# Polyhomogeneous expansions at the boundary for some blowing-up solutions of a class of semi-linear elliptic equations

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Let $g$ be mooth Riemannian metric on a smooth compact manifold with boundary $\overline{M}$, consider the problem

$$\Delta_g \varphi = \mid \varphi \mid^{\alpha-1} \varphi \;, \qquad \alpha > 1 \;, \tag{0.1}$$

$$\lim_{p \to \partial M} \varphi(p) = \infty \quad , \tag{0.2}$$

$\Delta_g$ -the Laplace-Beltrami operator on functions. It was shown in [2] that one then necessarily has (*cf.* also [4])

$$\lim_{p \to \partial M} \left[ \varphi(p) \;\, x(p)^{\frac{2}{\alpha-1}} \right] = C_\alpha \equiv \left( \frac{2(\alpha+1)}{(\alpha-1)^2} \right)^{\frac{1}{\alpha-1}} \quad . \tag{0.3}$$

Here $x(p)$ denotes the Riemannian distance from $p$ to $\partial M$. It follows from the results of [1] that the behavior of $\varphi$ near $\partial M$ can be described in a much more precise way. More precisely, we have the following:

**Theorem 0.1** *Let $(\overline{M}, g)$ be a smooth compact Riemannian manifold with boundary, let $\alpha > 1$, set $M = int\overline{M}$. Let $\varphi \in W_2^{p,loc}(M)$ satisfy*

$$\Delta_g \varphi = \mid \varphi \mid^{\alpha-1} \varphi,$$

$$\lim_{p \to \partial M} \left[ \varphi(p) \;\, x(p)^{\frac{2}{\alpha-1}} \right] = C_\alpha,$$

*where $x(p) = dist(p, \partial M)$. Define*

$$\mu_+ = \frac{2(\alpha+1)}{\alpha - 1}.$$

*Let $\overline{N}$ be a compact submanifold of $\overline{M}$ forming a neighborhood of $\partial M$ such that $\varphi \mid_{\overline{N}} > 0$, and $x \in C^\infty(\overline{N})$. There exist functions $\varphi_i \in C^\infty(\overline{N}), i \in \mathbb{N}$, such that the following hold : For any $k \in \mathbb{N}$ let $I \in \mathbb{N}$ be such that $I\mu_+ \geq k$. Then*

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*1. if $\mu_+ \notin \mathbb{N}$, then*

$$x^{\frac{2}{\alpha-1}}\varphi - \varphi_0 - x^{\mu_+}\varphi_1 - ... - x^{I\mu_+}\varphi_I \in C^k(\overline{N}).$$

*2. If $\mu_+ \in \mathbb{N}$, then*

$$x^{\frac{2}{\alpha-1}}\varphi - \varphi_0 - x^{\mu_+}\ln x\varphi_1 - ... - x^{I\mu_+}\ln^I x\varphi_I \in C^k(\overline{N}).$$

*If $\varphi_1 \mid_{\partial M}= 0$, then (in both cases)*

$$x^{\frac{2}{\alpha-1}}\varphi \in C^\infty(\overline{N}).$$

**Remarks:** 1. Similar results can be established using [1] when $(M, g)$ is only of finite degree of differentiability, with some finite number of functions $\varphi_i$ of finite degree of differentiability.
2. For $j \le Int(\mu_+)$ and $j < \mu_+$ the functions $\left(\partial^j_x\varphi_0\right) \mid_{\partial M}$ are uniquely determined by $g_{ij} \mid_{\partial M}$ and $\left(\partial^k_x g_{ij}\right) \mid_{\partial M}$, for some finite number of transverse derivatives of $g_{ij}$ at $\partial M$. Similarly when $\mu_+ \in \mathbb{N}$ the function $\varphi_1 \mid_{\partial M}$ is uniquely determined by the boundary values of a finite number of derivatives of the metric in directions transverse to $\partial M$.

PROOF: By elliptic regularity on $N = int\overline{N}$ we have $\varphi \in C^\infty(N)$. Set

$$u = \frac{\varphi x^{\frac{2}{\alpha-1}}}{C_\alpha} - 1.$$

$u \in C^\infty(N)$ and satisfies the equation

$$\begin{gathered}
Lu = \omega + F(u) \ , \\
F(u) = C_\alpha^{\alpha-1}[(1+u)^\alpha - 1 - \alpha u] \ , \quad \omega = \tfrac{2x\Delta_g x}{\alpha-1} \ , \\
Lu = x^2\Delta_g u - \tfrac{4x}{\alpha-1}g(Du, Dx) - \tfrac{2}{\alpha-1}[\alpha + 1 + x\Delta x]u \ .
\end{gathered}$$

$L$ is of the form considered in [1] with indicial exponents

$$\mu_- = -1 \ , \qquad \mu_+ = \frac{2(\alpha+1)}{\alpha-1} \ . \tag{0.4}$$

Let $\psi = \varphi \mid_{\partial N \cap M}$ . By the arguments of proof of [1, Proposition 7.3.1] there exists a solution $\widehat{\varphi}$ of

$$\begin{gathered}
\Delta\widehat{\varphi} = |\widehat{\varphi}|^{\alpha-1}\widehat{\varphi} \ , \\
\widehat{\varphi}|_{\partial N \cap M} = \psi \ , \\
\widehat{\varphi}x^{\alpha-1} - C_\alpha = O(x) \ .
\end{gathered}$$

(The function $u_0$ required in [1, Proposition 7.3.1] can be taken as $u_0 \equiv 0$; *cf.* also Remark 2 following [1, Proposition 7.3.1]). By the asymptotic maximum principle (*cf., e.g.,* [3, Theorem 3.5]) we have $\varphi \equiv \widehat{\varphi}$ which implies

$$u = O(x) \ .$$

Theorem 7.4.1 of [1] gives

$$u \in \mathscr{A}_{\mathrm{phg}}(N) \cap C^0(\overline{N}) \ .$$

By a closer examination of the structure of Equation (0.1) one obtains the claimed results. $\square$

Another example of equation to which the results of [1] apply is the equation

$$\Delta_g u = e^{\alpha u} \ , \qquad \alpha > 0 \ , \tag{0.5}$$

with the asymptotic condition

$$\lim_{p \to \partial M} u(p) = \infty \ . \tag{0.6}$$

It has been shown by Bandle and Marcus [M. Marcus, private communication] that $u$ then necessarily satisfies, for $x \le \varepsilon < 1$, for some $\varepsilon > 0$,

$$|u - \frac{2}{\alpha} \ln \frac{1}{x} - \frac{1}{\alpha} \ln \frac{2}{\alpha}| \le C x \ln \frac{1}{x} \ .$$

We have the following:

**Theorem 0.2** *Let $(\overline{M}, g)$ be a smooth Riemannian manifold with boundary, let $\alpha > 0$ and let $\varphi \in C^\infty(M)$ satisfy*

$$\Delta_g u = e^{\alpha u} \ .$$

*Suppose that there exists a constant $C$ such that*

$$|u - \frac{2}{\alpha} \ln \frac{1}{x}| \le C \ .$$

*Let $\overline{N}$ be a compact submanifold of $\overline{M}$ forming a neighborhood of $\partial M$ such that $x \in C^\infty(\overline{M})$. There exist functions $\varphi_i \in C^\infty(\overline{N})$, $i \in \mathbb{N}$, such that, for all $k \in \mathbf{N}$,*

$$u - \frac{2}{\alpha} \ln \frac{1}{x} - \frac{2}{\alpha} \ln \frac{2}{\alpha} - \varphi_1 x \ln x - \varphi_2 x^2 \ln^2 x - \ldots - \varphi_k x^k \ln^k x \in C^k(\overline{N}) \ .$$

*$\varphi_1|\partial M$ is proportional to the mean extrinsic curvature of $\partial M$ in $\overline{M}$. If that mean extrinsic curvature vanishes, then*

$$u - \frac{2}{\alpha} \ln \frac{1}{x} \in C^\infty(\overline{N}).$$

Proof: Write $u$ in the form $u = \frac{1}{\alpha}[2 \ln \frac{1}{x} + \ln \frac{2}{\alpha} + \omega]$. The function $\omega$ solves the equation

$$x^2 \Delta_g \omega = 2\Big(e^\omega - 1\Big) + 2x \Delta_g x.$$

The proof follows now that of Theorem 0.1 (in the present case we have $\mu_- = 0$, $\mu_+ = 1$). $\square$

## References


[1] L. Andersson and P.T. Chruściel, *On asymptotic behaviour of solutions of the constraint equations in general relativity with "hyperboloidal boundary conditions"*, Dissert. Math. **355** (1996), 1–100.

[2] C. Bandle and M. Marcus, *Asymptotic behaviour of solutions and their derivatives, for semilinear elliptic problems with blowup on the boundary*, Ann. Inst. H. Poincaré Anal. Non Linéaire **12** (1995), 155–171.

[3] C.R. Graham and J.M. Lee, *Einstein metrics with prescribed conformal infinity on the ball*, Adv. Math. **87** (1991), 186–225.

[4] L. Véron, *Semilinear elliptic equations with uniform blow-up on the boundary*, Jour. Anal. Math. **59** (1992), 231–250, Festschrift on the occasion of the 70th birthday of Shmuel Agmon.