thesis [20] and in [21], we only want to present the results.
Starting with the Bell state which has the correlation vector

$$\vec{c} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \tag{5.65}$$

we let **mode A** with the typical exponentional decay of the off-diagonal elements act on this state and get

$$\vec{c}_{mode\,A} = \begin{pmatrix} -e^{-\lambda_A t} \\ -e^{-\lambda_A t} \\ -1 \end{pmatrix}. \tag{5.66}$$

In Fig.5.6 the evolution of the Bell state under the decoherence mode A is illustrated. It starts at $t = 0$ in the corner of the positivity tetrahedron with the Bell state, goes along the edge which is given by a mixture of $|\Psi^-\rangle$ and $|\Psi^+\rangle$ and reach the peak of the doublepyramid for $t \longrightarrow \infty$.

If mode B ($GR \otimes R$) affects the Bell state, the evolution gives the correlation matrix

$$c_{mode\,B} = \begin{pmatrix} -e^{-\lambda_B t} & 0 & (-1 + e^{-\lambda_B t}) sin\theta\, cos\theta \\ 0 & -e^{-\lambda_B t} & 0 \\ 0 & 0 & -cos^2\theta - e^{-\lambda_B t} sin^2\theta \end{pmatrix}. \tag{5.67}$$

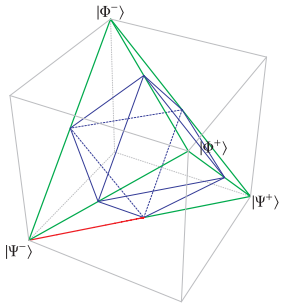


Figure 5.6: The red line shows the evolution of a Bell state under decoherence mode A in the spin geometry picture

The off-diagonal element comes from the special coupling of the differential equations and causes that the correlation vector $\vec{c}_{mode\,B} = (-c_1^B, -c_2^B, -c_3^B)^T$ contains also non-linear functions. The singular values of the matrix in Eq.(5.67) are

$$c_1^B = e^{-\lambda_B t} \tag{5.68}$$

$$c_{2,3}^B = \frac{e^{-\lambda_B t}}{2}\Big(3 + 2e^{2\lambda_B t}cos^2\theta - cos(2\theta) \mp \mp\sqrt{2}(e^{\lambda_B t} - 1)cos\theta\sqrt{5 - 3cos(2\theta) + 2e^{\lambda_B t}(2 + e^{\lambda_B t})cos^2\theta}\Big)^{\frac{1}{2}}. \tag{5.69}$$

This gives for the correlation vector up to the 2^{nd} order of θ

$$\vec{c}_{mode\,B} = \begin{pmatrix} -e^{-\lambda_B t} \\ -e^{-\lambda_B t} - \frac{e^{-\lambda_B t}}{2}\frac{1 - e^{\lambda_B t}}{1 + e^{\lambda_B t}}\theta^2 \\ -1 - \frac{e^{-\lambda_B t}}{2}\frac{1 - e^{\lambda_B t}}{1 + e^{\lambda_B t}}(2 + e^{\lambda t})\theta^2 \end{pmatrix}. \tag{5.70}$$

In Fig.5.7 this decoherence mode is illustrated for $\theta = \frac{\pi}{2}$ where the corresponding correlation vector is $\vec{c}_{mode\,B} = (-e^{-\lambda_B t}, -e^{-\lambda_B t}, -e^{-\lambda_B t})^T$. The state is on the line of the Werner state. The decoherence paths for other angles are not linear in general due to the deviation of $c_{2,3}^B$ from the exponential function (Eq.(5.70)), therefore the plane of all reachable states of this mode has a little bulge. The largest deviation arises for $\theta = \frac{3\pi}{8}$.

For a complete geometric discussion we must not forget the decoherence **mode C** where the projection operators are rotated in two subsystems. If in both systems a rotation of $\frac{\pi}{2}$ is performed, the state evolves on the line

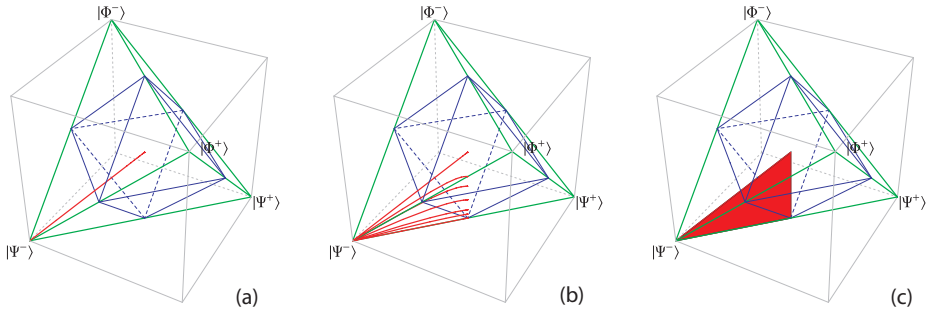


Figure 5.7: The decoherence mode B is represented by the red lines for different rotation angles θ. On the left side the evolution of the state is along the Werner state where the angle $\theta = \frac{\pi}{2}$. In (b) mode B is visualized for different θ-values in steps of $\frac{\pi}{12}$. Note that some lines are not linear, there is a little deviation depending on θ. On the right side the space approachable through mode B is illustrated by the red plane.

between the starting state $|\Psi^-\rangle$ and the other Bell state $|\Phi^-\rangle$, illustrated on the right side of Fig.5.8. If one uses also a phase shift of i and the rotation of $\frac{\pi}{2}$ in both subspaces then the system goes to another Bell state $|\Phi^+\rangle$, shown on the right side of Fig.5.8.

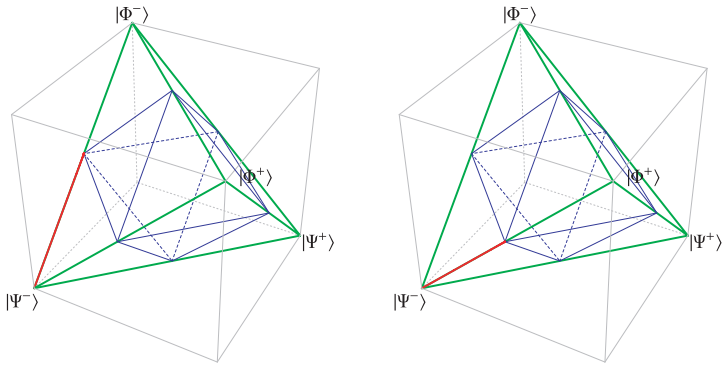


Figure 5.8: The red lines show decoherence paths for mode C, where the projection operators are rotated in both subsystems.

By tuning the decoherence modes in a proper way it is possible to reach any state in the picture starting from the respective Bell state.

5.2.5 Conclusion and unitary equivalence

We have considered a Markovian master equation where the Lindblad operators of the dissipator are given by projection operators. The various decoherence modes depended on the particular choice of the projections. We solved the arising differential equations and calculated the time evolution for different initial states. We startet our discussion with the Bell state $|\Psi^-\rangle$, followed by a set of pure states where we could adjust the starting amount of entanglement. Two extremal decoherence modes were found which build the upper and lower bounds of all possible evolutions of the system in the CP-diagram. Surprisingly these bounds were crossing when we started with a maximally entangled mixed state. This implies that any evolution of the state has to pass this crosspoint with given concurrence and purity.

At next we illustrated the different decoherence modes in the spin geometry picture and found that any state in the picture could be reached by an appropriate tuning of the modes and by the proper starting Bell state.

We have covered the whole CP-diagram with functions for mode A and mode B. Now we want to know if the evolution of a given state inside the diagram has again these functions as upper and lower borders or in other words: Are all the states with the same amount of entanglement and purity unitary equivalent? To answer this question we consider two states ρ_1 and ρ_2 which satisfy $C(\rho_1) = C(\rho_2)$ and $P(\rho_1) = P(\rho_2)$ and can be reached starting from two different pure states with mode A and mode B, illustrated by the solid green and red line in Fig.5.9. Now we "switch" from mode A to B and vice versa (dashed green and red line) and compare if the new modes are identical with the old ones (blackl line). We can see in Fig.5.9 that there is a difference between the modes which implies that the evolution depends on the particular state and is not always the same if we just fix the concurrence and the purity.

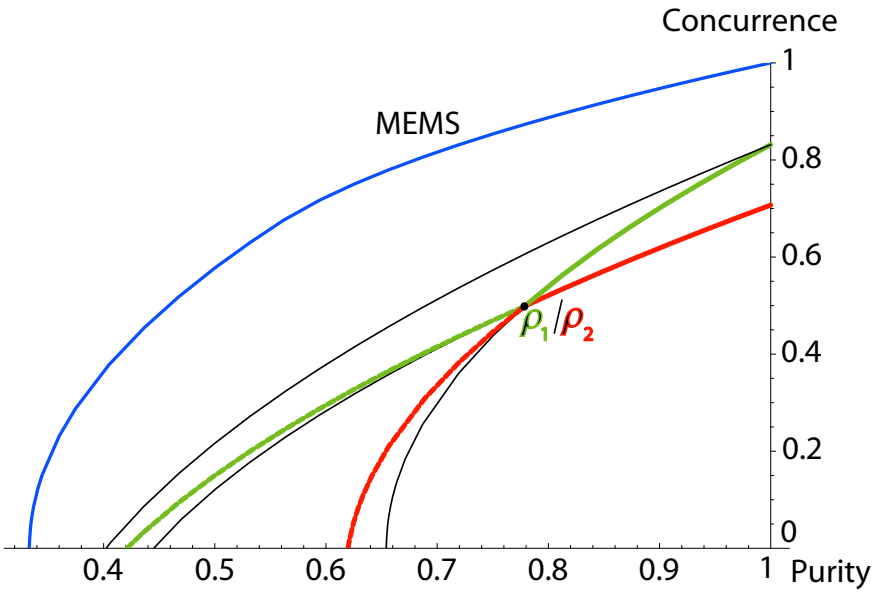


Figure 5.9: Mode A and B (solid green and red line) are switched when the states ρ_1 and ρ_2 are reached. These states have the same amount of concurrence and purity. Since the new (dashed lines) and old modes (black lines) are different, it seems that solely concurrence and purity does not define the evolution of the state under a certain decoherence.

Chapter 6

Decoherence of a bipartite system

By increasing the degrees of freedom of a bipartite quantum system to $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d$ we are confronted with several problems. In contrast to the two qubit case we have no measure of entanglement that holds for all properties a good measure should possess or its too complicated to work with. Even if we have a full analytical description of all possible decoherence modes, we would have a problem in characterising these states without a certain measure of entanglement. We could compare the behavior of the matrix elements under the different decoherence modes with the evolution of two qubits. For simplicity we will just focus on Markovian decoherence and start with a kind of Bell state for qutrits which should be maximally entangled and pure.

This chapter is split into two sections concerning different representations of the dissipator. In Sect.6.1 we consider again Lindblad operators which should be given by projections. We show that the calculations become far more complex even for two qutrits, although we will try to embed the different decoherence modes in the geometric picture for two qutrits, the magic simplex introduced in Sect.3.4. Then we present an analytical proof that the isotropic state is just reachable for states with an dimension of $d = 2^m$, where m is a natural number.
In the last part of this chapter we take a special set of Weyl operators instead of projection operators for the dissipator and we show that we can easily reach the isotropic states which suggests that the choice of projection operators is not fundamental enough. Further we are going to discuss the decoherence paths in the geometric picture.

6.1 Decoherence with projection operators

Analog to the previous chapter we are taking the dissipator in the projector form (see Sect.4.4.3) like

$$\mathcal{D}(\rho) = \lambda(\rho - \sum_{k=1}^{d^2} P_k \rho P_k) \tag{6.1}$$

and the evolution of the system is again given by

$$\frac{\partial}{\partial t}\rho(t) = -\mathcal{D}(\rho).$$

We start for simplicity with two qutrits and discuss what happens if mode A (Sect.6.1.2), mode B (Sect.6.1.3) or a mixture of both (Sect.6.1.4) is acting on the system. We recognize that we get in problems if we want to reach the isotropic state which is rather simply generated by a rotation of $\frac{\pi}{2}$ of the projector in the two qubit system. This problem is worked out explicitly for arbitrary dimensions in the last part of this section.

6.1.1 Notation

If we want to discuss a bipartite system with three degrees of freedom for each subsystem, we have to raise the dimension of the Hilbertspace to 9 according to $\mathcal{H} = \mathbb{C}^3 \otimes \mathbb{C}^3$. The three eigenstates of the subsystems are now described by 3 dimensional vectors. Analogical to the 2 qubit case we split the nine eigenstates of the Hamiltonian in the states of the subsystems like

$$\begin{aligned} |e_{1,4,7}\rangle &= |a_{1,2,3}\rangle|b_1\rangle \\ |e_{2,5,8}\rangle &= |a_{1,2,3}\rangle|b_2\rangle \\ |e_{3,6,9}\rangle &= |a_{1,2,3}\rangle|b_3\rangle, \end{aligned} \tag{6.2}$$

where the eigenstates of the first system are denoted by $\{|a_1\rangle, |a_2\rangle, |a_3\rangle\}$ and the states of the second one $\{|b_1\rangle, |b_2\rangle, |b_3\rangle\}$. For a general construction of the projection operators we again have to rotate the basis in one subsystem. There exist many different representations of a rotation in a 3-dimensional space and all of them have advantages and disadvantages. Here we want to use the most common representation with Euler angles which is of the form

$$R(\Psi, \Theta, \Phi) = R_Z(\Psi) R_x(\Theta) R_z(\Phi) =$$

$$\begin{pmatrix} \cos\Psi\cos\Phi - \sin\Psi\cos\Theta\sin\Phi & \cos\Psi\sin\Phi + \sin\Psi\cos\Theta\cos\Phi & \sin\Psi\sin\Theta \\ -\sin\Psi\cos\Phi - \cos\Psi\cos\Theta\sin\Phi & -\sin\Psi\sin\Phi + \cos\Psi\cos\Theta\cos\Phi & \cos\Psi\sin\Theta \\ \sin\Theta\sin\Phi & -\sin\Theta\cos\Phi & \cos\Theta \end{pmatrix} \tag{6.3}$$

where the vectors are rotated around the z-axis (angle Φ), then around x-axis (angle Θ) and finally around the z-axis again with an angle of Ψ. If we rotate one subsystem with the three Euler angles we get for the new vectors

$$\begin{pmatrix} |\alpha_1\rangle \\ |\alpha_2\rangle \\ |\alpha_3\rangle \end{pmatrix} = R(\Psi,\Theta,\Phi) \begin{pmatrix} |a_1\rangle \\ |a_2\rangle \\ |a_3\rangle \end{pmatrix} \tag{6.4}$$

So we can construct the set of the projection operators for a GR (general rotated) subspace like

$$P_k = |\hat{e}_k\rangle\langle\hat{e}_k|$$

$$\begin{pmatrix} |\hat{e}_{1,2,3}\rangle \\ |\hat{e}_{4,5,6}\rangle \\ |\hat{e}_{7,8,9}\rangle \end{pmatrix} = \begin{pmatrix} |\alpha_1\rangle|b_{1,2,3}\rangle \\ |\alpha_2\rangle|b_{1,2,3}\rangle \\ |\alpha_3\rangle|b_{1,2,3}\rangle \end{pmatrix} = R(\Psi,\Theta,\Phi)\cdot \begin{pmatrix} |e_{1,2,3}\rangle \\ |e_{4,5,6}\rangle \\ |e_{7,8,9}\rangle \end{pmatrix}. \tag{6.5}$$

In the next three sections we are reducing the rotation matrix $R(\Psi,\Theta,\Phi)$ since it is not possible to solve the equations analytically for all the angles Ψ,Θ and Φ.
For the initial state we choose the simplest possible entangled pure state which is given by

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle)$$

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{3}\begin{pmatrix} 1&0&0&0&1&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&1&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&1&0&0&0&1 \end{pmatrix}. \tag{6.6}$$

6.1.2 Mode A

We start with the simplest possible decoherence where we choose the euler angles to be zero. Under this assumption the projection operators are given by

$$P_k = |\hat{e}_k\rangle\langle\hat{e}_k| = |e_k\rangle\langle e_k| \tag{6.7}$$

The solution of the density matrix for the intial state Ψ (Eq.(6.6)) is analog to the 2 qubits characterized by the typical exponential decay of the off-diagonal elements of the state

$$\rho_{mode\,A} = \begin{pmatrix} 1 & 0 & 0 & 0 & e^{-\lambda_A t} & 0 & 0 & 0 & e^{-\lambda_A t} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ e^{-\lambda_A t} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & e^{-\lambda_A t} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ e^{-\lambda_A t} & 0 & 0 & 0 & e^{-\lambda_A t} & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.8}$$

Also in higher dimensional systems the entanglement is hidden in the off-diagonal elements of the density matrix if it is given in the computational basis. Therefore we can say that the state gets separable only for $t \to \infty$.

6.1.3 Mode B

Although we have already mentioned that an analytical solution for all Euler angles Ψ, θ and Φ is not possible, however, we want to analyse the general dissipator and the sets of differential equations.
If we express the rotated states $|\hat{e}_k\rangle$ by the eigenstates $|e_k\rangle$ of the system according to the eq.(6.5) we get

$$\begin{aligned} |\hat{e}_{1,2,3}\rangle &= |\alpha_1\rangle|b_{1,2,3}\rangle \\ &= (\cos\Psi\cos\Phi - \sin\Psi\cos\Theta\sin\Phi)\,|e_{1,2,3}\rangle \\ &+ (\cos\Psi\sin\Phi + \sin\Psi\cos\Theta\cos\Phi)\,|e_{4,5,6}\rangle \\ &+ \sin\Psi\sin\Theta\,|e_{7,8,9}\rangle \end{aligned} \tag{6.9}$$

where the first indeces belong together just as the second and so on. The second three states are

$$\begin{aligned} |\hat{e}_{4,5,6}\rangle &= |\alpha_2\rangle|b_{1,2,3}\rangle \\ &= (-\sin\Psi\cos\Phi - \cos\Psi\cos\Theta\sin\Phi)\,|e_{1,2,3}\rangle \\ &+ (-\sin\Psi\sin\Phi + \cos\Psi\cos\Theta\cos\Phi)\,|e_{4,5,6}\rangle \\ &+ \cos\Psi\sin\Theta\,|e_{7,8,9}\rangle \end{aligned} \tag{6.10}$$

and the last set of states is given by

$$\begin{aligned} |\hat{e}_{7,8,9}\rangle &= |\alpha_3\rangle|b_{1,2,3}\rangle \\ &= \sin\Theta\sin\Phi\ |e_{1,2,3}\rangle \\ &- \sin\Theta\cos\Phi\ |e_{4,5,6}\rangle \\ &+ \cos\Theta\ |e_{7,8,9}\rangle \end{aligned} \tag{6.11}$$

Note that every new state $|\hat{e}_k\rangle$ is a linear combination of just *three* eigenstates. If we would perform a rotation in both subspaces, mode C ($GR \otimes GR$), we would get a dependence of all eigenstates $|e_k\rangle$. But here we get a coupling of the states with indices $\{1,4,7\}$, $\{2,5,8\}$ and $\{3,6,9\}$ which is responsible for the particular structure of the dissipator given by

$$\mathcal{D}(\rho,\Psi,\Phi,\Theta) = \begin{pmatrix} \xi'_{11} & \rho_{12} & \rho_{13} & \xi'_{14} & \rho_{15} & \rho_{16} & \xi'_{17} & \rho_{18} & \rho_{19} \\ \rho_{21} & \xi''_{22} & \rho_{23} & \rho_{24} & \xi''_{25} & \rho_{26} & \rho_{27} & \xi''_{28} & \rho_{29} \\ \rho_{31} & \rho_{32} & \xi'''_{33} & \rho_{34} & \rho_{35} & \xi'''_{36} & \rho_{37} & \rho_{38} & \xi'''_{39} \\ \xi'_{41} & \rho_{42} & \rho_{43} & \xi'_{44} & \rho_{45} & \rho_{46} & \xi'_{47} & \rho_{48} & \rho_{49} \\ \rho_{51} & \xi''_{52} & \rho_{53} & \rho_{54} & \xi''_{55} & \rho_{56} & \rho_{57} & \xi''_{58} & \rho_{59} \\ \rho_{61} & \rho_{62} & \xi'''_{63} & \rho_{64} & \rho_{65} & \xi'''_{66} & \rho_{67} & \rho_{68} & \xi'''_{69} \\ \xi'_{71} & \rho_{72} & \rho_{73} & \xi'_{74} & \rho_{75} & \rho_{76} & \xi'_{77} & \rho_{78} & \rho_{79} \\ \rho_{81} & \xi''_{82} & \rho_{83} & \rho_{84} & \xi''_{85} & \rho_{86} & \rho_{87} & \xi''_{88} & \rho_{89} \\ \rho_{91} & \rho_{92} & \xi'''_{93} & \rho_{94} & \rho_{95} & \xi'''_{96} & \rho_{97} & \rho_{98} & \xi'''_{99} \end{pmatrix}. \tag{6.12}$$

The colors and primes of the functions ξ'_{ij},ξ''_{ij} and ξ'''_{ij} represent 3 different sets of couplings. These functions are dependent on the Euler angles and on certain matrix elements like

$$\xi'_{ij} = \xi'_{ij}(\Psi,\Phi,\Theta,\rho_{i'j'}) \qquad i,j,i',j' \in \{1,4,7\} \tag{6.13}$$

$$\xi''_{ij} = \xi''_{ij}(\Psi,\Phi,\Theta,\rho_{i'j'}) \qquad i,j,i',j' \in \{2,5,8\} \tag{6.14}$$

$$\xi'''_{ij} = \xi'''_{ij}(\Psi,\Phi,\Theta,\rho_{i'j'}) \qquad i,j,i',j' \in \{3,6,9\} \tag{6.15}$$

All other elements of the dissipator consist just of the respective density matrix element and therefore these particular elements of the physical system behaves analog to mode A with an exponential decay.
The 27 remaining elements are decoupling into 3 systems (ξ'_{ij},ξ''_{ij} , ξ'''_{ij}), where each system is a set of 9 coupled differential equations. We have focused on one system but it was still not possible to give an analytical solution. In addition even the functions of ξ'_{ij},ξ''_{ij} and ξ'''_{ij} are too complicated to present them here.
In the next section two of the Euler angles will be set to zero where the equations get simpler and solvable.

6.1.4 Rotation around an axis

Since we have no general solution of the system, we have to do simplifications. We set two of the Euler angles zero which means that we rotate the eigenstates of the subsystem around an axis. We choose a rotation around the 3^{rd} degree of freedom which means that $\theta = \Psi = 0$. This leads to the following rotation matrix

$$R(\Phi) = \begin{pmatrix} cos\Phi & -sin\Phi & 0 \\ sin\Phi & cos\Phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.16}$$

The rotated eigenstates are then given by

$$|\hat{e}_{1,2,3}\rangle = cos\Phi|e_{1,2,3}\rangle - sin\Phi|e_{4,5,6}\rangle \tag{6.17}$$
$$|\hat{e}_{4,5,6}\rangle = sin\Phi|e_{1,2,3}\rangle + cos\Phi|e_{4,5,6}\rangle \tag{6.18}$$
$$|\hat{e}_{7,8,9}\rangle = |e_{7,8,9}\rangle \tag{6.19}$$

Note that only the matrix elements $\{1,4\}$,$\{2,5\}$ and $\{3,6\}$ are coupled. This leads to the same differential equations which have been already solved for the 2 qubits, see Sect.5.2.2.
If the evolution starts with the maximally entangled Bell state , see eq.(6.6), we gain the density matrix

$$\rho_{mode\,A} = \begin{pmatrix} \rho_{11}(t) & 0 & 0 & \rho_{14}(t) & e^{-\lambda_A t} & 0 & 0 & 0 & e^{-\lambda_A t} \\ 0 & \rho_{22}(t) & 0 & 0 & \rho_{25}(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \rho_{41}(t) & 0 & 0 & \rho_{44}(t) & 0 & 0 & 0 & 0 & 0 \\ e^{-\lambda_A t} & \rho_{52}(t) & 0 & 0 & \rho_{55}(t) & 0 & 0 & 0 & e^{-\lambda_A t} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ e^{-\lambda_A t} & 0 & 0 & 0 & e^{-\lambda_A t} & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.20}$$

with

$$\rho_{11}(t) = \rho_{55}(t) = \frac{1}{6}(1 + 2cot^2(2\Phi) + e^{-\lambda t})sin^2(2\Phi) \tag{6.21}$$
$$\rho_{44}(t) = \rho_{22}(t) = \frac{1}{6}sin^2(2\Phi)(1 - e^{-\lambda t}) \tag{6.22}$$
$$\rho_{14}(t) = \rho_{41}(t) = \rho_{25}(t) = \rho_{52}(t) = -\frac{1}{12}sin(4\Phi)(1 - e^{-\lambda t}) \tag{6.23}$$

The form of the rotation matrix is directly reflected in the kind of evolution of the density matrix. The state evolutes in time like a mixture of mode B

and mode A for 2 qubits. All matrix elements with the unaffected 3^{rd} degree of freedom behaves like mode A, whereas the left subspace, which can be treated as a 2 qubit space ($\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$), is affected by the decoherence mode B with the rotation angle Φ.
Of course analog results are gained for a rotation around another axis or degree of freedom.

6.1.5 Embedding in the Weyl picture

Since the initial Bell state is included in the Weyl picture (see Sect.3.4), we can try to embed the different evolutions found in the previous sections. We begin with the decoherence mode A and try to calculate the cofficients c_{kl} for

$$\rho_{mode\,A} = \sum_{k,l=0}^{2} c_{kl}(t) P_{kl} \qquad \text{with} \qquad \sum_{kl=0}^{2} c_{kl} = 1. \tag{6.24}$$

If we compare the density matrix of mode A with the Bell type states (see Appendix A), we immediately see that all c_{k1} and c_{k2} are equal to zero. Therefore the only left 3 coefficients have to satisfy

$$\begin{aligned} c_{00} + \omega\, c_{10} + \omega^2 c_{20} &= e^{-\lambda_A t} && (6.25)\\ c_{00} + \omega^2 c_{10} + \omega^4 c_{20} &= e^{-\lambda_A t} && (6.26)\\ c_{00} + c_{10} + c_{20} &= 1, && (6.27) \end{aligned}$$

where $\omega = e^{\frac{2\pi i}{3}}$. The solution is given by

$$\left.\begin{array}{l} c_{00}(t) = \frac{1}{3}(1 + 2e^{-\lambda_A t}) \\ c_{10}(t) = \frac{1}{3}(1 - e^{-\lambda_A t}) \\ c_{20}(t) = \frac{1}{3}(1 - e^{-\lambda_A t}) \end{array}\right\}_{t=0} \quad \left.\begin{array}{l} c_{00}(0) = 1 \\ c_{10}(0) = 0 \\ c_{20}(0) = 0 \end{array}\right\} \quad \left.\begin{array}{l} c_{00} = \frac{1}{3} \\ c_{10} = \frac{1}{3} \\ c_{20} = \frac{1}{3} \end{array}\right\}_{t\to\infty} \tag{6.28}$$

We have also given the coefficients for $t = 0$ and $t \to \infty$. The state can be written as

$$\rho_{mode\,A} = \frac{1}{3}(1 + 2e^{-\lambda_A t})P_{00} + \frac{1}{3}(1 - e^{-\lambda_A t})P_{10} + \frac{1}{3}(1 - e^{-\lambda_A t})P_{20} \tag{6.29}$$

It starts at the maximally entangled state P_{00} and ends for $t \to \infty$ in σ_b which is separable and an equal mixture of all P_{k0}. The evolution of the state is illustrated in Fig.6.1.

The question that arises now is what would happen if the system is initially in the state $\frac{1}{2}(P_{10} + P_{20})$ and affected by mode A. We get the solution

$$\rho_{mode\,A} = \frac{1}{3}(1 - e^{-\lambda_A t})P_{00} + \frac{1}{6}(2 + e^{-\lambda_A t})P_{10} + \frac{1}{6}(2 + e^{-\lambda_A t})P_{20}, \tag{6.30}$$

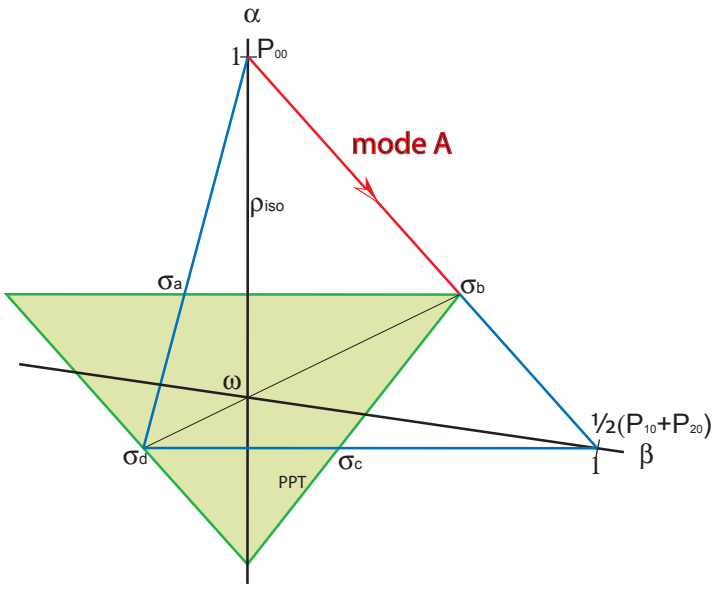


Figure 6.1: Decoherence mode A starting at the Bell state P_{00} and reaching the point σ_b for $t \to \infty$.

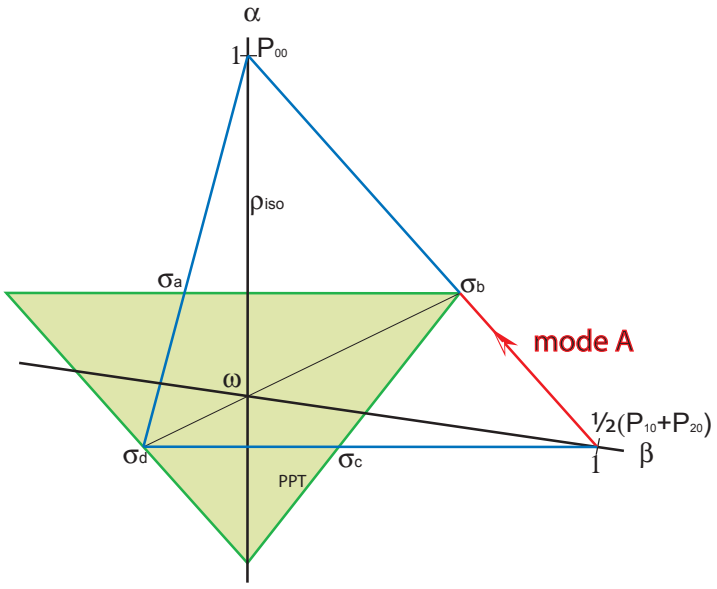


Figure 6.2: The evolution of the initial state $\frac{1}{2}(P_{10} + P_{20})$ ends for $t \to \infty$ in the state σ_b under mode A.

which is displayed in Fig.6.2.

If we want to embed the decoherence mode where we rotate around an axis, we have the problem that the created off-diagonal elements $\rho_{14}(t) = \rho_{41}(t) = \rho_{25}(t) = \rho_{52}(t)$ given by Eq.(6.23) cannot be represented by any P_{kl}. However, by setting the rotation angle $\phi = \frac{\pi}{4}$ no off-diagonal elements are generated and we have to claim for the coefficients according to the diagonal elements

$$c_{00} + c_{10} + c_{20} = 1 \tag{6.31}$$
$$c_{00} + c_{10} + c_{20} = 1 + e^{-\lambda t} \tag{6.32}$$

Since the equations cannot be solved, the evolution of the system under this decoherence mode has no representation in the Weyl picture.
Although all decoherence modes could be embedded in the spin geometry picture, it seems that the choice of projections for the Lindblad operators was not very beneficial. On the one hand we have problems solving the differential equations and on the other hand the gained decoherence modes cannot be illustrated in the Weyl picture. Nevertheless we have found a proof, presented in the next section, that the isotropic state can only be created in certain dimensions under Markovian decoherence with projection operators.

6.1.6 Isotropic states in arbitrary dimensions

We have found a certain decoherence mode for two qubits that generates the Werner state which means that the initial Bell state directly evolutes to the maximally mixed state, the identity. Although we used many different strategies to find this special decoherence mode also for the two qutrit system, we failed. We have expected that the isotropic state can easily be generated with projections due to the high symmetry of the state, but we will present a proof that such a decoherence mode can only be found for systems with $d = 2^m$ where $m \in \mathbb{N}$. We have considered systems where both subspaces have the same dimension d.
Since we have to use a more general notation, we will repeat some definitions and equations.
The isotropic state ρ_{iso} is defined by a mixture of the identity and the Bell state like

$$\rho_{iso} = \frac{1-\alpha}{d^2} I + \alpha |\Phi_+^d\rangle\langle\Phi_+^d| \qquad \alpha = e^{-\lambda t}, \tag{6.33}$$

where α is depending on the decoherence parameter λ and the time t. The state $|\Phi_+^d\rangle$ is the maximally entangled Bell state and given by

$$|\Phi_+^d\rangle = \frac{1}{\sqrt{d}} \sum_{s=0}^{d-1} |s\rangle \otimes |s\rangle. \tag{6.34}$$

In order to find conditions for the projection operators we have to claim that the isotropic state is a solution of the Markovian master equation where the unitary part of the evolution is neglected.,

$$\frac{d\rho_{iso}}{dt} = -\mathcal{D}\left(\rho_{iso}\right). \tag{6.35}$$

We use again the dissipator in the projector presentation which gives for ρ_{iso}

$$\mathcal{D}\left(\rho_{iso}\right) = \lambda \left(\rho_{iso} - \sum_{k=0}^{d^2-1} P_k \rho_{iso} P_k \right). \tag{6.36}$$

After substituting the isotropic state and some trivial calculations we get the relation

$$\sum_{k=0}^{d^2-1} P_k \left(|\Phi_+^d\rangle\langle\Phi_+^d| \right) P_k = \frac{1}{d^2} \mathbb{1}, \tag{6.37}$$

which has to be satisfied for a set of projection operators if the isotropic state is a solution of the Markovian decoherence.

Choice of Projection Operators

The main goal of this section is whether we can find a set of projection operators which can solve the condition given in Eq.(6.37). For this purpose we need to define a basis which should be given by the eigenstates of the Hamiltonian such as

$$H|e_k\rangle = E_k|e_k\rangle. \tag{6.38}$$

We can now split this basis for the bipartite system into a tensorproduct of the two subsystems like

$$|e_{di+j}\rangle = |a_i\rangle|b_j\rangle, \tag{6.39}$$

where $d = dim(\mathcal{H}_A) = dim(\mathcal{H}_B)$ and the indices i and j are running from 0 to $d-1$. Then the eigenstates satisfy

$$\langle e_{dk+l}|e_{di+j}\rangle = \delta_{ki}\delta_{lj}. \tag{6.40}$$

In order to express the projection operators in a general basis $|\hat{e}_k\rangle$ it is sufficient to rotate the set in one subspace so that the projection basis is given by

$$|\hat{e}_{dk+j}\rangle = \sum_{i=0}^{d-1} r_{ki}|a_i\rangle|b_j\rangle = \sum_{i=0}^{d-1} r_{ki}|e_{di+j}\rangle \tag{6.41}$$

This gives us for the projection operators

$$P_{dk+j} = |\hat{e}_{dk+j}\rangle\langle\hat{e}_{dk+j}| = \sum_{i,l=0}^{d-1} r_{ki}|e_{di+j}\rangle\langle e_{dl+j}|r_{kl}, \tag{6.42}$$

which satisfy the properties

$$P^2 = P \qquad and \qquad \sum_{k=0}^{d^2} P_k = \mathbb{1}, \tag{6.43}$$

and consequentially r_{ij} fulfills

$$\sum_{i=0}^{d-1} r_{ij}r_{ik} = \delta_{jk}. \tag{6.44}$$

If we define the matrix R by

$$R = \begin{pmatrix} r_{00} & r_{01} & \cdots & r_{0(d-1)} \\ \vdots & \vdots & \ddots & \vdots \\ r_{(d-1)0} & r_{(d-1)1} & \cdots & r_{(d-1)(d-1)} \end{pmatrix}, \tag{6.45}$$

we get from the projector properties the relation

$$RR^T = \mathbb{1}, \tag{6.46}$$

which characterizes a rotation matrix in d dimensions.
Further we substitute expression (6.42) and the definition of the maximally entangled state, Eq.(6.34), in Eq.(6.37) and get

$$\textstyle\sum_{k,j=0}^{d-1} P_{dk+j} \left(\sum_{s',s=0}^{d-1} |e_{ds+s}\rangle\langle e_{ds'+s'}|\right) P_{dk+j} = \frac{1}{d}\mathbb{1} \tag{6.47}$$

$$\textstyle\sum_{i,j,k,l,m,n,s,s'=0}^{d-1} r_{ki}|e_{di+j}\rangle \underbrace{\langle e_{dl+j}|r_{kl}|e_{ds+s}\rangle}_{r_{kl}\delta_{ls}\delta_{js}} \underbrace{\langle e_{ds'+s'}|r_{km}|e_{dm+j}\rangle}_{r_{km}\delta_{ms'}\delta_{js'}} \langle e_{dn+j}|r_{kn} = \frac{1}{d}\mathbb{1} \tag{6.48}$$

After summerizing over j,l,m and s' we get

$$\sum_{i,k,n,s=0}^{d-1} r_{ks}^2 r_{ki} r_{kn} |e_{di+s}\rangle\langle e_{dn+s}| = \frac{1}{d}\mathbb{1}, \tag{6.49}$$

which gives us the condition

$$\sum_{k=0}^{d-1} r_{ks}^2 r_{ki} r_{kn} = \frac{1}{d}\delta_{in}. \tag{6.50}$$

Now we are defining a new set of matrices M_s with

$$(M_{kn})_s \doteq r_{ks} r_{kn} = \begin{pmatrix} r_{00}r_{0s} & r_{01}r_{0s} & \cdots & r_{0d-1}r_{0s} \\ \vdots & \vdots & \ddots & \vdots \\ r_{d-1,0}r_{d-1s} & r_{d-1,1}r_{d-1s} & \cdots & r_{d-1,d-1}r_{d-1s} \end{pmatrix}, \tag{6.51}$$

where we can express the condition given by Eq.(6.50) with this matrix in the following way

$$M^T M = \sum_{k=0}^{d-1} r_{ks}^2 r_{ki} r_{kn} = \frac{1}{d}\delta_{in}. \tag{6.52}$$

We transpose both sides of the equation and get

$$MM^T \stackrel{!}{=} \frac{1}{d}\delta_{ni} \tag{6.53}$$

$$MM^T = \sum_{k=0}^{d-1} r_{ns} r_{nk} r_{is} r_{ik} = \frac{1}{d}\delta_{ni} \tag{6.54}$$

If we use the projection properties of Eq. (6.44) we get

$$r_{ns} r_{is} \delta_{ni} = \frac{1}{d}\delta_{ni} \tag{6.55}$$

$$\Rightarrow r_{ns}^2 = \frac{1}{d} \tag{6.56}$$

$$r_{ns} = \pm\frac{1}{\sqrt{d}} \tag{6.57}$$

All elements of the rotation matrix R and of the matrix M have the same absolute value. The only choice which is left is the sign of the elements. The question is whether we could choose the signs in that way that Eqs. (6.44) and (6.50) are satisfied.
To solve this question we define a set of new vectors $\vec{q}$ like

$$\vec{q_i} = (q_k)_i := r_{k0} r_{ki} = (M_{ki})_0 \tag{6.58}$$

According to Eq. (6.50) the scalar product of the vectors have to satisfy the relation

$$\vec{q}_i \cdot \vec{q}_n = \frac{1}{d}\delta_{in} \tag{6.59}$$

We have to find d different orthogonal vectors $\vec{q}_i$ whose elements have all the same absolute value. This certainly does not work in odd dimensions, so that we can claim $d \equiv 2m$ with $m \in \mathbb{N}$.
To check if such vectors can be found for any even dimension, we choose the signs of the first vector elements positive without loss of generality. Since the vector lives in the space $\mathbb{R}^d$ with the Cartesian coordinates $\{x_1, x_2, \ldots, x_d\}$, all other orthogonal vectors have to satisfy the equation

$$+x_1 + x_2 + x_3 + \ldots + x_d = 0. \tag{6.60}$$

Half the elements of the next vector must be negative and gives also the following condition for all other vectors

$$-x_1 - x_2 - \ldots - x_{\frac{d}{2}} + x_{\frac{d}{2}+1} + \ldots + x_d = 0. \tag{6.61}$$

All the $d-2$ left vectors have to satisfy both conditions and therefore

$$+x_1 + x_2 + \ldots + x_{\frac{d}{2}} = 0. \tag{6.62}$$

This equation can only be fulfilled if $\frac{d}{2}$ is odd. The whole procedure can be repeated till the last condition is

$$+x_1 + x_{\frac{d}{2^n}} = 0 \qquad \text{with} \qquad \frac{d}{2^n} = 2, \tag{6.63}$$

where $n \in \mathbb{N} \cup \{0\}$. Only for $d = 2^{n+1}$ we can find d different orthogonal vectors $\vec{q}_i$.

We have proven that it is only possible for systems with Hilbert spaces of the form $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d$ with $d = 2^m$ and $m \in \mathbb{N}$ to generate the isotropic state through Markovian decoherence with projections as Lindblad operators starting with the Bell state.

6.2 Decoherence with Weyl operators

Since the geometric picture for two qutrits is constructed with Weyl operators and we are not able to reach all the states with decoherence described with projection operators, see previous chapter, it is reasonable to use Weyl operators for the Lindblad terms. We are going to present a specific ansatz for two qutrits and discuss which possibilities we have in reducing the system to get analytical solutions. Although we show that we can quite simply construct the isotropic state, it is still not possible to reach any state in the picture.

6.2.1 Notation and master equation

We take for the Lindblad operators the ansatz

$$A_j := \sqrt{\lambda d_{nm}} W_{nm} \tag{6.64}$$

Due to the property of Weyl operators given by Eq. (3.21) the Lindblad operators satisfy

$$A_j^\dagger A_j = \lambda d_{nm}. \tag{6.65}$$

Then the dissipator reduces to

$$\begin{aligned} \mathcal{D}(\rho) &= \frac{1}{2}\sum_j \left(A_j^\dagger A_j \rho + \rho A_j^\dagger A_j - 2 A_j \rho A_j^\dagger \right) && (6.66) \\ &= -\lambda\rho + \sum_j A_j \rho A_j^\dagger && (6.67) \end{aligned}$$

For the simplification of Eq. (6.66) to Eq. (6.67) the weightings d_{nm} of the operators have to fulfill the condition

$$\sum_{n,m} d_{nm} = 1. \tag{6.68}$$

Futhermore we get for the dissipator if we allow only density matrices which are in the picture and hence of the form $\rho(t) = \sum_{n,m} c_{kl}(t) P_{kl}$

$$\mathcal{D}(\rho) = \lambda \sum_{n,m} c_{kl} P_{kl} - \lambda \sum_{n,m,k,l} d_{nm} c_{kl} W_{nm} P_{kl} W_{nm}^\dagger. \tag{6.69}$$

With

$$W_{nm} P_{kl} W_{nm}^\dagger = P_{n+k\; m+l} \tag{6.70}$$

we get for the master equation

$$\dot{\rho}(t) = -\lambda \sum_{n,m} c_{kl}(t) P_{kl} + \lambda \sum_{n,m,k,l} d_{nm} c_{kl}(t) P_{n+k\ m+l}. \tag{6.71}$$

If we want to express the master equation in terms of the coefficients $c_{kl}(t)$ we get

$$\frac{\dot{c}_{kl}(t)}{\lambda} + c_{kl}(t) = \sum_{n,m} d_{nm} c_{k-n\ l-m}(t). \tag{6.72}$$

Since this set of 9 coupled, first order differential equations cannot be solved analytically, we have to simplify the equations. We discuss in detail three possible methods in the next section.

6.2.2 Simplifications

To simplify the set of 9 coupled, first order differential equations (Eq. (6.72)), we present three different possible methods.

1. **Reduction of $\mathbf{c}_{kl}(t)$:**

   The 9 coefficients c_{kl} with the condition $\sum c_{kl} = 1$ draw a 8-dimensional picture but we need not be interested in the time evolution of each coefficient. It is sufficient to consider only a special set of states. Since we have already worked out two 2 dimensional pictures in Sect.3.4, we can use these special sets of states and still describe all possible decoherence modes created by the Weyl ansatz. For example we can set

   $$\begin{aligned} c_{01}(t) = c_{11}(t) = c_{21}(t) = c_{02}(t) = c_{12}(t) = c_{22}(t), \\ c_{10}(t) = c_{20}(t), \\ c_{00}(t). \end{aligned} \tag{6.73}$$

   This method simplifies the equations a lot without any loss of decoherence modes.

2. **Reduction of $\mathbf{d}_{nm}$:**

   By reducing the weights d_{nm} one have to be very tricky because you run the risk of loosing the whole set of possible decoherence modes. One alternative is to reduce them according to the use of a particular set of states

3. **Specific time evolution:**

   Another possible way of eliminating the differential equations is to specify how the coefficients $c_{kl}(t)$ explicitly depend on time. If we are only interested in linear solutions in the geometric picture, the coefficients must be of the form

   $$c_{kl}(t) = c'_{kl} + c''_{kl}e^{-\lambda t}, \tag{6.74}$$

   where c'_{kl} and c''_{kl} are constant in time. With that assumption the master equation reduces to a set of 9 ordinary equations which are far more easier to solve. Nevertheless not all decoherence modes are described

6.2.3 Reduced master equation

We try to reduce the master equation (Eq. (6.72)) with the possible simplifications discussed in the previous section. Since the question we are heading on is whether we could reach any state in the geometric picture starting with a maximally entangled state and since the allowed borders of the physical states, given through the positivity, are linear curves in the picture, it is sufficient just to search for linear decoherence modes. These modes should have the form

$$c_{kl}(t) = c'_{kl} + c''_{kl}e^{-\lambda t}, \tag{6.75}$$

where the new coefficients are constant in time. By explicitly giving the time evolution of the system we get two new time invariant master equations for c'_{kl} and c''_{kl}

$$\sum_{n,m} c'_{k-n,l-m}d_{nm} = c'_{kl}, \tag{6.76}$$

$$\sum_{n,m} c''_{k-n,l-m}d_{nm} = 0. \tag{6.77}$$

Since the normalization condition of the density matrix is given by

$$\sum_{k,l} c_{kl}(t) = 1, \tag{6.78}$$

we have for the new constant coefficients

$$\sum_{k,l} c'_{kl} = 1 \quad and \quad \sum_{k,l} c''_{kl} = 0. \tag{6.79}$$

Since it is still not easy to solve these two sets of 9 coupled equations, we do another simplification and consider a specific set of states. It is given together with the normalization (Eq. (6.79)) by

$$\begin{gathered} c'_{01} = c'_{11} = c'_{21} = c'_{02} = c'_{12} = c'_{22} = \tfrac{1}{6}(1 - 2c'_{10} - c'_{00}) \\ c'_{10} = c'_{20} \\ c'_{00}, \end{gathered} \tag{6.80}$$

and

$$\begin{gathered} c''_{01} = c''_{11} = c''_{21} = c''_{02} = c''_{12} = c''_{22} = \tfrac{1}{6}(-2c''_{10} - c''_{00}) \\ c''_{10} = c''_{20} \\ c''_{00}. \end{gathered} \tag{6.81}$$

To see how the equations look like we want to present them explicitly for c''_{kl}

$$\begin{aligned} -6\,c''_{00}d_{00} - 6\,c''_{10}d_{10} - 6\,c''_{10}d_{20} + \xi d_{01} + \xi d_{11} + \xi d_{21} + \xi d_{02} + \xi d_{12} + \xi d_{22} = 0 \\ -6\,c''_{10}d_{00} - 6\,c''_{00}d_{10} - 6\,c''_{10}d_{20} + \xi d_{01} + \xi d_{11} + \xi d_{21} + \xi d_{02} + \xi d_{12} + \xi d_{22} = 0 \\ -6\,c''_{10}d_{00} - 6\,c''_{10}d_{10} - 6\,c''_{00}d_{20} + \xi d_{01} + \xi d_{11} + \xi d_{21} + \xi d_{02} + \xi d_{12} + \xi d_{22} = 0 \\ \xi d_{00} + \xi d_{10} + \xi d_{20} - 6\,c''_{00}d_{01} - 6\,c''_{10}d_{11} - 6\,c''_{10}d_{21} + \xi d_{02} + \xi d_{12} + \xi d_{22} = 0 \\ \xi d_{00} + \xi d_{10} + \xi d_{20} - 6\,c''_{10}d_{01} - 6\,c''_{00}d_{11} - 6\,c''_{10}d_{21} + \xi d_{02} + \xi d_{12} + \xi d_{22} = 0 \\ \xi d_{00} + \xi d_{10} + \xi d_{20} - 6\,c''_{10}d_{01} - 6\,c''_{10}d_{11} - 6\,c''_{00}d_{21} + \xi d_{02} + \xi d_{12} + \xi d_{22} = 0 \\ \xi d_{00} + \xi d_{10} + \xi d_{20} + \xi d_{01} + \xi d_{11} + \xi d_{21} - 6\,c''_{00}d_{02} - 6\,c''_{10}d_{12} - 6\,c''_{10}d_{22} = 0 \\ \xi d_{00} + \xi d_{10} + \xi d_{20} + \xi d_{01} + \xi d_{11} + \xi d_{21} - 6\,c''_{10}d_{02} - 6\,c''_{00}d_{12} - 6\,c''_{10}d_{22} = 0 \\ \xi d_{00} + \xi d_{10} + \xi d_{20} + \xi d_{01} + \xi d_{11} + \xi d_{21} - 6\,c''_{10}d_{02} - 6\,c''_{10}d_{12} - 6\,c''_{00}d_{22} = 0 \end{aligned}$$

where

$$\xi = c''_{00} + 2\,c''_{10} \tag{6.82}$$

After substracting one equation of another we get three conditions for the decoherence parameters d_{nm}

$$d_{00} = d_{10} = d_{20} \quad := a \tag{6.83}$$

$$d_{01} = d_{11} = d_{21} \quad := b \tag{6.84}$$

$$d_{02} = d_{12} = d_{22} \quad := c \tag{6.85}$$

and the new parameters a,b and c according to Eq. (6.68) have to satisfy

$$a + b + c = \frac{1}{3}. \tag{6.86}$$

Now we want to summarize the results we gained through the simplifications. At first we started to give the explicit time dependence of the coefficients $c_{kl}(t)$ and reduced the coupled differential equations to ordinary

coupled equations for to new time invariant constants c'_{kl} and c''_{kl}. Afterwards we chose a specific set of states and obtained the result that several decoherence parameters have to have the same value. With this results we are going to solve the equations in the next section

6.2.4 Solutions of the reduced master equation

First mode

The first solution is analog to the decoherence mode A and generated by the weights d_{nm}

$$d_{nm} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6.87}$$

This immediately leads to the solution

$$\begin{aligned} c_{00}(t) &= \frac{1}{3}(c_{00}(0) + 2c_{10}(0) + 2(c_{00}(0) - c_{10}(0))e^{-\lambda t}) && (6.88)\\ c_{10}(t) = c_{20}(t) &= \frac{1}{3}(c_{00}(0) + 2c_{10}(0) - (c_{00}(0) - c_{10}(0))e^{-\lambda t}) && (6.89)\\ c_{l1}(t) = c_{k2}(t) &= c_{l1}(0) = c_{k2}(0) \qquad \text{for } \forall\, k,l \in \{0,1,2\} && (6.90) \end{aligned}$$

for any initial state in this set. We can transform the solution in the coordinates α and β used for the geometric picture. The transformation can be written as

$$\left.\begin{aligned} c_{00} &= \tfrac{1+8\alpha-\beta}{9} \\ c_{10} = c_{20} &= \tfrac{2-2\alpha+7\beta}{18} \end{aligned}\right\} \begin{aligned} \alpha &= \tfrac{7c_{00}+2c_{10}-1}{6} \\ \beta &= \tfrac{c_{00}+8c_{10}-1}{3} \end{aligned} \tag{6.91}$$

The time evolution of the new coordinates is then

$$\begin{aligned} \alpha(t) &= \frac{1}{3}(\alpha_0 + \beta_0 + (2\alpha_0 - \beta_0)e^{-\lambda t}) && (6.92)\\ \beta(t) &= \frac{1}{3}(2\alpha_0 + 2\beta_0 + (-2\alpha_0 + \beta_0)e^{-\lambda t}) && (6.93) \end{aligned}$$

where the initial state is characterized by α_0 and β_0. We can also express the coordinate α as a function of β which gives

$$\alpha(\beta) = (\alpha_0 + \beta_0) - \beta. \tag{6.94}$$

In Fig.6.3 the first decoherence mode is illustrated for different initial states. All decoherence paths are parallel and the evolution of any state ends for $t \to \infty$ in a mixture of the identity ω and σ_b. All states get separable in finite times, except the two initial states with $\alpha = 1$ or $\beta = 1$.

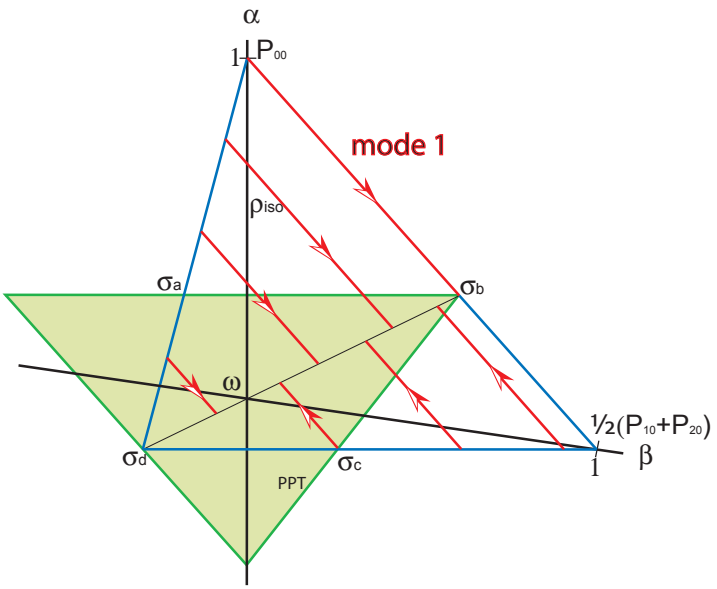


Figure 6.3: The first solution of the Markovian master equation with a specific Weyl ansatz for different initial states.

Second mode

We will see that the second solution can also generate the isotropic state. It is given by the parameters

$$d_{nm} = \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{6.95}$$

which lead to

$$c_{00}(t) = \frac{1}{9} + (c_{00}(0) - \frac{1}{9})e^{-\lambda t} \tag{6.96}$$

$$c_{10}(t) = c_{20}(t) = \frac{1}{9} + (c_{10}(0) - \frac{1}{9})^{-\lambda t} \tag{6.97}$$

$$c_{l1}(t) = c_{k2}(t) = \frac{1}{9} + (-\frac{1}{6}c_{00}(0) - \frac{1}{3}c_{10}(0) + \frac{1}{18})e^{-\lambda t} \tag{6.98}$$

where $l, k \in \{0, 1, 2\}$. We can transform the solution in the coordinates

$$\alpha(t) = \alpha_0 e^{-\lambda t} \tag{6.99}$$

$$\beta(t) = \beta_0 e^{-\lambda t} \tag{6.100}$$

with the implicit form

$$\alpha(\beta) = \frac{\alpha_0}{\beta_0}\beta. \tag{6.101}$$

This decoherence mode let the coordinates of the states exponentially decay. Thus all states are ending in the maximally mixed state ω for $t \to \infty$, which is graphically shown for different initial states in Fig.6.4. If we start for example with the Bell state P_{00}, this decoherence mode generates the isotropic state which was not possible with projection operators for 2 qutrits.

It seems that the choice of Weyl operators is much better than the projection

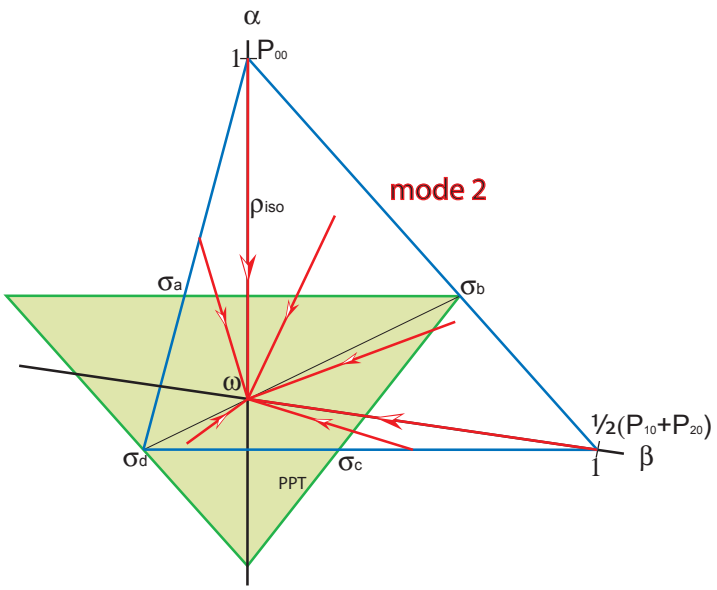


Figure 6.4: All decoherence paths for different initial states end in the maximally mixed state ω for infinite times

representation especially for describing the decoherence in this geometric picture.

Chapter 7

Conclusion and Outlook

Based on the work of Durstberger [20] we have studied the evolution of different bipartite quantum systems coupled to a Markovian environment. This dynamic is described with a master equation constructed by arbitrary Lindblad operators which were chosen to be projection operators and Weyl operators.

At first we considered the simplest possible bipartite system consisting of 2 qubits, whose decoherence should be generated by projection operators. We found two extremal modes in the CP-diagram, mode A and mode B with a projection rotation of $\theta = \frac{\pi}{2}$, which give us upper and lower borders for all possible evolutions of initially pure states. Surprisingly these borders are crossing in the diagram if the system starts at certain maximally entangled mixed states (MEMS). All other decoherence modes, varied by the rotation angle θ, are still between the two extremal modes which implies that *all* modes pass this crosspoint. A state starting at such MEMS will pass this point independent of the particular decoherence mode. All these states on the crosspoint produced by the different modes have the same amount of concurrence and purity.

Further we have shown that for starting density matrices with the same concurrence and purity a certain decoherence mode can vary. This means that the properties concurrence and purity alone are not enough for a full characterisation of a density matrix. There must be some other informations left in the state.

By investigating higher dimensional systems we were running in problems because even for qutrits an analytical solution could not be found. A typical decoherence process could be the evolution of a maximally entangled state to the identity. Therefore we analysed whether such an isotropic state could

be generated by Markovian decoherence for bipartite systems of the form $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d$. Surprisingly we found that starting with a Bell state it is not possible to generate the isotropic state in all dimensions with projection operators. In fact it is only possible for $d = 2^m$ where $m \in \mathbb{N}$. This result suggests that the choice of projection operators is too specific for a good characterisation of Markov decoherence especially for systems with higher degrees of freedom.

Finally we considered a two qutrit system and because of the fact that we have a powerful geometric picture constructed with Weyl operators, we used these operators in the Lindblad master equation. We found two different solutions for the equations of motion and illustrated them in the geometric picture for different initial states. These two solutions are analogous to mode A and mode B with $\theta = \frac{\pi}{2}$ for two qubits. In fact one of these solutions gives exactly the isotropic state if the system is initially in a Bell state. It seems that the use of Weyl operators instead of projection operators provides us with a more comprehensive set of decoherence modes.

For future investigations it would be more convenient to decompose the general Lindblad operators in terms of shift operators to get sure that no decoherence modes are lost. But the choice of Weyl operators is quite justified for the description of two qutrit systems since the geometric picture is constructed with this operators and includes many interesting states. Further investigations will reveal whether it is possible to reach the physically interesting regions of bound entangled states with the help of Markov decoherence.

Appendix A

Bell type states of the magic simplex

The 9 Bell type states P_{kl} of the magic simplex in the computational basis $\{|00\rangle, |01\rangle, |02\rangle, |10\rangle, |11\rangle, |12\rangle, |20\rangle, |21\rangle, |22\rangle\}$ are given by

$$P_{kl} = \frac{1}{3} \sum_{s,t=0}^{2} w^{k(s-t)} |s-l, s\rangle\langle t-l, t| \qquad \text{with } \omega = e^{\frac{2\pi i}{3}}, \tag{A.1}$$

where the indices $\{k, l, s, t\}$ are understood as modulo 3. The explicit form of the Bell type states is then

$$P_{k\,0} = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 & 0 & \omega^k & 0 & 0 & 0 & \omega^{2k} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \omega^{2k} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \omega^k \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \omega^k & 0 & 0 & 0 & \omega^{2k} & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A.2}$$

$$P_{k\,1} = \frac{1}{3} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \omega^k & \omega^{2k} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega^{2k} & 0 & 0 & 0 & 1 & \omega^k & 0 & 0 \\ 0 & \omega^k & 0 & 0 & 0 & \omega^{2k} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.3}$$

$$P_{k\,2} = \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \omega^k & 0 & 0 & 0 & \omega^{2k} & 0 \\ 0 & 0 & \omega^{2k} & 1 & 0 & 0 & 0 & \omega^k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^k & \omega^{2k} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.4}$$

It is clear that the linear combination of this states cannot describe the whole Hilbert space of two qutrits but also this simplex contains features of the entanglement that cannot be found for two qubits, e.g. the bound entangled states.

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