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# PROBLÈMES AVEC DONNÉES INITIALES CARACTÉRISTIQUES POUR DES ÉQUATIONS D'ONDES NON LINÉAIRES

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# Chapitre 1

# Présentation Générale

Cette thèse traite de l'existence de solution d'équations d'onde et d'un système symétrique hyperbolique dans un voisinage de deux hypersurfaces caractéristiques initiales transverses. Ce genre de problèmes est étudié depuis plus de quarante ans, il intervient notament en relativité générale, en particulier dans le cadre des équations d'Einstein. Jusqu'à présent, dans le cas d'équations non linéaires, les résultats connus donnaient l'existence d'une solution dans un voisinage de l'intersection des deux hypersurfaces caractéristiques (voir par exemple H. Müller zum Hagen et H.-J. Seifert [14], ou A. D. Rendall [15]). Ici, sous certaines conditions de structure, nous montrons l'existence et l'unicité d'une solution dans un voisinage de la totalité d'une ou des deux hypersurfaces caractéristiques initiales.

La première partie se situe dans le cadre d'une métrique plate et concerne une équation d'onde semilinéaire dont le second membre ne contient pas de gradient, avec données initiales sur deux hypersurfaces caractéristiques transverses. L'existence d'une solution est amenée par la méthode de Galerkin avec une décomposition spectrale suivant l'une des directions isotropiques. En effet dans un problème d'évolution classique considéré dans un espace-temps, avec données initiales sur l'hypersurface correspondant au temps égal à zéro, cette méthode permet d'obtenir l'existence d'une solution dans un voisinage de cette hypersurface jusqu'à un temps non nul. Le principe ici est de faire jouer le rôle du temps à une des directions isotropiques de sorte à obtenir l'existence d'une solution dans un voisinage de l'hypersurface caractéristique correspondant à l'autre direction isotropique. L'énergie est estimée sur des tranches d'espace-temps parallèles à la direction isotropique de l'hypersurface caractéristique au voisinage de laquelle nous obtiendrons l'existence, dans des espaces de Sobolev avec un nombre de dérivées non homogène suivant les variables. L'utilisation du tenseur d'énergie impulsion, et des résultats classiques sur l'unicité d'une solution d'équations d'onde dans le cône passé lumière d'un point, permet de démontrer l'unicité de la solution. La répétition de l'argument dans l'autre direction isotropique induit finalement l'existence et l'unicité d'une solution dans un voisinage de la totalité des deux hypersurfaces caractéristiques initiales. Notons que ce voisinage est situé d'un seul côté des hypersurfaces, dirigé vers le futur, et que son épaisseur le long des hypersurfaces diminue au fur et à mesure que le temps augmente.

Dans la deuxième partie, nous nous plaçons dans une métrique Lorentzienne où les deux directions isotropiques transverses sont supposées pouvoir être globalement paramétrées. Nous introduisons alors une équation d'onde semilinéaire dont le second membre dépend du gradient, de façon quelconque suivant toutes les directions de dérivation sauf dans une des directions isotropiques, où là la dépendance doit être linéaire. La démarche de la première partie ne s'étendant pas à ce problème (cf remarque 3.5.2), nous avons procédé différemment. La preuve est basée sur une méthode itérative analogue à celle utilisée par A. Majda [13]. L'existence de la solution du problème itéré linéaire découle de l'article d' A. D. Rendall [15] (elle peut aussi se déduire de H. Müller zum Hagen et H.-J. Seifert [14], ou L. Hörmander [9]). Les inégalités d'énergie sont établies à l'aide du tenseur d'énergie impulsion, sur des tranches d'espace-temps parallèles à l'une des directions isotropiques, dans des espaces de Sobolev exponentiellement pondérés en la variable paramétrant cette direction isotropique. Comme l'énergie sur une hypersurface n'admet pas de dérivées transverses (à cette hypersurface), nous avons dû contracter le tenseur d'énergie impulsion avec un vecteur convenable, permettant d'absorber ces dérivées transverses. L'existence et l'unicité de la solution sont déterminées dans un voisinage de la totalité de l'hypersurface caractéristique initiale dont la direction isotropique est transverse à celle correspondant à la dépendance linéaire du second membre de l'équation. Nous pouvons remarquer que même si l'équation d'onde considérée dans la deuxième partie englobe celle de la première, les hypothèses faites sur la régularité des fonctions données ne sont pas les mêmes, et que le résultat de la première partie conserve donc un intérêt.

Dans le but d'appliquer ce genre de résultat aux équations d'Einstein, la troisième partie concerne un système symétrique hyperbolique quasilinéaire dont la forme est inspirée par une décomposition des équations d'Einstein de type Newman-Penrose. L'approche est similaire à celle de la deuxième partie, et là encore nous obtenons l'existence et l'unicité dans un voisinage de la totalité de l'une des deux hypersurfaces caractéristiques initiales. La partie principale des systèmes symétriques hyperboliques étudiés ici est d'une forme qui semble s'appliquer aux équations d'Einstein. Néanmoins, en raison de certains termes de couplage d'ordre inférieur, l'application des résultats que nous avons obtenus aux systèmes d'équations associés aux équations d'Einstein (que ce soit dans la formulation harmonique, ou dans la formulation de Klainerman-Nicolo) ne s'avère pas évidente et nous étudions encore à l'heure actuelle ce problème.

# Chapitre 2

# Introduction

This thesis deals with the local existence of solutions of wave equations and hyperbolic symmetric systems in a neighborhood of two transverse characteristic initial hypersurfaces. This kind of problems has been studied for more than fourty years, notably in general relativity and, in particular for Einstein's equation. So far the known results in the nonlinear case gave the existence of a solution in a neighborhood of the intersection of characteristic initial hypersurfaces (see for example H. Müller zum Hagen and H.-J. Seifert [14], or A. D. Rendall [15]). Here, under certain structure conditions, we show the existence and uniqueness of a solution in a neighborhood of whole initial characteristic hypersurface, or of both.

In the first part of this work the metric is assumed to be flat, and one considers a semilinear wave equation which has no gradient on its right-hand-side, with initial values on two tranversely intersecting null hypersurfaces. The existence of a solution is provided by the Galerkin's method with a spectral decomposition along one of the isotropic directions. Indeed, in a classical evolution problem on a space-time with initial values on the hypersurface where the time vanishes, this method gives the existence of a solution in a neighborhood of this hypersurface for some non-vanishing time. The principle here is to make play the role of the time at one of the isotropic direction. In this way we get the existence of a solution in a neighborhood of the characteristic initial hypersurface which corresponds to the other isotropic direction. The energy is estimated on space-time slices which are tangential to the isotropic direction of the initial characteristic hypersurface in neighborhood of which we will have the existence. We work in Sobolev spaces with different orders partial derivatives according to the variables. We use the energy momentum tensor and classical results of uniqueness of a solution for a wave equation in the causal past of a point to prove the uniqueness of the solution. Finally, the repetition of the argument in the other isotropic direction induces existence and uniqueness of a solution in a neighborhood of the two entire characteristic initial hypersurfaces. We note that this neighborhood is one-sided future directed, and that its thickness decreases along the hypersurfaces as time increases.

In the second part, we consider a Lorentzian metric with two transverse isotropic directions which are assumed to be globally parametrized. Then, we introduce a semilinear wave equation with a right-hand side depending on gradient. This dependence is required to be linear in one of the isotropic directions of differentiation (there is no restriction on the other directions of differentiation). The process of the first part cannot be extended to this problem (see remark 3.5.2), hence we deal with it differently. The proof is based on an iterative method similar to the one used by A. Majda [13]. The existence of the solution of the linear iterative problem comes from the article of A. D. Rendall [15] (one can also deduce it from H. Müller zum Hagen and H.-J. Seifert [14], or L. Hörmander [9]). The energy estimates are stated on space-time slices which are tangential to one of the isotropic directions of the characteristic initial hypersurfaces, in weighted Sobolev spaces, by using the energy momentum tensor (the weight is an exponential of the variable which parametrizes the isotropic direction of the space-time slice). As the energy on a hypersurface doesn't contain transverse partial derivatives (to this hypersurface), we have to contract the energy momentum tensor with a suitable vector field which permits to absorb these transverse partial derivatives. The existence and uniqueness of the solution are obtained in a neighborhood of the whole initial characteristic hypersurface which is transverse to the isotropic direction in which the gradient's dependence of the right-hand-side equation is linear. Note that even if the wave equation of the first part is a particular case of the one of the second part, the assumptions on the regularity of the given functions are not the same, so the results of the first part keep some interest.

In the aim of applying analogous results to Einstein's equations, the third part treats a quasilinear symmetric hyperbolic system whose form is inspired by the Newman-Penrose decomposition of Einstein's equations. The approach is similar to the one of the second part, and once more we get existence and uniqueness in a neighborhood of one entire characteristic initial hypersurface. The principal part of the symmetric hyperbolic systems studied here is of a form which seems to apply to Einstein's equations. But, because of certain lower order terms, the application of our results on the systems associated to Einstein's equations (in harmonic formulation, or formulation of Klainerman-Nicolo) is not obvious. We are studying this problem actually.

# Chapitre 3

# A semilinear wave equation

**Abstract**

In this paper we are concerned with a semilinear wave equation with initial data given on two transversely intersecting null hypersurfaces in the Minkowski space $I\!R^{n+1}$. We prove existence and uniqueness of a solution in a (one-sided future directed) neighborhood of the initial data null hypersurfaces.

## Sommaire

## 3.1 Introduction

The problem we are interested in here is about a semilinear wave equation with data given on two transversely intersecting null hypersurfaces. Many problems with characteristic initial values have been studied in the last forty years. H. Friedrich [6] has written a few papers about characteristic initial value problem in the context of Einstein's vacuum field equations (his work consists essentialy in showing the way to apply the results of existence and uniqueness of solutions of wave equation with characteristic initial value). R. Courant and D. Hilbert [5] have shown the uniqueness of a solution of wave equation with data prescribed on a characteristic half-cone. Other works treat the Cauchy problem for quasi-linear equation with data on a characteristic conoid as F. Cagnac [2], F. Cagnac and M. Dossa [4]. In this article the initial characteristic hypersurfaces are $N_+, N_-$ defined in the Minkowsky space $I\!R^{n+1}$ by

$$\begin{aligned} N_+ &= \{t + x^1 = 0, t \geq 0, (x^2, ..., x^n) \in I\!R^{n-1}\} \\ N_- &= \{t - x^1 = 0, t \geq 0, (x^2, ..., x^n) \in I\!R^{n-1}\}. \end{aligned}$$

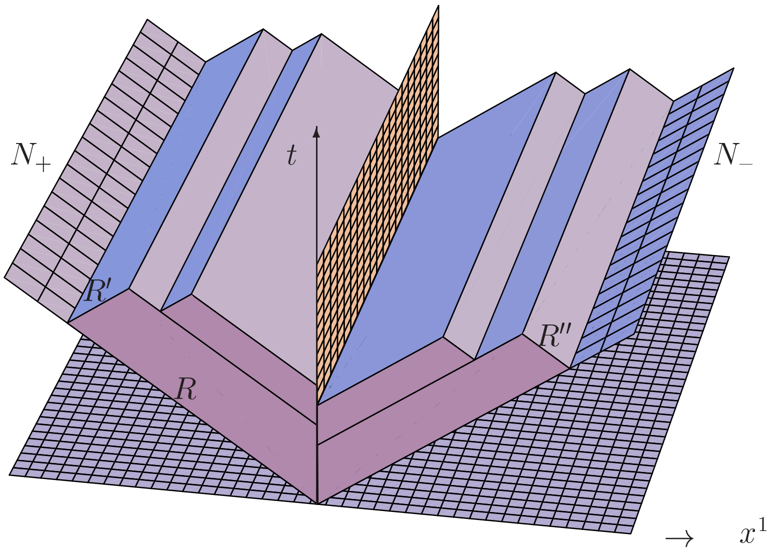
We know by standard results that there exists a global solution in the linear case. But in the case of a nonlinear hyperbolic equation, the published proofs give an existence (and uniqueness) of solutions in a neighborhood of the intersection of the null hypersurfaces, namely neighborhood with a finite time, as it is done in H. Müller zum Hagen and H.-J. Seifert [14] or A. D. Rendall [15].

In this paper we propose to demonstrate the existence and uniqueness of solutions in a one-sided neighborhood of both null hypersurfaces and not only of their intersection. More precisely, we consider in $I\!R^{n+1}$ the problem

$$\begin{cases} \Box\varphi(x,t) = F(\varphi(x,t), x, t) \\ \varphi|_{N_+} = \varphi_+ \\ \varphi|_{N_-} = \varphi_- \end{cases} \tag{3.1.1}$$

where $\Box = -\dfrac{\partial^2}{\partial t^2} + \Delta_x$
and $\varphi$ can be vector-valued .

We show, under certain conditions, that, for any positive real R, there exists positive reals R' and R" such that there exists a unique $C^2$ solution in the domain $\mathcal{V}_R := \{0 \leq t - x^1 \leq R,\ 0 \leq t + x^1 \leq R',\ (x^2, ..., x^n) \in I\!R^{n-1}\} \bigcup \{0 \leq t + x^1 \leq R,\ 0 \leq t - x^1 \leq R",\ (x^2, ..., x^n) \in I\!R^{n-1}\}$, then $\bigcup\limits_R \mathcal{V}_R$ gives a one-sided neighborhood of the initial data hypersurfaces. We can visualize a part of this neighborhood by the following figure :




The proof is based on the Galerkin method with estimates of energy in some special Sobolev spaces. The mathematics tools used in this article are very classical, but the originality here is to apply a standard method by considering a isotropic direction as the time direction. Moreover the implementation of the different parts of the proof are not so trivial.

The structure of this article is organised as follows.
We start in section 2 by a short presentation and results about the spaces in which we will work. In the third section, we give the assumptions on the functions $F, \tilde{\varphi}_+, \tilde{\varphi}_-$ and we transform the problem to obtain an equation more convenient with a new function $(\tilde{\varphi}, u, v, y) \mapsto \tilde{H}(\tilde{\varphi}, u, v, y)$ where $u = (t - x^1)/2$ , $v = (t + x^1)/2$ , $y = (x^2, ..., x^n)$ and $\tilde{H}$ vanishes at $(0, u, 0, y)$. In section 4, we construct a spectral approximation of a solution of the precedent equation. Then we estimate in the fifth section the energy of these solutions in the spaces introduced at the beginning. We deduce of this in section 6 the existence of a solution $\tilde{\varphi}$ and we discuss its regularity. After that in section 7 we come back to the first equation and discuss also the regularity and uniqueness of the solution of the problem (3.1.1), to prove the uniqueness we use a classical tool namely the energy-momentum tensor. In section 8, we resume the results obtained in the simpler case of dimension $1+1$ where we can work in Sobolev spaces $H^k$.

## 3.2 Spaces $\mathcal{H}_{m,k}$

Let $R$ be a strictly positive real, and $\mathbb{T}^{n-1}$ a torus of length $T$ in each direction. We will work in the spaces $\mathcal{H}_{m,k}$ where

$$\mathcal{H}_{m,k}([0;2R] \times \mathbb{T}^{n-1}) = \Bigg\{ \varphi \in L^2([0;2R] \times \mathbb{T}^{n-1}); \sum_{\substack{0 \leq a \leq k \\ 0 \leq |\nu| \leq m}} \int_0^{2R} \int_{\mathbb{T}^{n-1}} | \frac{\partial^a}{\partial v^a} \frac{\partial^{\nu_1}}{\partial y_1^{\nu_1}} \cdots \frac{\partial^{\nu_{n-1}}}{\partial y_{n-1}^{\nu_{n-1}}} \varphi |^2 \, dv \, d^{n-1}y < \infty \Bigg\}$$

with derivatives of $\varphi$ understood in the distribution sense. $\mathcal{H}_{m,k}$ is a Hilbert space hence it is reflexive.

We take a orthonormal basis of $L^2([0;2R]\times\mathbb{T}^{n-1})$. So we set

$$\Psi_\alpha(v,y) = (2R)^{-\frac{1}{2}}T^{-\frac{n-1}{2}}e^{i(\alpha_0 v\frac{\pi}{R}+\overline{\alpha}.y\frac{2\pi}{T})} \quad \text{with} \quad \alpha=(\alpha_0,\overline{\alpha})\in\mathbb{Z}^n$$
$$<\Psi_\alpha, f> = (2R)^{-\frac{1}{2}}T^{-\frac{n-1}{2}}\int_0^{2R}\int_{\mathbb{T}^{n-1}} e^{-i(\alpha_0 w\frac{\pi}{R}+\overline{\alpha}.z\frac{2\pi}{T})}f(w,z)dw\,d^{n-1}z\ .$$

We know that $f=\sum_{\alpha\in\mathbb{Z}^n} <\Psi_\alpha,f>\Psi_\alpha$ and we have

$$\| f \|^2_{\mathcal{H}_{m,k}} = \sum_{\substack{0\leq a\leq k\\ 0\leq|\nu|\leq m}} \| D_v^a D_y^\nu f \|^2_{L^2([0;2R]\times\mathbb{T}^{n-1})}\ .$$

The proofs of the following results are similar as in the classical Sobolev spaces $W^{s,p}$ and can be found in Appendix 3.9.

**Lemma 3.2.1** *We have the equivalence*
$\| f \|_{\mathcal{H}_{m,k}} \sim (\sum_{\alpha\in\mathbb{Z}^n} |<\Psi_\alpha,f>|^2\ (1+|\alpha_0|)^{2k}(1+|\overline{\alpha}|)^{2m})^{\frac{1}{2}}$ .

**Remark 3.2.1** : As it is done in the classical Sobolev spaces, we extend the spaces $\mathcal{H}_{m,k}$ to $m,k$ positive reals by the definition below :

$$\mathcal{H}_{m,k}([0;2R]\times\mathbb{T}^{n-1}) = \{f\in L^2([0;2R]\times\mathbb{T}^{n-1};\ \sum_{\alpha\in\mathbb{Z}^n} |<\psi_\alpha,f>|^2(1+|\alpha_0|)^{2k}(1+|\overline{\alpha}|)^{2m}<\infty\}$$

**Lemma 3.2.2** *Let $l$ a positive integer.*
*If* $\left\{\begin{array}{l} m>\frac{n-1}{2}+l\\ k>\frac{1}{2}+l\end{array}\right.$ *then* $\mathcal{H}_{m,k}([0;2R]\times\mathbb{T}^{n-1})\subset C^l([0;2R]\times\mathbb{T}^{n-1})$ .

**Lemma 3.2.3** *If $m<m'$ and $k<k'$ then $\mathcal{H}_{m',k'}\hookrightarrow\mathcal{H}_{m,k}$ with compact embedding.*

**Lemma 3.2.4** *If $f\in\mathcal{H}_{m,k}\cap\mathcal{H}_{m,k'}$ with $k<k'$ then $\forall\,\gamma\in[0;1]$,*
*$f\in\mathcal{H}_{m,\gamma k+(1-\gamma)k'}$ and $\| f \|_{\mathcal{H}_{m,\gamma k+(1-\gamma)k'}} \leq \| f \|^{\gamma}_{\mathcal{H}_{m,k}} \| f \|^{1-\gamma}_{\mathcal{H}_{m,k'}}$.*
*Similarly, if $f\in\mathcal{H}_{m,k}\cap\mathcal{H}_{m',k}$ with $m<m'$ then $\forall\,\gamma\in[0;1]$,*
*$f\in\mathcal{H}_{\gamma m+(1-\gamma)m',k}$ and $\| f \|_{\mathcal{H}_{\gamma m+(1-\gamma)m',k}} \leq \| f \|^{\gamma}_{\mathcal{H}_{m,k}} \| f \|^{1-\gamma}_{\mathcal{H}_{m',k}}$.*
*Furthermore, if $f\in\mathcal{H}_{m,k}\cap\mathcal{H}_{m',k'}$ with $m<m'$ and $k<k'$ then $\forall\,\gamma,\ \delta\in[0;1]$,*
*$f\in\mathcal{H}_{\gamma m+(1-\gamma)m',\delta k+(1-\delta)k'}$ and $\| f \|_{\mathcal{H}_{\gamma m+(1-\gamma)m',\delta k+(1-\delta)k'}} \leq \| f \|^{\gamma\delta}_{\mathcal{H}_{m,k}} \| f \|^{1-\gamma\delta}_{\mathcal{H}_{m',k'}}$.*

## 3.3 Transformation of the problem

In this section, we show how we transform the problem (3.1.1) to obtain a problem where the first equation is replaced by an equation of the form

$$\frac{\partial^2}{\partial u\partial v}\tilde{\varphi}(u,v,y) = \tilde{H}(\tilde{\varphi},u,v,y) + \Delta_y\tilde{\varphi}(u,v,y)$$

where $u = \dfrac{t-x^1}{2}$ , $v = \dfrac{t+x^1}{2}$ , $y = (x^2, ..., x^n)$ and $\tilde{H}(0,u,0,y)$ vanishes.
We notice that $N_+ = \{v = 0,\ u \geq 0,\ y \in I\!R^{n-1}\}$ and $N_- = \{u = 0,\ v \geq 0,\ y \in I\!R^{n-1}\}$.

If the function $\varphi$ satisfies $\dfrac{\partial^2}{\partial u \partial v}\varphi = \dfrac{\partial^2}{\partial v \partial u}\varphi$ the equation becomes :

$$\frac{\partial^2}{\partial u \partial v}\varphi = -F(\varphi, t, x^1, y) + \Delta_y \varphi =: H(\varphi, u, v, y) + \Delta_y \varphi. \tag{3.3.1}$$

Concerning regularity of the functions $F, \varphi_+, \varphi_-$ in the problem (3.1.1), we shall assume for the moment that there exists $m \in I\!N$ such that the following holds :

(i) $F : (\theta, t, x^1, y) \mapsto F(\theta, t, x^1, y)$ satisfies that for any $a, b = 0$ or 1, $\gamma \in I\!N$, $\mu \in I\!N^{n-1}$, $0 \leq \gamma + |\mu| \leq m+1$, $D_t^a D_{x^1}^b D_\theta^\gamma D_y^\mu F$ is continuous in all its variables.

(ii) $\varphi_+$ is of class $C^{m+5}$, $\varphi_-$ $C^{m+4}$ and $\varphi_+, \varphi_-$ satisfy the corner condition : $\varphi_+(0,y) = \varphi_-(0,y)$.

(iii) There exists a real $T > 0$ such that $F, \varphi_+, \varphi_-$ are $T$-periodic in each $y_i$.

**Remark 3.3.1** : The corner conditions are only those in (ii) because for the partial derivatives with respect to $u$ or $v$ separately, we have

$$\frac{\partial^k}{\partial u^k}\varphi(0,0,y) = \frac{\partial^k}{\partial u^k}\varphi_+(0,y)$$
$$\frac{\partial^k}{\partial v^k}\varphi(0,0,y) = \frac{\partial^k}{\partial v^k}\varphi_-(0,y)$$

and for the partial derivatives with respect to mixed $u$ and $v$, the corner conditions are assumed by the equation (3.3.1), namely

$$\frac{\partial^2}{\partial u \partial v}\varphi(0,0,y) = H(\varphi(0,0,y), 0, 0, y) + \Delta_y \varphi(0,0,y).$$

By induction, we get higher derivatives with respect to mixed $u$ and $v$ at $(0,0,y)$. $\triangle$

With the definitions of $H, u$ and $v$ above, we see that $H$ satisfies : for any $a, b = 0$ or 1, $0 \leq \gamma + |\mu| \leq m+1$, $D_u^a D_v^b D_\theta^\gamma D_y^\mu H$ continuous in all its variables.

After that we calculate $\frac{\partial}{\partial v}\varphi(u,0,y)$ with the initial values as follows : we know that

$$\frac{\partial^2}{\partial u \partial v}\varphi(u,0,y) = H(\varphi_+(u,y), u, 0, y) + \Delta_y \varphi_+(u,y)$$

(we can permute $\Delta_y$ and the limit in $v = 0$ because $\varphi$ is supposed $C^2$ in all its variables, for the same reason we will permute $\partial_v$ and the limit in $u = 0$ in the third line below). So by integrating in $u$, we obtain

$$\begin{aligned} \frac{\partial}{\partial v}\varphi(u,0,y) &= \frac{\partial}{\partial v}\varphi(0,0,y) + \int_0^u H(\varphi_+(s,y), s, 0, y) + \Delta_y \varphi_+(s,y) ds \\ &= \frac{\partial}{\partial v}\varphi_-(0,y) + \int_0^u H(\varphi_+(s,y), s, 0, y) + \Delta_y \varphi_+(s,y) ds. \end{aligned}$$

Then we set

$$\tilde{\varphi}(u,v,y) = \varphi(u,v,y) - (\varphi(u,0,y) + \frac{\partial}{\partial v}\varphi(u,0,y)\ v) =: \varphi(u,v,y) - \delta(\varphi_+, \varphi_-).$$

Thus $\tilde{\varphi}$ and its first derivative in $v$ vanish at $v = 0$. On another hand, if we take the equation (3.3.1) and put $\tilde{\varphi}$ in it, we obtain

$$\begin{aligned}
&\frac{\partial^2}{\partial u \partial v}\tilde{\varphi}(u,v,y)\\
&\quad = H(\tilde{\varphi}+\delta(\varphi_+,\varphi_-),u,v,y)+\Delta_y(\tilde{\varphi}+\delta(\varphi_+,\varphi_-))-\frac{\partial^2}{\partial u\partial v}\delta(\varphi_+,\varphi_-)\\
&\quad = H(\tilde{\varphi}+\delta(\varphi_+,\varphi_-),u,v,y)+\Delta_y(\tilde{\varphi}+\delta(\varphi_+,\varphi_-))-(H(\varphi_+(u,y),u,0,y)+\Delta_y\varphi_+(u,y))\\
&\quad =: \tilde{H}(\tilde{\varphi},u,v,y)+\Delta_y\tilde{\varphi}. \qquad (3.3.2)
\end{aligned}$$

$\tilde{H}$ has the same regularity as $H$ because $\delta(\varphi_+,\varphi_-)$, $\Delta_y\delta(\varphi_+,\varphi_-)$ and $\frac{\partial^2}{\partial u\partial v}\delta(\varphi_+,\varphi_-)$ are of class $C^{m+1}$.

If we look the value of $\tilde{H}+\Delta_y\tilde{\varphi}$ at $v=0$ we can see that it vanishes :

$$\begin{aligned}
&\tilde{H}(\tilde{\varphi}(u,0,y),u,0,y)+\Delta_y\tilde{\varphi}(u,0,y)\\
&\qquad = H(\varphi(u,0,y),u,0,y)+\Delta_y(\varphi)(u,0,y)-\frac{\partial^2}{\partial u\partial v}\delta(\varphi_+,\varphi_-)\\
&\qquad = H(\varphi(u,0,y),u,0,y)+\Delta_y(\varphi)(u,0,y)-\frac{\partial^2}{\partial u\partial v}\varphi(u,0,y)\\
&\qquad = 0.
\end{aligned}$$

But if $\varphi$ is supposed $C^2$ in all its variables, then $\tilde{\varphi}$ is continuous in all its variables, so we can permute $\Delta_y$ and the limit in $v=0$, thus $\Delta_y(\tilde{\varphi})(u,v,y)|_{v=0}=0$, hence we have

$$\tilde{H}(\tilde{\varphi}(u,0,y),u,0,y)=\tilde{H}(0,u,0,y)=0 \qquad (3.3.3)$$

So in setting

$$\tilde{\varphi}_-(v,y)=\varphi_-(v,y)-(\varphi_+(0,y)+\frac{\partial}{\partial v}\varphi_-(0,y)\ v) \qquad (3.3.4)$$

we want now to solve the problem :

$$\left\{\begin{array}{l}
\frac{\partial^2}{\partial u\partial v}\tilde{\varphi}(u,v,y)=\tilde{H}(\tilde{\varphi}(u,v,y),u,v,y)+\Delta_y\tilde{\varphi}(u,v,y)\\
\tilde{\varphi}(u,0,y)=0\\
\tilde{\varphi}(0,v,y)=\tilde{\varphi}_-(v,y)
\end{array}\right. \qquad (3.3.5)$$

where the assumptions of the regularity of the functions $\tilde{H}$ and $\tilde{\varphi}_-$ are the following :

$$\begin{array}{ll}
(i) & \tilde{H}:(\theta,u,v,y)\mapsto\tilde{H}(\theta,u,v,y) \text{ satisfies that } \forall\ a,b=0 \text{ or } 1\ ,\\
& \gamma\in I\!N\ ,\ \mu\in I\!N^{n-1},\ 0\le\gamma+|\mu|\le m+1,\\
& \quad D_u^aD_v^bD_\theta^\gamma D_y^\mu\tilde{H} \text{ is continuous in all its variables} \qquad (3.3.6)\\
(ii) & \tilde{\varphi}_- \text{ is of class } C^{m+4}\\
(iii) & \text{there exists a real } T>0 \text{ such that } \tilde{H},\tilde{\varphi}_- \text{ are } T\text{-periodic in each } y_i.
\end{array}$$

## 3.4 Spectral approximation of $\tilde{\varphi}$

We take an arbitrary real number $R>0$. Let

$$\hat{J}_\varepsilon\tilde{\varphi}=\sum_{|\alpha|\le\frac{1}{\varepsilon}}<\Psi_\alpha,\tilde{\varphi}>\Psi_\alpha.$$

We know that there exists a continuation of $\tilde{H}$ in $v$ from $[0;R]$ to $[0;2R]$ such that for any $a=0,1$, $0\leq\gamma+|\mu|\leq m+1$ we have $D_v^a D_\theta^\gamma D_y^\mu \tilde{H}$ continuous in all its variables (indeed, it suffices to set for $v>R$ , $\tilde{H}(\theta,u,v,y)=\tilde{H}(\theta,u,R,y)+(v-R)\frac{\partial}{\partial v}\tilde{H}(\theta,u,R,y))$ . The function $\tilde{H}$ in the following will be this function multiplied by a smooth cut off function $\phi_R$ of $v$ equal to 1 on $[0;R]$ and to 0 on $[\frac{3R}{4};2R]$. Similarly, there exists a continuation of $\tilde{\varphi}_-$ in $v$ from $[0;R]$ to $[0;2R]$ of class $C^k$ in all its variables. The function $\tilde{\varphi}_-$ in the following will be this function multiplied by $\phi_R$ .

We will build a solution $\tilde{\varphi}_\varepsilon$ of the problem :

$$\begin{cases} \hat{J}_\varepsilon\tilde{\varphi}_\varepsilon=\tilde{\varphi}_\varepsilon \\ \dfrac{\partial^2}{\partial u\partial v}\tilde{\varphi}_\varepsilon(u,v,y)=\hat{J}_\varepsilon\tilde{H}(\tilde{\varphi}_\varepsilon(u,v,y),u,v,y)+\Delta_y\tilde{\varphi}_\varepsilon(u,v,y) \\ \tilde{\varphi}_\varepsilon(u,0,y)=0 \\ \tilde{\varphi}_\varepsilon(0,v,y)=\hat{J}_\varepsilon\tilde{\varphi}_-(v,y) \end{cases} \tag{3.4.1}$$

We first show the existence of the $\tilde{\varphi}_\varepsilon$. By the first equation of problem (3.4.1), $\tilde{\varphi}_\varepsilon$ has a finite number of components $\tilde{\varphi}_{\varepsilon,\alpha}$ :
$\tilde{\varphi}_\varepsilon(u,v,y)=\sum\limits_{|\alpha|\leq\frac{1}{\varepsilon}}\tilde{\varphi}_{\varepsilon,\alpha}(u)\Psi_\alpha(v,y)$ with $\tilde{\varphi}_{\varepsilon,\alpha}(u)=<\Psi_\alpha(v,y),\tilde{\varphi}_\varepsilon(u,v,y)>$ .
We differentiate $\tilde{\varphi}_\varepsilon$ in $v$, after in $u$, on one hand, we have

$$\begin{aligned} \frac{\partial}{\partial v}\tilde{\varphi}_\varepsilon(u,v,y) &= \frac{\pi}{R}i\sum_{|\alpha|\leq\frac{1}{\varepsilon}}\alpha_0\tilde{\varphi}_{\varepsilon,\alpha}(u)\Psi_\alpha(v,y) \\ \frac{\partial^2}{\partial u\partial v}\tilde{\varphi}_\varepsilon(u,v,y) &= \frac{\pi}{R}i\sum_{|\alpha|\leq\frac{1}{\varepsilon}}\alpha_0\frac{\partial}{\partial u}\tilde{\varphi}_{\varepsilon,\alpha}(u)\Psi_\alpha(v,y). \end{aligned}$$

On another hand we have by using the second equation of the problem (3.4.1) :

$$\begin{aligned} &\frac{\partial^2}{\partial u\partial v}\tilde{\varphi}_\varepsilon(u,v,y) \\ &\quad=\sum_{|\beta|\leq\frac{1}{\varepsilon}}<\Psi_\beta,\tilde{H}(\sum_{|\gamma|\leq\frac{1}{\varepsilon}}\tilde{\varphi}_{\varepsilon,\gamma}\Psi_\gamma,u,v,y)+\Delta_y\sum_{|\gamma|\leq\frac{1}{\varepsilon}}\tilde{\varphi}_{\varepsilon,\gamma}\Psi_\gamma(v,y)>\Psi_\beta(v,y) \\ &\quad=\sum_{|\beta|\leq\frac{1}{\varepsilon}}<\Psi_\beta,\tilde{H}(\sum_{|\gamma|\leq\frac{1}{\varepsilon}}\tilde{\varphi}_{\varepsilon,\gamma}\Psi_\gamma,u,v,y)+\sum_{j=1}^{n-1}\frac{4\pi^2}{T^2}\sum_{|\gamma|\leq\frac{1}{\varepsilon}}\gamma_j^2\tilde{\varphi}_{\varepsilon,\gamma}\Psi_\gamma(v,y)>\Psi_\beta(v,y). \end{aligned}$$

Hence with these both results, by making scalar product by $\Psi_\alpha$ (recall that $(\Psi_\alpha)_{\alpha\in\mathbb{Z}^n}$ is an orthonormal basis), we can identify the components :

$$\frac{\pi}{R}i\alpha_0\frac{\partial}{\partial u}\tilde{\varphi}_{\varepsilon,\alpha}(u)=<\Psi_\alpha,\tilde{H}(\sum_{|\gamma|\leq\frac{1}{\varepsilon}}\tilde{\varphi}_{\varepsilon,\gamma}\Psi_\gamma,u,v,y)+\sum_{j=1}^{n-1}\frac{4\pi^2}{T^2}\sum_{|\gamma|\leq\frac{1}{\varepsilon}}\gamma_j^2\tilde{\varphi}_{\varepsilon,\gamma}\Psi_\gamma(v,y)>.$$

We can distinguish two cases. First if $\alpha_0\neq 0$ we obtain $\dfrac{\partial}{\partial u}\tilde{\varphi}_{\varepsilon,\alpha}(u)=F_\alpha((\tilde{\varphi}_{\varepsilon,\beta})_{|\beta|\leq\frac{1}{\varepsilon}},u)$ with $F_\alpha$ and $\dfrac{\partial}{\partial\tilde{\varphi}_{\varepsilon,\beta}}F_\alpha$ continuous in all their variables $((\tilde{\varphi}_{\varepsilon,\beta})_{|\beta|\leq\frac{1}{\varepsilon}},u)$ because $\tilde{H}$ and $D_\theta\tilde{H}$ are continuous in all their variables and $<,>$ is sesquilinear.

Now, if $\alpha_0 = 0$, to assume the third equation of problem (3.4.1) we want that $\tilde{\varphi}_\varepsilon(u,0,y) = \sum_{|\alpha|\leq\frac{1}{\varepsilon}} \tilde{\varphi}_{\varepsilon,\alpha}(u)(2R)^{-\frac{1}{2}}T^{-\frac{n-1}{2}}e^{i\overline{\alpha}.y\frac{2\pi}{T}} = 0$.

Recall that $\alpha = (\alpha_0, \overline{\alpha})$, we can decompose this sum in a sum on $\overline{\alpha}$ and a sum on $\alpha_0$, and as $\alpha_0$ just intervenes in $\tilde{\varphi}_{\varepsilon,\alpha}$ we obtain : $\sum_{|\overline{\alpha}|\leq\frac{1}{\varepsilon}} \big( \sum_{\{\alpha_0;|(\alpha_0,\overline{\alpha})|\leq\frac{1}{\varepsilon}\}} \tilde{\varphi}_{\varepsilon,\alpha}(u)\big)(2R)^{-\frac{1}{2}}T^{-\frac{n-1}{2}}e^{i\overline{\alpha}.y\frac{2\pi}{T}} = 0$.

As this holds for every $y$ in $\mathbb{T}^{n-1}$, we necessarily have
$\forall\overline{\alpha}$ such that $|\overline{\alpha}| \leq \frac{1}{\varepsilon}$, $\sum_{\{\alpha_0;|(\alpha_0,\overline{\alpha})|\leq\frac{1}{\varepsilon}\}} \tilde{\varphi}_{\varepsilon,\alpha}(u) = 0$
hence we define $\tilde{\varphi}_{\varepsilon,(0,\overline{\alpha})}$ by

$$\forall\overline{\alpha} \text{ such that } |\overline{\alpha}| \leq \frac{1}{\varepsilon}, \qquad \tilde{\varphi}_{\varepsilon,(0,\overline{\alpha})}(u) = -\sum_{\{\alpha_0\neq0;|(\alpha_0,\overline{\alpha})|\leq\frac{1}{\varepsilon}\}} \tilde{\varphi}_{\varepsilon,(\alpha_0,\overline{\alpha})}(u). \tag{3.4.2}$$

Finally all the $\tilde{\varphi}_{\varepsilon,(0,\overline{\alpha})}$ are $C^1$-function of the $\tilde{\varphi}_{\varepsilon,(\alpha_0,\overline{\alpha})}$ with $\alpha_0 \neq 0$ so we can express $\frac{\partial}{\partial u}\tilde{\varphi}_{\varepsilon,\alpha}(u)$ in function of $((\tilde{\varphi}_{\varepsilon,\beta})_{\{|\beta|\leq\frac{1}{\varepsilon};\beta_0\neq0\}}, u)$ as follows :

$$\forall\alpha_0 \neq 0, |\alpha| \leq \frac{1}{\varepsilon}, \qquad \frac{\partial}{\partial u}\tilde{\varphi}_{\varepsilon,\alpha}(u) = \tilde{F}_\alpha((\tilde{\varphi}_{\varepsilon,\beta})_{\{|\beta|\leq\frac{1}{\varepsilon};\beta_0\neq0\}}, u)$$

with $\tilde{F}_\alpha$ and $\frac{\partial}{\partial\tilde{\varphi}_{\varepsilon,\beta}}\tilde{F}_\alpha$ continuous in all their variables.

By the theorem of Cauchy-Lipschitz, we know that if a function f is continuous, locally Lipschitz with respect to its second variable, the problem $y' = f(t,y)$ with $y(t_0) = y_0$ has a unique $C^1$-solution $y(t)$ on a maximal open interval I. Here we take
$y = (\tilde{\varphi}_{\varepsilon,\alpha})_{\{|\alpha|\leq\frac{1}{\varepsilon};\alpha_0\neq0\}}$ $\qquad f = \left(\tilde{F}_\alpha\right)_{\{|\alpha|\leq\frac{1}{\varepsilon};\alpha_0\neq0\}}$
and $y(0) = \left((2R)^{-\frac{1}{2}}T^{-\frac{n-1}{2}}\int_0^{2R}\int_{\mathbb{T}^{n-1}} e^{-i(\alpha_0 w\frac{\pi}{R}+\overline{\alpha}.z\frac{2\pi}{T})}\tilde{\varphi}_-(w,z)dw\, d^{n-1}z\right)_{\{|\alpha|\leq\frac{1}{\varepsilon};\alpha_0\neq0\}}$.

For all $\varepsilon > 0$, there exists a maximal open interval $I_\varepsilon$ containing zero, in which we have a unique solution $\tilde{\varphi}_\varepsilon \equiv (\tilde{\varphi}_{\varepsilon,\alpha})_{\{|\alpha|\leq\frac{1}{\varepsilon};\alpha_0\neq0\}}$ $C^1$ in $u$ (the $(\tilde{\varphi}_{\varepsilon,\alpha})_{\{|\alpha|\leq\frac{1}{\varepsilon};\alpha_0=0\}}$ are given by (3.4.2)).

Moreover, $\tilde{\varphi}_\varepsilon$ is smooth in $(v,y)$ on $[0;2R]\times\mathbb{T}^{n-1}$, so we can commute all the partial derivatives in $v$ and $y_i$ at any order. And as for all $\beta$ in $\mathbb{N}$, $\gamma$ in $\mathbb{N}^{n-1}$ , $\frac{\partial^\beta}{\partial v^\beta}\frac{\partial^\gamma}{\partial y^\gamma}\tilde{\varphi}_\varepsilon$ is a finite sum of products of $C^1$-function in $u$ by $C^1$-function in $(v,y)$, we have $\frac{\partial^\beta}{\partial v^\beta}\frac{\partial^\gamma}{\partial y^\gamma}\tilde{\varphi}_\varepsilon$ in $C^1(I_\varepsilon\times[0;2R]\times\mathbb{T}^{n-1})$. So we can commute $\frac{\partial}{\partial u}$ with all the partial derivatives in $v$ and $y_i$ at any order.

**Remark 3.4.1** : In all this section if we keep the expression of $\tilde{H}$ with $H$ and $\delta(\varphi_+,\varphi_-)$, we see that we just need the following assumptions :

- $(i)$ $H : (\theta,u,v,y) \mapsto H(\theta,u,v,y)$ satisfies that
  $H$ and $\frac{\partial H}{\partial\theta}$ are continuous in all their variables
  $\forall i = 1,...,n-1, \quad \frac{\partial^2 H}{\partial\theta^2}, \frac{\partial^2 H}{\partial y_i\partial\theta}, \frac{\partial^2 H}{\partial\theta\partial y_i}, \frac{\partial^2 H}{\partial y_i^2},$ are continuous in variable $y_i$
- $(ii)$ $\varphi_+$ is of class $C^4$ or $H^s$ with $s > \frac{7}{2}+\frac{n}{2}$
- $(iii)$ $\varphi_-$ is of class $C^3$ or $H^{s-1}$
- $(iv)$ there exists a real $T > 0$ such that $H, \varphi_+, \varphi_-$ are $T$-periodic in each $y_i$.

(when we take $\varphi_+$ in $H^s$, the gain of an "half order" of derivative in comparison with the embedding $H^s \hookrightarrow C^4$ for $s > 4 + \frac{n}{2}$ comes from the fact that at a certain step we just need the continuity of $\varphi_+$ in variable $y$).

## 3.5 Estimation of $\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}}$

To estimate $\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}}$, we will first bound $\frac{d}{du} \| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}}$ by a continuous function of $\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}}$ and then we will use the Gronwall lemma.

### 3.5.1 bound of $\frac{d}{du} \| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}}$

**Proposition 3.5.1** *If $m > \frac{n-1}{2}$ , we have the following estimation*

$$\frac{d}{du} \| \tilde{\varphi}_\varepsilon(u) \|^2_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})} \leq \mathcal{F}(\| \tilde{\varphi}_\varepsilon(u) \|^2_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})}, u)$$

*with $\mathcal{F}$ continuous in both variables.*

**Remark 3.5.1** :

1) The assumption $m > \frac{n-1}{2}$ comes from the embedding $\mathcal{H}_{m,2}$ in $L^\infty$ and so we can bound $\tilde{H}(\tilde{\varphi}_\varepsilon, u, v, y)$ by a function of the norm $\mathcal{H}_{m,2}$ of $\tilde{\varphi}_\varepsilon(u)$.

2) By writing in details the partial derivatives of $\tilde{H}$ with the function $H$ and $\delta(\varphi_+, \varphi_-)$, we can reduce the assumptions on $\varphi_+, \varphi_-$. Then, for this proposition, we can replace assumptions on $\varphi_+, \varphi_-$ by the followings :
$\varphi_+ \in C^4 \cap H^{m+5}$ or $\varphi_+ \in H^s$ with $s > \frac{7}{2} + \frac{n}{2}$ and $s \geq m + 5$
$\varphi_- \in C^3 \cap H^{m+4}$ or $\varphi_- \in H^{s-1}$.

3) If the functions $\tilde{H}$ and $\tilde{\varphi}_-$ are not T-periodic in each $y_i$ or not defined on $\mathbb{R}^{n-1}$ in their variable $y$, we can get the existence (and uniqueness) of a solution of the problem (3.1.1) but in a smaller domain. We will see this in theorem 3.7.3.

**Proof of proposition 3.5.1 :**
The proof of the proposition is organised in five steps : estimation of $\frac{d}{du} \| \tilde{\varphi}_\varepsilon(u) \|^2_{L^2}$, estimation of $\frac{d}{du} \| \frac{\partial^\beta}{\partial y^\beta} \tilde{\varphi}_\varepsilon(u) \|^2_{L^2}$, estimation of $\frac{d}{du} \| \frac{\partial}{\partial v} \frac{\partial^\beta}{\partial y^\beta} \tilde{\varphi}_\varepsilon(u) \|^2_{L^2}$, estimation of $\frac{d}{du} \| \frac{\partial^2}{\partial v^2} \frac{\partial^\beta}{\partial y^\beta} \tilde{\varphi}_\varepsilon(u) \|^2_{L^2}$, conclusion.

**Estimation of $\frac{d}{du} \parallel \tilde{\varphi}_\varepsilon(u) \parallel^2_{L^2}$**

As $\tilde{\varphi}_\varepsilon$ is in $C^1(I_\varepsilon \times [0;2R] \times \mathbb{T}^{n-1})$ we can commute $\frac{d}{du}$ and $\int$ so

$$\begin{aligned}\frac{d}{du} \parallel \tilde{\varphi}_\varepsilon(u) \parallel^2_{L^2([0;2R]\times\mathbb{T}^{n-1})} &= \int_0^{2R}\int_{\mathbb{T}^{n-1}} \frac{\partial}{\partial u}(\tilde{\varphi}_\varepsilon)^2 dv\ d^{n-1}y \\ &= 2\int_0^{2R}\int_{\mathbb{T}^{n-1}} \tilde{\varphi}_\varepsilon(\frac{\partial}{\partial u}\tilde{\varphi}_\varepsilon) dv\ d^{n-1}y.\end{aligned}$$

As $\frac{\partial^2}{\partial v \partial u}\tilde{\varphi}_\varepsilon$ equals $\frac{\partial^2}{\partial u \partial v}\tilde{\varphi}_\varepsilon$ and so is continuous, we also have by integration in $v$ :

$$\frac{\partial}{\partial u}(\tilde{\varphi}_\varepsilon)(u,v,y) = \frac{\partial}{\partial u}(\tilde{\varphi}_\varepsilon)(u,0,y) + \int_0^v \frac{\partial^2}{\partial v \partial u}\tilde{\varphi}_\varepsilon(u,s,y)ds. \tag{3.5.1}$$

But $\tilde{\varphi}_\varepsilon$ is $C^1$ in variable $(u,v)$ so we can permute in the expression $\frac{\tilde{\varphi}_\varepsilon(u+h,0,y) - \tilde{\varphi}_\varepsilon(u,0,y)}{h}$ the limit in $v = 0$ and the limit in $h = 0$ corresponding to $\frac{\partial}{\partial u}$. As $\tilde{\varphi}_\varepsilon(u+h,0,y) = \tilde{\varphi}_\varepsilon(u,0,y) = 0$ given by the third equation in (3.4.1) we obtain $\frac{\partial}{\partial u}(\tilde{\varphi}_\varepsilon)(u,0,y) = 0$. Now, by using $\frac{\partial^2}{\partial v \partial u}\tilde{\varphi}_\varepsilon$ equals $\frac{\partial^2}{\partial u \partial v}\tilde{\varphi}_\varepsilon$ and the second equation of (3.4.1) we obtain

$$\begin{aligned}&\frac{d}{du} \parallel \tilde{\varphi}_\varepsilon(u) \parallel^2_{L^2([0;2R]\times\mathbb{T}^{n-1})} \\ &= 2\int_0^{2R}\int_{\mathbb{T}^{n-1}} \tilde{\varphi}_\varepsilon(u,v,y)\int_0^v \big(\hat{J}_\varepsilon\tilde{H}(\tilde{\varphi}_\varepsilon,u,s,y) + \Delta_y\tilde{\varphi}_\varepsilon\big) ds\ dv\ d^{n-1}y.\end{aligned} \tag{3.5.2}$$

On one hand, by using Cauchy-Schwarz inequality in $L^2([0;v])$ and the fact that $v$ is in $[0;2R]$ we have for the first term of the sum in the right member of (3.5.2)

$$\left|\int_0^v \hat{J}_\varepsilon\tilde{H}(\tilde{\varphi}_\varepsilon,u,s,y)ds\right| \leq (2R)^{\frac{1}{2}} \parallel \hat{J}_\varepsilon\tilde{H}(\tilde{\varphi}_\varepsilon,u,s,y) \parallel_{L^2([0;2R])}.$$

And so by definition of the norm $L^2$ we deduce

$$\parallel \int_0^v \hat{J}_\varepsilon\tilde{H}(\tilde{\varphi}_\varepsilon,u,s,y)ds \parallel_{L^2(\mathbb{T}^{n-1})} \leq (2R)^{\frac{1}{2}} \parallel \hat{J}_\varepsilon\tilde{H}(\tilde{\varphi}_\varepsilon,u,s,y) \parallel_{L^2([0;2R]\times\mathbb{T}^{n-1})}.$$

By using Cauchy-Schwarz inequality in $L^2(\mathbb{T}^{n-1})$ and the inequality above, we obtain

$$\begin{aligned}\left|\int_{\mathbb{T}^{n-1}} \tilde{\varphi}_\varepsilon \int_0^v \hat{J}_\varepsilon\tilde{H}(\tilde{\varphi}_\varepsilon,u,s,y)ds\right| &\leq \parallel \tilde{\varphi}_\varepsilon(u,v) \parallel_{L^2(\mathbb{T}^{n-1})} \parallel \int_0^v \hat{J}_\varepsilon\tilde{H}(\tilde{\varphi}_\varepsilon,u,s,y)ds \parallel_{L^2(\mathbb{T}^{n-1})} \\ &\leq (2R)^{\frac{1}{2}} \parallel \tilde{\varphi}_\varepsilon(u,v) \parallel_{L^2(\mathbb{T}^{n-1})} \parallel \hat{J}_\varepsilon\tilde{H}(\tilde{\varphi}_\varepsilon,u,s,y) \parallel_{L^2([0;2R]\times\mathbb{T}^{n-1})}.\end{aligned}$$

We know by Plancherel's theorem that for any $(2R \times T^{n-1})$-periodic function $f$ we have $\parallel f \parallel^2_{L^2} = \sum_{\alpha\in\mathbb{Z}^{n+1}} |<\Psi_\alpha, f>|^2$ so

$$\parallel \hat{J}_\varepsilon f \parallel_{L^2} \leq \parallel f \parallel_{L^2} \tag{3.5.3}$$

and as the function $\tilde{H}$ is continuous we can bound as follows

$$\begin{aligned}
&|\int_{\mathbb{T}^{n-1}} \tilde{\varphi}_\varepsilon \int_0^v \hat{J}_\varepsilon \tilde{H}(\tilde{\varphi}_\varepsilon, u, s, y) ds\ d^{n-1}y| \\
&\quad \leq (2R)^{\frac{1}{2}} \| \tilde{\varphi}_\varepsilon(u,v) \|_{L^2(\mathbb{T}^{n-1})} \| \tilde{H}(\tilde{\varphi}_\varepsilon, u, s, y) \|_{L^2([0;2R]\times\mathbb{T}^{n-1})} \\
&\quad \leq (2R) T^{\frac{n-1}{2}} \| \tilde{\varphi}_\varepsilon(u,v) \|_{L^2(\mathbb{T}^{n-1})} \max_{\substack{s\in[0;2R] \\ \theta\in\Theta_\varepsilon \\ y\in\mathbb{T}^{n-1}}} |\tilde{H}(\theta, u, s, y)|
\end{aligned}$$

where $\Theta_\varepsilon = [- \| \tilde{\varphi}_\varepsilon(u) \|_{L^\infty([0;2R]\times\mathbb{T}^{n-1})}, \| \tilde{\varphi}_\varepsilon(u) \|_{L^\infty([0;2R]\times\mathbb{T}^{n-1})}]$ so we obtain

$$|\int_{\mathbb{T}^{n-1}} \tilde{\varphi}_\varepsilon \int_0^v \hat{J}_\varepsilon(\tilde{\varphi}_\varepsilon, u, s, y) ds\ d^{n-1}y| \quad \leq \quad c_R(\| \tilde{\varphi}_\varepsilon(u) \|_{L^\infty([0;2R]\times\mathbb{T}^{n-1})}, u) \| \tilde{\varphi}_\varepsilon(u,v) \|_{L^2(\mathbb{T}^{n-1})}$$

with $c_R$ continuous in all its variables.

On another hand, for the second term of the sum in the right member of (3.5.2), we have in the same way

$$\begin{aligned}
|\int_{\mathbb{T}^{n-1}} \tilde{\varphi}_\varepsilon \int_0^v \Delta_y \tilde{\varphi}_\varepsilon ds\ d^{n-1}y| \quad &\leq \quad \| \tilde{\varphi}_\varepsilon(u,v) \|_{L^2(\mathbb{T}^{n-1})} \| \int_0^v \Delta_y \tilde{\varphi}_\varepsilon ds \|_{L^2(\mathbb{T}^{n-1})} \\
&\leq \quad (2R)^{\frac{1}{2}} \| \tilde{\varphi}_\varepsilon(u,v) \|_{L^2(\mathbb{T}^{n-1})} \| \Delta_y \tilde{\varphi}_\varepsilon(u) \|_{L^2([0;2R]\times\mathbb{T}^{n-1})} \\
&\leq \quad (2R)^{\frac{1}{2}} \| \tilde{\varphi}_\varepsilon(u,v) \|_{L^2(\mathbb{T}^{n-1})} \| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})} .
\end{aligned}$$

Finally, we integrate in $v$ and add these two estimations, so we obtain

$$\begin{aligned}
\frac{d}{du} \| \tilde{\varphi}_\varepsilon(u) \|^2_{L^2([0;2R]\times\mathbb{T}^{n-1}))} \quad \leq \quad & 2 \| \tilde{\varphi}_\varepsilon(u) \|_{L^2([0;2R]\times\mathbb{T}^{n-1})} c_R(\| \tilde{\varphi}_\varepsilon(u) \|_{L^\infty([0;2R]\times\mathbb{T}^{n-1})}, u) \\
& + 2(2R)^{\frac{1}{2}} \| \tilde{\varphi}_\varepsilon(u) \|_{L^2([0;2R]\times\mathbb{T}^{n-1})} \| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})}
\end{aligned}$$

hence as if $m > \frac{n-1}{2}$ we have $\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1}) \subset L^\infty([0;2R]\times\mathbb{T}^{n-1})$ (see lemma 3.2.2), and we can write

$$\frac{d}{du} \| \tilde{\varphi}_\varepsilon(u) \|^2_{L^2([0;2R]\times\mathbb{T}^{n-1}))} \leq c_{1R}(\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})}, u) \tag{3.5.4}$$

with $c_{1R}$ continuous in all its variables.

**Estimation of $\frac{d}{du} \| \frac{\partial^\beta}{\partial y^\beta} \tilde{\varphi}_\varepsilon(u) \|^2_{L^2}$**

Let $\beta \in \mathbb{N}^{n-1},\ 1 \leq |\beta| \leq m$ , we denote $\frac{\partial^\beta}{\partial y^\beta}$ where $\beta = (\beta_1, ..., \beta_{n-1})$ to mean that we differentiate $|\beta_i|$ times with respect to $y_i$.
As $\frac{\partial^\beta}{\partial y^\beta} \tilde{\varphi}_\varepsilon$ is in $C^1(I_\varepsilon \times [0;2R] \times \mathbb{T}^{n-1})$ we can commute $\frac{d}{du}$ and $\int$ , and after as we have done for $\frac{\partial}{\partial u} \tilde{\varphi}_\varepsilon$ in (3.5.1) we use that $\frac{\partial^{\beta+2}}{\partial v \partial u \partial y^\beta} \tilde{\varphi}_\varepsilon$ equals $\frac{\partial^{\beta+2}}{\partial u \partial v \partial y^\beta} \tilde{\varphi}_\varepsilon$ and so is continuous

, hence

$$\begin{aligned}\frac{d}{du} \parallel \frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon(u) \parallel^2_{L^2([0;2R]\times\mathbb{T}^{n-1})} &= 2\int_0^{2R}\int_{\mathbb{T}^{n-1}} \frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon \frac{\partial}{\partial u}\frac{\partial^\beta}{\partial y^\beta}(\tilde{\varphi}_\varepsilon) dv\ d^{n-1}y \\ &= 2\int_0^{2R}\int_{\mathbb{T}^{n-1}} \frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon [\frac{\partial}{\partial u}(\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon)(u,0,y) \\ &\qquad + \int_0^v \frac{\partial}{\partial v}\frac{\partial^{\beta+1}}{\partial u\partial y^\beta}(\tilde{\varphi}_\varepsilon)(u,s,y)\ ds] dv\ d^{n-1}y.\end{aligned}$$

We can show that $(\frac{\partial}{\partial u}(\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon))(u,0,y)$ equal zero in the same way as we have done for $\frac{\partial}{\partial u}(\tilde{\varphi}_\varepsilon)(u,0,y) = 0$ because for any $(u,y)$ in $I_\varepsilon \times \mathbb{T}^{n-1}$, we have $\tilde{\varphi}_\varepsilon(u,0,y) = 0$ , and for any $|\gamma| \leq |\beta|$ , $\frac{\partial^\gamma}{\partial y^\gamma}(\tilde{\varphi}_\varepsilon)$ is in $C^1(I_\varepsilon \times [0;2R] \times \mathbb{T}^{n-1})$ so we can permute the limits in $v = 0$ and in $h_1 = 0, \ldots, h_{\beta+1} = 0$ for the partial derivatives in $u$ and $y^\beta$ .

Finally, as $\frac{\partial^{\beta+2}}{\partial v\partial u\partial y^\beta}\tilde{\varphi}_\varepsilon$ equals $\frac{\partial^{\beta+2}}{\partial y^\beta\partial v\partial u}\tilde{\varphi}_\varepsilon$ and by using the second equation of (3.4.1), we obtain

$$\begin{aligned}&\frac{d}{du} \parallel \frac{\partial^\beta}{\partial y^\beta}\varphi_\varepsilon(u) \parallel^2_{L^2([0;2R]\times\mathbb{T}^{n-1})} \\ &\quad = 2\int_0^{2R}\int_{\mathbb{T}^{n-1}} \frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon \int_0^v \frac{\partial^\beta}{\partial y^\beta}(\hat{J}_\varepsilon\tilde{H}(\tilde{\varphi}_\varepsilon,u,s,y) + \Delta_y\tilde{\varphi}_\varepsilon) ds\ dv\ d^{n-1}y\end{aligned}$$

Now we will show that $\frac{\partial^\beta}{\partial y^\beta}(\hat{J}_\varepsilon(\tilde{H}(\tilde{\varphi}_\varepsilon,u,s,y)) = \hat{J}_\varepsilon(\frac{\partial^\beta}{\partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon,u,s,y))$. By the definition of $\hat{J}_\varepsilon$, and in the end by doing an integration by parts, we have

$$\begin{aligned}&\frac{\partial}{\partial y_i}\hat{J}_\varepsilon\tilde{H}(\tilde{\varphi}_\varepsilon,u,s,y) \\ &\quad = \frac{\partial}{\partial y_i}(\sum_{|\alpha|\leq\frac{1}{\varepsilon}}(2R)^{-\frac{1}{2}}T^{-\frac{n-1}{2}}\int_0^{2R}\int_{\mathbb{T}^{n-1}} e^{-i(\alpha_0 w\frac{\pi}{R}+\overline{\alpha}.z\frac{2\pi}{T})}\tilde{H}(\tilde{\varphi}_\varepsilon,u,w,z) dw\ d^{n-1}z\ \psi_\alpha(s,y)\ ) \\ &\quad = \sum_{|\alpha|\leq\frac{1}{\varepsilon}}\int_0^{2R}\int_{\mathbb{T}^{n-1}}(2R)^{-\frac{1}{2}}T^{-\frac{n-1}{2}}(\frac{2\pi}{T}\alpha_i)e^{-i(\alpha_0 w\frac{\pi}{R}+\overline{\alpha}.z\frac{2\pi}{T})}\tilde{H}(\tilde{\varphi}_\varepsilon,u,w,z) dw\ d^{n-1}z\ \psi_\alpha(s,y) \\ &\quad = \sum_{|\alpha|\leq\frac{1}{\varepsilon}} -\int_0^{2R}\int_{\mathbb{T}^{n-2}}(2R)^{-\frac{1}{2}}T^{-\frac{n-1}{2}}[e^{-i(\alpha_0 w\frac{\pi}{R}+\overline{\alpha}.z\frac{2\pi}{T})}\tilde{H}(\tilde{\varphi}_\varepsilon,u,w,z)]_{z_i\in\mathbb{T}}\ dw\ d^{n-2}z\ \psi_\alpha(s,y) \\ &\qquad + \hat{J}_\varepsilon(\frac{\partial}{\partial y_i}\tilde{H}(\tilde{\varphi}_\varepsilon,u,s,y))\end{aligned}$$

where $[f(z_i)]_{z_i\in\mathbb{T}}$ means $f(b) - f(a)$ if $\mathbb{T} = [a;b]$. We have supposed that $\tilde{H}$ and $\tilde{\varphi}_-$ are T-periodic in each $y_i$ , it implies that $\tilde{\varphi}_\varepsilon$ is T-periodic in each $y_i$ (by uniqueness of solution given by the Cauchy-Lipschitz theorem in section 4), thus the first part of the second member in the equation above vanishes and we have

$$\frac{\partial}{\partial y_i}\hat{J}_\varepsilon\tilde{H}(\tilde{\varphi}_\varepsilon,u,s,y) = \hat{J}_\varepsilon\frac{\partial}{\partial y_i}\tilde{H}(\tilde{\varphi}_\varepsilon,u,s,y)$$

For higher derivatives, we proceed by recurrence with the same method (we can notice that for any $|\gamma+\nu| \leq |\beta|$ , the functions $\frac{\partial^\beta}{\partial\theta^\gamma\partial y^\nu}\tilde{H}$ , $\frac{\partial^\gamma}{\partial y^\gamma}\tilde{\varphi}_\varepsilon$ are also T-periodic in each $y_i$).
So the following holds :
For any $\beta \in I\!N^{n-1}$, $1 \leq \beta \leq m$ ,

$$\frac{\partial^\beta}{\partial y^\beta}(\hat{J}_\varepsilon \tilde{H}(\tilde{\varphi}_\varepsilon, u, s, y)) = \hat{J}_\varepsilon(\frac{\partial^\beta}{\partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon, u, s, y)). \tag{3.5.5}$$

Hence we obtain

$$\begin{aligned}
&\frac{d}{du} \| \frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon(u) \|^2_{L^2([0;2R]\times\mathbb{T}^{n-1})} \\
&= 2\int_0^{2R}\int_{\mathbb{T}^{n-1}} (\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon) \int_0^v [\hat{J}_\varepsilon \frac{\partial^\beta}{\partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon, u, s, y) + \Delta_y\tilde{\varphi}_\varepsilon]ds\ dv\ d^{n-1}y \\
&= 2\int_0^{2R}\int_{\mathbb{T}^{n-1}}\int_0^v (\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon)[\hat{J}_\varepsilon \frac{\partial^\beta}{\partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon, u, s, y) + \Delta_y\tilde{\varphi}_\varepsilon]ds\ dv\ d^{n-1}y
\end{aligned}$$

(we can put $\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon$ under $\int_0^v$ by continuity of the functions on $[0; 2R] \times \mathbb{T}^{n-1}$).

Now for the first part, as we have done before, by using the fact that $v$ is in $[0; 2R]$, Cauchy-Schwarz inequality, and (3.5.3) we can bound as follows

$$\begin{aligned}
&|\int_{\mathbb{T}^{n-1}}\int_0^v (\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon)\quad \hat{J}_\varepsilon \tfrac{\partial^\beta}{\partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon, u, s, y)ds\ d^{n-1}y| \\
&\leq \int_{\mathbb{T}^{n-1}}\int_0^{2R} |(\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon)\hat{J}_\varepsilon\frac{\partial^\beta}{\partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon, u, s, y)|ds\ d^{n-1}y \\
&\leq \| \frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon(u) \|_{L^2([0;2R]\times\mathbb{T}^{n-1})} \| \hat{J}_\varepsilon\frac{\partial^\beta}{\partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon, u, s, y) \|_{L^2([0;2R]\times\mathbb{T}^{n-1})} \\
&\leq \| \frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon(u) \|_{L^2([0;2R]\times\mathbb{T}^{n-1})} \| \frac{\partial^\beta}{\partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon, u, s, y) \|_{L^2([0;2R]\times\mathbb{T}^{n-1})} .
\end{aligned}$$

Therefore we notice that $\frac{\partial^\beta}{\partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon, u, s, y)$ is a sum of $(\frac{\partial^{\delta+\mu}}{\partial\theta^\delta\partial y^\mu}\tilde{H})(\tilde{\varphi}_\varepsilon, u, s, y)\prod_\nu \frac{\partial^\nu}{\partial y^\nu}\tilde{\varphi}_\varepsilon(u, s, y)$ with $|\delta+\mu| \leq |\beta|$ and $\sum|\nu| \leq |\beta|$. By assumption (3.3.6) we know that $\frac{\partial^{\delta+\mu}}{\partial\theta^\delta\partial y^\mu}\tilde{H}$ is continuous, so when we take the norm $L^2$ of $\frac{\partial^\beta}{\partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon, u, s, y)$, we can extract it, thus we obtain

$$\begin{aligned}
&\| \frac{\partial^\beta}{\partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon, u, s, y) \|_{L^2([0;2R]\times\mathbb{T}^{n-1})} \\
&\leq \sum_{|\delta+\mu|\leq|\beta|} \max_{\substack{s\in[0;2R]\\ \theta\in\Theta_\varepsilon\\ y\in\mathbb{T}^{n-1}}} |\frac{\partial^{\delta+\mu}}{\partial\theta^\delta\partial y^\mu}\tilde{H}(\theta, u, s, y)| \, \| \prod_\nu \frac{\partial^\nu}{\partial y^\nu}\tilde{\varphi}_\varepsilon(u) \|_{L^2([0;2R]\times\mathbb{T}^{n-1})}
\end{aligned}$$

where $\Theta_\varepsilon = [- \| \tilde{\varphi}_\varepsilon(u) \|_{L^\infty([0;2R]\times\mathbb{T}^{n-1})}, \| \tilde{\varphi}_\varepsilon(u) \|_{L^\infty([0;2R]\times\mathbb{T}^{n-1})}]$. Then as we know that $\tilde{\varphi}_\varepsilon(u, v)$ is in $C^0(\mathbb{T}^{n-1}) \cap H^m(\mathbb{T}^{n-1})$, we can apply the proposition 3.6 page 9 of Taylor

[16] (which is still available with $\mathbb{T}^{n-1}$ instead of $I\!R^n$) with $f = g = \tilde{\varphi}_\varepsilon(u,v)$, thus we get

$$\| \prod_\nu \frac{\partial^\nu}{\partial y^\nu} \tilde{\varphi}_\varepsilon(u,v) \|_{L^2(\mathbb{T}^{n-1})} \leq c \| \tilde{\varphi}_\varepsilon(u,v) \|_{L^\infty(\mathbb{T}^{n-1})} \| \tilde{\varphi}_\varepsilon(u,v) \|_{H^m(\mathbb{T}^{n-1})} .$$

Now we integrate the square of this inequality in $v$ on $[0;2R]$, it gives

$$\| \prod_\nu \frac{\partial^\nu}{\partial y^\nu} \tilde{\varphi}_\varepsilon(u) \|^2_{L^2([0;2R]\times\mathbb{T}^{n-1})} \leq c^2 \| \tilde{\varphi}_\varepsilon(u) \|^2_{L^\infty([0;2R]\times\mathbb{T}^{n-1})} \| \tilde{\varphi}_\varepsilon(u) \|^2_{\mathcal{H}_{m,0}([0;2R]\times\mathbb{T}^{n-1})} .$$

Hence we have

$$\begin{aligned} &\| \frac{\partial^\beta}{\partial y^\beta} \tilde{H}(\tilde{\varphi}_\varepsilon, u, s, y) \|_{L^2([0;2R]\times\mathbb{T}^{n-1})} \\ &\quad \leq c \sum_{|\delta+\mu|\leq|\beta|} \max_{\substack{s\,\in\,[0;2R]\\ \theta\,\in\,\Theta_\varepsilon \\ y\,\in\,\mathbb{T}^{n-1}}} |\frac{\partial^{\delta+\mu}}{\partial\theta^\delta \partial y^\mu} \tilde{H}(\theta,u,s,y)| \| \tilde{\varphi}_\varepsilon(u) \|_{L^\infty} \| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,0}} . \end{aligned} \tag{3.5.6}$$

Therefore if $m > \frac{n-1}{2}$, we obtain

$$|\int_{\mathbb{T}^{n-1}} \int_0^v (\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon) \hat{J}_\varepsilon \frac{\partial^\beta}{\partial y^\beta} \tilde{H}(\tilde{\varphi}_\varepsilon, u, s, y) ds\, d^{n-1}y| \quad\leq\quad C_{2R}(\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})}, u)$$

with $C_{2R}$ continuous in all its variables.

**Remark 3.5.2** : Here we see that in our argument we can't consider a more general class of wave equations with $\tilde{H}$ depending on the gradient of $\tilde{\varphi}$. Indeed, if we insert even just a $\frac{\partial}{\partial y_i}\tilde{\varphi}_\varepsilon$ into $\tilde{H}$, when we will calculate $\frac{\partial^m}{\partial y^m}\tilde{H}(\tilde{\varphi}_\varepsilon, u, s, y)$, it will provide a term with a factor $\frac{\partial^{m+1}}{\partial y^{m+1}}\tilde{\varphi}_\varepsilon$. This term does not suit because we need a bound depending on $\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}}$ and not on $\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m+1,2}}$ to apply the nonlinear Gronwall's lemma at the next step.

Then by integrating in $v$ on $[0;2R]$

$$\begin{aligned} 2\int_0^{2R} \int_{\mathbb{T}^{n-1}} \int_0^v (\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon) \quad & \hat{J}_\varepsilon \tfrac{\partial^\beta}{\partial y^\beta} \tilde{H}(\tilde{\varphi}_\varepsilon, u, s, y) ds\, dv\, d^{n-1}y \\ & \leq 2RC_{2R}(\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})}, u). \end{aligned}$$

On another hand, for the second part, by continuity of the functions we can commute $\int_{\mathbb{T}^{n-1}}$ and $\int_0^v$, and as $\frac{\partial^{\beta-1}}{\partial y^{\beta-1}}\tilde{\varphi}_\varepsilon$ is T-periodic in each $y_i$ , we have by integrating by parts in each $y_i$ on $\mathbb{T}$ :

$$\begin{aligned} 2\int_0^{2R} \int_{\mathbb{T}^{n-1}} \int_0^v & \frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon \frac{\partial^\beta}{\partial y^\beta} \Delta_y \tilde{\varphi}_\varepsilon ds\, dv\, d^{n-1}y \\ & = 2\int_0^{2R} \int_0^v - \sum_{j=1}^{n-1} \int_{\mathbb{T}^{n-1}} \frac{\partial}{\partial y_j}(\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon) \frac{\partial}{\partial y_j}(\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon) dv\, ds\, d^{n-1}y \\ & = -2\sum_{j=1}^{n-1} \int_0^{2R} \int_0^v \int_{\mathbb{T}^{n-1}} (\frac{\partial}{\partial y_j}\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon)^2 dv\, ds\, d^{n-1}y \\ & \leq 0. \end{aligned}$$

Thus

$$\frac{d}{du} \| \frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon(u) \|^2_{L^2([0;2R]\times\mathbb{T}^{n-1})} \leq 2RC_{2R}(\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})}, u) \qquad (3.5.7)$$

**Estimation of** $\frac{d}{du} \| \frac{\partial}{\partial v}\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon(u) \|^2_{L^2}$

For any $\beta \in I\!N^{n-1},\ 0 \leq |\beta| \leq m$ , as $\frac{\partial}{\partial v}\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon$ is in $C^1(I_\varepsilon \times [0;2R] \times \mathbb{T}^{n-1})$ we can commute $\frac{d}{du}$ and $\int$ , and $\frac{\partial^{\beta+2}}{\partial u\partial v\partial y^\beta}\tilde{\varphi}_\varepsilon$ equals $\frac{\partial^{\beta+2}}{\partial y^\beta\partial u\partial v}\tilde{\varphi}_\varepsilon$ , we have

$$\frac{d}{du} \| \frac{\partial}{\partial v}\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon(u) \|^2_{L^2([0;2R]\times\mathbb{T}^{n-1})} = 2\int_0^{2R}\int_{\mathbb{T}^{n-1}}(\frac{\partial^{\beta+1}}{\partial v\partial y^\beta}\tilde{\varphi}_\varepsilon)\frac{\partial^\beta}{\partial y^\beta}(\frac{\partial^2}{\partial u\partial v}\tilde{\varphi}_\varepsilon)dv\ d^{n-1}y.$$

Then by using the second equation of (3.4.1) and (3.5.5) , we obtain

$$\begin{aligned}&\frac{d}{du} \| \frac{\partial}{\partial v}\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon(u) \|^2_{L^2([0;2R]\times\mathbb{T}^{n-1})}\\ &\qquad = 2\int_0^{2R}\int_{\mathbb{T}^{n-1}}(\frac{\partial^{\beta+1}}{\partial v\partial y^\beta}\tilde{\varphi}_\varepsilon)(\hat{J}_\varepsilon\frac{\partial^\beta}{\partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon,u,v,y) + \frac{\partial^\beta}{\partial y^\beta}\Delta_y\tilde{\varphi}_\varepsilon)dv\ d^{n-1}y.\end{aligned}$$

As we have done in (3.5.6), we can deduce that

$$|\int_0^{2R}\int_{\mathbb{T}^{n-1}}(\frac{\partial^{\beta+1}}{\partial v\partial y^\beta}\tilde{\varphi}_\varepsilon)\hat{J}_\varepsilon\frac{\partial^\beta}{\partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon,u,s,y)dv\ d^{n-1}y| \leq C_{3R}(\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})}, u)$$

with $C_{3R}$ continuous in all its variables.

For the second part, by integrating by parts in each $y_i$ on T , as $\frac{\partial^{\beta+1}}{\partial v\partial y^\beta}\tilde{\varphi}_\varepsilon$ and $\frac{\partial^{\beta+1}}{\partial y_j\partial y^\beta}\tilde{\varphi}_\varepsilon$ are T-periodic in each $y_i$ , we have :

$$\int_0^{2R}\int_{\mathbb{T}^{n-1}}(\frac{\partial^{\beta+1}}{\partial v\partial y^\beta}\tilde{\varphi}_\varepsilon)\frac{\partial^\beta}{\partial y^\beta}\Delta_y\tilde{\varphi}_\varepsilon dv\ d^{n-1}y = \int_0^{2R} -\sum_{j=1}^{n-1}\int_{\mathbb{T}^{n-1}}\frac{\partial}{\partial y_j}(\frac{\partial^{\beta+1}}{\partial v\partial y^\beta}\tilde{\varphi}_\varepsilon)\frac{\partial}{\partial y_j}\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon dv\ d^{n-1}y.$$

We know that $\frac{\partial}{\partial y_j}\frac{\partial^{\beta+1}}{\partial v\partial y^\beta}\tilde{\varphi}_\varepsilon = \frac{\partial}{\partial v}\frac{\partial^{\beta+1}}{\partial y_j\partial y^\beta}\tilde{\varphi}_\varepsilon$ , thus

$$\begin{aligned}\int_0^{2R}\int_{\mathbb{T}^{n-1}}(\frac{\partial^{\beta+1}}{\partial v\partial y^\beta}\tilde{\varphi}_\varepsilon)\frac{\partial^\beta}{\partial y^\beta}\Delta_y\tilde{\varphi}_\varepsilon dv\ d^{n-1}y &= -\sum_{j=1}^{n-1}\int_{\mathbb{T}^{n-1}}\int_0^{2R}\frac{\partial}{\partial v}(\frac{\partial^{\beta+1}}{\partial y_j\partial y^\beta}\tilde{\varphi}_\varepsilon)\frac{\partial^{\beta+1}}{\partial y_j\partial y^\beta}\tilde{\varphi}_\varepsilon dv\ d^{n-1}y\\ &= -\sum_{j=1}^{n-1}\int_{\mathbb{T}^{n-1}}\frac{1}{2}[(\frac{\partial^{\beta+1}}{\partial y_j\partial y^\beta}\tilde{\varphi}_\varepsilon)^2]d^{n-1}y.\end{aligned}$$

But $\frac{\partial^{\beta+1}}{\partial y_j\partial y^\beta}\tilde{\varphi}_\varepsilon(u,0,y) = 0$ , indeed it comes from the third equation of (3.4.1) and the continuity of all the functions $\frac{\partial^\gamma}{\partial y^\gamma}\tilde{\varphi}_\varepsilon$ on $[0;2R]\times\mathbb{T}^{n-1}$ , so we get

$$\begin{aligned}\int_0^{2R}\int_{\mathbb{T}^{n-1}}(\frac{\partial^{\beta+1}}{\partial v\partial y^\beta}\tilde{\varphi}_\varepsilon)\frac{\partial^\beta}{\partial y^\beta}\Delta_y\tilde{\varphi}_\varepsilon dv\ d^{n-1}y &= -\sum_{j=1}^{n-1}\int_{\mathbb{T}^{n-1}}\frac{1}{2}\big(\frac{\partial^{\beta+1}}{\partial y_j\partial y^\beta}\tilde{\varphi}_\varepsilon(u,2R,y)\big)^2 d^{n-1}y\\ &\leq 0.\end{aligned}$$

Finally, we have if $m > n-1$,

$$\frac{d}{du} \parallel \frac{\partial}{\partial v}\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon(u) \parallel^2_{L^2([0;2R]\times\mathbb{T}^{n-1})} \leq C_{3R}(\parallel \tilde{\varphi}_\varepsilon(u) \parallel_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})}, u) \quad (3.5.8)$$

with $C_{3R}$ continuous in all its variables.

**Estimation of $\frac{d}{du} \parallel \frac{\partial^2}{\partial v^2}\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon(u) \parallel^2_{L^2}$**

For any $\beta \in I\!N^{n-1}$, $0 \leq |\beta| \leq m$ , as $\frac{\partial^2}{\partial v^2}\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon$ is in $C^1(I_\varepsilon \times [0;2R]\times\mathbb{T}^{n-1})$ and $\frac{\partial^{\beta+3}}{\partial u\partial v^2\partial y^\beta}\tilde{\varphi}_\varepsilon$ equals $\frac{\partial^{\beta+3}}{\partial v\partial y^\beta\partial u\partial v}\tilde{\varphi}_\varepsilon$ we can proceed as before, so

$$\begin{aligned}
&\frac{d}{du} \parallel \frac{\partial^2}{\partial v^2}\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon(u) \parallel^2_{L^2([0;2R]\times\mathbb{T}^{n-1})} \\
&\qquad = 2\int_0^{2R}\int_{\mathbb{T}^{n-1}}(\frac{\partial^{\beta+2}}{\partial v^2\partial y^\beta}\tilde{\varphi}_\varepsilon)\frac{\partial^{\beta+1}}{\partial v\partial y^\beta}(\frac{\partial^2}{\partial u\partial v}\tilde{\varphi}_\varepsilon)dv\ d^{n-1}y \\
&\qquad = 2\int_0^{2R}\int_{\mathbb{T}^{n-1}}(\frac{\partial^{\beta+2}}{\partial v^2\partial y^\beta}\tilde{\varphi}_\varepsilon)\frac{\partial^{\beta+1}}{\partial v\partial y^\beta}(\hat{J}_\varepsilon\tilde{H}(\tilde{\varphi}_\varepsilon,u,v,y)+\Delta_y\tilde{\varphi}_\varepsilon)dv\ d^{n-1}y. \quad (3.5.9)
\end{aligned}$$

We estimate the first part of (3.5.9) corresponding to the first term in the sum beyond. By (3.5.5) we have

$$\frac{\partial^{\beta+1}}{\partial v\partial y^\beta}(\hat{J}_\varepsilon\tilde{H}(\tilde{\varphi}_\varepsilon,u,v,y)) = \frac{\partial}{\partial v}(\hat{J}_\varepsilon\frac{\partial^\beta}{\partial y^\beta}(\tilde{H}(\tilde{\varphi}_\varepsilon,u,v,y))).$$

Now by integrating by parts on $[0;2R]$, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial v}(\hat{J}_\varepsilon\frac{\partial^\beta}{\partial y^\beta}(\tilde{H}(\tilde{\varphi}_\varepsilon,u,v,y))) \\
&\quad = \sum_{|\alpha|\leq\frac{1}{\varepsilon}}\int_0^{2R}\int_{\mathbb{T}^{n-1}}(2R)^{-\frac{1}{2}}T^{-\frac{n-1}{2}}(\frac{\pi}{R}\alpha_0)e^{-i(\alpha_0 w\frac{\pi}{R}+\overline{\alpha}.z\frac{2\pi}{T})}\frac{\partial^\beta}{\partial y^\beta}(\tilde{H}(\tilde{\varphi}_\varepsilon,u,w,z))dw\ d^{n-1}z\ \psi_\alpha(s,y) \\
&\quad = \sum_{|\alpha|\leq\frac{1}{\varepsilon}} -\int_{\mathbb{T}^{n-1}}(2R)^{-\frac{1}{2}}T^{-\frac{n-1}{2}}[e^{-i(\alpha_0 w\frac{\pi}{R}+\overline{\alpha}.z\frac{2\pi}{T})}\frac{\partial^\beta}{\partial y^\beta}(\tilde{H}(\tilde{\varphi}_\varepsilon,u,w,z))]_0^{2R}\ d^{n-1}z\ \psi_\alpha(s,y) \\
&\qquad\quad + \hat{J}_\varepsilon\frac{\partial^{\beta+1}}{\partial v\partial y^\beta}(\tilde{H}(\tilde{\varphi}_\varepsilon,u,v,y))
\end{aligned}$$

where $[f(w)]_0^{2R}$ means $f(2R)-f(0)$.

But $\frac{\partial^\beta}{\partial y^\beta}(\tilde{H}(\tilde{\varphi}_\varepsilon(u,2R,z),u,2R,z)) = 0$, indeed $\tilde{H}$ is a product of a function $f$ by $v \mapsto \phi_R(v)$, so for any $\nu$ such that $|\nu| \leq |\beta|$, $\frac{\partial^\nu}{\partial y^\nu}\tilde{H} = \phi_R\frac{\partial^\nu}{\partial y^\nu}f$ and $\phi_R(2R)=0$.

On another hand, $\frac{\partial^\beta}{\partial y^\beta}(\tilde{H}(\tilde{\varphi}_\varepsilon(u,0,z),u,0,z)) = 0$. Indeed

$$\begin{aligned}
&\frac{\partial}{\partial y_i}(\tilde{H}(\tilde{\varphi}_\varepsilon(u,v,y),u,v,y)) \\
&\qquad = (\frac{\partial}{\partial y_i}\tilde{H})(\tilde{\varphi}_\varepsilon(u,v,y),u,v,y) + (\frac{\partial}{\partial\theta}\tilde{H})(\tilde{\varphi}_\varepsilon(u,v,y),u,v,y)(\frac{\partial}{\partial y_i}\tilde{\varphi}_\varepsilon)(u,v,y).
\end{aligned}$$

For the first term, $\tilde{\varphi}_\varepsilon(u,0,z) = 0$ and we can permute in the expression $(\tilde{H}(\theta,u,v,y+he_i) - \tilde{H}(\theta,u,v,y))/h$ the limit in $(\theta,v) = (0,0)$ and the limit in $h = 0$ corresponding to $\frac{\partial}{\partial y_i}$ because of the regularity of $\tilde{H}$. As $\tilde{H}(0,u,0,y) = 0$ for any $(u,y)$ (see (3.3.3)), this first term vanishes at $v = 0$. For the second term, we already have seen that $(\frac{\partial}{\partial y_i}\tilde{\varphi}_\varepsilon)(u,0,y) = 0$ so it vanishes at $v = 0$. For higher derivatives we proceed similarly.

Hence we obtain

$$\frac{\partial^{\beta+1}}{\partial v \partial y^\beta}(\hat{J}_\varepsilon \tilde{H}(\tilde{\varphi}_\varepsilon,u,v,y)) = \hat{J}_\varepsilon(\frac{\partial^{\beta+1}}{\partial v \partial y^\beta}(\tilde{H}(\tilde{\varphi}_\varepsilon,u,v,y))). \tag{3.5.10}$$

**Remark 3.5.3** : We can see here that we can't get an estimation with higher derivatives than two in $v$. Indeed, in $\dfrac{\partial^2}{\partial v^2}\hat{J}_\varepsilon \tilde{H}(\tilde{\varphi}_\varepsilon,u,v,y)$ appears a term $\dfrac{\partial}{\partial \theta}\tilde{H}(\tilde{\varphi}_\varepsilon(u,0,y),u,0,y)\dfrac{\partial}{\partial v}\tilde{\varphi}_\varepsilon(u,0,y)$ under the sum on $|\alpha| \leq \frac{1}{\varepsilon}$ and there's no reason for it to vanish. Then if we keep it, the estimation contains a factor of type $c(\frac{1}{\varepsilon})$ which is not uniformly bounded as $\varepsilon$ goes to 0.

Now we can write that if $m > \frac{n-1}{2}$,

$$\begin{aligned}
&2\int_0^{2R}\int_{\mathbb{T}^{n-1}}(\frac{\partial^{\beta+2}}{\partial v^2 \partial y^\beta}\tilde{\varphi}_\varepsilon)\hat{J}_\varepsilon(\frac{\partial^{\beta+1}}{\partial v \partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon,u,v,y))dv\, d^{n-1}y \\
&\quad \leq 2 \,\| \frac{\partial^{\beta+2}}{\partial v^2 \partial y^\beta}\tilde{\varphi}_\varepsilon \|_{L^2([0;2R]\times\mathbb{T}^{n-1})} \| \hat{J}_\varepsilon \frac{\partial^{\beta+1}}{\partial v \partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon,u,v,y) \|_{L^2([0;2R]\times\mathbb{T}^{n-1})} \\
&\quad \leq 2 \,\| \frac{\partial^{\beta+2}}{\partial v^2 \partial y^\beta}\tilde{\varphi}_\varepsilon \|_{L^2([0;2R]\times\mathbb{T}^{n-1})} \| \frac{\partial^{\beta+1}}{\partial v \partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon,u,v,y) \|_{L^2([0;2R]\times\mathbb{T}^{n-1})} \\
&\quad \leq C_{4R}(\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})}, u)
\end{aligned}$$

because of the assumptions (3.3.6) on $\tilde{H}$, with $C_{4R}$ continuous in all its variables. Indeed we bound the second factor of the right member above as we have done in (3.5.6), by applying the proposition 3.6 page 9 of Taylor [16] with $f = \tilde{\varphi}_\varepsilon(u,v)$ and $g = \frac{\partial}{\partial v}\tilde{\varphi}_\varepsilon(u,v)$, it gives

$$\begin{aligned}
&\| \frac{\partial^{\nu_1}}{\partial y^{\nu_1}}\tilde{\varphi}_\varepsilon(u,v)\frac{\partial^{\nu_2+1}}{\partial y^{\nu_2}\partial v}\tilde{\varphi}_\varepsilon(u,v) \|_{L^2(\mathbb{T}^{n-1})} \\
&\quad \leq c \,\| \tilde{\varphi}_\varepsilon(u,v) \|_{L^\infty(\mathbb{T}^{n-1})} \| \frac{\partial}{\partial v}\tilde{\varphi}_\varepsilon(u,v) \|_{H^m(\mathbb{T}^{n-1})} \\
&\qquad + c' \,\| \frac{\partial}{\partial v}\tilde{\varphi}_\varepsilon(u,v) \|_{L^\infty(\mathbb{T}^{n-1})} \| \tilde{\varphi}_\varepsilon(u,v) \|_{H^m(\mathbb{T}^{n-1})} .
\end{aligned}$$

We integrate the square of this inequality in $v$ on $[0;2R]$, use that $(A+B)^2 \leq 2(A^2+B^2)$, thus we obtain by taking the square root and as $\sqrt(A+B) \leq \sqrt(A) + \sqrt(B)$,

$$\begin{aligned}
&\| \frac{\partial^{\nu_1}}{\partial y^{\nu_1}}\tilde{\varphi}_\varepsilon(u,v)\frac{\partial^{\nu_2+1}}{\partial y^{\nu_2}\partial v}\tilde{\varphi}_\varepsilon(u,v) \|_{L^2([0;2R]\times\mathbb{T}^{n-1})} \\
&\quad \leq \sqrt(2)c \,\| \tilde{\varphi}_\varepsilon(u) \|_{L^\infty([0;2R]\times\mathbb{T}^{n-1})} \| \frac{\partial}{\partial v}\tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,0}([0;2R]\times\mathbb{T}^{n-1})} \\
&\qquad + \sqrt(2)c' \,\| \frac{\partial}{\partial v}\tilde{\varphi}_\varepsilon(u) \|_{L^\infty([0;2R]\times\mathbb{T}^{n-1})} \| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,0}([0;2R]\times\mathbb{T}^{n-1})} .
\end{aligned}$$

Then as $\frac{\partial}{\partial v}\tilde{\varphi}_\varepsilon(u)$ is in $\mathcal{H}_{m,1}$ and as if $m > \frac{n-1}{2}$, we have the embedding $\mathcal{H}_{m,1}$ in $L^\infty$, we get

$$\| \frac{\partial^{\beta+1}}{\partial v \partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon, u, v, y) \|_{L^2([0;2R]\times\mathbb{T}^{n-1})} \leq \overline{c}(\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}})$$

with $\overline{c}$ continuous.

Now, we estimate the second part of (3.5.9) corresponding to the second term. We know that we can commute any partial derivatives in $v$ and in $y_i$ on $\tilde{\varphi}_\varepsilon$. By integrating by parts in each $y_i$ on $\mathbb{T}$, as $\frac{\partial^{\beta+2}}{\partial v^2 \partial y^\beta}\tilde{\varphi}_\varepsilon$ and $\frac{\partial^{\beta+2}}{\partial v \partial y^\beta \partial y_i}\tilde{\varphi}_\varepsilon$ are T-periodic, we obtain

$$\begin{aligned}
&2\int_0^{2R}\int_{\mathbb{T}^{n-1}}(\frac{\partial^{\beta+2}}{\partial v^2 \partial y^\beta}\tilde{\varphi}_\varepsilon)\frac{\partial^{\beta+1}}{\partial v \partial y^\beta}(\Delta_y(\tilde{\varphi}_\varepsilon))dv\ d^{n-1}y \\
&\quad = -2\int_0^{2R}\sum_{i=1}^{n}\int_{\mathbb{T}^{n-1}}\frac{\partial}{\partial v}(\frac{\partial^{\beta+2}}{\partial v \partial y_i \partial y^\beta}\tilde{\varphi}_\varepsilon)\ \ (\frac{\partial^{\beta+2}}{\partial v \partial y_i \partial y^\beta}\tilde{\varphi}_\varepsilon)dv\ d^{n-1}y \\
&\quad = -\int_0^{2R}\sum_{i=1}^{n}\int_{\mathbb{T}^{n-1}}\frac{\partial}{\partial v}\ (\frac{\partial^{\beta+2}}{\partial v \partial y_i \partial y^\beta}\tilde{\varphi}_\varepsilon)^2 dv\ d^{n-1}y \\
&\quad = -\sum_{i=1}^{n}\int_{\mathbb{T}^{n-1}}(\frac{\partial^{\beta+2}}{\partial v \partial y_i \partial y^\beta}\tilde{\varphi}_\varepsilon)^2(u, 2R, y) - (\frac{\partial^{\beta+2}}{\partial v \partial y_i \partial y^\beta}\tilde{\varphi}_\varepsilon)^2(u, 0, y) d^{n-1}y.
\end{aligned}$$

The first term is less or equal to zero. For the second one, as $\dfrac{\partial}{\partial v}\dfrac{\partial}{\partial y_i}\dfrac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon$ is in $C^1(I_\varepsilon \times [0; 2R] \times \mathbb{T}^{n-1})$ we can write

$$(\frac{\partial^{\beta+2}}{\partial v \partial y_i \partial y^\beta}\tilde{\varphi}_\varepsilon)(u, 0, y) \ \ = \ \ (\frac{\partial^{\beta+2}}{\partial v \partial y_i \partial y^\beta}\tilde{\varphi}_\varepsilon)(0, 0, y) + \int_0^u (\frac{\partial^{\beta+3}}{\partial u \partial v \partial y_i \partial y^\beta}\tilde{\varphi}_\varepsilon)(s, 0, y)ds.$$

Then as $\tilde{\varphi}_\varepsilon(0, v, y) = \hat{J}_\varepsilon\tilde{\varphi}_-(v, y)$, and by the fact that we can commute the partial derivatives, we have

$$\begin{aligned}
&(\frac{\partial^{\beta+2}}{\partial v \partial y_i \partial y^\beta}\tilde{\varphi}_\varepsilon)(u, 0, y) \\
&\quad = (\frac{\partial^{\beta+2}}{\partial v \partial y_i \partial y^\beta}\hat{J}_\varepsilon\tilde{\varphi}_-)(0, y) + \int_0^u (\frac{\partial^{\beta+1}}{\partial y_i \partial y^\beta}\hat{J}_\varepsilon\tilde{H}(\tilde{\varphi}_\varepsilon, s, 0, y) + \frac{\partial^{\beta+1}}{\partial y_i \partial y^\beta}\Delta_y\tilde{\varphi}_\varepsilon(s, 0, y)\big)ds.
\end{aligned}$$

We have seen in (3.5.5) that we can commute $\hat{J}_\varepsilon$ with the partial derivatives with respect to $y$, and $\dfrac{\partial^{\beta+1}}{\partial y_i \partial y^\beta}\tilde{H}(\tilde{\varphi}_\varepsilon, u, v, y)$ is a sum of $(\dfrac{\partial^{\delta+\mu}}{\partial \theta^\delta \partial y^\mu}\tilde{H})(\tilde{\varphi}_\varepsilon, u, v, y)\prod_\nu \dfrac{\partial^\nu}{\partial y^\nu}\tilde{\varphi}_\varepsilon(u, v, y)$ with $|\delta + \mu| \leq |\beta| + 1$ and $\sum|\nu| \leq |\beta| + 1$. But we know that $\tilde{\varphi}_\varepsilon(u, 0, y) = 0$ so $\dfrac{\partial^\nu}{\partial y^\nu}\tilde{\varphi}_\varepsilon(u, 0, y) = 0$ and for the term $\big(\dfrac{\partial^{\beta+1}}{\partial y_i \partial y^\beta}\tilde{H}\big)(\tilde{\varphi}_\varepsilon(s, 0, y), s, 0, y) = \big(\dfrac{\partial^{\beta+1}}{\partial y_i \partial y^\beta}\tilde{H}\big)(0, s, 0, y)$, as $\tilde{H}(0, s, 0, y) = 0$ it vanishes. Thus it only stays $(\dfrac{\partial^{\beta+2}}{\partial v \partial y_i \partial y^\beta}\hat{J}_\varepsilon\tilde{\varphi}_-)(0, y)$. We show that

$$\frac{\partial^{\beta+2}}{\partial v \partial y_i \partial y^\beta}\hat{J}_\varepsilon\tilde{\varphi}_- \ \ = \ \ \hat{J}_\varepsilon\frac{\partial^{\beta+2}}{\partial v \partial y_i \partial y^\beta}\tilde{\varphi}_-$$

by proceeding as we have done in (3.5.10) because $\tilde{\varphi}_-$ is T-periodic, $\tilde{\varphi}_-$ is a product with a factor $\phi_R$ and $\frac{\partial^{\beta+1}}{\partial y_i \partial y^\beta}\tilde{\varphi}_-(0,z) = 0$. Indeed by (3.3.4)

$$\frac{\partial^{\beta+1}}{\partial y_i \partial y^\beta}\tilde{\varphi}_-(w,z) \;=\; \frac{\partial^{\beta+1}}{\partial y_i \partial y^\beta}\varphi_-(w,z) - \frac{\partial^{\beta+1}}{\partial y_i \partial y^\beta}\varphi_+(0,z) - \frac{\partial^{\beta+1}}{\partial y_i \partial y^\beta}\frac{\partial}{\partial v}\varphi_-(0,z)w$$

hence

$$\frac{\partial^{\beta+1}}{\partial y_i \partial y^\beta}\tilde{\varphi}_-(0,z) \;=\; \frac{\partial^{\beta+1}}{\partial y_i \partial y^\beta}\varphi_-(0,z) - \frac{\partial^{\beta+1}}{\partial y_i \partial y^\beta}\varphi_+(0,z) \;=\; 0$$

by the corner condition $\varphi_-(0,y) = \varphi_+(0,y)$. Now as $\| \hat{J}_\varepsilon f \|_{L^2(\mathbb{T}^{n-1})} \leq \| f \|_{L^2(\mathbb{T}^{n-1})}$, we get

$$\begin{aligned}\sum_{i=1}^n \int_{\mathbb{T}^{n-1}} \left(\frac{\partial^{\beta+2}}{\partial v \partial y_i \partial y^\beta}\tilde{\varphi}_\varepsilon\right)^2(u,0,y)d^{n-1}y \;&=\; \sum_{i=1}^n \int_{\mathbb{T}^{n-1}} \left(\frac{\partial^{\beta+2}}{\partial v \partial y_i \partial y^\beta}\tilde{\varphi}_-\right)^2(0,y)d^{n-1}y \\ &\leq\; \sum_{i=1}^n \| \frac{\partial^{\beta+2}}{\partial v \partial y_i \partial y^\beta}\tilde{\varphi}_-(0) \|^2_{L^2(\mathbb{T}^{n-1})} \;\leq\; \tilde{c}\end{aligned}$$

by the assumptions on $\tilde{\varphi}_-$.

Finally, we obtain if $m > \frac{n-1}{2}$,

$$\frac{d}{du} \| \frac{\partial^2}{\partial v^2}\frac{\partial^\beta}{\partial y^\beta}\tilde{\varphi}_\varepsilon(u) \|^2_{L^2([0;2R]\times\mathbb{T}^{n-1})} \;\leq\; C_{4R}(\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})}, u) \quad (3.5.11)$$

with $C_{4R}$ continuous in all its variables.

**Conclusion**

Now it suffices to add (3.5.4), (3.5.7), (3.5.8), (3.5.11), and we can conclude that if $m > \frac{n-1}{2}$,

$$\frac{d}{du} \| \tilde{\varphi}_\varepsilon(u) \|^2_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})} \leq \mathcal{F}(\| \tilde{\varphi}_\varepsilon(u) \|^2_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})}, u)$$

with $\mathcal{F}$ continuous in both variables.

### 3.5.2 bound of $\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})}$

**Proposition 3.5.2** *If $m > \frac{n-1}{2}$ , there exists a interval $[0; B_R[$ and a function $h_R : [0; B_R[ \to I\!R$ such that*

*(i) $\tilde{\varphi}_\varepsilon$ exist on $[0; B_R[\times[0; 2R] \times \mathbb{T}^{n-1}$*

*(ii) we have the following estimation for all $u$ in $[0; B_R[$*

$$\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})} \leq h_R(u)$$

*with $h_R$ continuous in its variable.*

**Proof :**
We first apply the nonlinear differential Gronwall's lemma, recall if $f$ is $C^1(I)$ with $I$ real interval including 0, $f(0) \leq M$ , $\dfrac{df}{dt} \leq F(f,t)$ , and $F$ continuous then there exists $I(M)$ including 0 and a continuous function $G_M : t \mapsto G_M(t)$ defined on $I(M)$ such that $f(t) \leq G_M(t)$ on $I \bigcap I(M) \bigcap I\!R^+$.
Here $f(u) = \| \tilde{\varphi}_\varepsilon(u) \|^2_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})}$ , $f(0) = \| \tilde{\varphi}_- \|^2_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})} \leq c(R)$ and $I = I_\varepsilon$.
So there exists $I\big(c(R)\big)$ including 0 and $G_R : u \mapsto G_R(u)$ continuous and defined on $I\big(c(R)\big)$ such that $\| \tilde{\varphi}_\varepsilon(u) \|^2_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})} \leq G_R(u)$ for all $u$ in $I_\varepsilon \bigcap I(R) \bigcap I\!R^+$.

Let $[0; B_R[= I\big(c(R)\big) \bigcap I\!R^+$. Now we want to show that $[0; B_R[$ is included in $I_\varepsilon$. Let $I_\varepsilon = ] - T_\varepsilon^-; T_\varepsilon^+[$ the maximal interval of existence of $\tilde{\varphi}_\varepsilon$ with respect to its variable $u$. Suppose that $T_\varepsilon^+ < B_R$ , we set $c^2 = \max\limits_{0\leq u\leq T_\varepsilon^+} G_R(u)$ then we have

$$\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})} \leq c \text{ on } [0; T_\varepsilon^+ - \frac{T}{2}] \qquad (\text{for any } T < 2T_\varepsilon^+).$$

Let $K = [0; 2T_\varepsilon^+]$ , $c > 0$ , by the theorem of Cauchy-Lipschitz, there exists $T_{c,K} > 0$ such that the solution of

$$(*) \qquad \left(\frac{\partial \tilde{\varphi}_{\varepsilon,\alpha}}{\partial u}\right)_{\{|\alpha|\leq\frac{1}{\varepsilon};\alpha_0\neq 0\}} = \left(\tilde{F}_\alpha\big((\tilde{\varphi}_{\varepsilon,\beta})_{\{|\beta|\leq\frac{1}{\varepsilon};\beta_0\neq0\}}, u\big)\right)_{\{|\alpha|\leq\frac{1}{\varepsilon};\alpha_0\neq0\}}$$

with the initial value $\tilde{\varphi}_\varepsilon(t_0) \quad (t_0 \in K)$ satisfying $\| \tilde{\varphi}_\varepsilon(t_0) \|_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})} \leq c$, exists on $[t_0; t_0 + T_{c,K}]$.
Let $v_\varepsilon(u) = \tilde{\varphi}_\varepsilon(u)$ for all $u$ in $[0; T_\varepsilon^+ - \frac{T_{c,K}}{2}]$ , and $v_\varepsilon(u)$ solution of $(*)$ with, at $t_0 = T_\varepsilon^+ - \frac{T_{c,K}}{2}$ $v_\varepsilon(T_\varepsilon^+ - \frac{T_{c,K}}{2}) = \tilde{\varphi}_\varepsilon(T_\varepsilon^+ - \frac{T_{c,K}}{2})$ ( indeed $\| \tilde{\varphi}_\varepsilon(T_\varepsilon^+ - \frac{T_{c,K}}{2}) \|_{\mathcal{H}_{m,2}([0;2R]\times\mathbb{T}^{n-1})} \leq c$ ).
Hence $v_\varepsilon$ exists on $[T_\varepsilon^+ - \frac{T_{c,K}}{2}; T_\varepsilon^+ + \frac{T_{c,K}}{2}]$, $v_\varepsilon$ is a solution on $[0; T_\varepsilon^+ + \frac{T_{c,K}}{2}]$, which is contrary of maximality of $] - T_\varepsilon^-; T_\varepsilon^+[$. So we obtain that $[0; B_R[$ is included in $I_\varepsilon$.

## 3.6 Existence of $\tilde{\varphi}$

We can show now the following proposition

**Proposition 3.6.1** *If $m > \frac{n-1}{2} + 2$ , there exists a solution $\tilde{\varphi}$, defined on $I \times [0; R] \times \mathbb{T}^{n-1}$ in variable $(u, v, y)$, for the problem (3.3.5) with assumptions (3.3.6), and for any $u$ in $I$, $\tilde{\varphi}(u)$ is in $C^0([0; R] \times \mathbb{T}^{n-1})$.*
*Moreover, if $m > \max(n - 1, \frac{n-1}{2} + 2)$ then $\tilde{\varphi}$ is in $C^0(I \times [0; R] \times \mathbb{T}^{n-1})$.*
*Moreover, for all $l \geq 2$, if $m > \max(n - 1, \frac{n-1}{2} + 4 + l)$, and if for any $0 \leq a, b \leq l - 1$, $0 \leq \gamma + |\mu| \leq m + 1$ , $D_t^a D_{x^1}^b D_\theta^\gamma D_y^\mu F$ is continuous in all its variables. then $\tilde{\varphi}$ is in $C^l(I \times [0; R] \times \mathbb{T}^{n-1})$.*

**Remark 3.6.1** : We suppose that $n \geq 2$, the results for the case $n = 1$ state in section 8.

### 3.6.1 Proof of the proposition 3.6.1

In the first step we prove the existence of a solution $\tilde{\varphi}$, then in the second step we study its regularity.

**Existence of a solution of the problem (3.3.5)**

We have shown in the proposition 3.5.2 that for any $\varepsilon > 0$ , $\tilde{\varphi}_\varepsilon$ exist on $[0; B_R[\times[0;R]\times \mathbb{T}^{n-1}$ and $\forall u \in [0; B_R[, \quad \| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}([0;R]\times\mathbb{T}^{n-1})} \leq h_R(u)$ with $h_R$ continuous.
So on $I = [0; \frac{B_R}{2}]$ we have $\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}([0;R]\times\mathbb{T}^{n-1})} \leq \max_I h_R = c$.
Thus for any u in I , $\tilde{\varphi}_\varepsilon(u)$ is bounded in $\mathcal{H}_{m,2}([0;R]\times\mathbb{T}^{n-1})$. As this space is reflexive, we can extract a sub-sequence $\tilde{\varphi}_{\varepsilon'}(u)$ which weakly converges to $\tilde{\varphi}(u)$ in $\mathcal{H}_{m,2}$ and $\| \tilde{\varphi}(u) \|_{\mathcal{H}_{m,2}} \leq \liminf \| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}} \leq c$ so $\tilde{\varphi}$ is in $L^\infty(I, \mathcal{H}_{m,2}([0;R]\times\mathbb{T}^{n-1}))$.

By compactness of embedding $\mathcal{H}_{m,2} \hookrightarrow \mathcal{H}_{m'',0}$ with $0 < m'' < m$ (see lemma 3.2.3 ), if $(\tilde{\varphi}_{\varepsilon'}(u))$ weakly converges to $\tilde{\varphi}(u)$ in $\mathcal{H}_{m,2}$ , then $(\tilde{\varphi}_{\varepsilon'}(u))$ strongly converges to $\tilde{\varphi}(u)$ in $\mathcal{H}_{m'',0}$ . By interpolation (see lemma (3.2.4)) , if $m'' < m' < m$ and $0 < k < 2$ we have

$$\begin{aligned}
\| \tilde{\varphi}_{\varepsilon'}(u) - \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m',k}} &\leq \| \tilde{\varphi}_{\varepsilon'}(u) - \tilde{\varphi}_\varepsilon(u) \|^{\nu}_{\mathcal{H}_{m'',0}} \| \tilde{\varphi}_{\varepsilon'}(u) - \tilde{\varphi}_\varepsilon(u) \|^{1-\nu}_{\mathcal{H}_{m,2}} \\
&\leq \| \tilde{\varphi}_{\varepsilon'}(u) - \tilde{\varphi}_\varepsilon(u) \|^{\nu}_{\mathcal{H}_{m'',0}} \left(\| \tilde{\varphi}_{\varepsilon'}(u) \|_{\mathcal{H}_{m,2}} + \| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,2}}\right)^{1-\nu} \\
&\leq \| \tilde{\varphi}_{\varepsilon'}(u) - \tilde{\varphi}_\varepsilon(u) \|^{\nu}_{\mathcal{H}_{m'',0}} (2c)^{1-\nu}
\end{aligned}$$

with $\nu = \gamma k/2$, where $\gamma$ is such that $m' = \gamma m'' + (1-\gamma)m$.
From this we can deduce that $(\tilde{\varphi}_{\varepsilon'}(u))$ strongly converges to $\tilde{\varphi}(u)$ in $\mathcal{H}_{m',k}$.
In particular, if $\frac{n-1}{2} < m' < m$ and $k = 1$ , by inclusion $\mathcal{H}_{m',1} \subset C^0$ (see lemma 3.2.2 ) we see that $(\tilde{\varphi}_{\varepsilon'}(u))$ strongly converges to $\tilde{\varphi}(u)$ in $C^0([0;R]\times\mathbb{T}^{n-1})$.
Then by continuity of $\tilde{H}$ , we get $(\tilde{H}(\tilde{\varphi}_{\varepsilon'}(u), u))$ strongly converges to $\tilde{H}(\tilde{\varphi}(u), u)$ in $C^0([0;R]\times\mathbb{T}^{n-1})$.

Now by observing that

$$\begin{aligned}
\| \hat{J}_{\varepsilon'}\tilde{H}(\tilde{\varphi}_{\varepsilon'}(u), u) - \tilde{H}(\tilde{\varphi}(u), u) \|_{C^0} \leq{}& \| \hat{J}_{\varepsilon'}\tilde{H}(\tilde{\varphi}_{\varepsilon'}(u), u) - \tilde{H}(\tilde{\varphi}_{\varepsilon'}(u), u) \|_{C^0} \\
&+ \| \tilde{H}(\tilde{\varphi}_{\varepsilon'}(u), u) - \tilde{H}(\tilde{\varphi}(u), u) \|_{C^0}
\end{aligned}$$

and that

$$\| \hat{J}_{\varepsilon'}\tilde{H}(\tilde{\varphi}_{\varepsilon'}(u), u) - \tilde{H}(\tilde{\varphi}_{\varepsilon'}(u), u) \|_{C^0} \quad \leq \quad \| \hat{J}_{\varepsilon'} - Id \|_{\mathcal{L}(L^2,H^1)} \| \tilde{H}(\tilde{\varphi}_{\varepsilon'}(u), u) \|_{C^0}$$

with $\| \hat{J}_{\varepsilon'} - Id \|_{\mathcal{L}(L^2,H^1)} \to 0$ and $\| \tilde{H}(\tilde{\varphi}_{\varepsilon'}(u), u) \|_{C^0}$ bounded, we can show that

$$\hat{J}_{\varepsilon'}\tilde{H}(\tilde{\varphi}_{\varepsilon'}(u), u) \to \tilde{H}(\tilde{\varphi}(u), u) \quad in \quad C^0([0;R]\times\mathbb{T}^{n-1}). \tag{3.6.1}$$

Now we show the convergence of the partial derivative of $\tilde{\varphi}_{\varepsilon'}$ with respect to v. We have

$$\| \frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon'}(u) \|_{\mathcal{H}_{m,1}} \leq \| \tilde{\varphi}_{\varepsilon'}(u) \|_{\mathcal{H}_{m,2}} \leq \ c$$

and $\mathcal{H}_{m,1}$ is reflexive so we can extract a subsequence $(\frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon''}(u))$ of $(\frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon'}(u))$ which weakly converges in $\mathcal{H}_{m,1}$ (then strongly in $\mathcal{H}_{m-1,\frac{3}{4}}$ by compactness of the embedding $\mathcal{H}_{m,1} \hookrightarrow \mathcal{H}_{m-1,\frac{3}{4}}$) to $\tilde{\tilde{\varphi}}(u) \in \mathcal{H}_{m,1}$ and $\| \tilde{\tilde{\varphi}}(u) \|_{\mathcal{H}_{m,1}} \leq c$.
Now we verify that $\tilde{\tilde{\varphi}}(u) = \frac{\partial}{\partial v}\tilde{\varphi}(u)$. Weak convergence in $\mathcal{H}_{m,1}([0;R]\times\mathbb{T}^{n-1})$ implies weak convergence in $L^2([0;R]\times\mathbb{T}^{n-1})$ , itself implies convergence in $\mathcal{D}'([0;R]\times\mathbb{T}^{n-1})$. So on one hand, $\frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon''}(u) \to \tilde{\tilde{\varphi}}(u)$ in $\mathcal{D}'([0;R]\times\mathbb{T}^{n-1})$ and on another hand $\tilde{\varphi}_{\varepsilon''}(u) \to \tilde{\tilde{\varphi}}(u)$ in

$\mathcal{D}'([0;R]\times\mathbb{T}^{n-1})$ , hence $\frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon''}(u) \to \frac{\partial}{\partial v}\tilde{\varphi}(u)$ in $\mathcal{D}'([0;R]\times\mathbb{T}^{n-1})$. By uniqueness of the limit in $\mathcal{D}'([0;R]\times\mathbb{T}^{n-1})$ we get $\tilde{\tilde{\varphi}}(u) = \frac{\partial}{\partial v}\tilde{\varphi}(u)$.

In the following we see that $\frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon''}(u)$ converges to $\frac{\partial}{\partial v}\tilde{\varphi}(u)$ in $C^0([0;R]\times\mathbb{T}^{n-1})$. It suffices to apply the argument of interpolation :
for all $\mu$ such that $m-1<\mu<m$ and $\frac{3}{4}<\kappa<1$, let $\sigma$ defined by $\mu=\sigma(m-1)+(1-\sigma)m$, and $\sigma'$ by $\kappa=\sigma'\frac{3}{4}+(1-\sigma')1$, we have

$$\begin{aligned}\| \frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon''}(u) - \frac{\partial}{\partial v}\tilde{\varphi}(u) \|_{\mathcal{H}_{\mu,\kappa}} &\leq \| \frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon''}(u) - \frac{\partial}{\partial v}\tilde{\varphi}(u) \|^{\sigma\sigma'}_{\mathcal{H}_{m-1,\frac{3}{4}}} \| \frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon''}(u) - \frac{\partial}{\partial v}\tilde{\varphi}(u) \|^{1-\sigma\sigma'}_{\mathcal{H}_{m,1}} \\ &\leq \| \frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon''}(u) - \frac{\partial}{\partial v}\tilde{\varphi}(u) \|^{\sigma\sigma'}_{\mathcal{H}_{m-1,\frac{3}{4}}} (2c)^{1-\sigma\sigma'}.\end{aligned}$$

Thus

$$\| \frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon''}(u) - \frac{\partial}{\partial v}\tilde{\varphi}(u) \|_{\mathcal{H}_{\mu,\kappa}} \to 0.$$

In particular, as $\mu > m-1 > \frac{n-1}{2}$ and $\kappa > \frac{3}{4}$, the embedding $\mathcal{H}_{\mu,\kappa} \hookrightarrow C^0$ holds, so

$$\| \frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon''}(u) - \frac{\partial}{\partial v}\tilde{\varphi}(u) \|_{C^0([0;R]\times\mathbb{T}^{n-1})} \to 0. \tag{3.6.2}$$

Similarly, we can show the following lemma that we need for the moment with $D^\alpha_y = \Delta_y$ :

**Lemma 3.6.1** *If $m-|\alpha| > \frac{n-1}{2}$ , $D^\alpha_y\tilde{\varphi}(u)$ is in $C^0([0;R]\times\mathbb{T}^{n-1})$.*

Proof of lemma 3.6.1 :
For all $|\alpha| < m$ ,we have $\| D^\alpha_y\tilde{\varphi}_{\varepsilon''}(u) \|_{\mathcal{H}_{m-|\alpha|,2}} \leq \| \tilde{\varphi}_{\varepsilon''}(u) \|_{\mathcal{H}_{m,2}} \leq c$ . So we can extract a subsequence (that we will denote also $\tilde{\varphi}_{\varepsilon''}$ for more commodity) weakly convergent in $\mathcal{H}_{m-|\alpha|,2}$ then strongly in $\mathcal{H}_{m'-|\alpha|,1}$ for all $0<m'<m$ . Arguing by uniqueness of the limit in $\mathcal{D}'([0;R]\times\mathbb{T}^{n-1})$, we show that its limit is $D^\alpha_y\tilde{\varphi}(u)$.
By interpolation, for all $\frac{n-1}{2} < m'-|\alpha| < m-|\alpha|$ , $0<k<2$ , $\| D^\alpha_y\tilde{\varphi}_{\varepsilon''}(u) - D^\alpha_y\tilde{\varphi}(u) \|_{\mathcal{H}_{m'-|\alpha|,k}} \to 0$. In particular, if $k=1$ , as $m'-|\alpha| > \frac{n-1}{2}$ by embedding $\mathcal{H}_{m'-|\alpha|,1} \hookrightarrow C^0$, we get

$$\| D^\alpha_y\tilde{\varphi}_{\varepsilon''}(u) - D^\alpha_y\tilde{\varphi}_{\varepsilon}(u) \|_{C^0([0;R]\times\mathbb{T}^{n-1})} \to 0.$$

Then as $C^0([0;R]\times\mathbb{T}^{n-1})$ is a complete space, we get that $D^\alpha_y\tilde{\varphi}(u)$ is in $C^0([0;R]\times\mathbb{T}^{n-1})$.
$\triangle$
By applying this lemma with $D^\alpha_y = \frac{\partial^2}{\partial y_i^2}$ and adding on $i=1,...,n-1$, we obtain that if $m > 2+\frac{n-1}{2}$ , $\Delta_y\tilde{\varphi}(u)$ is in $C^0([0;R]\times\mathbb{T}^{n-1})$. We will deduce from these results that $\tilde{\varphi}$ is a solution of the problem (3.3.5). Indeed,on one hand, from the continuity of $\frac{\partial^2}{\partial u\partial v}\tilde{\varphi}_{\varepsilon''}(u)$ we have
$\frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon''}(u) - \frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon''}(0) = \int_0^u \frac{\partial^2}{\partial u\partial v}\tilde{\varphi}_{\varepsilon''}(\sigma)d\sigma$. Therefore we use the theorem of dominated convergence of Lebesgue. By the convergence of $\Delta_y\tilde{\varphi}_{\varepsilon''}(\sigma)$ and (3.6.1) we can say that, for all $\sigma$ in $I$, $\frac{\partial^2}{\partial u\partial v}\tilde{\varphi}_{\varepsilon''}(\sigma) = \hat{J}_{\varepsilon''}\tilde{H}(\tilde{\varphi}_{\varepsilon''}(\sigma),\sigma) + \Delta_y\tilde{\varphi}_{\varepsilon''}(\sigma)$ converges to $\tilde{H}(\tilde{\varphi}(\sigma),\sigma) + \Delta_y\tilde{\varphi}(\sigma)$ in $C^0([0;R]\times\mathbb{T}^{n-1})$.
And $\| \frac{\partial^2}{\partial u\partial v}\tilde{\varphi}_{\varepsilon''} \|_{L^\infty(I,C^0([0;R]\times\mathbb{T}^{n-1}))} = \max_{\sigma\in I} \| \hat{J}_{\varepsilon''}\tilde{H}(\tilde{\varphi}_{\varepsilon''}(\sigma),\sigma) + \Delta_y\tilde{\varphi}_{\varepsilon''}(\sigma) \|_{C^0([0;R]\times\mathbb{T}^{n-1})} \leq \tilde{c}_R$ which is in $L^1([0,u])$.

So $\int_0^u \frac{\partial^2}{\partial u \partial v}\tilde{\varphi}_{\varepsilon"}(\sigma)d\sigma \to \int_0^u \tilde{H}(\tilde{\varphi}(\sigma),\sigma)+\Delta_y\tilde{\varphi}(\sigma)d\sigma$ in $C^0([0;R]\times\mathbb{T}^{n-1})$. Furthermore, $\frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon"}(u)-\frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon"}(0)\to\int_0^u \tilde{H}(\tilde{\varphi}(\sigma),\sigma)+\Delta_y\tilde{\varphi}(\sigma)d\sigma$ in $C^0([0;R]\times\mathbb{T}^{n-1})$.
On another hand, by (3.6.2) $\frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon"}(u)-\frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon"}(0)\to\frac{\partial}{\partial v}\tilde{\varphi}(u)-\frac{\partial}{\partial v}\tilde{\varphi}(0)$ in $C^0([0;R]\times\mathbb{T}^{n-1})$.
Hence by uniqueness of the limit in $C^0([0;R]\times\mathbb{T}^{n-1})$ we get
$\frac{\partial}{\partial v}\tilde{\varphi}(u)-\frac{\partial}{\partial v}\tilde{\varphi}(0)=\int_0^u \tilde{H}(\tilde{\varphi}(\sigma),\sigma)+\Delta_y\tilde{\varphi}(\sigma)d\sigma$.
Then we differentiate with respect to $u$ and we obtain

$$\frac{\partial^2}{\partial u\partial v}\tilde{\varphi}(u,v,y)=\tilde{H}(\tilde{\varphi}(u,v,y),u,v,y)+\Delta_y\tilde{\varphi}(u,v,y). \tag{3.6.3}$$

We notice that $\tilde{\varphi}(u,0,y)=0$ is given by $\tilde{\varphi}_{\varepsilon"}(u,0,y)=0$ and the convergence of $\tilde{\varphi}_{\varepsilon"}(u)$ in $C^0([0;R]\times\mathbb{T}^{n-1})$.
For the last equation of the problem (3.3.5), we recall that $\tilde{\varphi}_{\varepsilon"}(0,v,y)=\hat{J}_{\varepsilon"}\tilde{\varphi}_-(v,y)$ , and as

$$\| \hat{J}_{\varepsilon"}\tilde{\varphi}_- - \tilde{\varphi}_- \|_{C^0} \quad\leq\quad \| \hat{J}_{\varepsilon"} - Id \|_{\mathcal{L}(L^2,H^1)} \| \tilde{\varphi}_- \|_{C^0}$$

with $\| \hat{J}_{\varepsilon'} - Id \|_{\mathcal{L}(L^2,H^1)}\to 0$ and $\| \tilde{\varphi}_- \|_{C^0}$ finite, we can show that

$$\hat{J}_{\varepsilon"}\tilde{\varphi}_- \to \tilde{\varphi}_- \quad in \quad C^0([0;R]\times\mathbb{T}^{n-1}).$$

Now with the convergence of $\tilde{\varphi}_{\varepsilon"}(0)$ in $C^0([0;R]\times\mathbb{T}^{n-1})$ and the uniqueness of the limit we can conclude that $\tilde{\varphi}(0,v,y)=\tilde{\varphi}_-(v,y)$.

**Regularity of $\tilde{\varphi}$**

Now we are going to show that $\tilde{\varphi}$ is $C^0(I\times[0;R]\times\mathbb{T}^{n-1})$. To reach this goal, we will show that $\tilde{\varphi}$ is in $C^{0,1}(I,\mathcal{H}_{m',1}([0;R]\times\mathbb{T}^{n-1}))$ with $m'>(n-1)/2$. By the continuity in $v$ of $\frac{\partial}{\partial v}\tilde{\varphi}$ , we can write :

$$\tilde{\varphi}(u+h,v,y)-\tilde{\varphi}(u,v,y) \quad=\quad \tilde{\varphi}(u+h,0,y)-\tilde{\varphi}(u,0,y)+\int_0^v \frac{\partial}{\partial v}(\tilde{\varphi}(u+h,\sigma,y)-\tilde{\varphi}(u,\sigma,y))d\sigma.$$

Let $m'=m/2$ , as we have seen beyond $\tilde{\varphi}(u+h,0,y)=0$ and $\tilde{\varphi}(u,0,y)=0$, so

$$\| \tilde{\varphi}(u+h)-\tilde{\varphi}(u) \|_{\mathcal{H}_{m',1}([0;R]\times\mathbb{T}^{n-1})}=\| \int_0^v \frac{\partial}{\partial v}(\tilde{\varphi}(u+h,\sigma,y)-\tilde{\varphi}(u,\sigma,y))d\sigma \|_{\mathcal{H}_{m',1}([0;R]\times\mathbb{T}^{n-1})} \tag{3.6.4}$$

Here we need the following lemma, the proof of which can be found in appendix 3.10 :

**Lemma 3.6.2** *Suppose that $f$ is a function of $(s,y)$ such that for all $0\leq\nu\leq m'$ , $D_y^\nu f$ is in $C^0([0;R]\times\mathbb{T}^{n-1})$, then*

$$\| \int_0^v f(s,y)ds \|_{\mathcal{H}_{m',1}([0;R]\times\mathbb{T}^{n-1})} \quad\leq\quad (R^{\frac{3}{2}}+1) \| f(s,y) \|_{\mathcal{H}_{m',0}([0;R]\times\mathbb{T}^{n-1})}$$

Here by using lemma 3.6.1 with $\alpha=\nu$, we get that if $m-m'>(n-1)/2$ i.e. $m>n-1$, then for all $0\leq|\nu|\leq m'$ , $D_y^\nu\tilde{\varphi}(u)$ is in $C^0([0;R]\times\mathbb{T}^{n-1})$. So we can apply the lemma 3.6.2 on $\frac{\partial}{\partial v}\tilde{\varphi}(u+h,s,y)-\frac{\partial}{\partial v}\tilde{\varphi}(u,s,y)$, we obtain

$$\begin{aligned}&\| \int_0^v \frac{\partial}{\partial v}\tilde{\varphi}(u+h,\sigma,y)-\frac{\partial}{\partial v}\tilde{\varphi}(u,\sigma,y)d\sigma \|_{\mathcal{H}_{m',1}([0;R]\times\mathbb{T}^{n-1})}\\ &\qquad\leq (R^{\frac{3}{2}}+1) \| \frac{\partial}{\partial v}\tilde{\varphi}(u+h,\sigma,y)-\frac{\partial}{\partial v}\tilde{\varphi}(u,\sigma,y) \|_{\mathcal{H}_{m',0}([0;R]\times\mathbb{T}^{n-1})} .\end{aligned} \tag{3.6.5}$$

On another hand we know that for all $1 < \mu < m$,

$$\| \frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon"}(u) - \frac{\partial}{\partial v}\tilde{\varphi}(u) \|_{\mathcal{H}_{\mu,0}([0;R]\times\mathbb{T}^{n-1})} \to 0.$$

Hence

$$\| \frac{\partial}{\partial v}\tilde{\varphi}(u+h) - \frac{\partial}{\partial v}\tilde{\varphi}(u) \|_{\mathcal{H}_{\mu,0}} = \lim_{\varepsilon"\to 0} \| \frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon"}(u+h) - \frac{\partial}{\partial v}\tilde{\varphi}_{\varepsilon"}(u) \|_{\mathcal{H}_{\mu,0}} .$$

Recall that $\frac{\partial^2}{\partial u\partial v}\tilde{\varphi}_{\varepsilon"}$ is continuous in all its variables $(u,v,y)$, so we have

$$\| \frac{\partial}{\partial v}\tilde{\varphi}(u+h) - \frac{\partial}{\partial v}\tilde{\varphi}(u) \|_{\mathcal{H}_{\mu,0}} = \lim_{\varepsilon"\to 0} \| \int_u^{u+h} \frac{\partial^2}{\partial u\partial v}\tilde{\varphi}_{\varepsilon"}(\sigma)\, d\sigma \|_{\mathcal{H}_{\mu,0}} .$$

Then we need the following lemma, the proof of which can be found in appendix 3.10, (here we will just use the result with the norm $\mathcal{H}_{\mu,0}$ but as we will need it further we give also the result with the norm $\mathcal{H}_{\mu,1}$ in this lemma).

**Lemma 3.6.3** *If $(u,v,y) \mapsto f(u,v,y)$ is a function such that for all $0 \leq |\nu| \leq \mu,\ 0 \leq a \leq 1$, $D_v^a D_y^\nu f$ is continuous in all its variables, then*

$$\| \int_u^{u+h} f(\sigma)d\sigma \|_{\mathcal{H}_{\mu,0}} \leq \int_u^{u+h} \| f(\sigma)d\sigma \|_{\mathcal{H}_{\mu,0}}\, d\sigma\ ,$$
$$\| \int_u^{u+h} f(\sigma)d\sigma \|_{\mathcal{H}_{\mu,1}} \leq \int_u^{u+h} \| f(\sigma)d\sigma \|_{\mathcal{H}_{\mu,1}}\, d\sigma\ .$$

Now we apply the lemma above to $f = \frac{\partial^2}{\partial u\partial v}\tilde{\varphi}_{\varepsilon"}$ , so we obtain

$$\| \frac{\partial}{\partial v}\tilde{\varphi}(u+h) - \frac{\partial}{\partial v}\tilde{\varphi}(u) \|_{\mathcal{H}_{\mu,0}} \leq \lim_{\varepsilon"\to 0} \int_u^{u+h} \| \frac{\partial^2}{\partial u\partial v}\tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{\mu,0}}\ d\sigma.$$

But

$$\begin{aligned} \| \frac{\partial^2}{\partial u\partial v}\tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{\mu,0}} &= \| \hat{J}_{\varepsilon"}\tilde{H}(\tilde{\varphi}_{\varepsilon"}(\sigma),\sigma) + \Delta_y\tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{\mu,0}} \\ &\leq \| \tilde{H}(\tilde{\varphi}_{\varepsilon"}(\sigma),\sigma) \|_{\mathcal{H}_{\mu,0}} + \| \Delta_y\tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{\mu,0}} \\ &\leq c_R(\| \tilde{\varphi}_{\varepsilon"}(\sigma) \|_{L^\infty},\sigma)(1+ \| \tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{\mu,0}})+ \| \tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{\mu+2,0}} \end{aligned}$$

with $\| \tilde{\varphi}_{\varepsilon"}(\sigma) \|_{L^\infty},\sigma, \| \tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{\mu,0}}, \| \tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{\mu+2,0}}$ bounded on $I$. Hence,

$$\| \frac{\partial^2}{\partial u\partial v}\tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{\mu,0}} \leq \overline{c}_R.$$

Thus, we have with $\mu = m'$ (as $n \geq 2$ and $m' = \frac{m}{2} = \frac{1}{2}\max(n-1; \frac{n-1}{2}+2)$ we get $1 < m' < m$)

$$\| \frac{\partial}{\partial v}\tilde{\varphi}(u+h) - \frac{\partial}{\partial v}\tilde{\varphi}(u) \|_{\mathcal{H}_{\mu,0}([0;R]\times\mathbb{T}^{n-1})} \leq \lim_{\varepsilon"\to 0} \int_u^{u+h} \overline{c}_R d\sigma = \overline{c}_R h. \tag{3.6.6}$$

From (3.6.4), (3.6.5), (3.6.6), we can deduce that

$$\| \tilde{\varphi}(u+h) - \tilde{\varphi}(u) \|_{\mathcal{H}_{m',1}([0;R]\times\mathbb{T}^{n-1})} \leq (R^{\frac{3}{2}}+1)\overline{c}_R h.$$

It means that $\tilde{\varphi}$ is in $C^{0,1}(I, \mathcal{H}_{m',1}([0;R] \times \mathbb{T}^{n-1}))$.
But $C^{0,1}(I, \mathcal{H}_{m',1}([0;R] \times \mathbb{T}^{n-1})) \subset C^0(I, \mathcal{H}_{m',1}([0;R] \times \mathbb{T}^{n-1}))$, and as $m' > (n-1)/2$ i.e. $m > n-1$ we have $C^0(I, \mathcal{H}_{m',1}([0;R] \times \mathbb{T}^{n-1})) \subset C^0(I, C^0([0;R] \times \mathbb{T}^{n-1})) = C^0(I \times [0;R] \times \mathbb{T}^{n-1})$, which allows us to conclude that

$$\tilde{\varphi} \in C^0(I \times [0;R] \times \mathbb{T}^{n-1}).$$

Now we show that under certain conditions $\tilde{\varphi}$ is in $C^2(I \times [0;R] \times \mathbb{T}^{n-1})$. We start by getting $\frac{\partial^2}{\partial u \partial v}\tilde{\varphi}$ in $C^0(I \times [0;R] \times \mathbb{T}^{n-1})$. As $\tilde{H}$ is continuous in all its variables, we have $(u,v,y) \mapsto \tilde{H}(\tilde{\varphi}, u, v, y)$ is in $C^0(I \times [0;R] \times \mathbb{T}^{n-1})$. So it suffices to prove that $\Delta_y \tilde{\varphi}$ is continuous. Here we introduce a lemma because we will need it later too. Its proof can be found at the end of the section.

**Lemma 3.6.4** *If $m - |\alpha| - 2 > \frac{n-1}{2}$ , $D_y^\alpha \tilde{\varphi}$ is in $C^0(I \times [0;R] \times \mathbb{T}^{n-1})$.*

We apply this lemma to $\Delta_y$ and finally, we obtain that if $m > \max(n-1, \frac{n-1}{2} + 4)$ $\tilde{H}(\tilde{\varphi}) + \Delta_y \tilde{\varphi}$ is in $C^0(I \times [0;R] \times \mathbb{T}^{n-1})$. Now by the equality (3.6.3), we get

$$\frac{\partial^2}{\partial u \partial v}\tilde{\varphi} \in C^0(I \times [0;R] \times \mathbb{T}^{n-1}). \tag{3.6.7}$$

Then we show that $\tilde{\varphi}$ is in $C^2(I \times [0;R] \times \mathbb{T}^{n-1})$. First we can deduce from the result above that $\frac{\partial}{\partial v}\tilde{\varphi}$ is continuous in all its variables. Indeed

$$\begin{aligned}\frac{\partial}{\partial v}\tilde{\varphi}(u,v,y) &= \frac{\partial}{\partial v}\tilde{\varphi}(0,v,y) + \int_0^u \frac{\partial^2}{\partial u \partial v}\tilde{\varphi}(s,v,y)ds \\ &= \frac{\partial}{\partial v}\tilde{\varphi}_-(v,y) + \int_0^u \frac{\partial^2}{\partial u \partial v}\tilde{\varphi}(s,v,y)ds. \end{aligned} \tag{3.6.8}$$

By the definition of $\tilde{\varphi}_-$ we see that $\frac{\partial}{\partial v}\tilde{\varphi}_-(v,y) = \frac{\partial}{\partial v}\varphi_-(v,y) - \frac{\partial}{\partial v}\varphi(0,0,y) = \frac{\partial}{\partial v}\varphi_-(v,y) - \frac{\partial}{\partial v}\varphi_-(0,y)$. As $\varphi_-$ is $C^{m+4}$, we get

$$\frac{\partial}{\partial v}\tilde{\varphi} \in C^0(I \times [0;R] \times \mathbb{T}^{n-1}).$$

Now by this continuity of $\frac{\partial}{\partial v}\tilde{\varphi}$ we can write that

$$\begin{aligned}\tilde{\varphi}(u,v,y) &= \tilde{\varphi}(u,0,y) + \int_0^v \frac{\partial}{\partial v}\tilde{\varphi}(u,s,y)ds \\ &= \int_0^v \frac{\partial}{\partial v}\tilde{\varphi}(u,s,y)ds. \end{aligned} \tag{3.6.9}$$

We differentiate in $u$ and with (3.6.7), we get

$$\frac{\partial}{\partial u}\tilde{\varphi}(u,v,y) = \int_0^v \frac{\partial^2}{\partial u \partial v}\tilde{\varphi}(u,s,y)ds. \tag{3.6.10}$$

So

$$\frac{\partial}{\partial u}\tilde{\varphi} \in C^0(I \times [0;R] \times \mathbb{T}^{n-1}).$$

If we differentiate this equality in $v$, we obtain

$$\frac{\partial^2}{\partial v \partial u}\tilde{\varphi} = \frac{\partial^2}{\partial u \partial v}\tilde{\varphi} \in C^0(I \times [0; R] \times \mathbb{T}^{n-1}). \tag{3.6.11}$$

For derivatives in $y_i$ of first and second order we just have to apply the lemma 3.6.4. We differentiate (3.6.10) in $y_i$ and as $\tilde{\varphi}$ satisfies the equation (3.6.3), it gives

$$\frac{\partial^2}{\partial y_i \partial u}\tilde{\varphi} = \frac{\partial}{\partial y_i}\int_0^v \tilde{H}(\tilde{\varphi}, u, s, y) + \Delta_y \tilde{\varphi}(u, s, y) ds. \tag{3.6.12}$$

If $\frac{\partial}{\partial y_i}\tilde{H}$ and $\frac{\partial}{\partial \theta}\tilde{H}$ are continuous in all their variables that it is the case by the assumptions on $\tilde{H}$, and if $\frac{\partial}{\partial y_i}\tilde{\varphi}$ is continuous in all its variables that it is the case if $m > \max(n-1, \frac{n-1}{2} + 5)$, we have $\frac{\partial}{\partial y_i}\big(\tilde{H}(\tilde{\varphi}, u, s, y) + \Delta_y \tilde{\varphi}(u, s, y)\big) = \big(\frac{\partial}{\partial y_i}\tilde{H}\big)(\tilde{\varphi}, u, s, y) + \big(\frac{\partial}{\partial \theta}\tilde{H}\big)(\tilde{\varphi}, u, s, y)\frac{\partial}{\partial y_i}\tilde{\varphi} + \frac{\partial}{\partial y_i}\Delta_y \tilde{\varphi}(u, s, y)\big)$ continuous in all its variables. So we can commute $\int_0^v$ and $\frac{\partial}{\partial y_i}$ and conclude that

$$\frac{\partial^2}{\partial y_i \partial u}\tilde{\varphi} \in C^0(I \times [0; R] \times \mathbb{T}^{n-1}).$$

By the continuity of $\frac{\partial}{\partial u}\tilde{\varphi}$ we can write

$$\begin{aligned} \tilde{\varphi}(u, v, y) &= \tilde{\varphi}(0, v, y) + \int_0^u \frac{\partial}{\partial u}\tilde{\varphi}(s, v, y) ds \\ &= \tilde{\varphi}_-(v, y) + \int_0^u \frac{\partial}{\partial u}\tilde{\varphi}(s, v, y) ds. \end{aligned}$$

As we have shown that $\frac{\partial^2}{\partial y_i \partial u}\tilde{\varphi}$ is continuous, we have

$$\frac{\partial}{\partial y_i}\tilde{\varphi}(u, v, y) = \frac{\partial}{\partial y_i}\tilde{\varphi}_-(v, y) + \int_0^u \frac{\partial^2}{\partial y_i \partial u}\tilde{\varphi}(s, v, y) ds. \tag{3.6.13}$$

We differentiate this equality in $u$, thus

$$\frac{\partial^2}{\partial u \partial y_i}\tilde{\varphi} = \frac{\partial^2}{\partial y_i \partial u}\tilde{\varphi} \in C^0(I \times [0; R] \times \mathbb{T}^{n-1}).$$

For $\frac{\partial^2}{\partial y_i \partial v}\tilde{\varphi}$ we differentiate (3.6.8) in $y_i$ and as we have done for $\frac{\partial^2}{\partial y_i \partial u}\tilde{\varphi}$ we obtain that if $m > \max(n-1, \frac{n-1}{2} + 5)$

$$\frac{\partial^2}{\partial y_i \partial v}\tilde{\varphi} \in C^0(I \times [0; R] \times \mathbb{T}^{n-1}).$$

Now we differentiate the equality (3.6.9) first in $y_i$ , then in $v$, hence

$$\frac{\partial^2}{\partial v \partial y_i}\tilde{\varphi} = \frac{\partial^2}{\partial y_i \partial v}\tilde{\varphi} \in C^0(I \times [0; R] \times \mathbb{T}^{n-1}).$$

It remains to show that $\frac{\partial^2}{\partial u^2}\tilde{\varphi}$ and $\frac{\partial^2}{\partial v^2}\tilde{\varphi}$ are continuous. For this we will see that we need the continuity of $\frac{\partial^2}{\partial y_i^2}\Delta_y \tilde{\varphi}$ and so we must take $m > \max(n-1, \frac{n-1}{2} + 6)$ . We start by differentiating in $u$ the equality (3.6.10) and as $\tilde{\varphi}$ satisfies the equation (3.6.3), we obtain

$$\frac{\partial^2}{\partial u^2}\tilde{\varphi} = \frac{\partial}{\partial u}\Big(\int_0^v \tilde{H}(\tilde{\varphi}, u, s, y) + \Delta_y \tilde{\varphi}(u, s, y) ds\Big).$$

We notice that $\frac{\partial}{\partial u}\big(\tilde{H}(\tilde{\varphi},u,s,y)+\Delta_y\tilde{\varphi}(u,s,y)\big) = \big(\frac{\partial}{\partial u}\tilde{H}\big)(\tilde{\varphi},u,s,y)+\big(\frac{\partial}{\partial \theta}\tilde{H}\big)(\tilde{\varphi},u,s,y)\frac{\partial}{\partial u}\tilde{\varphi}+ \frac{\partial}{\partial u}\Delta_y\tilde{\varphi}(u,s,y)\big)$. The both first terms of the right member are continuous by the assumptions on $\tilde{H}$ and the results above. For $\frac{\partial}{\partial u}\Delta_y\tilde{\varphi}$ we look at $\frac{\partial^3}{\partial y_i^2\partial u}\tilde{\varphi}$. If $m > \max(n-1, \frac{n-1}{2}+6)$, $\frac{\partial^2}{\partial y_i^2}\Delta_y\tilde{\varphi}$ is continuous and by the assumptions on $\tilde{H}$, we have the continuity of

$$\begin{aligned}\frac{\partial^2}{\partial y_i^2}\big(\tilde{H}(\tilde{\varphi},u,s,y)+\Delta_y\tilde{\varphi}(u,s,y)\big) =&\big(\frac{\partial^2}{\partial y_i^2}\tilde{H}\big)(\tilde{\varphi},u,s,y)+\big(\frac{\partial^2}{\partial\theta\partial y_i}\tilde{H}\big)(\tilde{\varphi},u,s,y)\frac{\partial}{\partial y_i}\tilde{\varphi}\\ &+\big(\frac{\partial^2}{\partial y_i\partial\theta}\tilde{H}\big)(\tilde{\varphi},u,s,y)\frac{\partial}{\partial y_i}\tilde{\varphi}+\big(\frac{\partial^2}{\partial\theta^2}\tilde{H}\big)(\tilde{\varphi},u,s,y)\big(\frac{\partial}{\partial y_i}\tilde{\varphi}\big)^2\\ &+\big(\frac{\partial}{\partial\theta}\tilde{H}\big)(\tilde{\varphi},u,s,y)\frac{\partial^2}{\partial y_i^2}\tilde{\varphi}+\frac{\partial^2}{\partial y_i^2}\Delta_y\tilde{\varphi}(u,s,y).\end{aligned}$$

So by differentiating the equality (3.6.12) in $y_i$, we get

$$\frac{\partial^3}{\partial y_i^2\partial u}\tilde{\varphi} = \int_0^v \frac{\partial^2}{\partial y_i^2}\big(\tilde{H}(\tilde{\varphi},u,s,y)+\Delta_y\tilde{\varphi}(u,s,y)\big)ds.$$

Hence

$$\frac{\partial^3}{\partial y_i^2\partial u}\tilde{\varphi}\in C^0(I\times[0;R]\times\mathbb{T}^{n-1}).$$

Now we differentiate the equality (3.6.13) first in $y_i$, then in $u$, and so we obtain

$$\frac{\partial^3}{\partial u\partial y_i^2}\tilde{\varphi}=\frac{\partial^3}{\partial y_i^2\partial u}\tilde{\varphi}\in C^0(I\times[0;R]\times\mathbb{T}^{n-1}).$$

It suffices to add on $y_i$ to get the continuity of $\frac{\partial}{\partial u}\Delta_y\tilde{\varphi}$ . Finally we can say that if $m > \max(n-1, \frac{n-1}{2}+6)$,

$$\frac{\partial^2}{\partial u^2}\tilde{\varphi}\in C^0(I\times[0;R]\times\mathbb{T}^{n-1}).$$

We proceed similarly for $\frac{\partial^2}{\partial v^2}\tilde{\varphi}$ (we have supplementary terms, $\frac{\partial^2}{\partial v^2}\tilde{\varphi}_-$ and $\frac{\partial^3}{\partial y_i^2\partial v}\tilde{\varphi}_-$ which are continuous by assumptions on $\tilde{\varphi}_-$). If $m > \max(n-1, \frac{n-1}{2}+6)$,

$$\frac{\partial^2}{\partial v^2}\tilde{\varphi}\in C^0(I\times[0;R]\times\mathbb{T}^{n-1}).$$

At the end, all the results above allow us to conclude that if $m > \max(n-1, \frac{n-1}{2}+6)$,

$$\tilde{\varphi}\in C^2(I\times[0;R]\times\mathbb{T}^{n-1}).$$

For the class $C^l$ we follow the same method and so we take $m > \max(n-1, \frac{n-1}{2}+l+4)$, but we need also greater assumptions on $\tilde{H}$ and so on $H$, that is to say
for any $0\le a,b\le l-1$, $0\le\gamma+|\mu|\le m+1$, $D_u^aD_v^bD_\theta^\gamma D_y^\mu H$ continuous in all its variables.
This is equivalent to the assumptions on F :
for any $0\le a,b\le l-1$, $0\le\gamma+|\mu|\le m+1$ , $D_t^aD_{x^1}^bD_\theta^\gamma D_y^\mu F$ is continuous in all its variables.
□

**Proof of the Lemma 3.6.4 :**
In the proof of the lemma 3.6.1 we have seen that $\| D_y^\alpha \tilde{\varphi}_{\varepsilon"}(u) - D_y^\alpha \tilde{\varphi}(u) \|_{\mathcal{H}_{m-|\alpha|,1}} \to 0$. So we can write

$$\begin{aligned}\| D_y^\alpha \tilde{\varphi}(u+h) - D_y^\alpha \tilde{\varphi}(u) \|_{\mathcal{H}_{m-|\alpha|-2,1}} &= \lim_{\varepsilon" \to 0} \| D_y^\alpha \tilde{\varphi}_{\varepsilon"}(u+h) - D_y^\alpha \tilde{\varphi}_{\varepsilon"}(u) \|_{\mathcal{H}_{m-|\alpha|-2,1}} \\ &= \lim_{\varepsilon" \to 0} \| \int_u^{u+h} D_y^\alpha \frac{\partial}{\partial u} \tilde{\varphi}_{\varepsilon"}(\sigma)\, d\sigma \|_{\mathcal{H}_{m-|\alpha|-2,1}}\end{aligned}$$

because of the continuity in all its variables $(u, v, y)$ of $\frac{\partial}{\partial u} D_y^\alpha \tilde{\varphi}_{\varepsilon"}$.
Hence by lemma 3.6.3 and by taking the limit, we obtain

$$\| D_y^\alpha \tilde{\varphi}(u+h) - D_y^\alpha \tilde{\varphi}(u) \|_{\mathcal{H}_{m-|\alpha|-2,1}} \leq \lim_{\varepsilon" \to 0} \int_u^{u+h} \| D_y^\alpha \frac{\partial}{\partial u} \tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{m-|\alpha|-2,1}}\ d\sigma.$$

But by the continuity of $\frac{\partial}{\partial v} D_y^\alpha \frac{\partial}{\partial u} \tilde{\varphi}_{\varepsilon"}$ and the fact that we can commute the partial derivatives, we have

$$D_y^\alpha \frac{\partial}{\partial u} \tilde{\varphi}_{\varepsilon"}(\sigma, v, y) = D_y^\alpha \frac{\partial}{\partial u} \tilde{\varphi}_{\varepsilon"}(\sigma, 0, y) + \int_0^v D_y^\alpha \frac{\partial^2}{\partial u \partial v} \tilde{\varphi}_{\varepsilon"}(\sigma, s, y) ds.$$

The first term of right member vanishes (indeed the third equation of the problem (3.4.1) gives that $\tilde{\varphi}_{\varepsilon"}(u, 0, y)$ vanishes, so by differentiating in $u$ and in $y$, it also vanishes). By using the second equation of the problem (3.4.1) and the result (3.5.5), we get

$$D_y^\alpha \frac{\partial}{\partial u} \tilde{\varphi}_{\varepsilon"}(\sigma, v, y) = \int_0^v \hat{J}_{\varepsilon"} D_y^\alpha \tilde{H}(\tilde{\varphi}_{\varepsilon"}, \sigma, s, y) + (D_y^\alpha \Delta_y) \tilde{\varphi}_{\varepsilon"}(\sigma, s, y) ds.$$

Now, we take the norm $\mathcal{H}_{m-|\alpha|-2,1}$ of the both members and we apply lemma 3.6.2 on the right one, so

$$\begin{aligned}\| D_y^\alpha \frac{\partial}{\partial u} \tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{m-|\alpha|-2,1}} &\leq (R^{\frac{3}{2}} + 1) \| \hat{J}_{\varepsilon"} \Delta_y \tilde{H}(\tilde{\varphi}_{\varepsilon"}(\sigma), \sigma) + (D_y^\alpha \Delta_y) \tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{m-|\alpha|-2,0}} \\ &\leq (R^{\frac{3}{2}} + 1)(\| \Delta_y \tilde{H}(\tilde{\varphi}_{\varepsilon"}(\sigma), \sigma) \|_{\mathcal{H}_{m-|\alpha|-2,0}} + \| \tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{m,0}})\end{aligned}$$

Then by the assumptions on the regularity of $\tilde{H}$, we obtain

$$\begin{aligned}\| D_y^\alpha \frac{\partial}{\partial u} \tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{m-|\alpha|-2,1}} &\leq (R^{\frac{3}{2}} + 1)\big(c(\| \tilde{\varphi}_{\varepsilon"}(\sigma) \|_{L^\infty}, \sigma)(1 + \| \tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{m-|\alpha|,0}}) + \| \tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{m,0}}\ \big) \\ &\leq c_R\end{aligned}$$

because $\| \tilde{\varphi}_{\varepsilon"}(\sigma) \|_{L^\infty}, \sigma, \| \tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{m-|\alpha|,0}}, \| \tilde{\varphi}_{\varepsilon"}(\sigma) \|_{\mathcal{H}_{m,0}}$ are bounded on $I$.
Hence

$$\| D_y^\alpha \tilde{\varphi}(u+h) - D_y^\alpha \tilde{\varphi}(u) \|_{\mathcal{H}_{m-|\alpha|-2,1}} \leq \lim_{\varepsilon" \to 0} \int_u^{u+h} c_R d\sigma = c_R h$$

It means that $D_y^\alpha \tilde{\varphi}$ is in $C^{0,1}(I, \mathcal{H}_{m-|\alpha|-2,1}([0; R] \times \mathbb{T}^{n-1}))$.
But $C^{0,1}(I, \mathcal{H}_{m-|\alpha|-2,1}([0; R] \times \mathbb{T}^{n-1})) \subset C^0(I, \mathcal{H}_{m-|\alpha|-2,1}([0; R] \times \mathbb{T}^{n-1}))$, and as $m - |\alpha| - 2 > (n-1)/2$ we have $C^0(I, \mathcal{H}_{m-|\alpha|-2,1}([0; R] \times \mathbb{T}^{n-1})) \subset C^0(I, C^0([0; R] \times \mathbb{T}^{n-1})) = C^0(I \times [0; R] \times \mathbb{T}^{n-1})$, which allows us to conclude that

$$D_y^\alpha \tilde{\varphi} \in C^0(I \times [0; R] \times \mathbb{T}^{n-1}).$$

$\triangle$

## 3.7 Existence and uniqueness of solution of the first problem

### 3.7.1 Existence of $\varphi$

We can show now the following theorem

**Theorem 3.7.1** *If $m > \max(n-1, \frac{n-1}{2}+4)$, and*
*(i) $F : (\theta, t, x^1, y) \mapsto F(\theta, t, x^1, y)$ satisfies that for any $a, b = 0$ or $1$,*
*$0 \leq \gamma + |\mu| \leq m+1$ , $D_t^a D_{x^1}^b D_\theta^\gamma D_y^\mu F$ is continuous in all its variables*
*(ii) $\varphi_+, \varphi_-$ are of class $H^{m+5}$, and $\varphi_+, \varphi_-$ satisfy the corner condition :*
*$\varphi_+(0, y) = \varphi_-(0, y)$.*
*(iii) There exists a real $T > 0$ such that $F, \varphi_+, \varphi_-$ are $T$-periodic in each $y_i$.*
*then for all real $R > 0$ , there exist some reals $R' > 0$ and $R" > 0$ such that there exists a solution $\varphi$ for the problem (3.1.1) in the domain $\Omega = \{0 \leq t - x^1 \leq R,\ 0 \leq t + x^1 \leq R',\ (x^2, ..., x^n) \in \mathbb{T}^{n-1}\} \bigcup \{0 \leq t + x^1 \leq R,\ 0 \leq t - x^1 \leq R",\ (x^2, ..., x^n) \in \mathbb{T}^{n-1}\}$ where $\mathbb{T}^{n-1}$ is the torus of dimension $n-1$ and of length $T$ in each direction, and this solution is in $C^0(\Omega)$.*
*Moreover, for all $l \geq 2$, if $m > max(n-1, \frac{n-1}{2}+4+l)$, and if for any $0 \leq a, b \leq l-1$, $0 \leq \gamma + |\mu| \leq m+1$ , $D_t^a D_{x^1}^b D_\theta^\gamma D_y^\mu F$ is continuous in all its variables, then $\varphi$ is in $C^l(\Omega)$.*

**Proof of the theorem 3.7.1 :**
In the first step we prove the existence of a solution $\varphi$ satisfying equation (3.3.1), then in the second one we study its regularity , after that we show that we can do the same along $N_+$.

#### Existence of a solution $\varphi$

We set $I = [0; R']$ and

$$\varphi(u, v, y) = \tilde{\varphi}(u, v, y) + \varphi_+(u, y) + \Big(\frac{\partial}{\partial v}\varphi_-(0, y) + \int_0^u H(\varphi_+(s, y), s, 0, y) + \Delta_y \varphi_+(s, y) ds\Big) \tag{3.7.1}$$

We notice that

$$\varphi(u, 0, y) = \tilde{\varphi}(u, 0, y) + \varphi_+(u, y) = \varphi_+(u, y)$$

and

$$\begin{aligned}
\varphi(0, v, y) &= \tilde{\varphi}(0, v, y) + \varphi_+(0, y) + \big(\frac{\partial}{\partial v}\varphi_-(0, y)\big)v \\
&= \tilde{\varphi}_-(v, y) + \varphi_+(0, y) + \big(\frac{\partial}{\partial v}\varphi_-(0, y)\big)v \\
&= \varphi_-(v, y)
\end{aligned}$$

by the definition of $\tilde{\varphi}_-$ given in (3.3.4).

Now as we know that

$$\frac{\partial^2}{\partial u \partial v}\tilde{\varphi} = \tilde{H}(\tilde{\varphi}, u, v, y) + \Delta_y \tilde{\varphi}$$

and by the regularity of the functions $H, \varphi_+, \varphi_-$, we obtain

$$\frac{\partial^2}{\partial u \partial v}\varphi = \tilde{H}(\tilde{\varphi}, u, v, y) + \Delta_y \tilde{\varphi} + H(\varphi_+(u,y), u, 0, y) + \Delta_y \varphi_+(u,y).$$

By the definition of $\tilde{H}$ given in (3.3.2) we get

$$\frac{\partial^2}{\partial u \partial v}\varphi(u,v,y) = H(\varphi, u, v, y) + \Delta_y \varphi(u,v,y).$$

To obtain $\varphi$ solution of the problem (3.1.1) it remains to show that $\frac{\partial^2}{\partial u \partial v}\varphi = \frac{\partial^2}{\partial v \partial u}\varphi$. we differentiate the equality (3.7.1) first in $u$, then in $v$, hence

$$\frac{\partial^2}{\partial v \partial u}\varphi = \frac{\partial^2}{\partial v \partial u}\tilde{\varphi} + H(\varphi_+(u,y), u, 0, y) + \Delta_y \varphi_+(u,y).$$

But we know (see (3.6.11)) that if $m > max(n-1, \frac{n-1}{2}+4)$,

$$\begin{aligned} \frac{\partial^2}{\partial v \partial u}\tilde{\varphi} &= \frac{\partial^2}{\partial u \partial v}\tilde{\varphi} \\ &= \tilde{H}(\tilde{\varphi}, u, v, y) + \Delta_y \tilde{\varphi}. \end{aligned}$$

Thus we have

$$\frac{\partial^2}{\partial u \partial v}\varphi = \frac{\partial^2}{\partial v \partial u}\varphi$$

which gives that $\varphi$ is a solution of the problem (3.1.1).

### Regularity of $\varphi$

To study the regularity of $\varphi$ it suffices to study the regularity of $\delta(\varphi_+, \varphi_-) = \varphi_+(u,y) + \big(\frac{\partial}{\partial v}\varphi_-(0,y) + \int_0^u H(\varphi_+(s,y), s, 0, y) + \Delta_y \varphi_+(s,y) ds\big)v$ because we have already results about the regularity of $\tilde{\varphi}$ by the proposition (3.6.1).

We start by the derivative of first order of $\delta(\varphi_+, \varphi_-)$. We have

$$\frac{\partial}{\partial u}\delta(\varphi_+, \varphi_-) = \frac{\partial}{\partial u}\varphi_+(u,y) + \big(H(\varphi_+(u,y), u, 0, y) + \Delta_y \varphi_+(u,y) ds\big)v$$

and

$$\frac{\partial}{\partial v}\delta(\varphi_+, \varphi_-) = \frac{\partial}{\partial v}\varphi_-(0,y) + \int_0^u H(\varphi_+(s,y), s, 0, y) + \Delta_y \varphi_+(s,y) ds$$

which are in $C^0(I \times [0;R] \times \mathbb{T}^{n-1})$ by the assumptions on the functions $H, \varphi_+, \varphi_-$. At least, these assumptions on the functions $H, \varphi_+, \varphi_-$ allow us to commute $\int_0^u$ and $\frac{\partial}{\partial y_i}$, so we can write

$$\begin{aligned} \frac{\partial}{\partial y_i}\delta(\varphi_+, \varphi_-) = \frac{\partial}{\partial y_i}\varphi_+(u,y) + \Big(\frac{\partial^2}{\partial y_i \partial v}\varphi_-(0,y) + \int_0^u \Big(\frac{\partial}{\partial \theta}H\Big)(\varphi_+(s,y), s, 0, y)\frac{\partial}{\partial y_i}\varphi_+(s,y) \\ + \Big(\frac{\partial}{\partial y_i}\tilde{H}\Big)(\varphi_+(s,y), s, 0, y) + \frac{\partial}{\partial y_i}\Delta_y \varphi_+(s,y) ds\Big)v. \end{aligned}$$

So $\frac{\partial}{\partial y_i}\delta(\varphi_+,\varphi_-)$ is in $C^0(I\times[0;R]\times\mathbb{T}^{n-1})$.

For the derivative of second order of $\delta(\varphi_+,\varphi_-)$ we get similarly

$$\begin{aligned}\frac{\partial^2}{\partial y_i\partial v}\delta(\varphi_+,\varphi_-) &= \frac{\partial^2}{\partial y_i\partial v}\varphi_-(0,y)+\int_0^u\big(\frac{\partial}{\partial\theta}H\big)(\varphi_+(s,y),s,0,y)\frac{\partial}{\partial y_i}\varphi_+(s,y)\\ &\quad+\big(\frac{\partial}{\partial y_i}\tilde{H}\big)(\varphi_+(s,y),s,0,y)+\frac{\partial}{\partial y_i}\Delta_y\varphi_+(s,y)ds\\ &= \frac{\partial^2}{\partial v\partial y_i}\delta(\varphi_+,\varphi_-).\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial y_i\partial u}\delta(\varphi_+,\varphi_-) &= \frac{\partial^2}{\partial y_i\partial u}\varphi_+(u,y)+\big(\big(\frac{\partial}{\partial\theta}H\big)(\varphi_+(s,y),s,0,y)\frac{\partial}{\partial y_i}\varphi_+(s,y)\\ &\quad+\big(\frac{\partial}{\partial y_i}\tilde{H}\big)(\varphi_+(s,y),s,0,y)+\frac{\partial}{\partial y_i}\Delta_y\varphi_+(s,y)\big)v\\ &= \frac{\partial^2}{\partial u\partial y_i}\delta(\varphi_+,\varphi_-).\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial u\partial v}\delta(\varphi_+,\varphi_-) &= H(\varphi_+(u,y),u,0,y)+\Delta_y\varphi_+(u,y)\\ &= \frac{\partial^2}{\partial v\partial u}\delta(\varphi_+,\varphi_-)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial u^2}\delta(\varphi_+,\varphi_-) &= \frac{\partial^2}{\partial u^2}\varphi_+(u,y)+\big(\big(\frac{\partial}{\partial\theta}H\big)(\varphi_+(u,y),u,0,y)\frac{\partial}{\partial u}\varphi_+(u,y)\\ &\quad+\big(\frac{\partial}{\partial u}\tilde{H}\big)(\varphi_+(u,y),u,0,y)+\frac{\partial}{\partial u}\Delta_y\varphi_+(u,y)\big)v.\end{aligned}$$

We can see that they are all in $C^0(I\times[0;R]\times\mathbb{T}^{n-1})$. Now

$$\frac{\partial^2}{\partial v^2}\delta(\varphi_+,\varphi_-) = 0.$$

$$\begin{aligned}\frac{\partial^2}{\partial y_i^2}\delta(\varphi_+,\varphi_-) &= \frac{\partial^2}{\partial y_i^2}\varphi_+(u,y)+\big(\frac{\partial^3}{\partial y_i^2\partial v}\varphi_-(0,y)+\int_0^u\big(\frac{\partial^2}{\partial\theta^2}H\big)(\varphi_+(s,y),s,0,y)\frac{\partial}{\partial y_i}\varphi_+(s,y)\\ &\quad+\big(\frac{\partial^2}{\partial y_i\partial\theta}H\big)(\varphi_+(s,y),s,0,y)\frac{\partial}{\partial y_i}\varphi_+(s,y)+\big(\frac{\partial}{\partial\theta}H\big)(\varphi_+(s,y),s,0,y)\frac{\partial^2}{\partial y_i^2}\varphi_+(s,y)\\ &\quad+\big(\frac{\partial^2}{\partial\theta\partial y_i}H\big)(\varphi_+(s,y),s,0,y)\frac{\partial}{\partial y_i}\varphi_+(s,y)+\big(\frac{\partial^2}{\partial y_i^2}\tilde{H}\big)(\varphi_+(s,y),s,0,y)\\ &\quad+\frac{\partial^2}{\partial y_i^2}\Delta_y\varphi_+(s,y)\;ds\big)v.\end{aligned}$$

So $\frac{\partial^2}{\partial y_i^2}\delta(\varphi_+,\varphi_-)$ is also in $C^0(I\times[0;R]\times\mathbb{T}^{n-1})$.

Thus we can conclude without adding assumptions, that if $m>max(n-1,\frac{n-1}{2}+6)$ the solution $\varphi$ is in $C^2(I\times[0;R]\times\mathbb{T}^{n-1})$. We come back to the variables $t$ and $x^1$ by the fact that $t=u+v$ and $x^1=v-u$, so we get the same regularity.

We proceed similarly for higher derivatives and we see that the assumptions necessarily to obtain $\varphi$ of class $C^l$ are not stronger that those necessarily to obtain $\tilde{\varphi}$ of class $C^l$.

**Conclusion**

So we have finally the existence of the solution of the problem (3.1.1) in a one-sided future neighborhood of a compact $([0;R] \times \mathbb{T}^{n-1}) \subset N_-$ where $[0;R]$ is as large as we want.

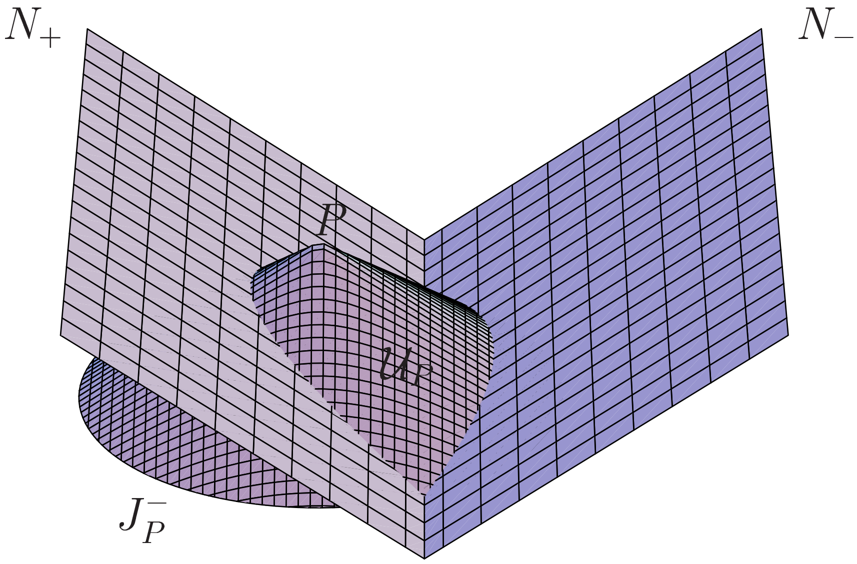
To obtain the existence of the solution of the problem (3.1.1) in a future timelike neighborhood of a compact $([0;R] \times \mathbb{T}^{n-1}) \subset N_+$ it suffices to exchange the role of $u$ and $v$ and to apply the same method.

### 3.7.2 Uniqueness of $\varphi$

For the uniqueness of the solution $\varphi$ we take a piece of time to examine the geometry of the problem.
Let $\tau = \frac{R}{\sqrt{2}}$, $\tilde{y} \in I\!R^{n-1}$ and $P$ the point of coordinates $(\tau + R', -\tau + R', \tilde{y})$ in $I\!R^{n+1}$. We consider $J_P^-$ a part of the past light cone issued of $P$, precisely

$$J_P^- = \{(t, x^1, y) \in I\!R^{n+1} / 0 \leq t \leq \tau + R', \quad \big(t - (\tau + R')\big)^2 = \big(x^1 - (-\tau + R')\big)^2 + |y - \tilde{y}|^2\}.$$

We recall that $N_+$ is the hypersurface $N_+ = \{(t, x^1, y) \in I\!R^{n+1} / t + x^1 = 0,\ t \geq 0\}$. It is easy to see that $J_P^- \bigcap N_+$ is a part of the parabola $\mathcal{P}$ of top $P'(\tau, -\tau, \tilde{y})$, $\mathcal{P} = \{(t, x^1, y) \in I\!R^{n+1} / |y - \tilde{y}|^2 = 4R'(x^1 + \tau)\}$. We call $\mathcal{U}_P$ the set $J_P^-$ intersected with the future of $N_+$ and the future of $N_- = \{(t, x^1, y) \in I\!R^{n+1} / t - x^1 = 0,\ t \geq 0\}$. We can visualize the situation by the following figure .




We're going to prove the uniqueness of the solution of the problem (3.1.1) found before, in $\mathcal{U}_P$. Then we call $\mathcal{U}_{P,\tau'}$ the set $\mathcal{U}_P$ intersected with the past of the hypersurface $\{(t, x^1, y) \in I\!R^{n+1} / t = \tau'\}$ which we denote simply $\{t = \tau'\}$.

Let $\varphi_1, \varphi_2$ be two solutions of the problem (3.1.1). We set $\varphi = \varphi_1 - \varphi_2$, so we have

$$\begin{cases} \Box\varphi = F(\varphi_1, t, x^1, y) - F(\varphi_2, t, x^1, y) \\ \varphi|_{N_+} = 0 \\ \varphi|_{N_-} = 0 \end{cases}$$

As $\frac{\partial F}{\partial \theta}$ is continuous (recall that $\theta$ is the first variable of $F$) and $\varphi_1, \varphi_2$ bounded (indeed $(u,v,y) \mapsto \varphi_1(u,v,y)$ and $(u,v,y) \mapsto \varphi_2(u,v,y)$ are $C^2$ so continuous on $[0;R] \times [0;R'] \times \mathbb{T}^{n-1}$), we can write that

$$\begin{aligned} |F(\varphi_1(t,x^1,y),t,x^1,y) - F(\varphi_2(t,x^1,y),t,x^1,y)| &\leq \parallel \varphi_1 - \varphi_2 \parallel \max_{0\leq s\leq 1} \parallel \frac{\partial F}{\partial \theta}\big((1-s)\varphi_1 + s\varphi_2\big) \parallel \\ &\leq c \parallel \varphi \parallel . \end{aligned}$$

Furthermore

$$|\Box\varphi| \leq c \parallel \varphi \parallel . \tag{3.7.2}$$

To prove that $\varphi$ vanishes in $\mathcal{U}_P$, we first estimate for any $0 \leq \tau' \leq \tau + R'$ some energy $E(\tau')$ of $\varphi$, namely

$$E(\tau') = \int_{\mathcal{U}_{P,\tau'} \bigcap \{t=\tau'\}} \frac{1}{2}\big(\varphi^2 + |\nabla\varphi|^2\big) dS$$

$$\text{where } |\nabla\varphi|^2 = \big(\frac{\partial\varphi}{\partial t}\big)^2 + \big(\frac{\partial\varphi}{\partial x^1}\big)^2 + \sum_{i=1}^{n-1}\big(\frac{\partial\varphi}{\partial y_i}\big)^2.$$

Then we show that $E(\tau')$ vanishes for any $0 \leq \tau' \leq \tau + R'$.

For this we use some notions of physics sciences and so introduce a tensor, called tensor of impulsive energy. As it is usually denoted in differential geometry literature, we set

$$X = \sum_{\mu} X^\mu \partial_\mu$$

where $\{\partial_\mu\}$ is a basis of local coordinates system of dimension $n+1$.
We denote $\nabla_\mu$ a covariant derivative with respect to $\partial_\mu$ and $\nabla^\mu := \sum_\nu \eta^{\mu\nu}\nabla_\nu$ where $\eta$ is the diagonal matrix of dimension $n+1$ of diagonal : $(-1, 1, ..., 1)$.
Now we consider the tensor T acting on one-vector field, namely

$$\begin{aligned} T(X) &= \sum_{\mu,\nu} T^\mu_{\ \nu} X^\nu \partial_\mu \\ \text{with } T^\mu_{\ \nu} &= \nabla^\mu\varphi\nabla_\nu\varphi - \frac{1}{2}\big((\sum_\alpha \nabla^\alpha\varphi\nabla_\alpha\varphi) + \varphi^2\big)\delta^\mu_{\ \nu} \end{aligned}$$

($\delta^\mu_{\ \nu}$ is the Kronecker symbol i.e. $\delta^\mu_{\ \nu}$ vanishes if $\mu \neq \nu$ and equals to 1 if $\mu = \nu$).

Notice that for $\mu = \nu = 0$ we obtain

$$\begin{aligned} T^0_{\ 0} &= -(\partial_t\varphi)^2 - \frac{1}{2}\big(-(\partial_t\varphi)^2 + (\partial_{x_1}\varphi)^2 + ... + (\partial_{x_n}\varphi)^2 + \varphi^2\big) \\ &= -\frac{1}{2}\big((\partial_t\varphi)^2 + (\partial_{x_1}\varphi)^2 + ... + (\partial_{x_n}\varphi)^2 + \varphi^2\big) \\ &= -\frac{1}{2}\big(\varphi^2 + |\nabla\varphi|^2\big). \end{aligned}$$

By the theorem of Stokes we know that, for every open set $\Omega$,

$$\int_{\partial\Omega} T(X) dS = \int_\Omega div\big(T(X)\big) dV$$

where $dS$ is the infinitesimal element of surface on $\partial\Omega$, more precisely $T(X)dS = \sum_\mu T^\mu(X)dS_\mu$, $dV$ is the infinitesimal element of volume on $\Omega$, and as we will take a constant vector $X$ (more precisely $X = \partial_0$), $div\big(T(X)\big) = \sum_{\mu,\nu} \nabla_\mu\big(T^\mu{}_\nu X^\nu\big)$.

Therefore we calculate $\nabla_\mu T^\mu{}_\nu$.

$$\begin{aligned}\nabla_\mu T^\mu{}_\nu &= \nabla_\mu\big(\nabla^\mu\varphi\nabla_\nu\varphi - \frac{1}{2}\big((\sum_\alpha \nabla^\alpha\varphi\nabla_\alpha\varphi) + \varphi^2\big)\delta^\mu{}_\nu\big)\\ &= \big(\nabla_\mu\nabla^\mu\varphi\big)\nabla_\nu\varphi + \nabla^\mu\varphi\big(\nabla_\mu\nabla_\nu\varphi\big) - \frac{1}{2}\delta^\mu{}_\nu\nabla_\mu\big((\sum_\alpha \nabla^\alpha\varphi\nabla_\alpha\varphi) + \varphi^2\big).\end{aligned}$$

Now we sum on $\mu$ :

$$\sum_\mu \nabla_\mu T^\mu{}_\nu = \sum_\mu \big(\nabla_\mu\nabla^\mu\varphi\big)\nabla_\nu\varphi + \sum_\mu \nabla^\mu\varphi\big(\nabla_\mu\nabla_\nu\varphi\big) - \frac{1}{2}\nabla_\nu\big(\sum_\alpha \nabla^\alpha\varphi\nabla_\alpha\varphi\big) - \varphi\nabla_\nu\varphi.$$

For the first term of the right member of the equality above, we can notice that

$$\big(\sum_\mu \nabla_\mu\nabla^\mu\varphi\big)\nabla_\nu\varphi = \big(\sum_{\mu,\alpha} \eta^{\mu\alpha}\nabla_\mu\nabla_\alpha\varphi\big)\nabla_\nu\varphi = \Box\varphi\nabla_\nu\varphi.$$

For the second and third one, we have

$$\sum_\mu \nabla^\mu\varphi\big(\nabla_\mu\nabla_\nu\varphi\big) = \sum_{\mu,\alpha} \eta^{\mu\alpha}\nabla_\alpha\varphi\big(\nabla_\mu\nabla_\nu\varphi\big)$$

and

$$-\frac{1}{2}\nabla_\nu\big(\sum_\alpha \nabla^\alpha\varphi\nabla_\alpha\varphi\big) = -\frac{1}{2}\sum_{\mu,\alpha} \big(\eta^{\mu\alpha}\big(\nabla_\nu\nabla_\mu\varphi\big)\nabla_\alpha\varphi + \eta^{\mu\alpha}\nabla_\mu\varphi\big(\nabla_\nu\nabla_\alpha\varphi\big)\big).$$

Then if $\varphi$ is of class $C^2$, it is easy to see that

$$\sum_\mu \nabla_\mu T^\mu{}_\nu = \big(\Box\varphi - \varphi\big)\nabla_\nu\varphi.$$

In particular if $\nu = 0$,

$$\sum_\mu \nabla_\mu T^\mu{}_0 = \big(\Box\varphi - \varphi\big)\nabla_0\varphi = \big(\Box\varphi - \varphi\big)\partial_t\varphi. \tag{3.7.3}$$

We apply the theorem of Stokes with $\Omega = \mathcal{U}_{P,\tau'}$. By looking the intersection of $\mathcal{U}_{P,\tau'}$ with the hypersurfaces $N_-, N_+$ and $\{t = \tau'\}$, we can decompose $\partial\mathcal{U}_{P,\tau'}$ in four parts as it follows :

$$\partial\mathcal{U}_{P,\tau'} = \big(\mathcal{U}_{P,\tau'} \cap N_-\big) \cup \big(\mathcal{U}_{P,\tau'} \cap N_+\big) \cup \big(\mathcal{U}_{P,\tau'} \cap \{t = \tau'\}\big) \cup \mathcal{C}_{\tau'}$$

where $\mathcal{C}_{\tau'}$ is the only curved part of $\partial\mathcal{U}_{P,\tau'}$.

As $\varphi$ vanishes on $N_-$ and $N_+$, when we integrate on $\partial\mathcal{U}_{P,\tau'}$ it only remains the integrals on $\mathcal{U}_{P,\tau'} \cap \{t = \tau'\}$ and on $\mathcal{C}_{\tau'}$.

For the integral on $\mathcal{C}_{\tau'}$, we integrate on characteristic hypersurface, by elementary lorrentzian geometry, we know that integrate on a characteristic hypersurface is equivalent to integrate only the component in isotropic vector tangent to this characteristic hypersurface, but $\sum T_{\mu\nu}Y^\mu Z^\nu \geq 0$ when $Y, Z$ are timelike or isotropic future directed vectors.Hence this integral is less or equal to zero.

For the integral on $\mathcal{U}_{P,\tau'} \cap \{t = \tau'\}$, as the time is constant, all the elements of surface $dS_\mu$ vanishes except of $dS_0$.

So we obtain

$$\begin{aligned} \int_{\mathcal{U}_{P,\tau'}\cap\{t=\tau'\}} T^0{}_0 dS_0 &\geq \int_{\mathcal{U}_{P,\tau'}} \sum_\mu \nabla_\mu T^\mu{}_0 dV \\ E(\tau') &\leq -\int_{\mathcal{U}_{P,\tau'}} \sum_\mu \nabla_\mu T^\mu{}_0 dV. \end{aligned}$$

On another hand by using (3.7.3) and (3.7.2), we have

$$\begin{aligned} \Big|\int_{\mathcal{U}_{P,\tau'}} (\nabla T)(X) dV\Big| &= \Big|\int_{\mathcal{U}_{P,\tau'}} (\Box\varphi - \varphi)\partial_t\varphi dV\Big| \\ &\leq \int_{\mathcal{U}_{P,\tau'}} c|\varphi||\partial_t\varphi| dV \\ &\leq \frac{1}{2}c\int_{\mathcal{U}_{P,\tau'}} |\varphi|^2 + |\partial_t\varphi|^2 dV \\ &\leq \frac{1}{2}c\int_{\mathcal{U}_{P,\tau'}} |\varphi|^2 + |\nabla\varphi|^2 dV. \end{aligned}$$

By the theorem of Fubini, as $\mathcal{U}_{P,\tau'} = \bigcup_{s\in[0;\tau']} \big(\mathcal{U}_{P,\tau'} \cap \{t = s\}\big)$, we get

$$\begin{aligned} \frac{1}{2}c\int_{\mathcal{U}_{P,\tau'}} |\varphi|^2 + |\nabla\varphi|^2 dV &= \frac{1}{2}c\int_0^{\tau'} \Big(\int_{\mathcal{U}_{P,\tau'}\cap\{t=s\}} |\varphi|^2 + |\nabla\varphi|^2 dS\Big) ds \\ &= c\int_0^{\tau'} E(s)ds. \end{aligned}$$

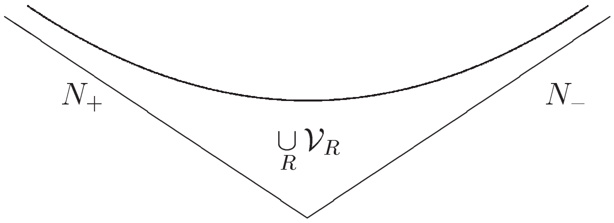
Finally for any $0 \leq \tau' \leq \tau + \lambda$,

$$E(\tau') \leq c\int_0^{\tau'} E(s)ds.$$

Then we set $h(t) = e^{-ct}\int_0^t E(s)ds$. We have $h'(t) = -ce^{-ct}\int_0^t E(s)ds + e^{-ct}E(t) \leq 0$ so for any $0 \leq t \leq \tau + \lambda$, $h(t) \leq h(0) = 0$, it means that for any $0 \leq t \leq \tau + \lambda$, $\int_0^t E(s)ds \leq 0$. Hence $E(t) \leq 0$ almost everywhere on $[0; \tau + \lambda]$, and as E is continuous, we can conclude that for any $0 \leq t \leq \tau + \lambda$, $E(t) = 0$. This implies that $\varphi$ vanishes almost everywhere in $\mathcal{U}_{P,\tau'}$, then everywhere by continuity of $\varphi$.

Hence if the functions $F, \varphi_+, \varphi_-$ are periodic in $y$, we get the uniqueness in $\bigcup_R \mathcal{V}_R$, where $\mathcal{V}_R := \{0 \leq t - x^1 \leq R,\ 0 \leq t + x^1 \leq R'_R,\ (x^2, ..., x^n) \in \mathbb{T}^{n-1}\} \bigcup \{0 \leq t + x^1 \leq R,\ 0 \leq t - x^1 \leq R''_R,\ (x^2, ..., x^n) \in \mathbb{T}^{n-1}\}$ ($R'_R$ and $R''_R$ are the reals found at each $R$ see theorem

3.7.1). Notice that $\bigcup_R \mathcal{V}_R$ is a set of length $T$ in each $y_i$ with a transversal section in $(u,v)$ which looks like a strip limited from below by $N_+ \cup N_-$, limited from above by an hyperbola, we can visualize it by the following figure.




We resume all the results in the following theorem :

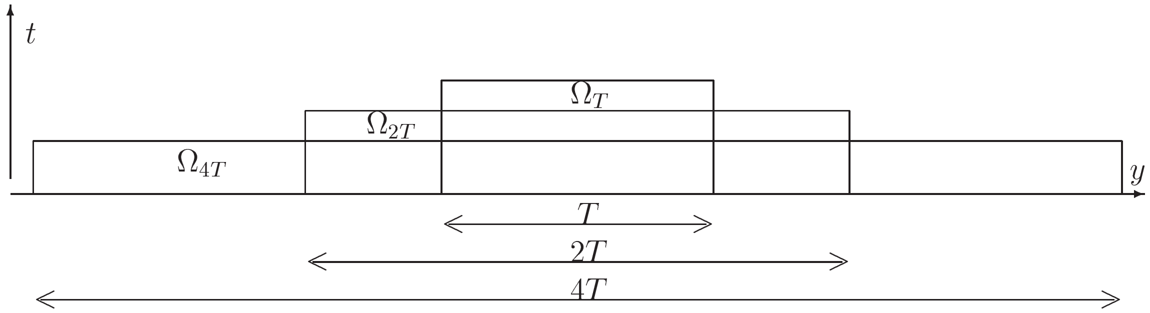
**Theorem 3.7.2** *If $m > max(n-1, \frac{n-1}{2}+4)$, and*
*(i) $F : (\theta, t, x^1, y) \mapsto F(\theta, t, x^1, y)$ satisfies that for any $a, b = 0$ or $1$,*
$0 \le \gamma + |\mu| \le m+1$ , $D_t^a D_{x^1}^b D_\theta^\gamma D_y^\mu F$ *is continuous in all its variables*
*(ii) $\varphi_+, \varphi_-$ are of class $H^{m+5}$, and $\varphi_+, \varphi_-$ satisfy the corner condition :*
$\varphi_+(0,y) = \varphi_-(0,y)$.
*(iii) There exists a real $T > 0$ such that $F, \varphi_+, \varphi_-$ are $T$-periodic in each $y_i$.*
*then there exists a unique $C^0$-solution $\varphi$ for the problem (3.1.1) in one-sided future neighborhood $\bigcup_R \mathcal{V}_R$ of the initial data hypersurfaces $N_+$ and $N_-$ .*
*Moreover, for all $l \ge 2$, if $m > max(n-1, \frac{n-1}{2}+4+l)$, and if for any $0 \le a, b \le l-1$, $0 \le \gamma + |\mu| \le m+1$ , $D_t^a D_{x^1}^b D_\theta^\gamma D_y^\mu F$ is continuous in all its variables, then $\varphi$ is in $C^l$.*

**Remark 3.7.1** : We have worked with the same periodicity $T$ in each $y_i$, but we can proceed similarly with different periodicities in each $y_i$, the functions $\Psi_\alpha(v,y)$, and $< \Psi_\alpha, f >$ will be a little more complicated, but we will get the sames results.

Now we remove the assumption of periodicity in $y$. We can consider two cases : first $Y = I\!R^{n-1}$, then $Y$ open set strictly included in $I\!R^{n-1}$.

If $\tilde{H}$ and $\tilde{\varphi}_-$ are defined on a set $Y = I\!R^{n-1}$ in their variable $y$ (which is equivalent to $F, \varphi_+, \varphi_-$ defined on $Y = I\!R^{n-1}$ in their variable $y$ ), in a first step, we work in a hypercube $\mathbb{T}'^{n-1}$ of length $2T$ in each $y_i$ and we multiply the functions $F, \varphi_+, \varphi_-$ by a cut off function in $y$ equal to 1 on a hypercube $\mathbb{T}^{n-1}$ of length $T$ in each $y_i$ strictly included in $\mathbb{T}'^{n-1}$, vanishing outside of $\mathbb{T}'^{n-1}$, with its partial derivatives also vanishing outside of $\mathbb{T}'^{n-1}$. Then if we replace $\mathbb{T}^{n-1}$ and $T$ by $\mathbb{T}'^{n-1}$ and $2T$ (length of $\mathbb{T}'$), we notice that as the data and their partial derivatives vanish on $\partial\mathbb{T}'^{n-1}$, all the argument still works. So we get by the theorem 3.7.2 a solution on a one-sided future neighborhood $V_T$ of $N_+ \cup N_-$ with variable $y$ defined in $\mathbb{T}'^{n-1}$. Now we consider $W_T$ the restriction of $V_T$ in variable $y$ to $\mathbb{T}^{n-1}$, and we define $\Omega_T$ as $\Omega_T = \{P \in W_T \, ; \big(J^-(P) \cap J^+(N_+ \cup N_-)\big) \subset W_T \}$ ($J^-(P)$ denotes the past light cone of $P$, $J^+(P)$ the future one). We keep the solution obtained on $V_T$ only on $\Omega_T$. We do it again with a hypercube $\mathbb{T}''^{n-1}$ of length $4T$ in each $y_i$ strictly including the hypercube $\mathbb{T}'^{n-1}$, we get another solution on a neighborhood $\Omega_{2T}$. As the initial data are the same on the restriction of $N_+ \cup N_-$ in variable $y$ to $\mathbb{T}^{n-1}$, the uniqueness of a solution of a wave equation in the past light cone of a point (notice that the proof of the uniqueness of $\varphi$ in subsection 3.7.2 still holds without periodic data) gives that the solution on $\Omega_{2T}$ is the same as the one on $\Omega_T$ on the intersection of both neighborhoods $\Omega_T \cap \Omega_{2T}$. So we obtain a solution on $\Omega_T \cup \Omega_{2T}$. By induction we construct a solution on

$\bigcup_{k\in I\!N} \Omega_{2^kT}$. We can visualize the process on the following figure where we show the section in $(t, y)$ of $\bigcup_{k\in I\!N} \Omega_{2^kT}$ , ($x_1$ is fixed).




Now if $Y$ is an open set strictly included in $I\!R^{n-1}$, we can consider some hypercube $\mathbb{T}^{n-1} \subset\subset \mathbb{T}'^{n-1} \subset Y$ (where $A \subset\subset B$ means $\overline{A} \subset B$). We multiply the functions $\tilde{H}$ and $\tilde{\varphi}_-$ by a cut off function equal to 1 on $\mathbb{T}^{n-1}$ and vanishing outside of $\mathbb{T}'^{n-1}$ and we replace $\mathbb{T}^{n-1}$ and $T$ by $\mathbb{T}'^{n-1}$ and $T'$ (length of $\mathbb{T}'$) in all the arguments, so we get a solution. We can't enlarge the hypercube as much as we want, but we can remark that when we consider again the intersection of the past light cone issued from $P(u, v, \tilde{y})$ ($u$ as large as necessary) with $N_+$, it's a part of parabola $\mathcal{P}$, which limit when $v \to 0$ is a segment $\{(s, 0, \tilde{y}); 0 \le s \le u\}$. This means that for any $u \ge 0$, we can find a $v \ge 0$ small enough such that the intersection of the past light cone issued of $P(u, v, \tilde{y})$ with the future of $N_+ \cup N_-$ is a set of points $Q(u', v', y')$ with $y'$ in $\mathbb{T}^{n-1}$. So by eventually reducing the thickness of the neighborhood obtained in theorem 3.7.2, the uniqueness of a solution of a wave equation in the past light cone of a point assures that the solution obtained in our argument is the right one. Hence we will obtain a neighborhood of $N_+ \cup N_-$ which becomes thiner and thiner when we reach the boundary of each connex component of $Y$. So we finally get the following theorem :

**Theorem 3.7.3** *If $m > max(n-1, \frac{n-1}{2} + 4)$, and*
*(i) The functions $F, \varphi_+, \varphi_-$ are defined on $I\!R^{n-1}$ in $y$.*
*(ii) $F : (\theta, t, x^1, y) \mapsto F(\theta, t, x^1, y)$ satisfies that for any $a, b = 0$ or $1$,*
$0 \le \gamma + |\mu| \le m+1$ , $D_t^a D_{x^1}^b D_\theta^\gamma D_y^\mu F$ *is continuous in all its variables*
*(iii) $\varphi_+, \varphi_-$ are of class $H^{m+5}$, and $\varphi_+, \varphi_-$ satisfy the corner condition :*
$\varphi_+(0, y) = \varphi_-(0, y)$.
*then there exists a unique $C^0$-solution $\varphi$ for the problem (3.1.1) in one-sided future neighborhood of the initial data hypersurfaces $N_+$ and $N_-$ .*
*Moreover, for all $l \ge 2$, if $m > max(n-1, \frac{n-1}{2} + 4 + l)$, and if for any $0 \le a, b \le l-1$, $0 \le \gamma + |\mu| \le m+1$ , $D_t^a D_{x^1}^b D_\theta^\gamma D_y^\mu F$ is continuous in all its variables, then $\varphi$ is in $C^l$.*

## 3.8 Case $I\!R^{1+1}$

We consider the same problem as (3.1.1) with $n = 1$, namely

$$\begin{cases} \Box\varphi(x, t) = F(\varphi(x, t), x, t) \\ \varphi|_{N_+} = \varphi_+ \\ \varphi|_{N_-} = \varphi_- \end{cases} \tag{3.8.1}$$

$$\text{where } N_+ = \{t + x = 0,\ t \geq 0\}$$
$$N_- = \{t - x = 0,\ t \geq 0\}$$
$$\Box = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}.$$

We proceed similarly as we have done for the case $I\!R^{n+1}$. Indeed, we first change variable $(t, x)$ to $(u, v)$, then we deal with a new equation in $\tilde{\varphi}$, and we approximate spectrally $\tilde{\varphi}$ by $\tilde{\varphi}_\varepsilon$. But in order to estimate $\| \tilde{\varphi}_\varepsilon(u) \|_{\mathcal{H}_{m,k}}$ we work with the norm $H^2([0, 2R]) = W^{2,2}([0, 2R])$. The estimations are similar but considerably simpler and we need weaker assumptions on the functions $F, \varphi_+, \varphi_-$. We obtain the following theorem.

**Theorem 3.8.1** *For all $l \geq 2$, if $F$ is of class $C^{l-1}$ , $\varphi_+, \varphi_-$ are of class $C^l$, and $\varphi_+, \varphi_-$ satisfy the corner condition :*
$\varphi_+(0, y) = \varphi_-(0, y)$,
*then for all real $R > 0$ , there exist some reals $R' > 0$ and $R" > 0$ such that there exists a unique solution $\varphi$ for the problem (3.8.1) in the domain $\Omega = \{0 \leq t - x \leq R,\ 0 \leq t + x \leq R'\} \bigcup \{0 \leq t + x \leq R,\ 0 \leq t - x \leq R"\}$ and this solution is in $C^l(\Omega)$.*

## 3.9 Appendix A

### 3.9.1 Proof of $\mathcal{H}_{m,k}$ Hilbert space

We set for any $f, g$ in $\mathcal{H}_{m,k}([0, 2R] \times \mathbb{T}^{n-1})$,

$$(f, g) = \sum_{\substack{0 \leq a \leq k \\ 0 \leq |\nu| \leq m}} \int_0^{2R} \int_{\mathbb{T}^{n-1}} (D_v^a D_y^\nu f)(D_v^a D_y^\nu g) dv\, d^{n-1}y.$$

$(.,.)$ is a symmetric and positive definite real valued bilinear form. We show that $\mathcal{H}_{m,k}$ is complete for the associated norm $\| f \| = (f, f)^{\frac{1}{2}}$.
Indeed, let $(u_n)$ be a Cauchy sequence in $\mathcal{H}_{m,k}([0, 2R] \times \mathbb{T}^{n-1})$, namely for all $0 \leq a \leq k$, for all $0 \leq |\nu| \leq m$, $(D_v^a D_y^\nu u_n)$ is a Cauchy sequence in $L^2([0; 2R] \times \mathbb{T}^{n-1})$. As $L^2([0; 2R] \times \mathbb{T}^{n-1})$ is a complete space, we know that for any $0 \leq a \leq k$, any $0 \leq |\nu| \leq m$, $(D_v^a D_y^\nu u_n)$ converges to a $L^2$-function $g_{a\nu}$. It remains to state that $g_{a\nu} = D_v^a D_y^\nu u$. We recall that $(D_v^a D_y^\nu u_n)$ converges to $(D_v^a D_y^\nu u)$ in $\mathcal{D}'([0; 2R] \times \mathbb{T}^{n-1})$ (we denote by $\mathcal{D}'$ the set of real-valued linear function defined on $\mathcal{D}$ the set of smooth compact-supported functions). On another hand, for any $\phi$ in $\mathcal{D}([0; 2R] \times \mathbb{T}^{n-1})$, by the Cauchy-Schwarz inequality it is clear that

$$\left| \int_{[0;2R] \times \mathbb{T}^{n-1}} (D_v^a D_y^\nu u_n - g_{a\nu})\phi \right| \leq \| D_v^a D_y^\nu u_n - g_{a\nu} \|_{L^2([0;2R] \times \mathbb{T}^{n-1})} \| \phi \|_{L^2([0;2R] \times \mathbb{T}^{n-1})} .$$

So $(D_v^a D_y^\nu u_n)$ converges to $g_{a\nu}$ in $\mathcal{D}'([0; 2R] \times \mathbb{T}^{n-1})$. By the uniqueness of the limit in $\mathcal{D}'([0; 2R] \times \mathbb{T}^{n-1})$, we can say that $g_{a\nu} = D_v^a D_y^\nu u$. The sequence $(u_n)$ converges to $u$ in $\mathcal{H}_{m,k}$, so $\mathcal{H}_{m,k}$ is a complete space. $\triangle$

### 3.9.2 Proof of lemma 3.2.1

We keep the notations introduced in section "Spaces $\mathcal{H}_{m,k}$". Our goal here is to prove the equivalence of the $\mathcal{H}_{m,k}$-norm defined above and the following one :

$$\rceil f \lceil = \Big( \sum_{\alpha \in \mathbb{Z}^n} |<\psi_\alpha, f>|^2 (1+|\alpha_0|)^{2k}(1+|\overline{\alpha}|)^{2m} \Big)^{\frac{1}{2}}.$$

We first show that

$$\| f \|^2_{\mathcal{H}_{m,k}} = \sum_{\substack{0 \le a \le k \\ 0 \le j \le m}} \Big( \sum_{\alpha \in \mathbb{Z}^n} |<\psi_\alpha, f>|^2 \big(\frac{\pi}{R}\big)^{2a} \big(\frac{2\pi}{T}\big)^{2j} |\alpha_0|^{2a} |\overline{\alpha}|^{2j} \Big). \tag{3.9.1}$$

(N.B. : in this paragraph, for more convenient we set by convention $0^0 = 1$, it avoids to distinguish the cases $a = 0$, $j = 0$ ...)
It suffices for that to show that

$$\sum_{|\gamma|=j} \| D_v^a D_y^\gamma f \|^2_{L^2} = \sum_{\alpha \in \mathbb{Z}^n} |<\psi_\alpha, f>|^2 \big(\frac{\pi}{R}\big)^{2a} \big(\frac{2\pi}{T}\big)^{2j} |\alpha_0|^{2a} |\overline{\alpha}|^{2j}.$$

But we know that

$$f(v,y) = \sum_{\alpha \in \mathbb{Z}^n} <\psi_\alpha, f> \psi_\alpha(v,y).$$

So by differentiating in $v$ and $y_i$, we have for any $l_1, ..., l_j$ in $\{1, ..., n-1\}$

$$D_v^a D_y^{l_1 \dots l_j} f(v,y) = \sum_{\alpha \in \mathbb{Z}^n} <\psi_\alpha, f> \big(i\frac{\pi}{R}\big)^a \big(i\frac{2\pi}{T}\big)^j (\alpha_0)^a \alpha_{l_1} \dots \alpha_{l_j} \psi_\alpha(v,y).$$

Hence

$$\| D_v^a D_y^{l_1 \dots l_j} f(v,y) \|^2_{L^2} = \sum_{\alpha \in \mathbb{Z}^n} |<\psi_\alpha, f>|^2 \big(\frac{\pi}{R}\big)^{2a} \big(\frac{2\pi}{T}\big)^{2j} |(\alpha_0)^a \alpha_{l_1} \dots \alpha_{l_j}|^2.$$

Then we notice that

$$\begin{aligned} \sum_{l_1,\dots,l_j \in \{1,\dots,n-1\}} |\alpha_0|^{2a} |\alpha_{l_1}|^2 \dots |\alpha_{l_j}|^2 &= |\alpha_0|^{2a} \sum_{l_1,\dots,l_j \in \{1,\dots,n-1\}} |\alpha_{l_1}|^2 \dots |\alpha_{l_j}|^2 \\ &= |\alpha_0|^{2a} \big( |\alpha_1|^2 + \dots + |\alpha_{n-1}|^2 \big)^j \\ &= |\alpha_0|^{2a} |\overline{\alpha}|^{2j}. \end{aligned}$$

Thus we get (3.9.1).
Now to obtain the equivalence of the norms it remains to find two constants $K$ and $K'$ such that

$$K(1+|\alpha_0|)^{2k}(1+|\overline{\alpha}|)^{2m} \le \sum_{\substack{0 \le a \le k \\ 0 \le j \le m}} \big(\frac{\pi}{R}\big)^{2a} \big(\frac{2\pi}{T}\big)^{2j} |\alpha_0|^{2a} |\overline{\alpha}|^{2j} \le K'(1+|\alpha_0|)^{2k}(1+|\overline{\alpha}|)^{2m}.$$

Let $K' = \max(1, (\frac{\pi}{R})^2, (\frac{\pi}{R})^{2k}, (\frac{2\pi}{T})^2, (\frac{2\pi}{T})^{2m}, (\frac{\pi}{R})^2(\frac{2\pi}{T})^2, (\frac{\pi}{R})^2(\frac{2\pi}{T})^{2m}, (\frac{\pi}{R})^{2k}(\frac{2\pi}{T})^2, (\frac{\pi}{R})^{2k}(\frac{2\pi}{T})^{2m})$.
Therefore

$$\begin{aligned} 1+(\frac{\pi}{R})^2|\alpha_0|^2+...+(\frac{\pi}{R})^{2k}(\frac{2\pi}{T})^{2m}|\alpha_0|^{2k}|\overline{\alpha}|^{2m} &\leq K'(1+|\alpha_0|^2+...+|\alpha_0|^{2k}|\overline{\alpha}|^{2m}) \\ &\leq K'\sum_{l=0}^{2k} C_{2k}^l|\alpha_0|^l\sum_{h=0}^{2m} C_{2m}^h|\overline{\alpha}|^h \\ &\leq K'(1+|\alpha_0|)^{2k}(1+|\overline{\alpha}|)^{2m}. \end{aligned}$$

Thus we can write

$$\| f \|^2_{\mathcal{H}m,k} \leq K' \sum_{\alpha\in\mathbb{Z}^n} |<\psi_\alpha, f>|^2(1+|\alpha_0|)^{2k}(1+|\overline{\alpha}|)^{2m}.$$

We denote
$\tilde{K} = \min(1, (\frac{\pi}{R})^2, (\frac{\pi}{R})^{2k}, (\frac{2\pi}{T})^2, (\frac{2\pi}{T})^{2m}, (\frac{\pi}{R})^2(\frac{2\pi}{T})^2, (\frac{\pi}{R})^2(\frac{2\pi}{T})^{2m}, (\frac{\pi}{R})^{2k}(\frac{2\pi}{T})^2, (\frac{\pi}{R})^{2k}(\frac{2\pi}{T})^{2m})$.
So

$$\tilde{K}(1+|\alpha_0|^2+...+|\alpha_0|^{2k}|\overline{\alpha}|^{2m}) \leq 1+(\frac{\pi}{R})^2|\alpha_0|^2+...+(\frac{\pi}{R})^{2k}(\frac{2\pi}{T})^{2m}|\alpha_0|^{2k}|\overline{\alpha}|^{2m}$$

By induction, we can calculate $c_i$ such that

$$\begin{aligned} c_k(1+|\alpha_0|)^{2k} &\leq 1+|\alpha_0|^2+...+|\alpha_0|^{2k} \\ c_m(1+|\overline{\alpha}|)^{2m} &\leq 1+|\overline{\alpha}|^2+...+|\overline{\alpha}|^{2m} \end{aligned}$$

(take $c_1 = \frac{1}{2}, c_{i+1} = \frac{c_i}{4}$). Furthermore,

$$\begin{aligned} c_k(1+|\alpha_0|)^{2k}(1+|\overline{\alpha}|^2+...+|\overline{\alpha}|^{2m}) &\leq (1+|\alpha_0|^2+...+|\alpha_0|^{2k}|\overline{\alpha}|^{2m}) \\ c_k(1+|\alpha_0|)^{2k}c_m(1+|\overline{\alpha}|)^{2m} &\leq (1+|\alpha_0|^2+...+|\alpha_0|^{2k}|\overline{\alpha}|^{2m}). \end{aligned}$$

We deduce from this that,

$$\tilde{K}c_kc_m\sum_{\alpha\in\mathbb{Z}^n} |<\psi_\alpha, f>|^2(1+|\alpha_0|)^{2k}(1+|\overline{\alpha}|)^{2m} \leq \| f \|^2_{\mathcal{H}m,k}.$$

△

### 3.9.3 Proof of lemma 3.2.2

We begin by establishing the following embedding .
If $\begin{cases} m > \frac{n-1}{2} \\ k > \frac{1}{2} \end{cases}$ then $\mathcal{H}_{m,k}([0;2R]\times\mathbb{T}^{n-1}) \subset L^\infty([0;2R]\times\mathbb{T}^{n-1})$.
We recall that

$$f = \sum_{\alpha\in\mathbb{Z}^n} f_\alpha(2R)^{-\frac{1}{2}}T^{-\frac{n-1}{2}}e^{i(\alpha_0 v\frac{\pi}{R}+\overline{\alpha}.y\frac{2\pi}{T})}$$

where $f_\alpha =< \Psi_\alpha, f >$. Therefore

$$\| f \|_{L^\infty} \leq (2R)^{-\frac{1}{2}}T^{-\frac{n-1}{2}}\sum_{\alpha\in\mathbb{Z}^n}|f_\alpha|$$

By the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^n}|f_\alpha| &= \sum_{\alpha\in\mathbb{Z}^n}\Big(|f_\alpha|(1+|\alpha_0|)^k(1+|\overline{\alpha}|)^m\times\frac{1}{(1+|\alpha_0|)^k(1+|\overline{\alpha}|)^m}\Big)\\ &\leq \rceil f \lceil \Big(\sum_{\alpha\in\mathbb{Z}^n}\frac{1}{(1+|\alpha_0|)^{2k}(1+|\overline{\alpha}|)^{2m}}\Big)^{\frac{1}{2}}.\end{aligned}$$

But we know that

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^n}\frac{1}{(1+|\alpha_0|)^{2k}(1+|\overline{\alpha}|)^{2m}} &= \int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}^{n-1}}\frac{1}{(1+|x|)^{2k}(1+|y|)^{2m}}dx\ d^{n-1}y\\ &= \int_{x\in\mathbb{R}}\frac{1}{(1+|x|)^{2k}}dx\int_{y\in\mathbb{R}^{n-1}}\frac{1}{(1+|y|)^{2m}}d^{n-1}y.\end{aligned}$$

These both integrals are convergent if $2k>1$ and $2m>n-1$ i.e. $k>\frac{1}{2}$ and $m>\frac{n-1}{2}$. At last, by using the equivalence of the norms above, we obtain

$$\| f \|_{L^\infty}\leq c\,\| f \|_{\mathcal{H}m,k}\,.$$

Now we show that

if $\begin{cases} m>\frac{n-1}{2}\\ k>\frac{1}{2}\end{cases}$ then $\mathcal{H}_{m,k}([0;2R]\times\mathbb{T}^{n-1})\subset C^0([0;2R]\times\mathbb{T}^{n-1})$.

Let $f$ in $\mathcal{H}_{m,k}$, for every $n$ in $\mathbb{N}^*$, we set $f_n=\hat{J}_{\frac{1}{n}}f$ ($\hat{J}$ has been defined in section "Spectral approximation of $\tilde{\varphi}$). It is clear that $f_n$ are in $\mathcal{H}_{m,k}$, and that

$$\| f_n \|_{L^\infty}\leq c\,\| f_n \|_{\mathcal{H}m,k}\,. \tag{3.9.2}$$

Then by the theorem of Plancherel we have $\| \hat{J}_{\frac{1}{n}}v-v \|_{L^2}\to 0$, if we apply this to $v=f,...,D_v^kD_y^mf$, we get

$$\| f-f_n \|_{\mathcal{H}m,k}\to 0.$$

The sequence $(f_n)$ converges to $f$ in $\mathcal{H}_{m,k}$, hence $(f_n)$ is a Cauchy sequence in $\mathcal{H}_{m,k}$, and in $L^\infty$ by (3.9.2). Moreover the functions $f_n$ are continuous, so $(f_n)$ is a Cauchy sequence in $C^0([0;2R]\times\mathbb{T}^{n-1})$. As this space is complete it implies that $(f_n)$ converges to $g$ in $C^0([0;2R]\times\mathbb{T}^{n-1})$.
It remains to show that $f=g$ almost everywhere. $(f_n)$ converges to $g$ in $L^2$, indeed

$$\| f_n-g \|_{L^2([0;2R]\times\mathbb{T}^{n-1})}\leq(2R\times T^{n-1})^{\frac{1}{2}}\,\| f_n-g \|_{L^\infty([0;2R]\times\mathbb{T}^{n-1})}\to 0.$$

But $(f_n)$ converges to $f$ in $\mathcal{H}_{m,k}$, in particular $(f_n)$ converges to $f$ in $L^2$, by the uniqueness of the limit in $L^2$, we can write that $f=g$ almost everywhere.
For the class $C^l$, it suffices to apply the result above to $\frac{\partial}{\partial v}f,\frac{\partial}{\partial y_i}f,...,D_v^lD_y^lf$. $\triangle$

### 3.9.4 Proof of lemma 3.2.3

We want to show that if $m<m'$ , $k<k'$ then the embedding $\mathcal{H}_{m',k'}\hookrightarrow\mathcal{H}_{m,k}$ is compact. We deal with the equivalent norm $\rceil f \lceil$ defined above and we will denote it also $\| f \|_{\mathcal{H}_{m,k}}$. As $(1+|\overline{\alpha}|)^{2m}\leq(1+|\overline{\alpha}|)^{2m'}$ and $(1+|\alpha_0|)^{2k}\leq(1+|\alpha_0|)^{2k'}$ it is clear

that $\| \dots \|_{\mathcal{H}m,k} \leq \| \dots \|_{\mathcal{H}m',k'}$. Set $i : \mathcal{H}_{m',k'} \hookrightarrow \mathcal{H}_{m,k}$, $i$ is a compact operator if it changes a bounded set in a relatively compact set. Let $(f_n)$ a bounded sequence of $\mathcal{H}_{m',k'}$. We have seen that $\mathcal{H}_{m',k'}$ is reflexive so we can extract a subsequence $(f_{n'})$ of $(f_n)$ which weakly converges to $f$ in $\mathcal{H}_{m',k'}$, and $\| f \|_{\mathcal{H}_{m',k'}} \leq \liminf \| f_{n'} \|_{\mathcal{H}_{m',k'}} \leq M$. We consider $\| f_{n'} - f \|^2_{\mathcal{H}_{m,k}}$ and cut the sum on $\alpha \in \mathbb{Z}^n$ in two parts, namely $I$ and $II$, as it follows

$$\| f_{n'} - f \|^2_{\mathcal{H}_{m,k}} \quad = \quad I + II$$

with

$$\begin{aligned} I &= \sum_{|\alpha| \leq A} | < \psi_\alpha, f_{n'} - f > |^2 (1 + |\alpha_0|)^{2k} (1 + |\overline{\alpha}|)^{2m} \\ II &= \sum_{|\alpha| > A} | < \psi_\alpha, f_{n'} - f > |^2 \frac{(1 + |\alpha_0|)^{2k'}}{(1 + |\alpha_0|)^{2(k'-k)}} \frac{(1 + |\overline{\alpha}|)^{2m'}}{(1 + |\overline{\alpha}|)^{2(m'-m)}} \end{aligned}$$

The function $f \mapsto < \psi_\alpha, f >$ is a continuous linear form on $\mathcal{H}_{m',k'}$, hence $< \psi_\alpha, f_{n'} > \to < \psi_\alpha, f >$ i.e. $< \psi_\alpha, f_{n'} - f > \to 0$. It implies that for all $\varepsilon_1 > 0$ there exists $\eta > 0$ such that for all $n' > \eta$ , $\displaystyle\sum_{|\alpha| \leq A} | < \psi_\alpha, f_{n'} - f > |^2 < \varepsilon_1^2$ . So

$$I \quad \leq \quad \varepsilon_1^2 (1 + A)^{2k+2m}.$$

We treat now the second term $II$. We notice that

$$\begin{aligned} \frac{1}{(1 + |\alpha_0|)^{2(k'-k)}} \frac{1}{(1 + |\overline{\alpha}|)^{2(m'-m)}} &\leq \frac{1}{\big[(1 + |\alpha_0|)(1 + |\overline{\alpha}|)\big]^{2\min(m'-m,k'-k)}} \\ &\leq \frac{1}{(1 + |\alpha|)^{2\min(m'-m,k'-k)}} . \end{aligned}$$

Thus

$$\begin{aligned} II &\leq \frac{1}{(1 + A)^{2\min(m'-m,k'-k)}} \| f_{n'} - f \|^2_{\mathcal{H}_{m',k'}} \\ &\leq \frac{1}{(1 + A)^{2\min(m'-m,k'-k)}} \big( \| f_{n'} \|_{\mathcal{H}_{m',k'}} + \| f \|_{\mathcal{H}_{m',k'}} \big)^2 \\ &\leq \frac{4M^2}{(1 + A)^{2\min(m'-m,k'-k)}}. \end{aligned}$$

Therefore for all $\varepsilon > 0$, we choose $A$ tall enough to get $\frac{4M^2}{(1+A)^{2\min(m'-m,k'-k)}} \leq \frac{\varepsilon^2}{2}$. Then we set $\varepsilon_1 = \frac{\varepsilon}{\sqrt{2}(1+A)^{2k+2m}}$. So there exists $\eta$ in $\mathbb{N}$ such that for all $n' \geq \eta$,

$$\begin{aligned} \| f_{n'} - f \|^2_{\mathcal{H}_{m,k}} &\leq \varepsilon_1^2 (1 + A)^{2k+2m} + \frac{4M^2}{(1 + A)^{2\min(m'-m,k'-k)}} \\ &\leq \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2. \end{aligned}$$

We obtain that $(f_{n'})$ converges to $f$ in $\mathcal{H}_{m,k}$. It means that $i(f_n)$ is a compact set *a fortiori* a relatively compact set.

### 3.9.5 Proof of lemma 3.2.4

Here we suppose that $f \in \mathcal{H}_{m,k} \cap \mathcal{H}_{m,k'}$ with $k < k'$, Let $\gamma \in [0;1]$, it is clear that $\mathcal{H}_{m,k'} \subset \mathcal{H}_{m,\gamma k+(1-\gamma)k'}$, so f is in $\mathcal{H}_{m,\gamma k+(1-\gamma)k'}$. We know that

$$\| f \|^2_{\mathcal{H}_{m,\gamma k+(1-\gamma)k'}} = \sum_{\alpha\in\mathbb{Z}^n} |<\psi_\alpha, f>|^2 (1+|\alpha_0|)^{2(\gamma k+(1-\gamma)k')}(1+|\overline{\alpha}|)^{2m}.$$

If we set

$$\begin{aligned} g(\alpha) &= \left(|<\psi_\alpha, f>|^2(1+|\alpha_0|)^{2k}(1+|\overline{\alpha}|)^{2m}\right)^\gamma \\ h(\alpha) &= \left(|<\psi_\alpha, f>|^2(1+|\alpha_0|)^{2k'}(1+|\overline{\alpha}|)^{2m}\right)^{1-\gamma} \end{aligned}$$

we can write that

$$\| f \|^2_{\mathcal{H}_{m,\gamma k+(1-\gamma)k'}} = \sum_{\alpha\in\mathbb{Z}^n} g(\alpha)h(\alpha).$$

Then by using Hölder inequality, we get

$$\sum_{\alpha\in\mathbb{Z}^n} g(\alpha)h(\alpha) \leq \big(\sum_{\alpha\in\mathbb{Z}^n} |g(\alpha)|^{\frac{1}{\gamma}}\big)^\gamma \big(\sum_{\alpha\in\mathbb{Z}^n} |h(\alpha)|^{\frac{1}{1-\gamma}}\big)^{1-\gamma}.$$

As

$$\begin{gathered} \big(\sum_{\alpha\in\mathbb{Z}^n} |<\psi_\alpha, f>|^2(1+|\alpha_0|)^{2k}(1+|\overline{\alpha}|)^{2m}\big)^\gamma = \| f \|^{2\gamma}_{\mathcal{H}_{m,k}} \\ \big(\sum_{\alpha\in\mathbb{Z}^n} |<\psi_\alpha, f>|^2(1+|\alpha_0|)^{2k'}(1+|\overline{\alpha}|)^{2m}\big)^{1-\gamma} = \| f \|^{2(1-\gamma)}_{\mathcal{H}_{m,k'}} \end{gathered}$$

we finally obtain,

$$\| f \|^2_{\mathcal{H}_{m,\gamma k+(1-\gamma)k'}} \leq \| f \|^{2\gamma}_{\mathcal{H}_{m,k}} \| f \|^{2(1-\gamma)}_{\mathcal{H}_{m,k'}} .$$

We proceed similarly for the case $f \in \mathcal{H}_{m,k} \cap \mathcal{H}_{m',k}$ with $m < m'$, hence we can say that for all $\gamma$ in $[0;1]$ ,
$f$ is in $\mathcal{H}_{\gamma m+(1-\gamma)m',k}$ and $\| f \|_{\mathcal{H}_{\gamma m+(1-\gamma)m',k}} \leq \| f \|^\gamma_{\mathcal{H}_{m,k}} \| f \|^{1-\gamma}_{\mathcal{H}_{m',k}}$.

Now we suppose that $f \in \mathcal{H}_{m,k} \cap \mathcal{H}_{m',k'}$ with $m < m'$, $k < k'$. Let $\gamma,\ \delta \in [0;1]$, it is clear that $\mathcal{H}_{m',k'} \subset \mathcal{H}_{\gamma m+(1-\gamma)m',\delta k+(1-\delta)k'}$, so f is in $\mathcal{H}_{\gamma m+(1-\gamma)m',\delta k+(1-\delta)k'}$. By using the inequalities above, we get that

$$\begin{aligned} \| f \|_{\mathcal{H}_{\gamma m+(1-\gamma)m',\delta k+(1-\delta)k'}} &\leq \| f \|^\gamma_{\mathcal{H}_{m,\delta k+(1-\delta)k'}} \| f \|^{1-\gamma}_{\mathcal{H}_{m',\delta k+(1-\delta)k'}} \\ &\leq \| f \|^{\gamma\delta}_{\mathcal{H}_{m,k}} \| f \|^{\gamma(1-\delta)}_{\mathcal{H}_{m,k'}} \| f \|^{(1-\gamma)\delta}_{\mathcal{H}_{m',k}} \| f \|^{(1-\gamma)(1-\delta)}_{\mathcal{H}_{m',k'}} . \end{aligned}$$

As the norm $\mathcal{H}_{m,k'}$ and $\mathcal{H}_{m',k}$ of $f$ can be bounded by the norm $\mathcal{H}_{m',k'}$ of $f$, we finally obtain that

$$\| f \|_{\mathcal{H}_{\gamma m+(1-\gamma)m',\delta k+(1-\delta)k'}} \leq \| f \|^{\gamma\delta}_{\mathcal{H}_{m,k}} \| f \|^{1-\gamma\delta}_{\mathcal{H}_{m',k'}} .$$

$\triangle$

## 3.10 Appendix B

### 3.10.1 Proof of the lemma 3.6.2

We notice that

$$\begin{aligned}
&\| \int_0^v f(s,y)ds \|_{\mathcal{H}_{m',1}([0;R]\times\mathbb{T}^{n-1})} \\
&\quad = \sum_{\substack{0\le a\le k \\ 0\le|\nu|\le m'}} \| \frac{\partial^a}{\partial v^a} D_y^\nu \int_0^v f(s,y)ds \|_{L^2([0;R]\times\mathbb{T}^{n-1})} \\
&\quad = \sum_{0\le|\nu|\le m'} \| D_y^\nu \int_0^v f(s,y)ds \|_{L^2([0;R]\times\mathbb{T}^{n-1})} + \sum_{0\le|\nu|\le m'} \| D_y^\nu f(v,y) \|_{L^2([0;R]\times\mathbb{T}^{n-1})} .
\end{aligned}$$

If $D_y^\nu f$ is in $C^0([0;R]\times\mathbb{T}^{n-1})$ then

$$D_y^\nu \int_0^v f(s,y)ds = \int_0^v D_y^\nu f(s,y)ds$$

and by the inequality of Cauchy-Schwarz

$$|\int_0^v D_y^\nu f(s,y)ds| \le v^{\frac{1}{2}} \| D_y^\nu f(v,y) \|_{L^2([0;v])} \le R^{\frac{1}{2}} \| D_y^\nu f(v,y) \|_{L^2([0;R])} .$$

Thus

$$\begin{aligned}
\| D_y^\nu \int_0^v f(s,y)ds \|_{L^2}^2 &= \int_0^R \int_{\mathbb{T}^{n-1}} |D_y^\nu \int_0^v f(s,y)ds|^2 dv\ d^{n-1}y \\
&\le R^{\frac{3}{2}} \int_{\mathbb{T}^{n-1}} \int_0^R |D_y^\nu f(v,y)|^2 dv\ d^{n-1}y \\
&= R^{\frac{3}{2}} \| D_y^\nu f(v,y) \|_{L^2([0;R]\times\mathbb{T}^{n-1})}^2 .
\end{aligned}$$

Finally we obtain

$$\| \int_0^v f(s,y)ds \|_{\mathcal{H}_{m',1}([0;R]\times\mathbb{T}^{n-1})} \le (R^{\frac{3}{2}}+1) \| f(s,y) \|_{\mathcal{H}_{m',0}([0;R]\times\mathbb{T}^{n-1})} .$$

△

### 3.10.2 Proof of the lemma 3.6.3

As the proof for the result with the norm $\mathcal{H}_{\mu,0}$ is similar, we just give the proof for the result with the norm $\mathcal{H}_{\mu,1}$. By definition

$$\| \int_u^{u+h} f(\sigma)d\sigma \|_{\mathcal{H}_{\mu,1}} = \sum_{\substack{0\le a\le 1 \\ 0\le|\nu|\le\mu}} \| \frac{\partial^a}{\partial v^a} D_y^\nu \int_u^{u+h} f(\sigma)d\sigma \|_{L^2}$$

And if $D_v^a D_y^\nu f$ is continuous in all its variables, we have

$$\begin{aligned}
\| D_v^a D_y^\nu \int_u^{u+h} f(\sigma) d\sigma \|_{L^2}^2 &= \| \int_u^{u+h} D_v^a D_y^\nu f(\sigma) d\sigma \|_{L^2}^2 \\
&= \int_0^R \int_{\mathbb{T}^{n-1}} | \int_u^{u+h} D_v^a D_y^\nu f(\sigma) d\sigma |^2 dv \, d^{n-1} y \\
&= \int_0^R \int_{\mathbb{T}^{n-1}} ( \int_u^{u+h} D_v^a D_y^\nu f(\sigma) d\sigma)( \int_u^{u+h} D_v^a D_y^\nu f(\gamma) d\gamma) dv \, d^{n-1} y.
\end{aligned}$$

We can commute the integration in $\sigma$ and $(v, y)$ by using the theorem of Fubini, hence

$$\| \int_u^{u+h} D_v^a D_y^\nu f(\sigma) d\sigma \|_{L^2}^2 = \int_u^{u+h} ( \int_0^R \int_{\mathbb{T}^{n-1}} D_v^a D_y^\nu f(\sigma) ( \int_u^{u+h} D_v^a D_y^\nu f(\gamma) d\gamma) dv \, d^{n-1} y) \, d\sigma.$$

Then by the inequality of Cauchy-Schwarz used on the integration in $(v, y)$, we get

$$\begin{aligned}
&\| \int_u^{u+h} D_v^a D_y^\nu f(\sigma) d\sigma \|_{L^2([0;R]\times\mathbb{T}^{n-1})}^2 \\
&\quad \leq \int_u^{u+h} (\| D_v^a D_y^\nu f(\sigma) \|_{L^2([0;R]\times\mathbb{T}^{n-1})} \| \int_u^{u+h} D_v^a D_y^\nu f(\gamma) d\gamma \|_{L^2([0;R]\times\mathbb{T}^{n-1})}) \, d\sigma.
\end{aligned}$$

The second factor under the integral in $\sigma$ is independent of $\sigma$, so we can get it out, thus

$$\begin{aligned}
&\| \int_u^{u+h} D_v^a D_y^\nu f(\sigma) d\sigma \|_{L^2([0;R]\times\mathbb{T}^{n-1})}^2 \\
&\quad \leq \| \int_u^{u+h} D_v^a D_y^\nu f(\gamma) d\gamma \|_{L^2([0;R]\times\mathbb{T}^{n-1})} \int_u^{u+h} (\| D_v^a D_y^\nu f(\sigma) \|_{L^2([0;R]\times\mathbb{T}^{n-1})}) \, d\sigma.
\end{aligned}$$

Then if $\| \int_u^{u+h} D_v^a D_y^\nu f(\sigma) d\sigma \|_{L^2([0;R]\times\mathbb{T}^{n-1})}$ vanishes, the inequality we want to show is trivial. Else we can divide by this positive quantity and so obtain

$$\| D_v^a D_y^\nu \int_u^{u+h} f(\sigma) d\sigma \|_{L^2([0;R]\times\mathbb{T}^{n-1})} \leq \int_u^{u+h} (\| D_v^a D_y^\nu f(\sigma) \|_{L^2([0;R]\times\mathbb{T}^{n-1})}) \, d\sigma.$$

To conclude it suffices to add this inequality on every $0 \leq a \leq 1, \ 0 \leq | \nu | \leq \mu$. $\triangle$

# Chapitre 4

# A semilinear wave equation with gradient in right-hand-side

**Abstract**

We analyse an initial value problem for nonlinear wave equations with gradient in the second member, with data given on two transversely intersecting null hypersurfaces of a Lorentzian manifold. Existence and uniqueness of a solution is obtained in a (one–sided future) neighborhood of the initial data null hypersurfaces.

## Sommaire

## 4.1 Introduction

It is of interest to study characteristic initial value problems, notably for their applications in general relativity. For linear equations one has global solutions by standard results. For quasi-linear equations, F. Cagnac [2], [3], F. Cagnac and M. Dossa [4] treat the Cauchy problem with given values on a characteristic conoid. In the case of a nonlinear hyperbolic equation, H. Müller zum Hagen and H.-J. Seifert [14] and A. D. Rendall [15] give existence and uniqueness of solutions in a neighborhood of the intersection of the null hypersurfaces. The object of this work is to prove, under certain conditions, existence and uniqueness of solutions in a whole neighborhood of characteristic hypersurfaces.

More precisely, consider two transversal characteristic hypersurfaces of a Lorentzian manifold (considered as a space time). These hypersurfaces define two isotropic directions. We are interested in a semilinear wave equation which has a second member depending on the solution and his gradient but linearly in one of the isotropic directions. We show the existence and uniqueness in a one-sided future neighborhood of the hypersurface corresponding to the isotropic direction in which the second member is not necessary linear. Obviously if the second member of the equation is depending on the solution and his gradient linearly in both isotropic directions, we obtain existence and uniqueness of a solution in a one-sided future neighborhood of both hypersurfaces.

The proof is based on a iterative method of the same model as the one used by A. Majda [13]. The theorem that gives the existence at each step comes from the article of A. D. Rendall [15] (see also H. Müller zum Hagen and H.-J. Seifert [14] or L. Hörmander [9]). We work with estimates of energy in Sobolev spaces on some slices with the usual tool of the energy-momentum tensor contracted with a convenient vector field. This leads to a difficulty, that the energy on characteristic hypersurfaces does not control the norm of the derivatives in transverse directions. We show that this can be overcome by a good choice of the vector field used in the energy argument, which eventually allows one to obtain a whole neighborhood of the initial surface.

The structure of this article is as follows.
We start in section two by a presentation of the problem, first in a flat metric, then in a general smooth Lorentzian metric on a manifold satisfying certain causality conditions. In section three we present the iterative scheme. In section four, we describe the global process to get existence, regularity and uniqueness of the solution. In section five, we detail the estimations necessary in the previous section. In section six, we give the result with less regular given functions.

## 4.2 Presentation of the problem

### 4.2.1 The flat metric case

Before doing the presentation in a general metric, we can illustrate the problem in which we are interested by considering the flat metric case in the Minkowski space-time $\mathbb{R}^{n+1}$. After the change of variables

$$u = \frac{t - x^1}{2}\ , \quad v = \frac{t + x^1}{2}\ , \quad y = (x^2, ..., x^n),$$

the wave equation under study takes the form

$$\frac{\partial^2}{\partial u \partial v}\varphi - \Delta_y \varphi \tag{4.2.1}$$
$$= h(\varphi, \partial_v\varphi, \partial_{y^1}\varphi, ..., \partial_{y^{n-1}}\varphi, u, v, y)\ \frac{\partial}{\partial u}\varphi + H(\varphi, \partial_v\varphi, \partial_{y^1}\varphi, ..., \partial_{y^{n-1}}\varphi, u, v, y)$$

with initial values of characteristic hypersurfaces,

$$\varphi|_{N^+} = \varphi_+$$
$$\varphi|_{N^-} = \varphi_-$$

$$\text{where } N^+ = \{v = 0, y \in I\!R^{n-1}\}, \quad N^- = \{u = 0, y \in I\!R^{n-1}\}.$$

We assume that $h$, $H$, $\varphi_+$, $\varphi_-$ are $C^\infty$ in all their variables, and the corner condition $\varphi_+(0, y) = \varphi_-(0, y)$.
We will obtain that there exists a smooth unique solution in a future neighborhood of $N^-$.

It is of interest to enquire whether such a result can be established for non-linearities of a form more general than in (4.2.1). Let us show by a simple example that a non-linearity in $\frac{\partial}{\partial u}\varphi$ can imply explosion of the solution along $N^-$. Indeed, let $c, c'$ two non vanishing real numbers, consider in $I\!R^{1+1}$ with the flat metric the following problem :

$$\begin{cases} \dfrac{\partial^2\varphi}{\partial u \partial v} = c\Big(\dfrac{\partial\varphi}{\partial u}\Big)^2 \\ \\ \varphi|_{N^+} = \varphi_+(u) = c'u \\ \varphi|_{N^-} = \varphi_-(v) = c'v \end{cases} \tag{4.2.2}$$

This can be directly solved (under the assumption that we can commute the partial derivatives with respect to $u$ and $v$), by first integrating in $v$, then in $u$, to obtain

$$\frac{-1}{\frac{\partial\varphi}{\partial u}(u, v)} = cv - \frac{1}{c'}$$

then

$$\varphi(u, v) = \frac{-u}{cv - \frac{1}{c'}} + c'v.$$

So we have explosion of the solution at $v = \frac{1}{cc'}$ for all $u$.
But there are also examples with a non-linearity in $\frac{\partial}{\partial u}\varphi$ which induce a smooth solution along $N^-$. Indeed, let $c' > 0$, and consider the problem

$$\begin{cases} \dfrac{\partial^2\varphi}{\partial u \partial v} = \dfrac{1}{2\dfrac{\partial\varphi}{\partial u}} \\ \\ \varphi|_{N^+} = \varphi_+(u) = c'u \\ \varphi|_{N^-} = \varphi_-(v) = c'v \end{cases} \tag{4.2.3}$$

We get

$$(\frac{\partial\varphi}{\partial u})^2(u,v) = v + c'^2$$

then

$$\varphi(u,v) = \sqrt{v+c'^2}\; u + c'v.$$

$\varphi$ is a smooth solution of (4.2.3) on the future of $N^+ \cup N^-$ entirely.

**Remark 4.2.1** : notice that if we take an equation with a certain type of function linear in both gradients with respect to both isotropic directions, we find a class of equations somewhat similar to the ones satisfying the "Null Condition" of S. Klainerman [10]. More precisely, to illustrate this in the flat metric case, consider in $I\!R^{n+1}$

$$\begin{aligned}\frac{\partial^2\varphi}{\partial u\partial v} - \Delta_y\varphi &= c(\varphi) + d(\varphi)\partial_u\varphi + e(\varphi)\partial_v\varphi + f(\varphi).\nabla_y\varphi + k(\varphi)\big[-\partial_u\varphi\partial_v\varphi + (\nabla_y\varphi)^2\big] \\ &= F(\varphi, \partial_u\varphi, \partial_v\varphi, \nabla_y\varphi).\end{aligned}$$

We see that $(w, w_u, w_v, w_{y_1}, ..., w_{y_n}) \mapsto F(w, w_u, w_v, w_{y_1}, ..., w_{y_n})$ satisfies that for all isotropic vector $X = (X^u, X^v, X^{y_1}, ..., X^{y_n})$ (i.e. $-X^uX^v + \sum(X^{y_i})^2 = 0$),

$$\sum_{a,b\in\{u,v,y_1,...,y_n\}} \frac{\partial^2 F}{\partial w_a\partial w_b} X^aX^b = 0.$$

(In a general Lorentzian metric $g$, we would have considered $\Box\varphi = A(\varphi) + B(\varphi).\nabla\varphi + C(\varphi)g(\nabla\varphi,\nabla\varphi)$).

**Remark 4.2.2** : we could examine also an equation with a nonlinearity in $\partial_u\varphi$ which looks like "dissipation" type, for example, in the flat metric case,

$$\Box\varphi(u,v,y) = h(\varphi(u,v,y), \nabla\varphi(u,v,y), u, v, y)\partial_u\varphi(u,v,y)$$

with a prescribed sign of $h$. But with our method we can't get the estimations we need to prove existence of a solution. Indeed as we use an iterative system with a linear equation (to get existence of a solution at each step) we must take an iterative system of the form

$$\Box\varphi^k(u,v,y) = h(\varphi^{k-1}(u,v,y), \nabla\varphi^{k-1}(u,v,y), u, v, y)\partial_u\varphi^k(u,v,y).$$

As we need to differentiate the equation with respect to $v$ and $y_i$ ( $1 \le i \le n-1$) several times, we obtain terms which some are products of partial derivatives of $h$ by products of different partial derivatives with respect to $v$ and $y_i$ of $\partial_u\varphi^k$ and of $\partial_u\varphi^{k-1}$, so we can't conclude anything about sign. We can see that even in a simple example as the one given in appendix 4.8, which has an explicit solution, our method doesn't permit to obtain the existence of a solution.

### 4.2.2 In a Lorentzian metric

The problem (4.2.1) has a natural generalisation with a Lorentzian metric. Indeed as space-time, we take $(\mathcal{M}, g)$ a smooth Lorentzian manifold of dimension $n+1$. Let $\Psi_+, \Psi_-$ be two $C^\infty$-real-valued functions defined on $\mathcal{M}$, globally hyperbolic, such that $\nabla\Psi_+$ and $\nabla\Psi_-$ are isotropic (i.e. $g(\nabla\Psi_+, \nabla\Psi_+) = 0$ and $g(\nabla\Psi_-, \nabla\Psi_-) = 0$), never vanish and are not colinear ($\nabla$ means the gradient for the Levi-Civita connexion relatively to $g$, i.e. $\nabla g = 0$ and the torsion vanishes). Let $N^+, N^-$ be the two transversal characteristic hypersurfaces defined by

$$N^+ := \{P \in \mathcal{M}, \Psi_+(P) = 0\}$$
$$N^- := \{P \in \mathcal{M}, \Psi_-(P) = 0\}.$$

We know that $\nabla\Psi_+$ is the isotropic direction of $N^+$ and $\nabla\Psi_-$ that of $N^-$. Hence by the assumption $\nabla\Psi_+$ and $\nabla\Psi_-$ not colinear, so in particular on $N^+ \cap N^-$, we get $N^+$ and $N^-$ transversal (we need also $\nabla\Psi_+$ and $\nabla\Psi_-$ not colinear, to define the slices on which we will make energy estimates). Without losing generality, we can suppose that $\nabla\Psi_+$ and $\nabla\Psi_-$ are past directed (it suffices possibly to replace $\Psi_+$ and/or $\Psi_-$ by its opposite, as $\nabla\Psi_+$ and $\nabla\Psi_-$ never vanish and g is Lorentzian, $\nabla\Psi_+$ and $\nabla\Psi_-$ can't be both isotropic and spacelike, hence by the continuity of $\Psi$, they are everywhere past directed or future directed ). Then we know that if $X, Y$ are causal, past directed and not colinear, it implies that $g(X, Y) < 0$, in particular $g(\nabla\Psi_+, \nabla\Psi_-) < 0$.
We denote

$$l := \frac{-\nabla\Psi_-}{\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}} \qquad w := \frac{-\nabla\Psi_+}{\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}} \tag{4.2.4}$$

It will be convenient to have a product decomposition of a future neighborhood $M$ of $N^+ \cup N^-$, the construction proceeds as follows :

we suppose that $N^+ \cap N^-$ is compact. For any $q$ in $N^+ \cap N^-$, we define $q_+(u)$ $(u \geq 0)$ to be the point $p$ on the integral curve $\Gamma_q$ of $w$ starting at $q$ such that $\psi_-(p) = u$. (We note that
$d(\psi_- \circ \Gamma_q)/ds = g(\nabla\psi_-, w) = -g(\nabla\psi_-, \nabla\psi_+)/\sqrt{-2g(\nabla\psi_-, \nabla\psi_+)} > 0$, so $\Psi_-$ is strictly increasing along those integral curves.) The implicit function theorem leads then to a neighborhood $\mathscr{O}_+ \subset N^+$ of $N^+ \cap N^-$ which is diffeomorphic to a product

$$\mathscr{O}_+ = \{(u, y) \ : \ y \in Y \ , \ u \in [0, u_{\max}(y))\} \ ,$$

for some lower semi-continuous function $u_{\max} : Y \to \mathbb{R}^+$. Notice that as $N^+ \cap N^-$ is compact, $Y$ is compact. As a *causal regularity condition* on $N^+$ we require that $N^+ = \mathscr{O}_+$, so that :

$$N^+ = \{(u, y) \ : \ y \in Y \ , \ u \in [0, u_{\max}(y))\} \ .$$

We note that this can always be achieved by replacing $N^+$ by a subset thereof if necessary ; in such a case the neighborhood obtained in our main existence theorem will be a neighborhood of $\mathscr{O}_+$ rather than of $N^+$.

The following simple example shows that if $\mathscr{O}_+ \neq N^+$, then existence of continuous solutions will fail in general :

in the two-dimension flat metric case, let $N^+ = \{v = 0, u \in I\!R \backslash \{1\}\}$, and consider the following equation

$$\frac{\partial^2}{\partial u \partial v}\varphi = 0,$$

then all solutions are of the form

$$\varphi(u,v) = \varphi_1(u) + \varphi_2(v).$$

As the initial data are given on $N^+$ and $N^-$, we have

$$\begin{aligned} \varphi(u,0) &= \varphi_1(u) + \varphi_2(0) = \varphi_+(u) \\ \varphi(0,v) &= \varphi_1(0) + \varphi_2(v) = \varphi_-(v) \\ \varphi(0,0) &= \varphi_1(0) + \varphi_2(0) = \varphi_+(0) = \varphi_-(0). \end{aligned}$$

Hence we get

$$\varphi(u,v) = \varphi_+(u) - \varphi_2(0) + \varphi_-(v) - \varphi_1(0) = \varphi_+(u) + \varphi_-(v) - \varphi_+(0).$$

Thus if $\varphi_+$ is continuous on $N^+$ but has a discontinuity at $u = 1$, we obtain a solution $\varphi$ in the distribution sense, but not a continuous solution.

Similarly following integral curves of $l$ on $N^-$ we obtain a neighborhood $\mathscr{O}_- \subset N^-$ of $N^+ \cap N^-$ with an analogous product decomposition ; the requirement $\mathscr{O}_- = N^-$ leads then to

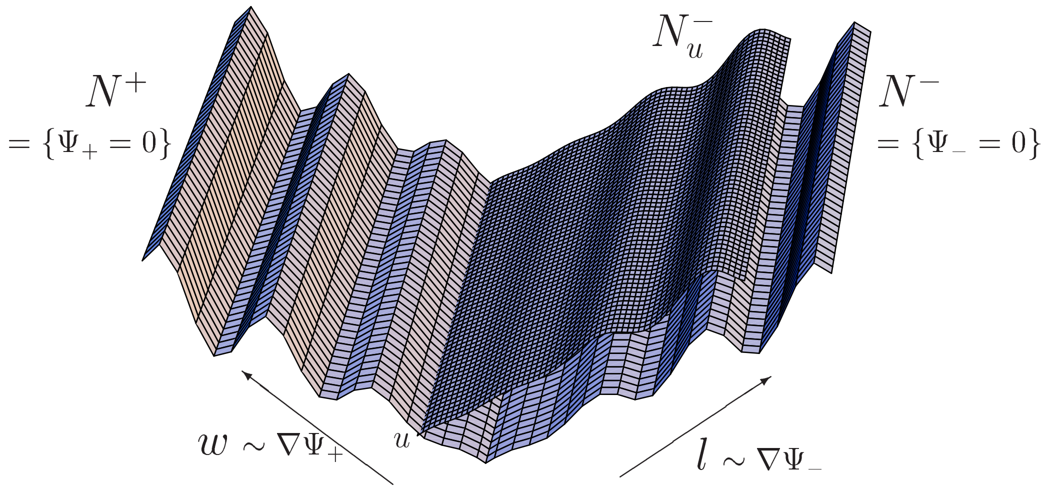
$$N^- = \{(v,y) \ : \ y \in Y \ , \ v \in [0, v_{\max}(y))\}$$

(for all $p = (v,y)$ in $N^-$, $v = \Psi_+(p) \geq 0$).

Next, we define the null hypersurface $N_u^-$ as the hypersurface obtained by following to the future the integral curves of $l$ starting at $(u,y) \in N^+$. If $p$ is a point lying on such a curve, we define $u(p)$ as the value of $u$ at the starting point, and $y(p)$ as the value of $y$ at the starting point. One easily checks that (as, if $\Gamma$ is the integral curve of $l$ starting at $(u,y)$, we have $d(\Psi_- \circ \Gamma)/ds = g(\nabla\Psi_-, l) = 0$ hence $\Psi_-$ is constant along $\Gamma$, equals to its value on $N^+$, namely $u$),

$$N_u^- \subset \{p : \psi_-(p) = u\} \ ,$$

so that the hypersurfaces $N_u^-$ are subsets of the level sets of $\psi_-$. We can visualize it on the following figure.



Analogously, we define the null hypersurface $N_v^+$ as the hypersurface obtained by following to the future the integral curves of $w$ starting at points in $N^-$ of the form $(v, y')$, $y' \in Y$. If $p$ is a point lying on such a curve, we define $v(p)$ as the value of $v$ at the starting point. (Here one could also define a second map $y'(p) \in Y$, but we have no use of that.) We have also

$$N_v^+ \subset \{p : \psi_+(p) = v\} .$$

The implicit function theorem shows that there exists a future neighborhood $M$ of $N^- \cup N^+$ on which the map

$$M \ni p \to (u(p), v(p), y(p)) \in \{(u, v, y)/y \in Y,\ u \in [0, u_{\max}(y)),\ v \in [0, v_{\max}(y))\} \tag{4.2.5}$$

is a diffeomorphism. In all the remaining considerations we restrict our attention to $M$.

Notice that $\nabla\Psi_+$ is normal to $N_v^+$, $\nabla\Psi_-$ to $N_u^-$. Indeed for any curve $\Gamma \subset N_v^+$, as $\Psi_+ \circ \Gamma = 0$, we have $d(\Psi_+ \circ \Gamma)/ds = g(\nabla\Psi_+, \dot{\Gamma}) = 0$, but any vector of $TN_v^+$, the tangent vector space of $N_v^+$, can be written as a $\dot{\Gamma}$, hence for any $X$ in $TN_v^+$, $g(\nabla\Psi_+, X) = 0$, and similarly for any $X$ in $TN_u^-$, $g(\nabla\Psi_-, X) = 0$.
$\nabla\Psi_+, \nabla\Psi_-$ are not colinear so we can locally complete them to obtain a local basis $(\nabla\Psi_+, \nabla\Psi_-, f_2, ..., f_n)$ of $TM$ the tangent vector space of $M$. As $N_v^+$ is of dimension $n$ and can't have two different isotropic directions we get locally $TN_v^+ = Vect\{\nabla\Psi_+, f_2, ..., f_n\}$, similarly $TN_u^- = Vect\{\nabla\Psi_-, f_2, ..., f_n\}$. Let $Y_{uv} = N_v^+ \cap N_u^-$ hence $TY_{uv} = TN_v^+ \cap TN_u^- = Vect\{f_2, ..., f_n\}$. Then as $\nabla\Psi_+$ is orthogonal to $TN_v^+$, $\nabla\Psi_-$ to $TN_u^-$, we obtain $TY_{uv} = (Vect\{\nabla\Psi_+, \nabla\Psi_-\})^\perp = (Vect\{w, l\})^\perp$ where $A^\perp = \{z \in TM, \forall a \in A \quad g(z, a) = 0\}$. We denote

$$Q := (Vect\{l, w\})^\perp = TY_{uv} \qquad D\varphi := P_Q(\nabla\varphi) \tag{4.2.6}$$

where $P_Q$ is the projection onto $Q$.

We consider the following problem

$$\begin{cases} \Box\varphi(P) = h(\varphi(P), l(\varphi)(P), D(\varphi)(P), P)\ w(\varphi)(P) + H(\varphi(P), l(\varphi)(P), D(\varphi)(P), P) \\ \varphi|_{N^+} = \varphi_+ \\ \varphi|_{N^-} = \varphi_- \end{cases} \tag{4.2.7}$$

where $\Box\varphi = g(\nabla\varphi, \nabla\varphi)$, with $h$ and $H$ are $C^\infty$ in all their variables, $\varphi_+,\ \varphi_-$ are $C^\infty$, and the corner condition $\varphi_+ = \varphi_-$ on $N^+ \cap N^-$.

We first get existence of a solution of (4.2.7) by an iterative method described in the next section.

## 4.3 Iterative scheme

For more convenience, we first "translate" the problem to one of vanishing initial data, but we'll see in section six that it is not indispensable. More precisely, let

$$\tilde{\varphi} := \varphi - \varphi_+ - \varphi_- + \varphi_0 \tag{4.3.1}$$

where $\varphi_0 = \varphi_+|_{N^+\cap N^-}$ (recall that $\varphi_+ = \varphi_-$ on $N^+ \cap N^-$). With the notations of (4.2.5), we can set

$$\varphi_+(P) := \varphi_+(u(P), y(P)) \qquad \varphi_-(P) := \varphi_-(v(P), y(P)) \qquad \varphi_0(P) := \varphi_0(y(P)).$$

Then we have

$$\begin{aligned}
\Box\tilde{\varphi}(P) &= \Box\varphi(P) - \Box\varphi_+(P) - \Box\varphi_-(P) + \Box\varphi_0(P) \\
&= h(\varphi(P), l(\varphi)(P), D(\varphi)(P), P)\ w(\varphi)(P) + H(\varphi(P), l(\varphi)(P), D(\varphi)(P), P) \\
&\qquad -\Box\varphi_+(P) - \Box\varphi_-(P) + \Box\varphi_0(P).
\end{aligned}$$

As

$$\begin{aligned}
l(\tilde{\varphi}) &= l(\varphi) - l(\varphi_+) - l(\varphi_-) + l(\varphi_0) \\
D(\tilde{\varphi}) &= D(\varphi) - D(\varphi_+) - D(\varphi_-) + D(\varphi_0) \\
w(\tilde{\varphi}) &= w(\varphi) - w(\varphi_+) - w(\varphi_-) + w(\varphi_0)
\end{aligned}$$

we can write

$$\Box\tilde{\varphi}(P) = \tilde{h}(\tilde{\varphi}(P), l(\tilde{\varphi})(P), D(\tilde{\varphi})(P), (P))\ w(\tilde{\varphi})(P) + \tilde{H}(\tilde{\varphi}(P), l(\tilde{\varphi})(P), D(\tilde{\varphi})(P), (P))$$

where

$$\begin{aligned}
\tilde{h}(\theta, \lambda, \delta, P) := h(\theta + \varphi_+(P) + \varphi_-(P) - \varphi_0(P)), \\
\lambda + l(\varphi_+)(P) + l(\varphi_-)(P) - l(\varphi_0)(P), \\
\delta + D(\varphi_+)(P) + D(\varphi_-)(P) - D(\varphi_0)(P), P
\end{aligned} \tag{4.3.2}$$

and

$$\begin{aligned}
\tilde{H}(\theta, \lambda, \delta, P) := \tilde{h}(\theta, \lambda, \delta, P)\big(w(\varphi_+)(P) + w(\varphi_-)(P) - w(\varphi_0)(P)\big) \\
+ H(\theta + \varphi_+(P) + \varphi_-(P) - \varphi_0(P), \\
\lambda + l(\varphi_+)(P) + l(\varphi_-)(P) - l(\varphi_0)(P), \\
\delta + D(\varphi_+)(P) + D(\varphi_-)(P) - D(\varphi_0)(P), P) \\
+ \Box\varphi_+(P) + \Box\varphi_-(P) - \Box\varphi_0(P).
\end{aligned} \tag{4.3.3}$$

Thus the problem (4.2.7) becomes :

$$\left\{\begin{array}{l}
\Box\tilde{\varphi} = \tilde{h}(\tilde{\varphi}(P), l(\tilde{\varphi})(P), D(\tilde{\varphi})(P), P)\ w(\tilde{\varphi})(P) + \tilde{H}(\tilde{\varphi}(P), l(\tilde{\varphi})(P), D(\tilde{\varphi})(P), P) \\
\tilde{\varphi}|_{N^+} = 0 \\
\tilde{\varphi}|_{N^-} = 0
\end{array}\right. \tag{4.3.4}$$

We want to construct approximate solutions $\left(\tilde{\varphi}^k\right)_{k\in I\!N}$ of the problem (4.3.4) by induction. Suppose that $\tilde{\varphi}^k$ is known, we can set

$$\begin{aligned}
\tilde{h}^{k+1}(P) &:= \tilde{h}(\tilde{\varphi}^k(P), l(\tilde{\varphi}^k)(P), D(\tilde{\varphi}^k)(P), P) \\
\tilde{H}^{k+1}(P) &:= \tilde{H}(\tilde{\varphi}^k(P), l(\tilde{\varphi}^k)(P), D(\tilde{\varphi}^k)(P), P),
\end{aligned}$$

then $\tilde{\varphi}^{k+1}$ is defined as the solution of a problem of type :

$$\left\{\begin{array}{l}
\Box\tilde{\varphi}^{k+1}(P) = h^{k+1}(P)\ w(\tilde{\varphi}^{k+1})(P) + H^{k+1}(P) \\
\tilde{\varphi}^{k+1}|_{N^+} = 0 \\
\tilde{\varphi}^{k+1}|_{N^-} = 0
\end{array}\right. \tag{4.3.5}$$

The existence of a solution $\tilde{\varphi}^{k+1}$ follows from the theorem 1 of A. D. Rendall in his article [15] which proves the existence and uniqueness of a solution of a quasilinear equation with prescribed data on two transversely hypersurfaces in a neighborhood $U$ of the intersection of these hypersurfaces (to apply it, set $x^1 = \Psi_- + \Psi_+$, $x^2 = \Psi_- - \Psi_+$, $N_1 = N^+$, $N_2 = N^-$). The neighborhood $U$ is first determined by solving the equations for functions satisfying the given values, and their partial derivatives according the problem, on the hypersurfaces, namely the propagation equations (indeed the values on $N_1 \cup N_2$ and the wave equation prescribe the values of the partial derivatives on $N_1 \cup N_2$), as in our case these equations are linear, we obtain $U$ as large as we want. In a second step $U$ is determined by the application of the standard theorems for solving the Cauchy problem, more precisely a suitable function solution of the propagation equations above defines initial data on a Cauchy surface containing the intersection of the hypersurfaces ($N_1 \cap N_2$), and the classical methods give the existence and uniqueness of a solution with this initial data in a certain neighborhood $U$. This gives the existence of the wanted solution on $U$ intersected with the future of $N_1 \cup N_2$. As in our case we have obtained the initial data on a set as large as we want and as (4.3.5) is linear we get a solution $\tilde{\varphi}^{k+1}$ on $M$. This solution is $C^\infty$ if all the remaining functions are $C^\infty$, hence we can continue the process (at each step $\tilde{\varphi}^k \in C^\infty$ implies $\tilde{h}^{k+1}, \tilde{H}^{k+1}$ $C^\infty$), and we obtain solutions $\tilde{\varphi}^k \in C^\infty$.

We start the iterative scheme with

$$\tilde{\varphi}^0(P) = 0.$$

N.B. : We don't take $w(\tilde{\varphi}^k)$ (instead of $w(\tilde{\varphi}^{k+1})$) in the right member of the first equation of the problem (4.3.5) because $w$ is transverse to the hypersurfaces on which we will get the energy estimates, so we don't control $w(\tilde{\varphi}^k)$. To neutralize this factor $w(\tilde{\varphi}^{k+1})$ in the estimations, when we will use the energy momentum tensor we will contract it with a particular timelike vector.

N.B. : In the case where $(\theta, \lambda, \gamma) \mapsto \tilde{h}(\theta, \lambda, \gamma, P)$ and $(\theta, \lambda, \gamma) \mapsto \tilde{H}(\theta, \lambda, \gamma, P)$ are not defined on $I\!R^{n+1}$, we can guarantee that $\tilde{h}^{k+1}(P) := \tilde{h}(\tilde{\varphi}^k(P), l(\tilde{\varphi}^k)(P), D(\tilde{\varphi}^k)(P), P)$ and $\tilde{H}^{k+1}(P) := \tilde{H}(\tilde{\varphi}^k(P), l(\tilde{\varphi}^k)(P), D(\tilde{\varphi}^k)(P), P)$ are well defined in (4.3.5). Indeed the lemma 4.4.1 will assure that $|\tilde{\varphi}^k(P)|$, $|l(\tilde{\varphi}^k)(P)|$ and $|D(\tilde{\varphi}^k)(P)|$ are contained in compacts independent of $k$.

Now we describe the global process to get the convergence of $\tilde{\varphi}^k$ to $\tilde{\varphi}$ solution of (4.3.4).

## 4.4 Global process to get existence

### 4.4.1 Definition of the Sobolev norm

In order to highlight the structure of our proof we have chosen to use in this section some arguments, the details of which are presented in section five. More precisely, the section five contains the proof of lemma 4.4.1 and 4.4.2. But as we need also further in this section certain inequalities which are similar to the ones necessary to obtain lemma 4.4.1 and 4.4.2, we don't have so much detailed them again.

We will work with some energy estimates on level sets of $\Psi_-$. Recall that $N_u^-$ is a set of geodesics issued from $P \in N^+ \cap \{P' \in M; \Psi_-(P') = u\}$ and of tangent vector

$l(P) \sim \nabla\Psi_-(P)$ at $P$. Let $0 < V < \min_{y\in Y}(v_{\max}(y))$ (by reducing $Y$ further, we will be allowed to increase $V$), we denote $N_u^-|_V$ the following part of $N_u^-$

$$N_u^-|_V := \{(s,v,y)/s = u,\ v \in [0;V],\ y \in Y\}$$

**Remark 4.4.1** : we need to consider this kind of slices on which we will estimate the energy, in sort of that the isotropic direction $w \sim \nabla\Psi_+$ will play the role of the time in the classical methods, and will permit us to obtain the existence of a solution on a neighborhood along the hypersurface $N^-$, and not only on a neighborhood on $N^+ \cap N^-$.

Now we define the weighted Sobolev spaces with which we will work. At all point of $M$, we can choose a local orthonormal basis $(e_0, e_1, ..., e_n)$ such that

$$e_0 := -\frac{\nabla\Psi_+ + \nabla\Psi_-}{\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}} = \frac{l+w}{2} \qquad e_1 := \frac{\nabla\Psi_+ - \nabla\Psi_-}{\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}} = \frac{l-w}{2} \tag{4.4.1}$$

(indeed $g(e_0, e_0) = -1$, $g(e_1, e_1) = 1$, $g(e_0, e_1) = 0$). We will denote

$$(\theta^0, \theta^1, ..., \theta^n) \text{ the dual basis associated to } (e_0, e_1, ..., e_n). \tag{4.4.2}$$

Then we have $g = -\theta^0 \otimes \theta^0 + \theta^1 \otimes \theta^1 + ... + \theta^n \otimes \theta^n$, and we set $k_- = \frac{1}{4}(\theta^0 + \theta^1) \otimes (\theta^0 + \theta^1) + \theta^2 \otimes \theta^2 + ... + \theta^n \otimes \theta^n$, $k_-$ is a Riemannian metric on hypersurfaces of $M$. We denote

$$\| \phi \|_{H^m([0;V]\times Y)} = \Big( \sum_{0\le\gamma\le m} \int_{N_u^-|_V} k_-(\nabla^\gamma(\phi), \nabla^\gamma(\phi)) dS' \Big)^{\frac{1}{2}}$$

where $\nabla^\gamma = \nabla \circ ... \circ \nabla$ $\gamma$-times. Here $dS'$ is the infinitesimal element of surface $dS$ on $N_u^-|_V$ multiplied by a factor $e^{-\lambda\Psi_+(P)}$, and $dS$ will be defined shortly. The necessity of the weight comes from the one-vector field on which the energy momentum tensor will act, to get the estimations. To visualize this norm we can notice the following. As $l = \frac{1}{2}(e_0 + e_1)$, we have globally (because $e_0$ and $e_1$ are globally defined),

$$k_-(\nabla\phi, \nabla\phi) = |l(\phi)|^2 + |D(\phi)|^2$$

hence, we obtain for $m = 1$,

$$\| \phi \|_{H^1([0;V]\times Y)} = \Big( \int_{N_u^-|_V} (|\phi|^2 + |l(\phi)|^2 + |D(\phi)|^2) dS' \Big)^{\frac{1}{2}}.$$

To do similarly for $m > 1$, we need to introduce some vectors fields. As $Y_{uv}$ is compact ($Y_{uv}$ is diffeomorphic to $Y$), from a suitable double covering of $Y_{uv}$ by open sets $(\mathcal{O}_i)$ in which we have a local basis $(e_a)_{2\le a\le n}$ of $Q = TY_{uv}$, we can extract a finite number $\nu$ of these $(\mathcal{O}_i)$ which will cover $Y_{uv}$ and such that we can take a convenient partition of unity $\chi$, $\chi_i$ equals 1 on a part of $\mathcal{O}_i$ and vanishes outside of $\mathcal{O}_i$, with $\sum_{1\le i\le\nu} \chi_i = 1$ everywhere on $Y_{uv}$. As $(e_a)_{2\le a\le n}$ is a local basis of $Q$ on $\mathcal{O}_i$, any vector of $Q$ can be written as a linear combination of the $\nu(n-1)$ vector fields $\chi_i e_a$. So we can find $r$ ($r \le \nu(n-1)$) vector fields $(q_1, ...., q_r)$ of $Q$ such that in any $\mathcal{O}_i$ $(1 \le i \le \nu)$, $D\varphi = P_Q(\nabla\varphi) = (q_{j_1(i)}(\varphi), ..., q_{j_{n-1}(i)}(\varphi))$ (i.e. $Q = Vect\{q_1, ...., q_r\}$). Then if we set

$$\hat{q} = (l, q_1, ...., q_r) \tag{4.4.3}$$

we can write

$$\| \phi \|_{H^m([0;V]\times Y)} = \Big( \sum_{0\le|\eta|\le m} \int_{N_u^-|_V} |\hat{q}^\eta(\phi)|^2 dS' \Big)^{\frac{1}{2}}$$

where for any $\eta \in \underset{k=0,\dots,m}{\cup} \{0,...,r\}^k, \eta = (\eta_0, ..., \eta_k)$, $\hat{q}^\eta = \hat{q}_{\eta_0} \circ ... \circ \hat{q}_{\eta_k}$, $|\eta| = k$.
We can notice that in this norm $H^m$, we don't take all the vector fields of $Q$, but as any vector field of $Q$ is a linear combination of $(q_1, ...., q_r)$ (with bounded coefficients by working with normal vector fields), we obtain a equivalent norm (see E. Hebey [7] chapter 2, Propositon 2.3).

## 4.4.2 Estimations of $\tilde{\varphi}^k$

If we choose $m > \frac{n}{2} + 2$, we have in particular $m > \frac{n}{2} + 1$, and so the embedding $H^m([0;V]\times Y) \hookrightarrow C^1([0;V]\times Y)$ holds (see for example T. Aubin [1] chapter 2, paragraph 11, Theorem 2.34) and there exists $c' > 0$ such that

$$\| f \|_{C^1([0;V]\times Y)} \le c' \| f \|_{H^m([0;V]\times Y)} \tag{4.4.4}$$

(this inequality is still available with a weight $e^{-\lambda\Psi_+(P)}$ in $H^m$, it suffices to multiply the constant c' obtained for a unweighted Sobolev space by $e^{\lambda V}$).
(N.B. : $m > \frac{n}{2} + 2$ will be necessary to get $H^m \hookrightarrow C^2$ and bound the norm $L^\infty([0;V]\times Y)$ of $\Delta_y \tilde{\varphi}^k$ by the norm $H^m([0;V]\times Y)$ of $\tilde{\varphi}^k$.)

We choose $\rho > 0$ so that $(\theta, \lambda, \gamma_1, ..., \gamma_{n-1}) \mapsto \tilde{h}(\theta, \lambda, \gamma_1, ..., \gamma_{n-1}, P)$ and $(\theta, \lambda, \gamma_1, ..., \gamma_{n-1}) \mapsto \tilde{H}(\theta, \lambda, \gamma_1, ..., \gamma_{n-1}, P)$ are defined on $[-c'\rho; c'\rho]^{n+1}$. Let $\tilde{\varphi}^k(u)$ the restriction of $\tilde{\varphi}^k$ on $N_u^-|_V$, the function $u \mapsto \tilde{\varphi}^{k+1}(u)$ will be well defined on $[0; u'_k]$ if

$$u'_k \text{ is the largest number such that } \max_{0\le u\le u'_k} \| \tilde{\varphi}^k(u) \|_{H^m([0;V]\times Y)} \le \rho. \tag{4.4.5}$$

The fact that there exists a $u_*$ such that for all $k$ in $I\!N$, $u'_k > u_*$ comes from the following lemma (to clarify the process we assume for the moment lemma 4.4.1 and 4.4.2, their proofs are given in the next section).

We first assume that $Y$ has no boundary and that $0 < V < \min_{y\in Y}(v_{\max}(y))$, but we will see at the end of this section that this can be overcome by reducing the set $Y$ and examining the case of $Y$ has a boundary.

**Lemma 4.4.1** *For any integer $m$, $m > n/2 + 2$, there exists $u_* > 0$ such that for all $k$ in $I\!N$,*

$$\max_{0\le u\le u_*} \| \tilde{\varphi}^k(u) \|_{H^m([0;V]\times Y)} \le \rho.$$

From this lemma, we can deduce the following one.

## 4.4.3 Convergence of $(\tilde{\varphi}^k)$

**Lemma 4.4.2** *There exists $0 < u_{**} \le u_*$, $\alpha < 1$, such that for all $k$ in $I\!N$,*

$$\max_{0\le u\le u_{**}} \| \tilde{\varphi}^{k+1}(u) - \tilde{\varphi}^k(u) \|_{H^1([0;V]\times Y)} \le \alpha \max_{0\le u\le u_{**}} \| \tilde{\varphi}^k(u) - \tilde{\varphi}^{k-1}(u) \|_{H^1([0;V]\times Y)} .$$

This lemma implies that

$$\sum_{k=1}^{N} \max_{0\le u\le u_{**}} \| \tilde{\varphi}^{k+1}(u) - \tilde{\varphi}^{k}(u) \|_{H^1([0;V]\times Y)} \le \sum_{k=1}^{N} \alpha^k \max_{0\le u\le u_{**}} \| \tilde{\varphi}^{1}(u) - \tilde{\varphi}^{0}(u) \|_{H^1([0;V]\times Y)} .$$

So, as $\alpha < 1$ we get, when $N \to \infty$, that

$$\sum_{k=1}^{\infty} \max_{0\le u\le u_{**}} \| \tilde{\varphi}^{k+1}(u) - \tilde{\varphi}^{k}(u) \|_{H^1([0;V]\times Y)} < \infty.$$

Thus there exists $\tilde{\varphi}(u)$ in $L^\infty\big([0;u_{**}], H^1([0;V]\times Y)\big)$ such that

$$\lim_{k\to\infty} \max_{0\le u\le u_{**}} \| \tilde{\varphi}^{k}(u) - \tilde{\varphi}(u) \|_{H^1([0;V]\times Y)} = 0. \tag{4.4.6}$$

Then by (4.4.6) and the bounds in lemma 4.4.1 we can show that $\big(\tilde{\varphi}^k(u)\big)$ converges to $\tilde{\varphi}(u)$ in some spaces $H^{m'}$ with $1 < m' < m$, $m' \in I\!N$. Indeed, first assume the following lemma (the proof of which can be found in appendix 4.7).

**Lemma 4.4.3** *Let $\mathcal{S}$ be a compact Riemannian manifold (with or without boundary), $m$ in $I\!N$, if $f$ is in $H^1(\mathcal{S}) \cap H^m(\mathcal{S})$ then for all $1 < m' < m$, $m' \in I\!N$, $f$ is in $H^{m'}(\mathcal{S})$ and there exists $c > 0$ such that*

$$\| f \|_{H^{m'}(\mathcal{S})} \;\le\; c \;\| f \|^{\sigma}_{H^1(\mathcal{S})} \| f \|^{1-\sigma}_{H^m(\mathcal{S})}$$

*where $m' = \sigma + (1-\sigma)m$.*

Let $0 < \sigma < 1$ such that $m' = \sigma + (1-\sigma)m$, we have for any $k, l$ in $I\!N$,

$$\begin{aligned}
&\max_{0\le u\le u_{**}} \| \tilde{\varphi}^{k}(u) - \tilde{\varphi}^{l}(u) \|_{H^{m'}} \\
&\quad\le c \max_{0\le u\le u_{**}} \| \tilde{\varphi}^{k}(u) - \tilde{\varphi}^{l}(u) \|^{\sigma}_{H^1} \max_{0\le u\le u_{**}} \| \tilde{\varphi}^{k}(u) - \tilde{\varphi}^{l}(u) \|^{1-\sigma}_{H^m} \\
&\quad\le c \max_{0\le u\le u_{**}} \| \tilde{\varphi}^{k}(u) - \tilde{\varphi}^{l}(u) \|^{\sigma}_{H^1} \Big( \max_{0\le u\le u_{**}} \| \tilde{\varphi}^{k}(u) \|_{H^m} + \max_{0\le u\le u_{**}} \| \tilde{\varphi}^{l}(u) \|_{H^m} \Big)^{1-\sigma} \\
&\quad\le c \;(2\rho)^{1-\sigma} \max_{0\le u\le u_{**}} \| \tilde{\varphi}^{k}(u) - \tilde{\varphi}^{l}(u) \|^{\sigma}_{H^1} \\
&\quad= \overline{c} \max_{0\le u\le u_{**}} \| \tilde{\varphi}^{k}(u) - \tilde{\varphi}^{l}(u) \|^{\sigma}_{H^1} .
\end{aligned}$$

From this and (4.4.6) we deduce that $\big(\tilde{\varphi}^k(u)\big)$ is a Cauchy sequence in $H^{m'}([0;V]\times Y)$ which is a complete space so $\big(\tilde{\varphi}^k(u)\big)$ converges to a function $f(u)$ of $H^{m'}([0;V]\times Y)$. This convergence implies also that $\big(\tilde{\varphi}^k(u)\big)$ converges to $f(u)$ in $H^1([0;V]\times Y)$, by the uniqueness of the limit in $H^1([0;V]\times Y)$, we obtain that $f(u) = \tilde{\varphi}(u)$. Moreover by taking the limit in lemma 4.4.1, we get

$$\max_{0\le u\le u_{**}} \| \tilde{\varphi} \|_{H^{m'}([0;V]\times Y)} \le \rho. \tag{4.4.7}$$

If we choose $\frac{n}{2}+2<m'<m$ we get by the embedding $H^{m'}([0;V]\times Y)\hookrightarrow C^2([0;V]\times Y)$ (see T. Aubin [1] chapter 2, paragraph 11, Theorem 2.34), so for all $0\leq u\leq u_{**}$, $\big(\tilde{\varphi}^k(u)\big)$ converges to $\tilde{\varphi}(u)$ in $C^2([0;V]\times Y)$ .

Hence we obtain that for all $u$ in $[0;u_{**}]$ the convergence in $C^0([0;V]\times Y)$ of $\big(\tilde{\varphi}^k(u)\big)$, $\big(l(\tilde{\varphi}^k)(u)\big)$, $\big(D\tilde{\varphi}^k(u)\big)$ and $\big(q_i\circ q_j(\tilde{\varphi}^k)(u)\big)$ (for all $1\leq i,j\leq r$) to respectively $\tilde{\varphi}(u)$, $l(\tilde{\varphi})(u)$, $D\tilde{\varphi}(u)$ and $q_i\circ q_j(\tilde{\varphi})(u)$. As $\tilde{h}$, $\tilde{H}$ are continuous, to show that $\tilde{\varphi}$ satisfies the first equation of the problem (4.3.5), we start by showing that $\big(w(\tilde{\varphi}^k)(u)\big)$ converges to $w(\tilde{\varphi})(u)$ in $C^0([0;V]\times Y)$.

For that we show that $\big(w(\tilde{\varphi}^k)(u)\big)$ is a Cauchy sequence in $C^0([0;V]\times Y)$. Indeed we proceed similarly as in (4.5.32) but with $\tilde{\varphi}^k(u,v,y)-\tilde{\varphi}^l(u,v,y)$ instead of $\tilde{\varphi}^k(u,v,y)$, we obtain

$$|w(\tilde{\varphi}^k)(u,v,y)-w(\tilde{\varphi}^l)(u,v,y)| \;\leq\; \tilde{C}\int_0^v |l\circ w(\tilde{\varphi}^k)(u,s,y)-l\circ w(\tilde{\varphi}^l)(u,s,y)|ds$$

with (if $\frac{n}{2}+2<m'<m$)

$$\begin{aligned}&|l\circ w(\tilde{\varphi}^k)(u,s,y)-l\circ w(\tilde{\varphi}^l)(u,s,y)|\\ &\leq |-\Box(\tilde{\varphi}^k-\tilde{\varphi}^l)|+C|w(\tilde{\varphi}^k)-w(\tilde{\varphi}^l)|+\hat{C}\parallel \tilde{\varphi}^k-\tilde{\varphi}^l\parallel_{H^{m'}([0;V]\times Y)}\end{aligned}$$

(because the coefficients $a,b,c_1,...,c_r$ in (4.5.36) just depend on $w,l$ and not on $\tilde{\varphi}^k$). Now if we write

$$\begin{aligned}&\tilde{h}(\tilde{\varphi}^{k-1},l(\tilde{\varphi}^{k-1}),D\tilde{\varphi}^{k-1},u,s,y)w(\tilde{\varphi}^k)(u,s,y)-\tilde{h}(\tilde{\varphi}^{l-1},l(\tilde{\varphi}^{l-1}),D\tilde{\varphi}^{l-1},u,s,y)w(\tilde{\varphi}^l)(u,s,y)\\ &=(\tilde{h}(\tilde{\varphi}^{k-1},l(\tilde{\varphi}^{k-1}),D(\tilde{\varphi}^{k-1}),u,s,y)-\tilde{h}(\tilde{\varphi}^{l-1},l(\tilde{\varphi}^{l-1}),D\tilde{\varphi}^{l-1},u,s,y))w(\tilde{\varphi}^k)(u,s,y)\\ &+\tilde{h}(\tilde{\varphi}^{l-1},l(\tilde{\varphi}^{l-1}),D\tilde{\varphi}^{l-1},u,s,y)(w(\tilde{\varphi}^k)(u,s,y)-w(\tilde{\varphi}^l)(u,s,y))\end{aligned}$$

by the lemma 4.4.1 and as the norm $L^\infty$ of $w(\tilde{\varphi}^k)$ is bounded by a constant depending on $\rho$ (by (4.5.39) and the lemma 4.4.1), we obtain that

$$\begin{aligned}&|w(\tilde{\varphi}^k)(u,v,y)-w(\tilde{\varphi}^l)(u,v,y)|\\ &\leq \tilde{C}(\overline{c}_1(\rho)+C)\int_0^v |w(\tilde{\varphi}^k)(u,s,y)-w(\tilde{\varphi}^l)(u,s,y)|ds\\ &+\tilde{C}\int_0^v \big[|\tilde{h}(\tilde{\varphi}^{k-1},l(\tilde{\varphi}^{k-1}),D\tilde{\varphi}^{k-1},u,s,y)-\tilde{h}(\tilde{\varphi}^{l-1},l(\tilde{\varphi}^{l-1}),D\tilde{\varphi}^{l-1},u,s,y)|\overline{c}_2(\rho)\\ &+|\tilde{H}(\tilde{\varphi}^{k-1},l(\tilde{\varphi}^{k-1}),D\tilde{\varphi}^{k-1},u,s,y)-\tilde{H}(\tilde{\varphi}^{l-1},l(\tilde{\varphi}^{l-1}),D\tilde{\varphi}^{l-1},u,s,y)|\\ &+\hat{C}\parallel \tilde{\varphi}^k-\tilde{\varphi}^l\parallel_{H^{m'}([0;V]\times Y)}\big]ds.\end{aligned}$$

Then as we have done in (4.5.38) by applying the linear Gronwall lemma we get

$$|w(\tilde{\varphi}^k)(u,v,y)-w(\tilde{\varphi}^l)(u,v,y)|\leq \overline{c}(\rho)e^{\tilde{C}(c_1(\rho)+C)v}$$

where

$$\begin{aligned}\overline{c}(\rho)=\tilde{C}\int_0^v &\big[|\tilde{h}(\tilde{\varphi}^{k-1},l(\tilde{\varphi}^{k-1}),D\tilde{\varphi}^{k-1},u,s,y)-\tilde{h}(\tilde{\varphi}^{l-1},l(\tilde{\varphi}^{l-1}),D\tilde{\varphi}^{l-1},u,s,y)|\overline{c}_2(\rho)\\ &+|\tilde{H}(\tilde{\varphi}^{k-1},l(\tilde{\varphi}^{k-1}),D\tilde{\varphi}^{k-1},u,s,y)-\tilde{H}(\tilde{\varphi}^{l-1},l(\tilde{\varphi}^{l-1}),D\tilde{\varphi}^{l-1},u,s,y)|\\ &+\hat{C}\parallel \tilde{\varphi}^k-\tilde{\varphi}^l\parallel_{H^{m'}([0;V]\times Y)}\big]ds.\end{aligned}$$

And so $\overline{c}(\rho) \to 0$ when $k, l \to \infty$ by applying the dominated convergence theorem of Lebesgue (as $(\tilde{\varphi}^k(u))$ converges to $\tilde{\varphi}(u)$ in $C^2([0;V] \times Y) \cap H^{m'}([0;V] \times Y)$ and by the lemma 4.4.1). Hence we get that $\big(w(\tilde{\varphi}^k)(u)\big)$ is a Cauchy sequence in $C^0([0;V] \times Y)$ and as $C^0$ is a complete space we get that $\big(w(\tilde{\varphi}^k)(u)\big)$ converges to a function $F$ in $C^0([0;V] \times Y)$. Then as $\big(w(\tilde{\varphi}^k)(u)\big)$ converges to $w(\tilde{\varphi})(u)$ in the distribution sense and as the convergence in $C^0([0;V] \times Y)$ implies the convergence in the distribution space, by the uniqueness of the limit in the distribution space, we obtain $F = w(\tilde{\varphi})(u)$.
Thus we get that $\tilde{h}(\tilde{\varphi}^{k-1}, l(\tilde{\varphi}^{k-1}), D\tilde{\varphi}^{k-1}, u, v, y)w(\tilde{\varphi}^k) + \tilde{H}(\tilde{\varphi}^{k-1}, l(\tilde{\varphi}^{k-1}), D\tilde{\varphi}^{k-1}, u, v, y)$ converges to $\tilde{h}(\tilde{\varphi}, l(\tilde{\varphi}), D\tilde{\varphi}, u, v, y)w(\tilde{\varphi}) + \tilde{H}(\tilde{\varphi}, l(\tilde{\varphi}), D\tilde{\varphi}, u, v, y)$ in $C^0([0;V] \times Y)$. Now if we integrate $w \circ l(\tilde{\varphi}^k)$ along the integral curve of $w$ starting at the point $Q = (0, v, y)$, with $(u, v, y)$ defined as in (4.2.5), we have

$$\begin{aligned} l(\tilde{\varphi}^k)(u,v,y) &= l(\tilde{\varphi}^k)(0,v,y) + \int_0^u w \circ l(\tilde{\varphi}^k)(s,v,y)\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}ds \\ &= \int_0^u w \circ l(\tilde{\varphi}^k)(s,v,y)\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}ds \end{aligned}$$

because $\tilde{\varphi}^k$ vanishes on $N^-$, and $l$ is in $TN^-$, so $l(\tilde{\varphi}^k)$ vanishes on $N^-$. Therefore with (4.5.33) and as in (4.5.34), we get locally

$$\begin{aligned} w \circ l(\tilde{\varphi}^k) &= \frac{1}{2} w \circ l(\tilde{\varphi}^k) + \frac{1}{2} l \circ w(\tilde{\varphi}^k) - \frac{1}{2}\varpi(\tilde{\varphi}^k) \\ &= -\Box\tilde{\varphi}^k + \sum_{2 \le a,b \le n} g^{ab}\nabla_{f_a}\nabla_{f_b}\tilde{\varphi}^k + \sum_{i,j=0}^n l^j w^i \sum_{\lambda=0}^n \Gamma^\lambda{}_{ji}\partial_\lambda\tilde{\varphi}^k - \frac{1}{2}\varpi(\tilde{\varphi}^k). \end{aligned}$$

But $\Box\tilde{\varphi}^k$ is equal to $\tilde{h}(\tilde{\varphi}^{k-1}, l(\tilde{\varphi}^{k-1}), D\tilde{\varphi}^{k-1}, u, v, y)w(\tilde{\varphi}^k) + \tilde{H}(\tilde{\varphi}^{k-1}, l(\tilde{\varphi}^{k-1}), D\tilde{\varphi}^{k-1}, u, v, y)$ and so converges to $\tilde{h}(\tilde{\varphi}, l(\tilde{\varphi}), D\tilde{\varphi}, u, v, y)w(\tilde{\varphi}) + \tilde{H}(\tilde{\varphi}, l(\tilde{\varphi}), D\tilde{\varphi}, u, v, y)$ in $C^0([0;V] \times Y)$. $\nabla_{f_a}\nabla_{f_b}\tilde{\varphi}^k$ converges to $\nabla_{f_a}\nabla_{f_b}\tilde{\varphi}$ because $Vect\{f_2, ..., f_n\} = TY_{uv} = Vect\{q_1, ..., q_r\}$ and $(\tilde{\varphi}^k(u))$ converges to $\tilde{\varphi}(u)$ in $C^2([0;V] \times Y)$. And locally, $\sum_{i,j=0}^n l^j w^i \sum_{\lambda=0}^n \Gamma^\lambda{}_{ji}\partial_\lambda\tilde{\varphi}^k - \frac{1}{2}\varpi(\tilde{\varphi}^k)$ converges to $\sum_{i,j=0}^n l^j w^i \sum_{\lambda=0}^n \Gamma^\lambda{}_{ji}\partial_\lambda\tilde{\varphi} - \frac{1}{2}\varpi(\tilde{\varphi})$ by recalling (4.5.36), the fact that $(\tilde{\varphi}^k(u))$ converges to $\tilde{\varphi}(u)$ in $C^1([0;V] \times Y)$ and that $\big(w(\tilde{\varphi}^k)(u)\big)$ converges to $w(\tilde{\varphi})(u)$ in $C^0([0;V] \times Y)$. Finally locally, $\big(w \circ l(\tilde{\varphi}^k)(u)\big)$ converges to $-\tilde{h}(\tilde{\varphi}, l(\tilde{\varphi}), D\tilde{\varphi}, u, v, y)w(\tilde{\varphi}) - \tilde{H}(\tilde{\varphi}, l(\tilde{\varphi}), D\tilde{\varphi}, u, v, y) + \sum_{2 \le a,b \le n} g^{ab}\nabla_{f_a}\nabla_{f_b}\tilde{\varphi} + \sum_{i,j=0}^n l^j w^i \sum_{\lambda=0}^n \Gamma^\lambda{}_{ji}\partial_\lambda\tilde{\varphi} - \frac{1}{2}\varpi(\tilde{\varphi})$ in $C^0([0;V] \times Y)$. As we work on compact sets we can apply the dominated convergence theorem of Lebesgue. As $\big(l(\tilde{\varphi}^k)(u)\big)$ converges also to $l(\tilde{\varphi})(u)$, it gives by the uniqueness of the limit in $C^0$, that

$$\begin{aligned} l(\tilde{\varphi})(u,v,y) = \int_0^u & [-\tilde{h}(\tilde{\varphi}, l(\tilde{\varphi}), D\tilde{\varphi}, s, v, y)w(\tilde{\varphi}) - \tilde{H}(\tilde{\varphi}, l(\tilde{\varphi}), D\tilde{\varphi}, s, v, y) \\ &+ \sum_{2 \le a,b \le n} g^{ab}\nabla_{f_a}\nabla_{f_b}\tilde{\varphi} + \sum_{i,j=0}^n l^j w^i \sum_{\lambda=0}^n \Gamma^\lambda{}_{ji}\partial_\lambda\tilde{\varphi} - \frac{1}{2}\varpi(\tilde{\varphi})]\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}ds. \end{aligned}$$

From which we can deduce that locally

$$\begin{aligned} w \circ l(\tilde{\varphi})(u,v,y) &= -\tilde{h}(\tilde{\varphi}, l(\tilde{\varphi}), D\tilde{\varphi}, u, v, y)w(\tilde{\varphi}) - \tilde{H}(\tilde{\varphi}, l(\tilde{\varphi}), D\tilde{\varphi}, u, v, y) \\ &+ \sum_{2 \le a,b \le n} g^{ab}\nabla_{f_a}\nabla_{f_b}\tilde{\varphi} + \sum_{i,j=0}^n l^j w^i \sum_{\lambda=0}^n \Gamma^\lambda{}_{ji}\partial_\lambda\tilde{\varphi} - \frac{1}{2}\varpi(\tilde{\varphi}). \end{aligned}$$

Furthermore we have globally,

$$\Box\tilde{\varphi}(u,v,y) = \tilde{h}(\tilde{\varphi}, l(\tilde{\varphi}), D\tilde{\varphi}, u, v, y)w(\tilde{\varphi})(u,v,y) + \tilde{H}(\tilde{\varphi}, l(\tilde{\varphi}), D\tilde{\varphi}, u, v, y).$$

As $\tilde{\varphi}$ vanishes on $N^+ \cup N^-$ (because $\tilde{\varphi}^k = 0$ on $N^+ \cup N^-$) $\tilde{\varphi}$ is a solution of (4.3.5).

### 4.4.4 Regularity of $\tilde{\varphi}$ solution of (4.3.5)

Now we take care of the regularity of $\tilde{\varphi}$. We can show that $u \mapsto \| \tilde{\varphi}(u) \|_{C^0([0;V]\times Y)}$ is in $C^0([0; u_{**}])$ . Indeed by (4.5.39) and the lemma 4.4.1 the norm $L^\infty$ of $w(\tilde{\varphi}^k)$ is bounded by a constant depending on $\rho$, hence

$$|w(\tilde{\varphi}^k)(u,v,y)| \leq \overline{c}(\rho)$$

furthermore by integrating in $u$ along the integral curve of $w$ between the points $P(u,v,y)$ and $Q(u+h,v,y)$, we get

$$|\tilde{\varphi}^k(u+h,v,y) - \tilde{\varphi}^k(u,v,y)| \leq \overline{c'}(\rho)|h|$$

and by taking the limit in $C^0([0;V]\times Y)$ when $k \to \infty$ we have

$$|\tilde{\varphi}(u+h,v,y) - \tilde{\varphi}(u,v,y)| \leq \overline{c'}(\rho)|h|.$$

Thus $\tilde{\varphi}$ is in $C^{0,1}\big([0;u_{**}], C^0([0;V]\times Y)\big) \subset C^0\big([0;u_{**}], C^0([0;V]\times Y)\big)$ hence $\tilde{\varphi}$ is in $C^0\big([0;u_{**}]\times[0;V]\times Y\big)$.
By proceeding as in (4.5.37) with $w \circ l$ instead of $l \circ w$ and $\tilde{\varphi}$ instead of $\tilde{\varphi}^k$, we obtain

$$|w \circ l(\tilde{\varphi})(u,v,y)| \leq |-\Box(\tilde{\varphi})(u,v,y)| + C|w(\tilde{\varphi})(u,v,y)| + \hat{C} \parallel \tilde{\varphi}(u) \parallel_{H^m([0;V]\times Y)}$$

with $m > \frac{n}{2} + 2$. Hence

$$|w \circ l(\tilde{\varphi})(u,v,y)| \leq c(\rho)$$

from which we deduce by integrating in $u$ along the integral curve of $w$ between the points $P(u,v,y)$ and $Q(u+h,v,y)$,

$$|l(\tilde{\varphi})(u+h,v,y) - l(\tilde{\varphi})(u,v,y)| \leq \overline{c}(\rho)|h|.$$

Thus $l(\tilde{\varphi})$ is in $C^{0,1}\big([0;u_{**}], C^0([0;V]\times Y)\big) \subset C^0\big([0;u_{**}], C^0([0;V]\times Y)\big)$ hence $l(\tilde{\varphi})$ is in $C^0\big([0;u_{**}]\times[0;V]\times Y\big)$.
Furthermore to achieve to show that $\tilde{\varphi}$ is in $C^1\big([0;u_{**}]\times[0;V]\times Y\big)$, it suffices to show that $q_i(\tilde{\varphi})$ $(1 \leq i \leq r)$ is in $C^{0,1}\big([0;u_{**}], C^0([0;V]\times Y)\big) \subset C^0\big([0;u_{**}], C^0([0;V]\times Y)\big)$ and $w(\tilde{\varphi})$ is in $C^{0,1}\big([0;u_{**}], C^0([0;V]\times Y)\big) \subset C^0\big([0;u_{**}], C^0([0;V]\times Y)\big)$ because $(l, w, q_1, ..., q_r)$ generate $TM$. For that we write that for any $1 \leq i \leq r$ (as in (4.5.32) but with $q_i(\tilde{\varphi}^k)$ instead of $\tilde{\varphi}^k$),

$$\begin{aligned} w \circ q_i(\tilde{\varphi}^k)(u,v,y) &= \int_0^v l \circ w \circ q_i(\tilde{\varphi}^k)(u,s,y)\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}ds \\ &= \int_0^v \big(q_i \circ l \circ w(\tilde{\varphi}^k)(u,s,y) + \vartheta(\tilde{\varphi}^k)(u,s,y)\big)\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}ds \end{aligned}$$

where $\vartheta(\tilde{\varphi}^k)$ is the remainder of the commutation and can be bound by a constant depending on $\rho$ . Hence by expressing $l \circ w(\tilde{\varphi}^k)$ in function of $\Box\tilde{\varphi}^k$ and a remainder, then by taking $q_i$ of this expression we can show that (as $H^m([0;V] \times Y) \subset C^3([0;V] \times Y)$ if $m > n/2 + 3$)

$$|w \circ q_i(\tilde{\varphi}^k)(u, v, y)| \leq c(\rho).$$

After that we conclude as we have done before that $q_i(\tilde{\varphi}^k)$ $(1 \leq i \leq r)$ is in $C^{0,1}\big([0; u_{**}], C^0([0; V]\times Y)\big)$ and then $q_i(\tilde{\varphi})$ $(1 \leq i \leq r)$ is in $C^{0,1}\big([0; u_{**}], C^0([0; V] \times Y)\big)$ by taking the limit in $C^0([0; V] \times Y)$ when $k \to \infty$. Now to show that $w \circ w(\tilde{\varphi})$ is bounded by a constant depending on $\rho$, we need to show that $w \circ q_i \circ q_j(\tilde{\varphi}^k)$ $(1 \leq i, j \leq r)$ is bounded by a constant depending on $\rho$. We obtain it by proceeding as we have already done for $w \circ q_i(\tilde{\varphi}^k)$ (take $m > n/2 + 4$). Therefore one can easily check that

$$\begin{aligned} w \circ w(\tilde{\varphi}^k)(u, v, y) &= \int_0^v l \circ w \circ w(\tilde{\varphi}^k)(u, s, y)\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}ds \\ &= \int_0^v \big(w \circ l \circ w(\tilde{\varphi}^k)(u, s, y) + \varrho(\tilde{\varphi}^k)(u, s, y)\big)\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}ds \end{aligned}$$

with

$$|\varrho(\tilde{\varphi}^k)| \leq c(\rho)(1 + w \circ w(\tilde{\varphi}^k)).$$

Then similarly as before, when we take $w$ of the expression of $l \circ w(\tilde{\varphi}^k)$ in function of $\Box\tilde{\varphi}^k$ and a remainder, we obtain some terms of the form $w \circ q_i \circ q_j(\tilde{\varphi}^k)$ $(1 \leq i, j \leq r)$ and $w \circ w(\tilde{\varphi}^k)$. Thus

$$|w \circ w(\tilde{\varphi}^k)(u, v, y)| \leq \overline{c}_1(\rho) \int_0^v |w \circ w(\tilde{\varphi}^k)(u, s, y)|ds + \overline{c}_2(\rho)$$

which gives by applying the linear Gronwall lemma a bound of $w \circ w(\tilde{\varphi}^k)$ by a constant depending on $\rho$ and permits us to reach our goal by the same method as above.

Finally we get that $\tilde{\varphi}$ is in $C^1\big([0; u_{**}] \times [0; V] \times Y\big)$. By repeating this method as much as it is necessary, we can show that (as we can take $m$ larger as we want), $\tilde{\varphi}$ is in $C^\infty\big([0; u_{**}] \times [0; V] \times Y\big)$.

### 4.4.5 Uniqueness of $\tilde{\varphi}$ solution of (4.3.5)

The question of the uniqueness is quickly solved. Indeed if we let $\tilde{\varphi}_1, \tilde{\varphi}_2$ be two $C^1$-solutions of the problem (4.3.5), it suffices to take again the inequalities of the proof of lemma 4.4.2 with $\tilde{\varphi}_1 - \tilde{\varphi}_2$ instead of $\tilde{\varphi}^{k+1} - \tilde{\varphi}^k$ to obtain that (with a choice of $\lambda$ large enough), for all $\tau$ in $[0; u_{**}]$,

$$\frac{1}{2} \parallel \tilde{\varphi}_1(\tau) - \tilde{\varphi}_2(\tau) \parallel^2_{H^1([0;V]\times Y)} \leq \int_0^\tau \overline{c}_1(\rho) \parallel \tilde{\varphi}_1(u) - \tilde{\varphi}_2(u) \parallel^2_{H^1([0;V]\times Y)} e^{-\lambda u} du.$$

Hence by using the linear Gronwall lemma we get

$$\parallel \tilde{\varphi}_1(\tau) - \tilde{\varphi}_2(\tau) \parallel^2_{H^1([0;V]\times Y)} \leq 0 \qquad \text{almost everywhere on } [0; u_{**}]$$

so

$$\tilde{\varphi}_1(\tau) - \tilde{\varphi}_2(\tau) = 0 \qquad \text{everywhere on } [0; u_{**}]$$

by the regularity of the solutions.
It gives the following proposition.

**Proposition 4.4.1** *Let $V > 0$, if $\tilde{h}$ and $\tilde{H}$ are $C^\infty$, there exists $u_{**} > 0$ and $\tilde{\varphi}$ such that $\tilde{\varphi}$ is a solution of the problem (4.3.5) on $[0; u_{**}] \times [0; V] \times Y$.*
*Moreover $\tilde{\varphi}$ is in $C^\infty([0; u_{**}] \times [0; V] \times Y)$ and is unique.*

### 4.4.6 Return to $\varphi$ solution of (4.2.7)

Now we come back to the solution of the problem (4.2.7).
Let

$$\varphi(u, v, y) := \tilde{\varphi}(u, v, y) + \varphi_+(u, y) + \varphi_-(v, y) - \varphi_0(y) \tag{4.4.8}$$

(recall that $\varphi_0 = \varphi_+|_{N^+ \cap N^-}$ and that $\varphi_+ = \varphi_-$ on $N^+ \cap N^-$). We get

$$\begin{aligned}
&\Box\varphi(u, v, y) \\
&\quad = \Box\tilde{\varphi}(u, v, y) - \Box\varphi_+(u, y) - \Box\varphi_-(v, y) + \Box\varphi_0(y) \\
&\quad = \tilde{h}(\tilde{\varphi}, \partial_v\tilde{\varphi}, \nabla_y\tilde{\varphi}, u, v, y)\partial_u\tilde{\varphi} + \tilde{H}(\tilde{\varphi}, \partial_v\tilde{\varphi}, \nabla_y\tilde{\varphi}, u, v, y) - \Box\varphi_+(u, y) - \Box\varphi_-(v, y) \\
&\qquad +\Box\varphi_0(y).
\end{aligned}$$

Then by the definition of $\tilde{h}$ and $\tilde{H}$ (see (4.3.2) and (4.3.3)) we obtain

$$\begin{aligned}
\Box\varphi(u, v, y) = {} & h(\varphi(u, v, y), l(\varphi)(u, v, y), D\varphi(u, v, y), u, v, y)w(\varphi)(u, v, y) \\
& +H(\varphi(u, v, y), l(\varphi)(u, v, y), D\varphi(u, v, y), u, v, y)
\end{aligned}$$

and on $N^+$, $N^-$,

$$\begin{aligned}
\varphi(0, v, y) = \tilde{\varphi}(0, v, y) + \varphi_+(0, y) + \varphi_-(v, y) - \varphi_0(y) = \varphi_-(v, y) \\
\varphi(u, 0, y) = \tilde{\varphi}(u, 0, y) + \varphi_+(u, y) + \varphi_-(0, y) - \varphi_0(y) = \varphi_+(u, y)
\end{aligned}$$

because $\tilde{\varphi}$ vanishes on $N^+ \cup N^-$ and $\varphi_0(y) = \varphi_-(0, y) = \varphi_+(0, y)$. So $\varphi$ is a solution of the problem (4.2.7). As $\varphi_+$ and $\varphi_-$ are $C^\infty$, we get that $\varphi$ is in $C^\infty(\bigcup_{u \in [0; u_{**}]} [0; V] \times Y)$, in particular we verify the regularity at the corner $(0, 0, y)$ in what follows. We show that the values obtained at the corner by the definition (4.4.8) are the same as the ones obtained by the propagation equations. Indeed if we integrate $w \circ l(\tilde{\varphi})$ along the integral curve of $w$ starting at the point $Q = (0, v, y)$, with $(u, v, y)$ defined as in (4.2.5), we have

$$\begin{aligned}
l(\tilde{\varphi})(u, v, y) &= l(\tilde{\varphi})(0, v, y) + \int_0^u w \circ l(\tilde{\varphi})(s, v, y)\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}ds \\
&= \int_0^u w \circ l(\tilde{\varphi})(s, v, y)\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}ds
\end{aligned}$$

hence $l(\tilde{\varphi})(0,0,y) = 0$ and so $l(\varphi)(0,0,y) = \lim_{u\to 0} l(\varphi_+)(u,y) + \lim_{v\to 0} l(\varphi_-)(v,y) - l(\varphi_0)(y) = l(\varphi_-)(0,y)$ (here we need $\varphi_+$ and $\varphi_-$ $C^1$, which is the case because they are supposed $C^\infty$). This is compatible with the fact that, as

$$l(\varphi)(u,v,y) \;=\; l(\varphi)(0,v,y) + \int_0^u w \circ l(\varphi)(s,v,y)\sqrt{-2g(\nabla\Psi_+,\nabla\Psi_-)}ds,$$

we also have $l(\varphi)(0,0,y) = l(\varphi_-)(0,y)$. We can proceed in analogous way for $w(\varphi)(0,0,y)$ and higher derivatives. Uniqueness of $\varphi$ can be obtained in the same way as we have done for $\tilde{\varphi}$. Indeed let $\varphi_1$ and $\varphi_2$ be two $C^1$-solutions of the problem (4.2.7), when we integrate on $\partial\Omega$ , the terms on $\Omega \cap N^+$ and on $\Omega \cap N^-$ vanish because $\varphi_1 = \varphi_2$ on $N^+ \cup N^-$ and so we get the same result as for $\tilde{\varphi}_1 - \tilde{\varphi}_2$ (the bound of the norm $H^m$ of $\varphi$ comes from the bound of the norm $H^m$ of $\tilde{\varphi}$), and we conclude that $\varphi_1 = \varphi_2$. It leads us to the following proposition.

**Proposition 4.4.2** *Let $0 < V < \min_{y\in Y}(v_{\max}(y))$, if $h$, $H$, $\varphi_+$, $\varphi_-$ are $C^\infty$, there exists $u_{**} > 0$ and a solution $\varphi$ of the problem (4.2.7) on $[0;u_{**}] \times [0;V] \times Y$.*
*Moreover $\varphi$ is in $C^\infty([0;u_{**}] \times [0;V] \times Y)$ and is unique.*

### 4.4.7 Improvement of the neighborhood of existence of $\varphi$

Notice that if $v_{\max}$ is constant (i.e. $N^-$ has a product structure $[0;V] \times Y$, where $V$ can be $+\infty$), we get a neighborhood of whole $N^-$. Otherwise, to obtain a neighborhood of the whole $N^-$, we must show in a first step that we can solve the same problem with a given function on the boundary $\partial\mathcal{Y} := \bigcup_{u,v} \partial Y_{uv}$ (compatible with the given functions on $N^+ \cup N^-$), and in a second step that for any given function on $\partial\mathcal{Y}$ (compatible with the given functions on $N^+ \cup N^-$), there's a neighborhood of $N^-$ (with a length in $v$ as large as we need) on which we have uniqueness of the solution. Then we will cut $Y$ in several sets $\tilde{Y}$, as much as necessary, and embed each set $\tilde{Y}$ in a larger set $Y'$ on which we will prescrib a boundary condition compatible with the given functions on $N^+ \cup N^-$, thus we will get a solution and by uniqueness it will be the one we look for on $\tilde{Y}$.

Let examine the case of $Y$ has a boundary. If $Y$ has a boundary and if $w$, $l$ are tangent to $\partial\mathcal{Y}$, we can add a boundary condition in the problem (4.2.7), namely prescribed data on $\partial\mathcal{Y}$. Let $\varphi_Y(u,v)$ be the prescribed value on $\partial\mathcal{Y}$. We assume that $\varphi_Y$ is $C^\infty$, that

$$\varphi_Y(u,0) = \varphi_+|_{\partial\mathcal{Y}} \qquad \varphi_Y(0,v) = \varphi_-|_{\partial\mathcal{Y}}$$

and that partial derivatives of $\varphi_Y$ at any $(u,0)$, $(0,v)$ are equal to the values obtained according the propagation equation. We first make this prescribed data vanishing by setting

$$\begin{aligned}\tilde{\varphi}(u,v,y) := \varphi(u,v,y) - &\varphi_+(u,y) - \varphi_-(v,y) + \varphi_+(0,y) - \varphi_Y(u,v) + \varphi_Y(u,0)\\ &+ \varphi_Y(0,v) - \varphi_Y(0,0)\end{aligned}$$

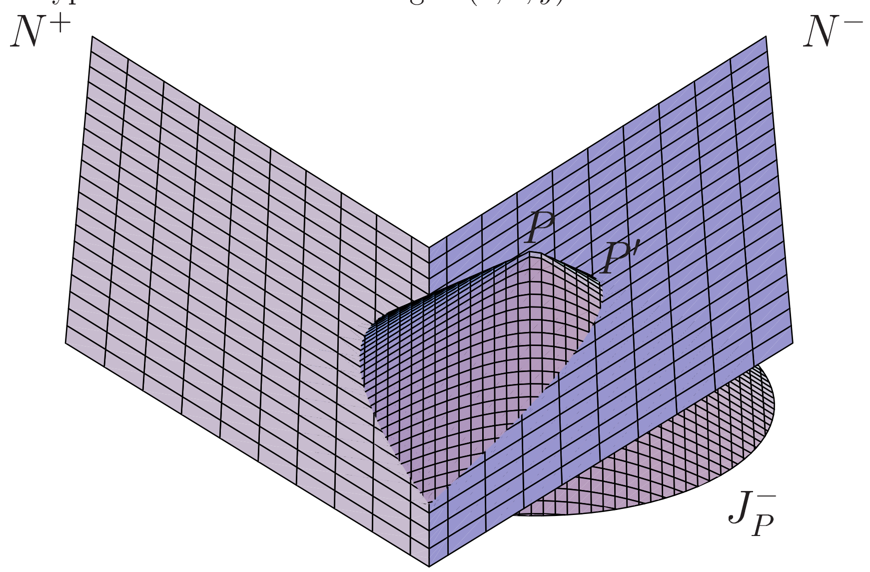
(recall that $\varphi_+(0,y) = \varphi_-(0,y)$, that $(u,v,y)$ is defined in (4.2.5)) and by defining the right functions $\tilde{h}$, $\tilde{H}$ in analogous way as we have already done in (4.3.2) and (4.3.3). Now in the theorem of Stokes, as the integral on $\Omega \cap \partial\mathcal{Y}$ will vanish, all will work similarly.

Hence we will get a smooth unique solution $\tilde{\varphi}$ of the problem with vanishing value on $\underset{u,v}{\cup}\,\partial Y_{uv}$, and then a solution $\varphi$ of the problem (4.2.7) with prescribed data on $\underset{u,v}{\cup}\,\partial Y_{uv}$ by setting

$$\varphi(u,v,y) := \tilde{\varphi}(u,v,y)+\varphi_+(u,y)+\varphi_-(v,y)-\varphi_+(0,y)+\varphi_Y(u,v)-\varphi_Y(u,0) \\ -\varphi_Y(0,v)+\varphi_Y(0,0).$$

Regularity and uniqueness are preserved.

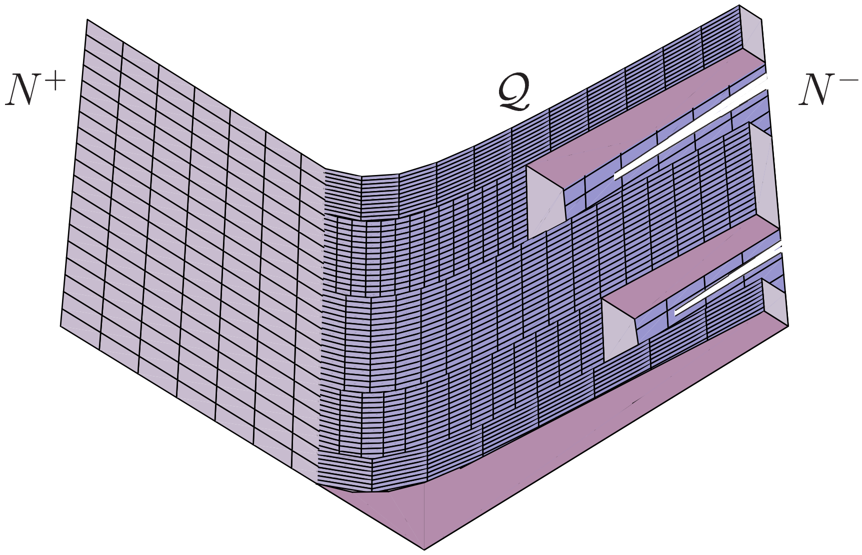
In a second step notice that we can work on sets with $V$ (the length in variable $v$) as long as we want by eventually reducing $Y$. More precisely we can take $Y'$, $\tilde{Y}$ compact sets such that $\tilde{Y} \subset \mathring{Y'}$ and $Y' \subset \mathring{Y}$ (where $\mathring{A}$ denotes the interior of $A$), and such that there exists $d > 0$ satisfying $\big(\underset{u\in[0;d]}{\cup} \tilde{N}_u^-|_V\big)$, where $\tilde{N}_u^-|_V = \{P(u,v,y); v \in [0;V]\ ,\ y \in \tilde{Y}\}$, is strictly included in a set which has a product structure $\mathcal{U}\times\mathcal{V}\times Y'$ (corresponding to $(u,v,y)$ defined in (4.2.5)). Indeed let $0 < \tilde{v} < v_{\max}(\tilde{y})\ ,\ v \le V,\ \tilde{y} \in Y$. If we look at the intersection $\mathcal{I}$ of the future of $N^- \cup N^+$ and the causal past $J_P^-$ of $P(d,\tilde{v},\tilde{y})$ with $P$ sufficiently close to $N^-$ ($P(u,v,y)$ defined in (4.2.5), $d > 0$) we see that the intersection of $\mathcal{I}$ and $N^-$ is an hypersurface with a boundary of the form $\tilde{Y} \cup \mathcal{P}$ where $\tilde{Y} \subset Y$ and $\mathcal{P}$ is an hypersurface of $\mathcal{I}$ containing $P'(0,\tilde{v},\tilde{y})$. We can visualize it on the following figure.




By continuity and global hyperbolicity the limit of $\mathcal{I}$ when $d \to 0$ is the part of the integral curve of $l$ starting at $(0,0,\tilde{y})$ until $P'(0,\tilde{v},\tilde{y})$. Hence by choosing $d$ little enough, we can have $\mathcal{I}$ strictly included in $\underset{y\in Y}{\cup}\{P(u,v,y); (u,v,y) \in [0;u_{\max}(y))\times[0;V]\times Y\}$. Now we can choose, by eventually reducing $d$ again, a compact set which strictly contains the intersection of the future of $N^+ \cup N^-$ with the causal past of $P(d,\tilde{v},\tilde{y})$, and which has a product structure $\mathcal{U}\times\mathcal{V}\times Y'$ with $Y'$ compact. In this way we are sure that $l$ and $w$ are tangent to $\mathcal{U}\times\mathcal{V}\times\partial Y'$. Then we prescribe any smooth function $\varphi_{Y'}$ on $\mathcal{U}\times\mathcal{V}\times\partial Y'$, such that

$$\varphi_{Y'}(u,0) = \varphi_+|_{\partial Y'} \qquad \varphi_{Y'}(0,v) = \varphi_-|_{\partial Y'}$$

and its partial derivatives at any $(u,0)$, $u \in [0;d]$, $(0,v)$, $v \in [0;\tilde{v}]$, are equal to the values obtained according the propagation equation. By extending functions $h$ and $H$ in a smooth way, we can apply the argument with $Y'$, compact with a boundary, instead of $Y$. So we will get a solution of the problem (4.2.7) on $\{P(u,v,y);\ (u,v,y)\in[0;u_{**}]\times[0;V]\times Y'\}$ and we consider its restriction to the causal past of $P(d,\tilde{v},\tilde{y})$. We know by classical arguments that we have uniqueness in the causal past of $P(d,\tilde{v},\tilde{y})$. Hence to get a neighborhood of the whole $N^-$, we choose $V>\min\limits_{y\in Y}(v_{\max}(y))$ and repeat this as much as necessary, taking the union over $V$ and over $\tilde{Y}$, one obtains a solution in a neighborhood of whole $N^-$. We can visualize this neighborhood $\mathcal{Q}$ by the following figure.




**Theorem 4.4.1** *If $h$, $H$, $\varphi_+$, $\varphi_-$ are $C^\infty$, there exists a unique $C^\infty$ solution $\varphi$ of the problem (4.2.7) on* $\bigcup\limits_{y\in Y}\ \bigcup\limits_{0<V<v_{\max}(y)}\{P(u,v,y);(u,v,y)\in[0;u_{**}(V,\tilde{Y}(V))]\times[0;V]\times\tilde{Y}(V)\}$.

**Remark 4.4.2** : If $Y$ is not compact, by the same argument as above, we can obtain a solution on a neighborhood of $N^-$.

## 4.5 Details of estimations in lemma 4.4.1 and 4.4.2

We will denote $c,\tilde{c}_i,\overline{c}_i$ some constants, they will be able to change at each lines . The constants which do not change are $c'$ (defined in (4.4.4)), $c''$ (defined in (4.5.35)), $\check{c}$ (defined in (4.5.5)), $\hat{c}$ (defined in (4.5.6)), $c_1(\rho)$ (defined in (4.5.11)), $c_2(\rho)$ (defined in (4.5.15)).

### 4.5.1 Proof of lemma 4.4.1

$\rho$ being fixed, we want to show that there exists a $u_*>0$ independent of $k$ such that

$$\max_{0\le u\le u_*}\ \|\ \tilde{\varphi}^k(u)\ \|_{H^m([0;V]\times Y)}\le\rho$$

We proceed by induction, as $\varphi^0=0$, the estimation is trivial for $k=0$. Then we suppose that the estimation is realised for $k-1$ in $\mathbb{N}$ and we will show that it is always true for $k$.
So let assume that

$$\max_{0\le u\le u_*}\ \|\ \tilde{\varphi}^{k-1}(u)\ \|_{H^m([0;V]\times Y)}\le\rho \tag{4.5.1}$$

We use the tool of energy momentum tensor, for that we need to introduce some notations. As it is usually denoted in differential geometry literature, we set for any vector field $X$,

$$X = \sum_{\mu} X^{\mu} \partial_{\mu}$$

where $\{\partial_\mu\}$ is a local basis of $T\mathcal{M}$ of dimension $n+1$.
We denote $\nabla_\mu$ the covariant derivative with respect to $\partial_\mu$ (where $\nabla$ is the Levi-Civita connexion of $(\mathcal{M}, g)$), and $\nabla^\mu := \sum_\nu g^{\mu\nu} \nabla_\nu$.
Now we consider the tensor T acting on one-vector field, namely

$$\begin{aligned} T(X) &= \sum_{\mu,\nu} T^{\mu}_{\ \nu} X^{\nu} \partial_{\mu} \\ \text{with } T^{\mu}_{\ \nu} &= \nabla^{\mu}\tilde{\varphi}^k \nabla_{\nu}\tilde{\varphi}^k - \frac{1}{2}\big((\sum_{\alpha} \nabla^{\alpha}\tilde{\varphi}^k \nabla_{\alpha}\tilde{\varphi}^k) + (\tilde{\varphi}^k)^2\big)\delta^{\mu}_{\ \nu} \end{aligned}$$

($\delta^{\mu}_{\ \nu}$ is the Kronecker symbol i.e. $\delta^{\mu}_{\ \nu}$ vanishes if $\mu \neq \nu$ and equals to 1 if $\mu = \nu$).
Notice that in any local orthonormal basis with (4.4.1), we have

$$\begin{aligned} T^{0}_{\ 0} &= -(\nabla_0\tilde{\varphi}^k)^2 - \frac{1}{2}\big( - (\nabla_0\tilde{\varphi}^k)^2 + (\nabla_1\tilde{\varphi}^k)^2 + \dots + (\nabla_n\tilde{\varphi}^k)^2 + (\tilde{\varphi}^k)^2\big) \\ &= -\frac{1}{2}\big((\nabla_0\tilde{\varphi}^k)^2 + (\nabla_1\tilde{\varphi}^k)^2 + \dots + (\nabla_n\tilde{\varphi}^k)^2 + (\tilde{\varphi}^k)^2\big) \\ &= -\frac{1}{2}\Big(\frac{1}{2}\big((\nabla_0\tilde{\varphi}^k)^2 + (\nabla_1\tilde{\varphi}^k)^2 - 2\nabla_0\tilde{\varphi}^k\nabla_1\tilde{\varphi}^k\big) \\ &\qquad + \frac{1}{2}\big((\nabla_0\tilde{\varphi}^k)^2 + (\nabla_1\tilde{\varphi}^k)^2 + 2\nabla_0\tilde{\varphi}^k\nabla_1\tilde{\varphi}^k\big) + |D\tilde{\varphi}^k|^2 + (\tilde{\varphi}^k)^2\Big) \\ &= -\frac{1}{2}\Big(2\big(w(\tilde{\varphi}^k)\big)^2 + 2\big(l(\tilde{\varphi}^k)\big)^2 + |D\tilde{\varphi}^k|^2 + (\tilde{\varphi}^k)^2\Big) \end{aligned} \tag{4.5.2}$$

and as in the last right member above $w(\tilde{\varphi}^k), l(\tilde{\varphi}^k), D\tilde{\varphi}^k, \tilde{\varphi}^k$ are globally defined, we obtain a global expression of $T^{0}_{\ 0}$.
By the theorem of Stokes we know that, for every open set $\Omega$,

$$\int_{\partial\Omega} \sum_{\mu,\nu} T^{\mu}_{\ \nu} X^{\nu} dS_{\mu} = \int_{\Omega} \sum_{\mu,\nu} \nabla_{\mu}\big(T^{\mu}_{\ \nu} X^{\nu}\big) dV$$

where $dS_\mu$ is the infinitesimal element of surface corresponding to $\partial_\mu$ on $\partial\Omega$, and $dV$ is the infinitesimal element of volume on $\Omega$. Of course we have

$$\sum_{\mu,\nu} \nabla_{\mu}\big(T^{\mu}_{\ \nu} X^{\nu}\big) = \sum_{\mu,\nu} \big(\nabla_{\mu} T^{\mu}_{\ \nu}\big) X^{\nu} + \sum_{\mu,\nu} T^{\mu}_{\ \nu}\big(\nabla_{\mu} X^{\nu}\big)$$

Therefore we can show the following lemma (its proof can be found in the appendix 4.7)

**Lemma 4.5.1**

$$\sum_{\mu} \nabla_{\mu} T^{\mu}_{\ \nu} = \big(\Box\tilde{\varphi}^k - \tilde{\varphi}^k\big)\nabla_{\nu}\tilde{\varphi}^k.$$

Now we set $t = \Psi_+ + \Psi_-$ ($\nabla t$ is timelike, indeed $g(\nabla t, \nabla t) = 2g(\nabla\Psi_+, \nabla\Psi_-) < 0$, we can notice here that the existence of $\Psi_+, \Psi_-$ globally defined implies that $(\mathcal{M}, g)$ is stably causal as $t$ is a global time function). Let $\lambda > 0$, we take

$$X = e^{-\lambda t} e_0 = e^{-\lambda t}(l + w)$$

($e_0$ has been defined in (4.4.1), $l, w$ in (4.2.4), notice also that $e_0 = -\nabla t / |\nabla t|$). Then

$$\begin{aligned} \sum_{\mu,\nu} (\nabla_\mu T^\mu_{\ \nu}) X^\nu &= \sum_\nu (\Box\tilde{\varphi}^k - \tilde{\varphi}^k)(\nabla_\nu \tilde{\varphi}^k) X^\nu \\ &= (\Box\tilde{\varphi}^k - \tilde{\varphi}^k) X(\tilde{\varphi}^k) \\ &= e^{-\lambda t}(\Box\tilde{\varphi}^k - \tilde{\varphi}^k)\big(l(\tilde{\varphi}^k) + w(\tilde{\varphi}^k)\big). \end{aligned} \tag{4.5.3}$$

On the other hand for $\sum_{\mu,\nu} T^\mu_{\ \nu}(\nabla_\mu X^\nu)$, we have

$$\nabla_\mu X^\nu = e_\mu(X^\nu) + \sum_\sigma {\Gamma^\nu}_{\mu\sigma} X^\sigma \tag{4.5.4}$$

where

$$\nabla_\mu e_\sigma = \sum_\nu {\Gamma^\nu}_{\mu\sigma} e_\nu$$

Hence if we take any local basis with (4.4.1), we get

$$X = e^{-\lambda t} e_0 \qquad \text{so } X^0 = e^{-\lambda t} \text{ and } X^i = 0 \ \forall i \neq 0$$

and we can write

$$e_\mu = {e_\mu}^0 \partial_t + \sum_{i=1}^n {e_\mu}^i \partial_i$$

where $(\partial_t, \partial_1, ..., \partial_n)$ is any local system of coordinates with just $\partial_t$ fixed (as $t$ has been fixed). Therefore

$$\begin{aligned} e_\mu(X^0) &= -\lambda e^{-\lambda t} {e_\mu}^0 \\ e_\mu(X^i) &= 0 \ \forall i \neq 0 \end{aligned}$$

with ${e_\mu}^0 = 0$ for all $\mu \neq 0$. Indeed recall that $(\theta^i)$ is the dual basis of $(e_i)$, and as

$$e_0 = -\frac{\nabla t}{\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}} \text{ we get } \theta^0 = -\frac{dt}{\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}}$$

(because $e_0$ and $\nabla t$ colinear implies $\theta^0$ and $dt$ proportional, but $\theta^0$ is normal so $\theta^0 = \dfrac{dt}{\sqrt{-g^\#(dt, dt)}}$ where $g^\#$ is the associated metric of $g$ for the one-forms i.e. $g^\#(dt, dt) = g(\nabla t, \nabla t) = 2g(\nabla\Psi_+, \nabla\Psi_-)$). Hence we have for all $\mu \neq 0$,

$$\begin{aligned} 0 = \theta^0(e_\mu) = \theta^0({e_\mu}^0 \partial_t + \sum_{j=1}^n {e_\mu}^j \partial_j) &= \frac{-1}{\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}} dt({e_\mu}^0 \partial_t + \sum_{j=1}^n {e_\mu}^j \partial_j) \\ &= \frac{-1}{\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}} {e_\mu}^0. \end{aligned}$$

Thus

$$\begin{array}{l}\nabla_{e_0} X^0 = \nabla_0 X^0 = -\lambda e^{-\lambda t} e_0{}^0 + \Gamma^0{}_{00} e^{-\lambda t} \\ \nabla_{e_\mu} X^0 = \nabla_\mu X^0 = \Gamma^0{}_{\mu 0} e^{-\lambda t} \ \forall \mu \neq 0 \\ \nabla_{e_\mu} X^i = \nabla_\mu X^i = \Gamma^i{}_{\mu 0} e^{-\lambda t} \ \forall i \neq 0.\end{array}$$

Finally

$$\sum_{\mu,\nu} T^\mu_\nu (\nabla_\mu X^\nu) = [T^0_0(-\lambda e_0{}^0 + \Gamma^0{}_{00}) + \sum_{\mu=1}^n T^\mu_0 \Gamma^0{}_{\mu 0} + \sum_{\mu=0}^n \sum_{i=1}^n T^\mu_i \Gamma^i{}_{\mu 0}] e^{-\lambda t}.$$

We know that for all $0 \leq i \leq n$, $0 \leq \mu \leq n$, $\Gamma^i{}_{\mu 0}$ is uniformly bounded on any compact so we can suppose that there exists $\check{c} \geq 0$ such that for all $0 \leq i \leq n$, $0 \leq \mu \leq n$,

$$|\Gamma^i{}_{\mu 0}| \leq \check{c} \qquad \text{on} \bigcup_{u \in [0;u_1']} N_u^-|_V. \tag{4.5.5}$$

We can calculate $e_0{}^0$, indeed

$$e_0 = e_0{}^0 \partial_t + \sum_{i=1}^n e_0{}^i \partial_i$$

Then

$$\theta^0(e_0) = -1 = -\frac{dt}{\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}}(e_0{}^0 \partial_t + \sum_{i=1}^n e_0{}^i \partial_i) = -\frac{e_0{}^0}{\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)}}$$

hence $e_0{}^0 = \sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)} > 0$. As $g(\nabla\Psi_+, \nabla\Psi_-) < 0$ and is uniformly bounded on any compact, there exists $\hat{c} > 0$ such that

$$\hat{c} \leq \sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)} \qquad \text{on} \bigcup_{u \in [0;u_1']} N_u^-|_V. \tag{4.5.6}$$

On another hand notice that in any local orthonormal basis with (4.4.1), we have for all $i \neq 0$,

$$T^0_i = \nabla^0 \tilde{\varphi}^k \nabla_i \tilde{\varphi}^k = -\nabla_0 \tilde{\varphi}^k \nabla_i \tilde{\varphi}^k$$

for all $0 \leq i \leq n$, $1 \leq \mu \leq n$, $i \neq \mu$,

$$T^\mu_i = \nabla^\mu \tilde{\varphi}^k \nabla_i \tilde{\varphi}^k = \nabla_\mu \tilde{\varphi}^k \nabla_i \tilde{\varphi}^k$$

for all $1 \leq \mu \leq n$,

$$\begin{aligned} T^\mu_\mu &= \nabla^\mu \tilde{\varphi}^k \nabla_\mu \tilde{\varphi}^k - \frac{1}{2}\big( - (\nabla_0 \tilde{\varphi}^k)^2 + (\nabla_1 \tilde{\varphi}^k)^2 + ... + (\nabla_n \tilde{\varphi}^k)^2 + (\tilde{\varphi}^k)^2 \big) \\ &= \frac{1}{2}\big( (\nabla_0 \tilde{\varphi}^k)^2 - (\nabla_1 \tilde{\varphi}^k)^2 - ... + (\nabla_\mu \tilde{\varphi}^k)^2 - ... - (\nabla_n \tilde{\varphi}^k)^2 - (\tilde{\varphi}^k)^2 \big). \end{aligned}$$

Hence by the second expression of $T^0_0$ in (4.5.2), for all $1 \leq \mu \leq n$, as

$$\begin{aligned} &-\frac{1}{2}\big( (\nabla_0 \tilde{\varphi}^k)^2 + (\nabla_1 \tilde{\varphi}^k)^2 + ... + (\nabla_n \tilde{\varphi}^k)^2 + (\tilde{\varphi}^k)^2 \big) \\ &\qquad \leq T^\mu_\mu \leq \frac{1}{2}\big( (\nabla_0 \tilde{\varphi}^k)^2 + (\nabla_1 \tilde{\varphi}^k)^2 + ... + (\nabla_n \tilde{\varphi}^k)^2 + (\tilde{\varphi}^k)^2 \big) \end{aligned}$$

we have also

$$|T^{\mu}_{\ \mu}| \le |T^{0}_{\ 0}| = -T^{0}_{\ 0}.$$

And as for all $0 \le i \le n$, $0 \le \mu \le n$, $i \ne \mu$,

$$|T^{\mu}_{\ i}| \le \frac{1}{2}\big((\nabla_\mu \tilde{\varphi}^k)^2 + (\nabla_i \tilde{\varphi}^k)^2\big)$$

we finally get

$$|\sum_{\mu=1}^{n} T^{\mu}_{\ 0}\Gamma^{0}_{\ \mu 0} + \sum_{\mu=0}^{n}\sum_{i=1}^{n} T^{\mu}_{\ i}\Gamma^{i}_{\ \mu 0}| \le -3n\check{c}T^{0}_{\ 0}. \tag{4.5.7}$$

This bound will be necessary later.

By the expression of $T^0_{\ 0}$ in (4.5.2), we obtain

$$\sum_{\mu,\nu} T^{\mu}_{\ \nu}\big(\nabla_\mu X^\nu\big) = \Big(-\frac{1}{2}\big(-\lambda\sqrt{-2g(\nabla\Psi_+,\nabla\Psi_-)} + \Gamma^{0}_{\ 00}\big) \tag{4.5.8}$$
$$\times\big[2\big(w(\tilde{\varphi}^k)\big)^2 + 2\big(l(\tilde{\varphi}^k)\big)^2 + |D\tilde{\varphi}^k|^2 + (\tilde{\varphi}^k)^2\big] + \big[\sum_{\mu=1}^{n} T^{\mu}_{\ 0}\Gamma^{0}_{\ \mu 0} + \sum_{\mu=0}^{n}\sum_{i=1}^{n} T^{\mu}_{\ i}\Gamma^{i}_{\ \mu 0}\big]\Big)e^{-\lambda t}.$$

Let $0 < \tau \le u'_1$, such that

$$\tau \le \min_{y\in Y}(u_{\max}(y)). \tag{4.5.9}$$

We apply the theorem of Stokes on the interior of $\Omega$ with $\Omega = \{P(u,v,y)\ ;\ (u,v,y) \in [0;\tau]\times[0;V]\times Y\}$. Recall that $TY_{uv} = Q = (Vect\{l,w\})^{\perp}$ and as we have assumed that $Y$ is compact without boundary, we get that $Y_{uv}$ is compact without boundary. By looking at the intersection of $\Omega$ with the hypersurfaces $N^-$, $N^+$, $N^-_\tau$, $N^+_V$, we can decompose $\partial\Omega$ in four parts as it follows :

$$\begin{aligned}\partial\Omega = \big(\Omega\cap N^-\big) \cup \big(\Omega \cap N^+\big) \cup \\ \big(\Omega\cap N^-_\tau\big) \cup \big(\Omega\cap N^+_V\big)\end{aligned}$$

As $\tilde{\varphi}^k$ vanishes on $N^-$ and $N^+$, the integrals on $\Omega\cap N^-$ and on $\Omega\cap N^+$ vanish. So when we integrate on $\partial\Omega$ it only remains the integrals on $\Omega\cap N^-_\tau$ and on $\Omega\cap N^+_V$.
For the integral on $\Omega\cap N^-_\tau$, and on $\Omega\cap N^+_V$, if we take any local orthonormal basis with (4.4.1) we notice that we have the following lemma (proved in the appendix 4.7).

**Lemma 4.5.2** *Let $Z$ be any vector field on $M$, we have*

$$\begin{aligned}\int_{\Omega\cap N^-_\tau}\sum_\mu Z^\mu dS_\mu &= \int_{\Omega\cap N^-_\tau}(Z^0 - Z^1)dS \\ \int_{\Omega\cap N^+_V}\sum_\mu Z^\mu dS_\mu &= \int_{\Omega\cap N^+_V}(Z^0 + Z^1)dS\end{aligned}$$

*where $dS = w\ \rfloor dV$ if we integrate on $N^-_\tau$ and $dS = l\ \rfloor dV$ if we integrate on $N^+_V$ ($dV$ is the infinitesimal element of volume).*

So we obtain

$$\begin{aligned}
\int_{\Omega\cap N_\tau^-}\sum_{\mu,\nu} T^\mu_{\ \nu}X^\nu dS_\mu &= \int_{\Omega\cap N_\tau^-}(T^0_{\ 0}-T^1_{\ 0})e^{-\lambda t}dS \\
\int_{\Omega\cap N_V^+}\sum_{\mu,\nu} T^\mu_{\ \nu}X^\nu dS_\mu &= \int_{\Omega\cap N_V^+}(T^0_{\ 0}+T^1_{\ 0})e^{-\lambda t}dS.
\end{aligned}$$

But, as in (4.5.2), we have

$$\begin{aligned}
T^0_{\ 0} &= -\frac{1}{2}\big((\nabla_0\tilde\varphi^k)^2+(\nabla_1\tilde\varphi^k)^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2\big) \\
T^1_{\ 0} &= \nabla_1\tilde\varphi^k\nabla_0\tilde\varphi^k.
\end{aligned}$$

Hence

$$\begin{aligned}
T^0_{\ 0}-T^1_{\ 0} &= -\frac{1}{2}\big((\nabla_0\tilde\varphi^k)^2+(\nabla_1\tilde\varphi^k)^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2+2\nabla_1\tilde\varphi^k\nabla_0\tilde\varphi^k\big) \\
&= -\frac{1}{2}\big((\nabla_0\tilde\varphi^k+\nabla_1\tilde\varphi^k)^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2\big) \\
&= -\frac{1}{2}\big(4(l(\tilde\varphi^k))^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2\big) \\
T^0_{\ 0}+T^1_{\ 0} &= -\frac{1}{2}\big(-2\nabla_1\tilde\varphi^k\nabla_0\tilde\varphi^k+(\nabla_0\tilde\varphi^k)^2+(\nabla_1\tilde\varphi^k)^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2\big) \\
&= -\frac{1}{2}\big((\nabla_0\tilde\varphi^k-\nabla_1\tilde\varphi^k)^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2\big) \\
&= -\frac{1}{2}\big(4(w(\tilde\varphi^k))^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2\big).
\end{aligned}$$

Thus

$$\int_{\Omega\cap N_\tau^-}\sum_{\mu,\nu} T^\mu_{\ \nu}X^\nu dS_\mu = \int_{\Omega\cap N_\tau^-}-\frac{1}{2}\big(4(l(\tilde\varphi^k))^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2\big)e^{-\lambda t}dS$$

and

$$\int_{\Omega\cap N_V^+}\sum_{\mu,\nu} T^\mu_{\ \nu}X^\nu dS_\mu = \int_{\Omega\cap N_V^+}-\frac{1}{2}\big(4(w(\tilde\varphi^k))^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2\big)e^{-\lambda t}dS.$$

Then if we include $e^{-\lambda t}=e^{-\lambda(\Psi_++\Psi_-)}$ as a weight in $dV'$, $e^{-\lambda\Psi_-}$ in $dS'$ if we integrate on $\Omega\cap N_\tau^-$, $e^{-\lambda\Psi_+}$ in $dS'$ if we integrate on $\Omega\cap N_V^+$, and if we apply the theorem of Stokes, as (4.5.3) and (4.5.8), we can write that

$$\begin{aligned}
&-e^{-\lambda\tau}\int_{\Omega\cap N_\tau^-}\frac{1}{2}\big(4(l(\tilde\varphi^k))^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2\big)dS' \\
&\quad - e^{-\lambda V}\int_{\Omega\cap N_V^+}\frac{1}{2}\big(4(w(\tilde\varphi^k))^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2\big)dS' \\
&\quad = \int_\Omega\Big[\big(\Box\tilde\varphi^k-\tilde\varphi^k\big)\big(l(\tilde\varphi^k)+w(\tilde\varphi^k)\big)-\frac{1}{2}(-\lambda\sqrt{-2g(\nabla\Psi_+,\nabla\Psi_-)}+\Gamma^0_{\ 00}) \\
&\qquad\times\big(2\big(w(\tilde\varphi^k)\big)^2+2\big(l(\tilde\varphi^k)\big)^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2\big)+\sum_{\mu=1}^n T^\mu_{\ 0}\Gamma^0_{\ \mu 0}+\sum_{\mu=0}^n\sum_{i=1}^n T^\mu_{\ i}\Gamma^i_{\ \mu 0}\Big]dV'.
\end{aligned}$$

Now we take the opposite of this inequality, and as $\lambda > 0, 0 < \tau \leq u'_1, V > 0$, we deduce from it that

$$\begin{aligned}
&\frac{e^{-\lambda u'_1}}{2}\int_{\Omega\cap N_\tau^-}\big(4(l(\tilde\varphi^k))^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2\big)dS'+\frac{e^{-\lambda V}}{2}\int_{\Omega\cap N_V^+}\big(4(w(\tilde\varphi^k))^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2\big)dS'\\
&\qquad+\frac{\lambda}{2}\int_\Omega\big[\sqrt{-2g(\nabla\Psi_+,\nabla\Psi_-)}\big(2\big(l(\tilde\varphi^k)\big)^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2\big)\big]dV'\\
&\qquad\leq\int_\Omega\big[\big(-\Box\tilde\varphi^k+\tilde\varphi^k\big)\big(l(\tilde\varphi^k)+w(\tilde\varphi^k)\big)+\big(-\lambda\sqrt{-2g(\nabla\Psi_+,\nabla\Psi_-)}+{\Gamma^0}_{00}\big)\big(w(\tilde\varphi^k)\big)^2\\
&\qquad+\frac{1}{2}{\Gamma^0}_{00}\big(2\big(l(\tilde\varphi^k)\big)^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2\big)-\sum_{\mu=1}^n T^\mu_{\ 0}{\Gamma^0}_{\mu0}+\sum_{\mu=0}^n\sum_{i=1}^n T^\mu_{\ i}{\Gamma^i}_{\mu0}\big]dV'.
\end{aligned}$$

Hence as $\lambda > 0$ and by the first equation of (4.3.5), by (4.5.5), (4.5.6), (4.5.7) we get

$$\begin{aligned}
&\frac{e^{-\lambda u'_1}}{2}\int_{\Omega\cap N_\tau^-}\big(4(l(\tilde\varphi^k))^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2\big)dS'\\
&\leq\int_\Omega\big[\big(-\tilde h^k w(\varphi^k)-\tilde H^k+\tilde\varphi^k\big)\big(w(\tilde\varphi^k)+l(\tilde\varphi^k)\big)+(-\lambda\hat c+(3n+1)\check c)\big(w(\tilde\varphi^k)\big)^2\\
&\quad+\frac{3n+1}{2}\check c\big(\,2\big(l(\tilde\varphi^k)\big)^2+|D\tilde\varphi^k|^2+(\tilde\varphi^k)^2\big)\big]dV'.
\end{aligned}\tag{4.5.10}$$

At the left side, we have the energy $\|\ \tilde\varphi^k(\tau)\ \|_{H^1([0;V]\times Y)}$, at the right side we want to obtain the integral with respect to $u$ from 0 to $\tau$ of this energy and by choosing a $\lambda$ large enough all which contains $w(\tilde\varphi^k)$ will be absorbed. For that we develop the first product under the integral on $\Omega$, we obtain six terms. We bound the first one as follows, we set

$$\max_{(\theta,\lambda,\gamma,u,v,y)\in Z}|\tilde h(\theta,\lambda,\gamma,u,v,y)|=c_1(\rho)\tag{4.5.11}$$

where

$$Z=[-c'\rho;c'\rho]^{n+1}\times[0;u'_1]\times[0;V]\times Y.\tag{4.5.12}$$

(we recall that $\tilde h^k(u,v,y)=\tilde h(\tilde\varphi^{k-1},l(\tilde\varphi^{k-1}),D\tilde\varphi^{k-1},u,v,y)$, that $c'$ is defined in (4.4.4), and $u'_1$ in (4.4.5) ). Thus we get

$$\int_\Omega-\tilde h^k(u,v,y)\big(w(\tilde\varphi^k)\big)^2dV'\leq\int_\Omega c_1(\rho)\big(w(\tilde\varphi^k)\big)^2dV'.\tag{4.5.13}$$

For the second term we use moreover $ab\leq\frac{1}{2}(a^2+b^2)$, hence

$$\int_\Omega-\tilde h^k(u,v,y)w(\tilde\varphi^k)l(\tilde\varphi^k)dV'\leq\int_\Omega\frac{1}{2}c_1(\rho)\big[\big(w(\tilde\varphi^k)\big)^2+\big(l(\tilde\varphi^k)\big)^2\big]dV'.\tag{4.5.14}$$

For the third and fourth term, by setting

$$\max_{(\theta,\lambda,\gamma,u,v,y)\in Z}|\tilde H(\theta,\lambda,\gamma,u,v,y)|=c_2(\rho)\tag{4.5.15}$$

and notice that $a\leq 1+a^2$, we get

$$\int_\Omega-\tilde H^k(u,v,y)w(\tilde\varphi^k)dV'\ \ \leq\ \ \int_\Omega c_2(\rho)\big[1+\big(w(\tilde\varphi^k)\big)^2\big]dV'\tag{4.5.16}$$

$$\int_\Omega -\tilde{H}^k(u,v,y) l(\tilde{\varphi}^k) dV' \;\; \leq \;\; \int_\Omega c_2(\rho)\left[1+\left(l(\tilde{\varphi}^k)\right)^2\right] dV'. \tag{4.5.17}$$

For the fifth and sixth term, we use again $ab \leq \frac{1}{2}(a^2+b^2)$, then

$$\int_\Omega \tilde{\varphi}^k w(\tilde{\varphi}^k) dV' \;\; \leq \;\; \int_\Omega \left[\frac{1}{2}(\tilde{\varphi}^k)^2 + \frac{1}{2}\left(w(\tilde{\varphi}^k)\right)^2\right] dV' \tag{4.5.18}$$

$$\int_\Omega \tilde{\varphi}^k l(\tilde{\varphi}^k) dV' \;\; \leq \;\; \int_\Omega \left[\frac{1}{2}(\tilde{\varphi}^k)^2 + \frac{1}{2}\left(l(\tilde{\varphi}^k)\right)^2\right] dV' \tag{4.5.19}$$

Finally by adding (4.5.13), (4.5.14), (4.5.16), (4.5.17), (4.5.18), (4.5.19) and with (4.5.10), we have

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega\cap N_\tau^-} \left(4(l(\tilde{\varphi}^k))^2 + |D\tilde{\varphi}^k|^2 + (\tilde{\varphi}^k)^2\right) dS' \\
&\quad \leq \int_\Omega \Big[2c_2(\rho) + \left(\frac{1}{2}c_1(\rho) + c_2(\rho) + \frac{1}{2} + (3n+1)\check{c}\right)\left(l(\tilde{\varphi}^k)\right)^2 + \frac{3n+1}{2}\check{c}|D\tilde{\varphi}^k|^2 \\
&\qquad + \left(\frac{1}{2}c_1(\rho) + c_2(\rho) + \frac{1}{2} + \frac{3n+1}{2}\check{c}\right)\left(\tilde{\varphi}^k\right)^2 \\
&\qquad\quad + \left(\frac{3}{2}c_1(\rho) + c_2(\rho) + \frac{1}{2} + (3n+1)\check{c} - \lambda\hat{c}\right)\left(w(\tilde{\varphi}^k)\right)^2\Big] dV'.
\end{aligned} \tag{4.5.20}$$

We already guess here the way to obtain the complete estimation. Indeed, we will do this inequality again but with some covariant derivatives with respect to the direction of $l, q_1, ..., q_r$ of $\tilde{\varphi}^k$ instead of $\tilde{\varphi}^k$. At the left side, we will always have a part of the energy $\| \tilde{\varphi}^k(\tau) \|_{H^m([0;V]\times Y)}$, at the right side the role of $\lambda$ will be to absorb all which contains $\left(w(\tilde{\varphi}^k)\right)^2$ (and the analogous with covariant derivatives with respect to the direction of $l, q_1, ..., q_r$ of $\tilde{\varphi}^k$ instead of $\tilde{\varphi}^k$) by choosing a $\lambda$ large enough. Thus it will remain at the right side a sum of a constant and of the integral in $u$ on $[0;\tau]$ of a part of the energy $\| \tilde{\varphi}^k(u) \|_{H^m([0;V]\times Y)}$. Then we will add all these inequalities and we will apply the Gronwall lemma. To get the complete energy, we must restart with $l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)$ (and all the possible commutations of $l$ and $q_i$ $(1 \leq i \leq r)$) instead of $\tilde{\varphi}^k$ for all $0 \leq \alpha_1 + |\alpha_2| \leq m-1$. We notice that vector fields $l$, $w$, $q_1$, ... , $q_r$ don't commute with each other. Here we just detail the estimate with $l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)$ (for any vector field $X$, any $\alpha$ in $I\!N$, $X^\alpha = X\circ ...\circ X$ $\alpha$-times), but it's exactly the same way for any commutation of $l$ and $q_i$ $(1 \leq i \leq r)$ in $l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)$. We take again (4.5.10), we can write

$$\Box(l^{\beta_1}\circ q^{\beta_2}(\tilde{\varphi}^k)) = l^{\beta_1}\circ q^{\beta_2}(\Box(\tilde{\varphi}^k)) + \sum_{\substack{|\nu_0|\leq 1 \\ |\nu_1|+|\nu_2|\leq|\beta|}} f_\nu \, w^{\nu_0}\circ l^{\nu_1}\circ q^{\nu_2}(\tilde{\varphi}^k)$$

where $f_\nu$ are smooth functions independent of $\tilde{\varphi}^k$. We obtain (for convenience we write

$q \circ l^{\alpha_1} \circ q^{\alpha_2}(\tilde{\varphi}^k)$ for $D(l^{\alpha_1} \circ q^{\alpha_2}(\tilde{\varphi}^k)))$

$$\begin{aligned}
&\frac{e^{-\lambda u'_1}}{2}\int_{\Omega\cap N_\tau^-}\left[\left(l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\right)^2+\left(l\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\right)^2+\left(q\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\right)^2\right]dS'\\
&\quad\leq\int_\Omega\Big[\Big(c\sum_{\substack{\beta_1+\gamma_1=\alpha_1\\ \beta_2+\gamma_2=\alpha_2}} l^{\beta_1}\circ q^{\beta_2}\big(-\tilde{h}(\tilde{\varphi}^{k-1},l(\tilde{\varphi}^{k-1}),D\tilde{\varphi}^{k-1},u,v,y)\big)l^{\gamma_1}\circ q^{\gamma_2}\circ w(\tilde{\varphi}^k)\\
&\qquad+l^{\alpha_1}\circ q^{\alpha_2}(-\tilde{H}(\tilde{\varphi}^{k-1},l(\tilde{\varphi}^{k-1}),D\tilde{\varphi}^{k-1},u,v,y))+\sum_{\substack{|\nu_0|\leq 1\\ |\nu_1|+|\nu_2|\leq|\beta|}} f_\nu\, w^{\nu_0}\circ l^{\nu_1}\circ q^{\nu_2}(\tilde{\varphi}^k)\\
&\qquad+l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\Big)\Big(w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\\
&\qquad+l\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\Big)+(-\lambda\hat{c}+(3n+1)\check{c})\left(w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\right)^2+\frac{3n+1}{2}\check{c}\Big(\,2\left(l\circ l^{\beta_1}\circ q^{\beta_2}(\tilde{\varphi}^k)\right)^2\\
&\qquad+\left(q\circ l^{\beta_1}\circ q^{\beta_2}(\tilde{\varphi}^k)\right)^2+\left(l^{\beta_1}\circ q^{\beta_2}(\tilde{\varphi}^k)\right)^2\Big)\Big]dV'. \qquad (4.5.21)
\end{aligned}$$

We develop the first product under the integral on $\Omega$, we obtain eight terms. For the last four terms, we use that $ab \leq \frac{1}{2}(a^2+b^2)$, hence

$$\begin{aligned}
&\int_\Omega\Big(\sum_{\substack{|\nu_0|\leq 1\\ |\nu_1|+|\nu_2|\leq|\beta|}} f_\nu\, w^{\nu_0}\circ l^{\nu_1}\circ q^{\nu_2}(\tilde{\varphi}^k)\Big)\left(w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\right)dV'\\
&\quad\leq\int_\Omega\sum_{\substack{|\nu_0|\leq 1\\ |\nu_1|+|\nu_2|\leq|\beta|}}\Big[\frac{1}{2}\left(f_\nu\, w^{\nu_0}\circ l^{\nu_1}\circ q^{\nu_2}(\tilde{\varphi}^k)\right)^2+\frac{1}{2}\left(w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\right)^2\Big]dV' \qquad (4.5.22)
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega\Big(\sum_{\substack{|\nu_0|\leq 1\\ |\nu_1|+|\nu_2|\leq|\beta|}} f_\nu\, w^{\nu_0}\circ l^{\nu_1}\circ q^{\nu_2}(\tilde{\varphi}^k)\Big)\left(l\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\right)dV'\\
&\quad\leq\int_\Omega\sum_{\substack{|\nu_0|\leq 1\\ |\nu_1|+|\nu_2|\leq|\beta|}}\Big[\frac{1}{2}\left(f_\nu\, w^{\nu_0}\circ l^{\nu_1}\circ q^{\nu_2}(\tilde{\varphi}^k)\right)^2+\frac{1}{2}\left(l\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\right)^2\Big]dV' \qquad (4.5.23)
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega\left(l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\right)\left(w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\right)dV'\\
&\quad\leq\int_\Omega\Big[\frac{1}{2}\left(l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\right)^2+\frac{1}{2}\left(w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\right)^2\Big]dV'. \qquad (4.5.24)
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega\left(l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\right)\left(l\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\right)dV'\\
&\quad\leq\int_\Omega\Big[\frac{1}{2}\left(l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\right)^2+\frac{1}{2}\left(l\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\right)^2\Big]dV'. \qquad (4.5.25)
\end{aligned}$$

We see that these four terms won't be a problem as we will first add the inequalities on $\alpha$, then choose a $\lambda$ large enough to absorb all terms which contain $w$.

For the third and fourth terms, we use the Cauchy-Schwarz inequality (which is still available with the weight $e^{-\lambda v}$, by writing $fge^{-\lambda v} = f\sqrt{e^{-\lambda v}}g\sqrt{e^{-\lambda v}}$, we will denote $\underline{L}^2$ to indicate the occurence of the weight), $\underline{L}^2$ means implicitly $\underline{L}^2([0;V]\times Y)$ (we don't always write $[0;V]\times Y$ because it takes too much place),

$$\int_\Omega l^{\alpha_1}\circ q^{\alpha_2}\big(-\tilde{H}(\tilde{\varphi}^{k-1}, l(\tilde{\varphi}^{k-1}), D\tilde{\varphi}^{k-1}, u, v, y)\big)\big(w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\big)dV' \tag{4.5.26}$$
$$\leq \int_{[0;\tau]} \| \, l^{\alpha_1}\circ q^{\alpha_2}\big(-\tilde{H}(\tilde{\varphi}^{k-1}, l(\tilde{\varphi}^{k-1}), D\tilde{\varphi}^{k-1}, s, v, y)\big) \, \|_{\underline{L}^2} \| \, w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)(s) \, \|_{\underline{L}^2} \, e^{-\lambda s}ds.$$

But we know by the Moser inequalities (see M. E. Taylor [16] chapter 13, paragraph 3, proposition 3.9) that if $F$ is $C^s$ ($F(0)=0$ is not needed here because we work on compacts) we have for all $w\in H^s\cap L^\infty$,

$$\| \, F(w) \, \|_{H^s} \leq c_s(\| \, w \, \|_{L^\infty})(1+ \| \, w \, \|_{H^s})$$

we set $w = (\tilde{\varphi}^{k-1}, l(\tilde{\varphi}^{k-1}), D\tilde{\varphi}^{k-1}, u, v, y)$, $s = m-1$, $F = \tilde{H}$, we get

$$\| -\tilde{H}(\tilde{\varphi}^{k-1}, l(\tilde{\varphi}^{k-1}), D\tilde{\varphi}^{k-1}, u, v, y) \, \|_{H^{m-1}} \leq c(\| \, \tilde{\varphi}^{k-1}(u) \, \|_{C^1})(1+ \| \, \tilde{\varphi}^{k-1}(u) \, \|_{H^m}) = c(\rho)$$

by the assumption (4.5.1) of induction. Therefore

$$\int_\Omega l^{\alpha_1}\circ q^{\alpha_2}\big(-\tilde{H}(\tilde{\varphi}^{k-1}, l(\tilde{\varphi}^{k-1}), D\tilde{\varphi}^{k-1}, u, v, y)\big)\big(w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\big)dV'$$
$$\leq c(\rho)\int_{[0;\tau]} (1+ \| \, w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)(s) \, \|^2_{\underline{L}^2([0;V]\times Y)})e^{-\lambda s}ds. \tag{4.5.27}$$

Similarly

$$\int_\Omega l^{\alpha_1}\circ q^{\alpha_2}\big(-\tilde{H}(\tilde{\varphi}^{k-1}, l(\tilde{\varphi}^{k-1}), D\tilde{\varphi}^{k-1}, u, v, y)\big)\big(l\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\big)dV'$$
$$\leq c(\rho)\int_{[0;\tau]} (1+ \| \, l\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)(s) \, \|^2_{\underline{L}^2([0;V]\times Y)})e^{-\lambda s}ds. \tag{4.5.28}$$

For both first terms, we see that we can't do the same as above, because as we have a factor $l^{\gamma_1}\circ q^{\gamma_2}\circ w(\tilde{\varphi}^k)$ more we would get in (4.5.26) a product in one of the norm $\underline{L}^2$, then we can't bound as we need here. We can't use anymore Moser inequalitie on whole $\Omega$ because we differentiate just one time in the direction of $w$. So we must detail $l^{\beta_1}\circ q^{\beta_2}(\tilde{h})$. We have

$$l^{\beta_1}\circ q^{\beta_2}\big(-\tilde{h}(\tilde{\varphi}^{k-1}, l(\tilde{\varphi}^{k-1}), D\tilde{\varphi}^{k-1}, u, v, y)\big) = \sum_{\beta+\delta+|\kappa|+\iota+|\sigma|\leq\beta_1+|\beta_2|}$$
$$c\,(-D^\beta_\theta D^\delta_\lambda D^\kappa_\gamma D^\iota_v D^\sigma_y\tilde{h})(\tilde{\varphi}^{k-1}, l(\tilde{\varphi}^{k-1}), D\tilde{\varphi}^{k-1}, u, v, y)\Big(\prod_i l^{\eta_i}\circ q^{\mu_i}\circ l^{\varsigma_i}\circ q^{\xi_i}(\tilde{\varphi}^{k-1})\Big)$$

with for all $i$, $\varsigma_i + |\xi_i| \leq 1$, $\sum_i \eta_i \leq \beta_1$ and $\sum_i |\mu_i| \leq |\beta_2|$ (hence $\sum_i \eta_i + |\mu_i| + \gamma_1 + |\gamma_2| \leq \alpha_1 + |\alpha_2|$). Now we consider the expression of $l^{\beta_1}\circ q^{\beta_2}(\tilde{h})$ above in

$$\int_\Omega \Big( \sum_{\substack{\beta_1+\gamma_1=\alpha_1\\ \beta_2+\gamma_2=\alpha_2}} c\, l^{\beta_1}\circ q^{\beta_2}(-\tilde{h}(\tilde{\varphi}^{k-1}, l(\tilde{\varphi}^{k-1}), D\tilde{\varphi}^{k-1}, u, v, y))l^{\gamma_1}\circ q^{\gamma_2}\circ w(\tilde{\varphi}^k) \; w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)dV'.$$

As $D^\beta_\theta D^\delta_\lambda D^\kappa_\gamma D^\iota_v D^\sigma_y \tilde{h}$ is continuous (recall that $\tilde{h}$ is $C^\infty$), we can bound it by its norm $L^\infty(Z)$ ($Z$ is defined in (4.5.12)), we obtain a constant $c(\rho)$. Then for the other factors, we use the Hölder inequality in $\underline{L}^1([0;V]\times Y)$ (this inequality is still available with a weight $e^{-\lambda v}$ as we have already noticed for the Cauchy-Schwarz inequality) with the $i+2$ exponents

$$\frac{2(\alpha_1+|\alpha_2|)}{\eta_i+|\mu_i|+\delta_i},\quad \frac{2(\alpha_1+|\alpha_2|)}{\gamma_1+|\gamma_2|},\quad \frac{2(\alpha_1+|\alpha_2|)}{\alpha_1+|\alpha_2|}$$

where $\delta_i\geq 0$ are chosen such that

$$\frac{\sum_i \eta_i+|\mu_i|+\delta_i+\gamma_1+|\gamma_2|+\alpha_1+|\alpha_2|}{2(\alpha_1+|\alpha_2|)}=1.$$

When we integrate on the part $\Omega\cap N_s^-$ of $\Omega$ we obtain (here we don't write the variable $s$ in each norm because it takes too much place, implicitly all the norms are on $\Omega\cap N_s^-=\{P(s,v,y)\ ;\ (v,y)\in[0;V]\times Y\}$ unless explicitly otherwise)

$$\begin{aligned}
&|\int_{\Omega\cap N_s^-}\Big(\prod_i l^{\eta_i}\circ q^{\mu_i}\circ l^{\varsigma_i}\circ q^{\xi_i}(\tilde{\varphi}^{k-1})\Big) l^{\gamma_1}\circ q^{\gamma_2}\circ w(\tilde{\varphi}^k)\ w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)dS'|\\
&\quad\leq\Big(\prod_i \parallel l^{\eta_i}\circ q^{\mu_i}\circ l^{\varsigma_i}\circ q^{\xi_i}(\tilde{\varphi}^{k-1})\parallel_{\underline{L}^{\frac{2(\alpha_1+|\alpha_2|)}{\eta_i+|\mu_i|+\delta_i}}}\Big)\\
&\qquad\times\parallel l^{\gamma_1}\circ q^{\gamma_2}\circ w(\tilde{\varphi}^k)\parallel_{\underline{L}^{\frac{2(\alpha_1+|\alpha_2|)}{\gamma_1+|\gamma_2|}}}\parallel w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\parallel_{\underline{L}^2}.
\end{aligned}$$

As we work on a compact $\Omega\cap N_s^-$, we have the embedding $\underline{L}^{\frac{2(\alpha_1+|\alpha_2|)}{\eta_i+|\mu_i|}}\hookrightarrow\underline{L}^{\frac{2(\alpha_1+|\alpha_2|)}{\eta_i+|\mu_i|+\delta_i}}$ and we can bound the factors under $\prod_i$ above by replacing the norm $\underline{L}^{\frac{2(\alpha_1+|\alpha_2|)}{\eta_i+|\mu_i|+\delta_i}}$ by the sum of the norm $\underline{L}^{\frac{2(\alpha_1+|\alpha_2|)}{\eta_i+|\mu_i|}}$ and a constant depending on $V$ and $Y$ , i.e.

$$\prod_i \parallel l^{\eta_i}\circ q^{\mu_i}\circ l^{\varsigma_i}\circ q^{\xi_i}(\tilde{\varphi}^{k-1})\parallel_{\underline{L}^{\frac{2(\alpha_1+|\alpha_2|)}{\eta_i+|\mu_i|+\delta_i}}}\leq\prod_i(\tilde{c}+\parallel l^{\eta_i}\circ q^{\mu_i}\circ l^{\varsigma_i}\circ q^{\xi_i}(\tilde{\varphi}^{k-1})\parallel_{\underline{L}^{\frac{2(\alpha_1+|\alpha_2|)}{\eta_i+|\mu_i|}}})$$

Furthermore we use one of the Gagliardo-Nirenberg-Ni inequalities (see M. E. Taylor [16] chapter 13 paragraph 3 Proposition 3.5 or T. Aubin [1] chapter 3 paragraph 7.6 Theorem 3.70), namely if $l<s$

$$\parallel D^l w\parallel_{L^{2\frac{s}{l}}}\leq c\parallel w\parallel^{1-\frac{l}{s}}_{L^\infty}\parallel D^s w\parallel^{\frac{l}{s}}_{L^2}.$$

We apply it with $w=\tilde{\varphi}^{k-1}$, $s=\alpha_1+|\alpha_2|+1$, $l=\eta_i+|\mu_i|+\varsigma_i+|\xi_i|$ on one hand and with $w=w(\tilde{\varphi}^k)$, $s=\alpha_1+|\alpha_2|$, $l=\gamma_1+|\gamma_2|$ on another hand. We get

$$\begin{aligned}
&|\int_{\Omega\cap N_s^-}\Big(\prod_i l^{\eta_i}\circ q^{\mu_i}\circ l^{\varsigma_i}\circ q^{\xi_i}(\tilde{\varphi}^{k-1})\Big) l^{\gamma_1}\circ q^{\gamma_2}\circ w(\tilde{\varphi}^k)\ w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)dS'|\\
&\quad\leq\Big(\prod_i(c\parallel\tilde{\varphi}^{k-1}\parallel^{1-\frac{\eta_i+|\mu_i|+\varsigma_i+|\xi_i|}{\alpha_1+|\alpha_2|+1}}_{L^\infty}\parallel l^{\alpha_{i3}}\circ q^{\alpha_{i4}}\circ l^{\varsigma_i}\circ q^{\xi_i}(\tilde{\varphi}^{k-1})\parallel^{\frac{\eta_i+|\mu_i|+\varsigma_i+|\xi_i|}{\alpha_1+|\alpha_2|+1}}_{\underline{L}^2}+\tilde{c})\Big)\\
&\qquad c\parallel w(\tilde{\varphi}^k)\parallel^{1-\frac{\gamma_1+|\gamma_2|}{\alpha_1+|\alpha_2|}}_{L^\infty}\parallel l^{\alpha_5}\circ q^{\alpha_6}\circ w(\tilde{\varphi}^k)\parallel^{\frac{\gamma_1+|\gamma_2|}{\alpha_1+|\alpha_2|}}_{\underline{L}^2}\parallel w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\parallel_{\underline{L}^2}.
\end{aligned}$$

with $\alpha_{i3}+|\alpha_{i4}| = \alpha_1+|\alpha_2|$, $\alpha_5+|\alpha_6| = \alpha_1+|\alpha_2|$. Indeed we notice that we don't necessary get the norm $\underline{L}^2$ of $l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^{k-1})$, because to obtain interpolation inequalities one integrates by parts, so for example if we bound the norm $\underline{L}^{2\frac{\alpha_1+|\alpha_2|}{\beta_2}}$ of $q^{\beta_2}(\tilde{\varphi}^{k-1})$ we can't obtain some covariant derivatives of $\tilde{\varphi}^{k-1}$ with respect to the direction of $l$.
Finally by using (4.5.1) and $ab\le\frac{1}{2}(a^2+b^2)$, we can write

$$|\int_{\Omega\cap N_s^-}\big(\prod_i l^{\eta_i}\circ q^{\mu_i}\circ l^{\varsigma_i}\circ q^{\xi_i}(\tilde{\varphi}^{k-1})\big)l^{\gamma_1}\circ q^{\gamma_2}\circ w(\tilde{\varphi}^k)\ w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)dS'| \tag{4.5.29}$$
$$\le c(\rho)\big(\prod_i \|\ w(\tilde{\varphi}^k)\ \|_{L^\infty}^{1-\sigma_i}\ \big)\big(1+\|\ l^{\alpha_5}\circ q^{\alpha_6}\circ w(\tilde{\varphi}^k)\ \|_{\underline{L}^2}^2+\|\ w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\ \|_{\underline{L}^2}^2\ \big)$$

where $0\le\sum_i\sigma_i\le 1$.
N.B. : the term 1 in (4.5.29) comes from $\tilde{c}$ on one hand and from the bound of the exponent $2\frac{\gamma_1+|\gamma_2|}{\alpha_1+|\alpha_2|}$ by 2 on another, in the case of $|l^{\alpha_5}\circ q^{\alpha_6}\circ w(\tilde{\varphi}^k)|$ is less than 1.
Thus as

$$\prod_i\|\ w(\tilde{\varphi}^k)\ \|_{L^\infty}^{1-\sigma_i}\le\|\ w(\tilde{\varphi}^k)\ \|_{L^\infty}^{\sigma}$$

(with $\sigma>0$ rational), we can write

$$|\int_{\Omega\cap N_s^-}\big(\prod_i l^{\eta_i}\circ q^{\mu_i}\circ l^{\varsigma_i}\circ q^{\xi_i}(\tilde{\varphi}^{k-1})\big)l^{\gamma_1}\circ q^{\gamma_2}\circ w(\tilde{\varphi}^k)\ w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)dS'|$$
$$\le c(\rho)\ \|\ w(\tilde{\varphi}^k)\ \|_{L^\infty}^{\sigma}\ \big(1+\|\ l^{\alpha_5}\circ q^{\alpha_6}\circ w(\tilde{\varphi}^k)\ \|_{\underline{L}^2}^2+\|\ w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\ \|_{\underline{L}^2}^2\ \big).$$

Therefore when we come back to the entire first term under the integral on $\Omega$, we obtain

$$\int_\Omega\big(c\sum_{\substack{\beta_1+\gamma_1=\alpha_1\\ \beta_2+\gamma_2=\alpha_2}} l^{\beta_1}\circ q^{\beta_2}(-\tilde{h}(\tilde{\varphi}^{k-1},l(\tilde{\varphi}^{k-1}),D\tilde{\varphi}^{k-1},u,v,y))l^{\gamma_1}\circ q^{\gamma_2}\circ w(\tilde{\varphi}^k)\ w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)dV'$$
$$\le\int_0^\tau\overline{c}(\rho)\ \|\ w(\tilde{\varphi}^k)\ \|_{L^\infty}^{\sigma} \tag{4.5.30}$$
$$\sum_{\alpha_5+|\alpha_6|=\alpha_1+|\alpha_2|}\big(1+\|\ l^{\alpha_5}\circ q^{\alpha_6}\circ w(\tilde{\varphi}^k)\ \|_{\underline{L}^2}^2+\|\ w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\ \|_{\underline{L}^2}^2\ \big)e^{-\lambda u}du.$$

We proceed similarly for the term with $l\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)$ ( instead of $w\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)$), hence

$$\int_\Omega\big(c\sum_{\substack{\beta_1+\gamma_1=\alpha_1\\ \beta_2+\gamma_2=\alpha_2}} l^{\beta_1}\circ q^{\beta_2}(-\tilde{h}(\tilde{\varphi}^{k-1},l(\tilde{\varphi}^{k-1})D\tilde{\varphi}^{k-1},u,v,y))l^{\gamma_1}\circ q^{\gamma_2}\circ w(\tilde{\varphi}^k)\ l\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)dV'$$
$$\le\int_0^\tau\overline{c}(\rho)\ \|\ w(\tilde{\varphi}^k)\ \|_{L^\infty}^{\sigma} \tag{4.5.31}$$
$$\sum_{\alpha_5+|\alpha_6|=\alpha_1+|\alpha_2|}\big(1+\|\ l^{\alpha_5}\circ q^{\alpha_6}\circ w(\tilde{\varphi}^k)\ \|_{\underline{L}^2}^2+\|\ l\circ l^{\alpha_1}\circ q^{\alpha_2}(\tilde{\varphi}^k)\ \|_{\underline{L}^2}^2\ \big)e^{-\lambda u}du.$$

Now it remains to estimate $\|\ w(\tilde{\varphi}^k)\ \|_{L^\infty}$. We will show that on $\Omega$, for any $m>n/2+2$,

$$|w(\tilde{\varphi}^k)|\le\tilde{c}_1(\rho)+\tilde{c}_2(\rho)\ \|\ \tilde{\varphi}^k(u)\ \|_{H^m(\Omega\cap N_u^-|_V)}\ .$$

Indeed, as $\tilde{\varphi}^k$ is $C^2$, if we integrate $l \circ w(\tilde{\varphi}^k)$ along the integral curve of $l$ starting at the point $Q = (u, 0, y)$, with $(u, v, y)$ defined as in (4.2.5), we have

$$\begin{aligned} w(\tilde{\varphi}^k)(u,v,y) &= w(\tilde{\varphi}^k)(u,0,y) + \int_0^v l \circ w(\tilde{\varphi}^k)(u,s,y)\sqrt{-2g(\nabla\Psi_+,\nabla\Psi_-)}ds \\ &= \int_0^v l \circ w(\tilde{\varphi}^k)(u,s,y)\sqrt{-2g(\nabla\Psi_+,\nabla\Psi_-)}ds \qquad (4.5.32) \end{aligned}$$

because $\tilde{\varphi}^k$ vanishes on $N^+$, and $w$ is in $TN^+$, so $w(\tilde{\varphi}^k)$ vanishes on $N^+$. We want to express $l \circ w(\tilde{\varphi}^k)$ in terms of $\Box\tilde{\varphi}^k$ to use the first equation of the problem (4.3.5). We can choose a local basis of $TM$ of the form $(w, l, f_2, ..., f_n)$. We calculate

$$g(w,l) = g(l,w) = g\left(\frac{-\nabla\Psi_-}{\sqrt{-2g(\nabla\Psi_+,\nabla\Psi_-)}}, \frac{-\nabla\Psi_+}{\sqrt{-2g(\nabla\Psi_+,\nabla\Psi_-)}}\right) = \frac{g(\nabla\Psi_-,\nabla\Psi_+)}{-2g(\nabla\Psi_+,\nabla\Psi_-)} = -1/2,$$

and as $w, l$ are orthogonal to $TY_{uv}$ with $TY_{uv} = Vect\{f_2, ..., f_n\}$ (indeed $Y_{uv} = N_{u+} \cap N_{v-}$ so $Y_{uv}$ is of dimension $n-1$, and $w \notin TN_{v-}$, $l \notin TN_{u+}$ imply $(w,l) \notin TY_{uv} = TN_{u+} \cap TN_{v-}$), we get that for all $2 \le i \le n$, $g(w, f_i) = g(f_i, w) = 0$ and $g(l, f_i) = g(f_i, l) = 0$. Thus

$$\begin{aligned} \Box\tilde{\varphi}^k &= \sum_{\mu,\nu} g^{\mu\nu}\nabla_\mu\nabla_\nu\tilde{\varphi}^k \\ &= -\frac{1}{2}\nabla_w\nabla_l\tilde{\varphi}^k - \frac{1}{2}\nabla_l\nabla_w\tilde{\varphi}^k + \sum_{2\le a,b\le n} g^{ab}\nabla_{f_a}\nabla_{f_b}\tilde{\varphi}^k. \end{aligned}$$

In any local coordinates, we can write

$$w(\tilde{\varphi}^k) = \sum_{i=0}^n w^i\partial_i\tilde{\varphi}^k.$$

So we obtain locally,

$$l \circ w(\tilde{\varphi}^k) = \sum_{j=0}^n l^j \sum_{i=0}^n [(\partial_j w^i)(\partial_i\tilde{\varphi}^k) + w^i\partial_j\partial_i\tilde{\varphi}^k]$$

hence

$$\begin{aligned} l \circ w(\tilde{\varphi}^k) &= w \circ l(\tilde{\varphi}^k) - \sum_{j=0}^n w^j \sum_{i=0}^n (\partial_j l^i)(\partial_i\tilde{\varphi}^k) + \sum_{j=0}^n l^j \sum_{i=0}^n (\partial_j w^i)(\partial_i\tilde{\varphi}^k) \\ &= w \circ l(\tilde{\varphi}^k) + \varpi(\tilde{\varphi}^k). \qquad (4.5.33) \end{aligned}$$

But

$$\nabla_l\nabla_w\tilde{\varphi}^k = l \circ w(\tilde{\varphi}^k) - \sum_{i,j=0}^n l^j w^i \sum_{\lambda=0}^n {\Gamma^\lambda}_{ji}\partial_\lambda\tilde{\varphi}^k$$

where $\nabla_{f_j} f_i = \sum_\lambda {\Gamma^\lambda}_{ji}\partial_\lambda$. As the torsion vanishes for the Levi Civita connexion, $\Gamma^\lambda_{ji} = \Gamma^\lambda_{ij}$, it gives

$$\begin{aligned} l \circ w(\tilde{\varphi}^k) &= \frac{1}{2} l \circ w(\tilde{\varphi}^k) + \frac{1}{2} w \circ l(\tilde{\varphi}^k) + \frac{1}{2}\varpi(\tilde{\varphi}^k) \\ &= \frac{1}{2}\nabla_l\nabla_w\tilde{\varphi}^k + \frac{1}{2}\nabla_w\nabla_l\tilde{\varphi}^k + \sum_{i,j=0}^n l^j w^i \sum_{\lambda=0}^n {\Gamma^\lambda}_{ji}\partial_\lambda\tilde{\varphi}^k + \frac{1}{2}\varpi(\tilde{\varphi}^k). \end{aligned}$$

Finally

$$l \circ w(\tilde{\varphi}^k) = -\Box\tilde{\varphi}^k + \sum_{2\le a,b\le n} g^{ab}\nabla_{f_a}\nabla_{f_b}\tilde{\varphi}^k + \sum_{i,j=0}^{n} l^j w^i \sum_{\lambda=0}^{n} \Gamma^{\lambda}{}_{ji}\partial_\lambda\tilde{\varphi}^k + \frac{1}{2}\varpi(\tilde{\varphi}^k). \quad (4.5.34)$$

As we work on a compact set and $Vect\{f_2, ..., f_n\} = TY_{uv} = Vect\{q_1, ..., q_r\}$, we can bound the terms $g^{ab}\nabla_{f_a}\nabla_{f_b}\tilde{\varphi}^k$ by the norm $C^2([0;V]\times Y)$ of $\tilde{\varphi}^k$, multiplied by a constant independent of $k$. Then if we take $m > \frac{n}{2}+2$, the embedding $H^m([0;V]\times Y) \hookrightarrow C^2([0;V]\times Y)$ holds and there exists $c'' > 0$ such that

$$\parallel \phi \parallel_{C^2([0;V]\times Y)} \le c'' \parallel \phi \parallel_{H^m([0;V]\times Y)} . \quad (4.5.35)$$

Therefore the terms $g^{ab}\nabla_{f_a}\nabla_{f_b}\tilde{\varphi}^k$ are less than or equal to the norm $H^m([0;V]\times Y)$ of $\tilde{\varphi}^k$, multiplied by a constant independent of $k$.

For the terms $l^j w^i \Gamma^{\lambda}{}_{ji}\partial_\lambda\tilde{\varphi}^k$ and $\varpi(\tilde{\varphi}^k)$, as $(l, w, q_1, ..., q_r)$ generate $TM$, there exists $a, b, c_1, ..., c_r$ such that

$$\sum_{i,j=0}^{n} l^j w^i \sum_{\lambda=0}^{n} \Gamma^{\lambda}{}_{ji}\partial_\lambda\tilde{\varphi}^k + \frac{1}{2}\varpi(\tilde{\varphi}^k) \quad\le\quad a\ w(\tilde{\varphi}^k) + b\ l(\tilde{\varphi}^k) + \sum_{i=1}^{r} c_i\ q_i(\tilde{\varphi}^k). \quad (4.5.36)$$

Hence we can bound them by the sum of $|w(\tilde{\varphi}^k)|$ and the norm $H^m([0;V]\times Y)$ of $\tilde{\varphi}^k$ (as $m > \frac{n}{2} + 2 > \frac{n}{2} + 1$ $H^m \subset C^1$), multiplied by a constant independent of $k$ . From this, we deduce

$$|l \circ w(\tilde{\varphi}^k)| \le |-\Box\tilde{\varphi}^k| + C|w(\tilde{\varphi}^k)| + \hat{C} \parallel \tilde{\varphi}^k \parallel_{H^m([0;V]\times Y)} . \quad (4.5.37)$$

Now by using the first equation of the problem (4.3.5), the bound of $h^k$, $H^k$, namely (4.5.11) and (4.5.15), and the fact that there exists $\tilde{C}$ such that $\sqrt{-2g(\nabla\Psi_+, \nabla\Psi_-)} \le \tilde{C}$, we get

$$\begin{aligned}&|w(\tilde{\varphi}^k)(u, v, y)| \\ &\quad\le \tilde{C}(c_1(\rho) + C)\int_0^v |w(\tilde{\varphi}^k(u, s, y)|ds + \tilde{C}Vc_2(\rho) + \tilde{C}V\hat{C} \parallel \tilde{\varphi}^k(u) \parallel_{H^m([0;V]\times Y)} .\end{aligned}$$

We apply the linear Gronwall lemma, namely if we have

$$f(v) \le c\int_0^v f(s)ds + \tilde{c}$$

$(c > 0)$ we set

$$h(v) = e^{-cv}(\int_0^v f(s)ds + \frac{\tilde{c}}{c})$$

then

$$h'(v) = -ce^{-cv}(\int_0^v f(s)ds + \frac{\tilde{c}}{c}) + e^{-cv}f(v) \quad\le\quad 0$$

from which we deduce that $h$ is decreasing and so $h(v) \le h(0) = \frac{\tilde{c}}{c}$, so

$$\int_0^v f(s)ds + \frac{\tilde{c}}{c} \le e^{cv}\frac{\tilde{c}}{c}$$

hence

$$\begin{aligned} \int_0^v f(s)ds - (e^{cv}-1)\frac{\tilde{c}}{c} &\leq 0 \\ \int_0^v (f(s) - e^{cs}\tilde{c})ds &\leq 0 \end{aligned}$$

for all $v$, thus

$$f(s) \leq e^{cs}\tilde{c} \quad \text{almost everywhere.} \tag{4.5.38}$$

Here, it gives

$$|w(\tilde{\varphi}^k)(u,v,y)| \leq \left(\tilde{C}Vc_2(\rho) + \tilde{C}V\hat{C} \parallel \tilde{\varphi}^k(u) \parallel_{H^m([0;V]\times Y)}\right)e^{\tilde{C}(c_1(\rho)+C)v} \quad \text{almost everywhere.}$$

(by the continuity of $w(\tilde{\varphi}^k)(u,v,y)$ we get the inequality everywhere but it's not necessary for us). Then as we know that $v \leq V$, we can write that for all $P=(u,v,y)$ in $\Omega$,

$$|w(\tilde{\varphi}^k)(u,v,y)| \leq \tilde{c}_1(\rho) + \tilde{c}_2(\rho) \parallel \tilde{\varphi}^k(u) \parallel_{H^m([0;V]\times Y)} . \tag{4.5.39}$$

Now we use this bound in (4.5.30) and (4.5.31), by noticing that for every $\sigma > 0$ there exists $\overline{c}$ such that for all $a, b \geq 0$, we have $(a+b)^\sigma \leq \overline{c}(a^\sigma + b^\sigma)$, we obtain

$$\begin{aligned} &\int_\Omega l^{\alpha_1} \circ q^{\alpha_2}\big(\tilde{h}(\tilde{\varphi}^{k-1}, l(\tilde{\varphi}^{k-1}), D\tilde{\varphi}^{k-1}, u, v, y)w(\tilde{\varphi}^k)\big) w \circ l^{\alpha_1} \circ q^{\alpha_2}(\tilde{\varphi}^k)dV' \\ &\quad \leq \int_0^\tau (\overline{c}_1(\rho) + \overline{c}_2(\rho) \parallel \tilde{\varphi}^k(u) \parallel_{H^m}^\sigma) \\ &\qquad \sum_{\alpha_5+|\alpha_6|=\alpha_1+|\alpha_2|} \left(1+ \parallel l^{\alpha_5} \circ q^{\alpha_6} \circ w(\tilde{\varphi}^k)(u) \parallel_{\underline{L}^2}^2 + \parallel w \circ l^{\alpha_1} \circ q^{\alpha_2}(\tilde{\varphi}^k)(u) \parallel_{\underline{L}^2}^2\right) e^{-\lambda u} du \end{aligned} \tag{4.5.40}$$

and

$$\begin{aligned} &\int_\Omega l^{\alpha_1} \circ q^{\alpha_2}\big(\tilde{h}(\tilde{\varphi}^{k-1}, l(\tilde{\varphi}^{k-1}), D\tilde{\varphi}^{k-1}, u, v, y)w(\tilde{\varphi}^k)\big) l \circ l^{\alpha_1} \circ q^{\alpha_2}(\tilde{\varphi}^k)dV' \\ &\quad \leq \int_0^\tau (\overline{c}_1(\rho) + \overline{c}_2(\rho) \parallel \tilde{\varphi}^k(u) \parallel_{H^m}^\sigma) \\ &\qquad \sum_{\alpha_5+|\alpha_6|=\alpha_1+|\alpha_2|} \left(1+ \parallel l^{\alpha_5} \circ q^{\alpha_6} \circ w(\tilde{\varphi}^k)(u) \parallel_{\underline{L}^2}^2 + \parallel l \circ l^{\alpha_1} \circ q^{\alpha_2}(\tilde{\varphi}^k)(u) \parallel_{\underline{L}^2}^2\right) e^{-\lambda u} du. \end{aligned} \tag{4.5.41}$$

Now we add the inequalities (4.5.40), (4.5.41), (4.5.27), (4.5.28), (4.5.22), (4.5.23), (4.5.24), (4.5.25), we obtain a bound of the right member of (4.5.21), then we get

$$\begin{aligned} &\frac{e^{-\lambda u_1'}}{2} \int_{\Omega\cap N_\tau^-} \big(|l^{\alpha_1} \circ q^{\alpha_2}(\tilde{\varphi}^k)|^2 + |l \circ l^{\alpha_1} \circ q^{\alpha_2}(\tilde{\varphi}^k)|^2 + |q \circ l^{\alpha_1} \circ q^{\alpha_2}(\tilde{\varphi}^k)|^2\big)dS' \\ &\quad \leq \int_0^\tau \Big[\tilde{c}_1(\rho) + \int_{\Omega\cap N_u^-|_V} \big(|l^{\alpha_1} \circ q^{\alpha_2}(\tilde{\varphi}^k)|^2 + \tilde{c}_2(\rho)(1+ \parallel \tilde{\varphi}^k \parallel_{H^m}^\sigma) \\ &\qquad + \tilde{c}_3(\rho)(1+ \parallel \tilde{\varphi}^k \parallel_{H^m}^\sigma)|l \circ l^{\alpha_1} \circ q^{\alpha_2}(\tilde{\varphi}^k)|^2 \\ &\qquad\quad + \tilde{c}_4(\rho)(1+ \parallel \tilde{\varphi}^k \parallel_{H^m}^\sigma) \sum_{\alpha_5+|\alpha_6|=\alpha_1+|\alpha_2|} |l^{\alpha_5} \circ q^{\alpha_6} \circ w(\tilde{\varphi}^k)|^2 \\ &\qquad\qquad + \tilde{c}_5(\rho)(1+ \parallel \tilde{\varphi}^k \parallel_{H^m}^\sigma |w \circ l^{\alpha_1} \circ q^{\alpha_2}(\tilde{\varphi}^k)|^2 - \frac{\lambda}{2}|w \circ l^{\alpha_1} \circ q^{\alpha_2}(\tilde{\varphi}^k)|^2\big)dS'\Big]e^{-\lambda u}du. \end{aligned} \tag{4.5.42}$$

Notice that the fact that $w$ is not at the beginning in $l^{\alpha_5} \circ q^{\alpha_6} \circ w(\tilde{\varphi}^k)$ is not a problem as

$$l^{\alpha_5} \circ q^{\alpha_6} \circ w(\tilde{\varphi}^k) = w \circ l^{\alpha_5} \circ q^{\alpha_6}((\tilde{\varphi}^k)) + \sum_{\substack{|\nu_0| \leq 1 \\ |\nu_1| + |\nu_2| \leq |\alpha|}} \tilde{f}_\nu w^{\nu_0} \circ l^{\nu_1} \circ q^{\nu_2}(\tilde{\varphi}^k).$$

If we sum all the inequalities (4.5.42) for $0 \leq \alpha_1 + |\alpha_2| \leq m-1$, we obtain

$$\begin{aligned} \| \tilde{\varphi}^k(\tau) \|_{H^m}^2 \leq \int_0^\tau & \Big[ \overline{c}_1(\rho) + \overline{c}_2(\rho)(1 + \| \tilde{\varphi}^k(u) \|_{H^m}^{\sigma+2}) \\ & + \big( \overline{c}_3(\rho)(1 + \| \tilde{\varphi}^k(u) \|_{H^m}^{\sigma}) - \frac{\lambda}{2} \big) \| w(\tilde{\varphi}^k)(u) \|_{H^{m-1}}^2 \Big] e^{-\lambda u} du. \end{aligned}$$

As $\tilde{\varphi}^k$ is $C^\infty$ we know that the norm $H^{m-1}$ of $w(\tilde{\varphi}^k)$ and the norm $H^m$ of $\tilde{\varphi}^k$ are finite, so we can choose $\lambda$ large enough so that $\overline{c}_3(\rho)(1 + \| \tilde{\varphi}^k \|_{H^m}^{\sigma}) - \frac{\lambda}{2} \leq 0$. Thus we can write

$$\| \tilde{\varphi}^k(\tau) \|_{H^m}^2 \leq \int_0^\tau \big( \overline{c}_1(\rho) + \overline{c}_2(\rho)(1 + \| \tilde{\varphi}^k(u) \|_{H^m}^{\sigma+2}) \big) e^{-\lambda u} du.$$

Now if we set

$$f(\tau) = \| \tilde{\varphi}^k(\tau) \|_{H^m([0;V]\times Y)}^2$$

we have for all $0 \leq \tau \leq u_1'$,

$$f(\tau) \leq \int_0^\tau F(f(s), s) ds$$

with $f \leq 0$, $F$ continuous, the nonlinear Gronwall lemma gives that there exists an interval $I$ including 0 such that

$$f(\tau) \leq G(\tau) \qquad \forall \tau \in [0; u_1'] \cap I.$$

Then

$$f(\tau) \leq \int_0^\tau F(f(s), s) ds \leq \tau \max_{\substack{\theta \in [0;\Gamma] \\ s \in [0; u_1'] \cap I}} |F(\theta, s)| = \tau M$$

where $\Gamma = \max_{\tau \in [0; u'1] \cap I} G(\tau)$. It remains to choose $u_*$ such that $[0; u_*] \subset [0; u_1'] \cap I$ and

$$u_* \, M \quad \leq \quad \rho^2$$

thus

$$\| \tilde{\varphi}^k(\tau) \|_{H^m([0;V]\times Y)}^2 \leq \rho^2 \qquad \forall \tau \in [0; u_*]$$

hence

$$\max_{0 \leq u \leq u_*} \| \tilde{\varphi}^k(u) \|_{H^m([0;V]\times Y)} \leq \rho.$$

N.B. : it's important here to notice that the choice of $u_*$ is independent of $k$ because $F$ is independent of $k$.

### 4.5.2 Proof of lemma 4.4.2

We take the inequality (4.5.10) again but with $\tilde{\varphi}^{k+1}-\tilde{\varphi}^k$ instead of $\tilde{\varphi}^k$, we obtain

$$\begin{aligned}
&\frac{e^{-\lambda u_1'}}{2}\int_{\Omega\cap N_\tau^-}\big(4(l(\tilde{\varphi}^{k+1}-\tilde{\varphi}^k))^2+|D(\tilde{\varphi}^{k+1}-\tilde{\varphi}^k)|^2+|\tilde{\varphi}^{k+1}-\tilde{\varphi}^k|^2\big)dS' \qquad (4.5.43)\\
&\le \int_0^\tau\int_{\Omega\cap N_u^-|_V}\big[\big(-\tilde{h}^{k+1}(u,v,y)w(\tilde{\varphi}^{k+1})+\tilde{h}^k(u,v,y)w(\tilde{\varphi}^k)-\tilde{H}^{k+1}(u,v,y)+\tilde{H}^k(u,v,y)+\\
&\quad\tilde{\varphi}^{k+1}-\tilde{\varphi}^k\big)\big(w(\tilde{\varphi}^{k+1}-\tilde{\varphi}^k)+l(\tilde{\varphi}^{k+1}-\tilde{\varphi}^k)\big)+(-\lambda\hat{c}+(3n+1)\check{c})\big(w(\tilde{\varphi}^{k+1}-\tilde{\varphi}^k)\big)^2\\
&\quad+\frac{3n+1}{2}\check{c}\big(2\big(l(\tilde{\varphi}^{k+1}-\tilde{\varphi}^k)\big)^2+|D(\tilde{\varphi}^{k+1}-\tilde{\varphi}^k)|^2+|\tilde{\varphi}^{k+1}-\tilde{\varphi}^k|^2\big)\big]dS'e^{-\lambda u}du.
\end{aligned}$$

At the left side we have the square of the norm $H^1([0;V]\times Y)$ of $\tilde{\varphi}^{k+1}(\tau)-\tilde{\varphi}^k(\tau)$. At the right side we want to make appear the square of the norm $H^1([0;V]\times Y)$ of $\tilde{\varphi}^k(u)-\tilde{\varphi}^{k-1}(u)$. But we can write that

$$\begin{aligned}
&\tilde{h}^{k+1}(u,v,y)w(\tilde{\varphi}^{k+1})-\tilde{h}^k(u,v,y)w(\tilde{\varphi}^k)\\
&\quad=\tilde{h}^{k+1}(u,v,y)(w(\tilde{\varphi}^{k+1})-w(\tilde{\varphi}^k))+(\tilde{h}^{k+1}(u,v,y)-\tilde{h}^k(u,v,y))w(\tilde{\varphi}^k).
\end{aligned}$$

Hence for the first term of the right member above, when we insert it in (4.5.43) we can get out $\tilde{h}^{k+1}(u,v,y)$ of the integral on $\Omega\cap N_u^-|_V$ by taking its norm $L^\infty$ which is a constant $c(\rho)$ (indeed it's a consequence of the lemma 4.4.1 and the embedding $H^m([0;V]\times Y)\hookrightarrow C^1([0;V]\times Y)$ as $m>\frac{n}{2}+1$). Thus when we will multiply it by $w(\tilde{\varphi}^{k+1}-\tilde{\varphi}^k)+l(\tilde{\varphi}^{k+1}-\tilde{\varphi}^k)$ it will give a bound

$$\int_{\Omega\cap N_u^-|_V}\big(c(\rho)|w(\tilde{\varphi}^{k+1}-\tilde{\varphi}^k)|^2+\frac{1}{2}c(\rho)(|w(\tilde{\varphi}^{k+1}-\tilde{\varphi}^k)|^2+|l(\tilde{\varphi}^{k+1}-\tilde{\varphi}^k)|^2)\big)dS' \qquad (4.5.44)$$

by using $ab\le\frac{1}{2}(a^2+b^2)$ for the second factor.
For the second term of the right member above we notice that

$$\begin{aligned}
&|\tilde{h}^{k+1}(u,v,y)-\tilde{h}^k(u,v,y)|=|\tilde{h}(\tilde{\varphi}^k,l(\tilde{\varphi}^k),D\tilde{\varphi}^k,u,v,y)-\tilde{h}(\tilde{\varphi}^{k-1},l(\tilde{\varphi}^{k-1}),D\tilde{\varphi}^{k-1},u,v,y)|\\
&\quad=|\int_0^1\frac{\partial}{\partial q}\tilde{h}\big(q(\tilde{\varphi}^k,l(\tilde{\varphi}^k),D\tilde{\varphi}^k,u,v,y)+(1-q)(\tilde{\varphi}^{k-1},l(\tilde{\varphi}^{k-1}),D\tilde{\varphi}^{k-1},u,v,y))dq|\\
&\quad=|\int_0^1\frac{\partial}{\partial q}\tilde{h}\big(q\tilde{\varphi}^k+(1-q)\tilde{\varphi}^{k-1},ql(\tilde{\varphi}^k)+(1-q)l(\tilde{\varphi}^{k-1}),qD\tilde{\varphi}^k+(1-q)D\tilde{\varphi}^{k-1},u,v,y\big)dq|\\
&\quad=|\int_0^1\big[\big(\frac{\partial}{\partial\theta}\tilde{h}\big)\big(q\tilde{\varphi}^k+(1-q)\tilde{\varphi}^{k-1},ql(\tilde{\varphi}^k)+(1-q)l(\tilde{\varphi}^{k-1}),qD\tilde{\varphi}^k+(1-q)D\tilde{\varphi}^{k-1},u,v,y\big)\\
&\qquad\qquad\big(\tilde{\varphi}^k-\tilde{\varphi}^{k-1}\big)\\
&\qquad+\big(\frac{\partial}{\partial\lambda}\tilde{h}\big)\big(q\tilde{\varphi}^k+(1-q)\tilde{\varphi}^{k-1},ql(\tilde{\varphi}^k)+(1-q)l(\tilde{\varphi}^{k-1}),qD\tilde{\varphi}^k+(1-q)D\tilde{\varphi}^{k-1},u,v,y\big)\\
&\qquad\qquad\big(l(\tilde{\varphi}^k)-l(\tilde{\varphi}^{k-1})\big)\\
&\qquad+\big(\nabla_\gamma\tilde{h}\big)\big(q\tilde{\varphi}^k+(1-q)\tilde{\varphi}^{k-1},ql(\tilde{\varphi}^k)+(1-q)l(\tilde{\varphi}^{k-1}),qD\tilde{\varphi}^k+(1-q)D\tilde{\varphi}^{k-1},u,v,y\big)\\
&\qquad\qquad\cdot\big(D\tilde{\varphi}^k-D\tilde{\varphi}^{k-1}\big)\big]dq|.
\end{aligned}$$

From this we deduce

$$\begin{aligned}
&|\tilde{h}^{k+1}(u,v,y)-\tilde{h}^k(u,v,y)|\\
&\quad\le\tilde{c}_1(\rho)|\tilde{\varphi}^k-\tilde{\varphi}^{k-1}|+\tilde{c}_2(\rho)|l(\tilde{\varphi}^k)-l(\tilde{\varphi}^{k-1})|+\tilde{c}_3(\rho)|D\tilde{\varphi}^k-D\tilde{\varphi}^{k-1}|. \qquad (4.5.45)
\end{aligned}$$

Now by (4.5.39) and the lemma 4.4.1 we can bound $w(\tilde{\varphi}^k)$ by a constant $c(\rho)$. Hence when we will multiply $(\tilde{h}^{k+1}(u,v,y) - \tilde{h}^k(u,v,y))w(\tilde{\varphi}^k)$ by $w(\tilde{\varphi}^{k+1} - \tilde{\varphi}^k) + l(\tilde{\varphi}^{k+1} - \tilde{\varphi}^k)$ under the integral on $\Omega \cap N_u^-|_V$, we can bound it, after using $ab \leq \frac{1}{2}(a^2+b^2)$, by

$$\begin{aligned}&\tilde{c}_4(\rho) \parallel \tilde{\varphi}^k(u) - \tilde{\varphi}^{k-1}(u) \parallel^2_{H^1([0;V]\times Y)}\\ &\quad +\tilde{c}_5(\rho)\int_{\Omega\cap N_u^-|_V} \big(|w(\tilde{\varphi}^{k+1} - \tilde{\varphi}^k)|^2 + |l(\tilde{\varphi}^{k+1} - \tilde{\varphi}^k)|^2\big)dS'. \qquad (4.5.46)\end{aligned}$$

The integral on $\Omega \cap N_u^-|_V$ of $\big(\tilde{H}^{k+1}(u,v,y) - \tilde{H}^k(u,v,y)\big)\big(w(\tilde{\varphi}^{k+1} - \tilde{\varphi}^k) + l(\tilde{\varphi}^{k+1} - \tilde{\varphi}^k)\big)$ will give a similar bound because (4.5.45) holds for $\tilde{H}$ instead of $\tilde{h}$.
It remains to bound the integral on $\Omega \cap N_u^-|_V$ of
$(\tilde{\varphi}^{k+1} - \tilde{\varphi}^k)\big(w(\tilde{\varphi}^{k+1} - \tilde{\varphi}^k) + l(\tilde{\varphi}^{k+1} - \tilde{\varphi}^k)\big)$, by using $ab \leq \frac{1}{2}(a^2+b^2)$, we get the bound

$$\int_{\Omega\cap N_u^-|_V} \big(|\tilde{\varphi}^{k+1} - \tilde{\varphi}^k|^2 + \frac{1}{2}|w(\tilde{\varphi}^{k+1} - \tilde{\varphi}^k)|^2 + \frac{1}{2}|l(\tilde{\varphi}^{k+1} - \tilde{\varphi}^k)|^2\big)dS'. \qquad (4.5.47)$$

Finally by taking back (4.5.43) and adding the bounds (4.5.44), (4.5.46), (4.5.47), we obtain

$$\begin{aligned}&\frac{e^{-\lambda u_1'}}{2} \parallel \tilde{\varphi}^{k+1}(\tau) - \tilde{\varphi}^k(\tau) \parallel^2_{H^1([0;V]\times Y)}\\ &\quad \leq \int_0^\tau \big[\overline{c}_1(\rho) \parallel \tilde{\varphi}^k(u) - \tilde{\varphi}^{k-1}(u) \parallel^2_{H^1([0;V]\times Y)} + \overline{c}_2(\rho) \parallel \tilde{\varphi}^{k+1}(u) - \tilde{\varphi}^k(u) \parallel^2_{H^1([0;V]\times Y)} +\\ &\qquad\quad \int_{\Omega\cap N_u^-|_V} \big(\frac{3}{2}c(\rho) + \tilde{c}_5(\rho) + \frac{1}{2} - \lambda\hat{c} - (3n+1)\check{c}\big)|w(\tilde{\varphi}^{k+1} - \tilde{\varphi}^k)|^2\big)dS'\big]e^{-\lambda u}du\end{aligned}$$

with $\overline{c}_2(\rho) > 0$. If we choose $\lambda$ large enough we get

$$\begin{aligned}&\frac{e^{-\lambda u_1'}}{2} \parallel \tilde{\varphi}^{k+1}(\tau) - \tilde{\varphi}^k(\tau) \parallel^2_{H^1}\\ &\quad \leq \int_0^\tau \big[\overline{c}_1(\rho) \parallel \tilde{\varphi}^k(u) - \tilde{\varphi}^{k-1}(u) \parallel^2_{H^1} + \overline{c}_2(\rho) \parallel \tilde{\varphi}^{k+1}(u) - \tilde{\varphi}^k(u) \parallel^2_{H^1}\big]e^{-\lambda u}du.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{e^{-\lambda u_1'}}{2} \parallel \tilde{\varphi}^{k+1}(\tau) - \tilde{\varphi}^k(\tau) \parallel^2_{H^1}\\ &\quad \leq \tau\overline{c}_1(\rho) \max_{0\leq s\leq\tau} \parallel \tilde{\varphi}^k(s) - \tilde{\varphi}^{k-1}(s) \parallel^2_{H^1} + \tau\overline{c}_2(\rho) \max_{0\leq s\leq\tau} \parallel \tilde{\varphi}^{k+1}(s) - \tilde{\varphi}^k(s) \parallel^2_{H^1}.\end{aligned}$$

This inequality holds for all $0 \leq \tau \leq u_{**}$ where $0 < u_{**} \leq u_*$, so

$$\begin{aligned}&\frac{e^{-\lambda u_1'}}{2} \max_{0\leq\tau\leq u_{**}} \parallel \tilde{\varphi}^{k+1}(\tau) - \tilde{\varphi}^k(\tau) \parallel^2_{H^1}\\ &\quad \leq u_{**}\overline{c}_1(\rho) \max_{0\leq s\leq u_{**}} \parallel \tilde{\varphi}^k(s) - \tilde{\varphi}^{k-1}(s) \parallel^2_{H^1} + u_{**}\overline{c}_2(\rho) \max_{0\leq s\leq u_{**}} \parallel \tilde{\varphi}^{k+1}(s) - \tilde{\varphi}^k(s) \parallel^2_{H^1}.\end{aligned}$$

Then we take $u_{**}\overline{c}_2(\rho) < \frac{1}{2}$ to pass the norm $H^1([0;V]\times Y)$ of $\tilde{\varphi}^{k+1}(s) - \tilde{\varphi}^k(s)$ of the right member above to the left side, after that we want

$$\frac{u_{**}\overline{c}_1(\rho)}{\frac{e^{-\lambda u_1'}}{2} - u_{**}\overline{c}_2(\rho)} \quad < \quad 1$$

it leads us to set

$$u_{**} \quad < \quad \min\Big(\frac{e^{-\lambda u'_1}}{2(\overline{c}_1(\rho)+\overline{c}_2(\rho))}, \frac{e^{-\lambda u'_1}}{2\overline{c}_2(\rho)}\Big).$$

Thus we have for all $k$ in $I\!N$,

$$\max_{0\leq u\leq u_{**}} \| \tilde{\varphi}^{k+1}(u) - \tilde{\varphi}^{k}(u) \|_{H^1(\Omega\cap N_u^-|_V)} \leq \alpha \max_{0\leq u\leq u_{**}} \| \tilde{\varphi}^{k}(u) - \tilde{\varphi}^{k-1}(u) \|_{H^1(\Omega\cap N_u^-|_V)}$$

with $\alpha = \frac{u_{**}\overline{c}_1(\rho)}{\frac{e^{-\lambda u'_1}}{2} - u_{**}\overline{c}_2(\rho)} < 1$.

## 4.6 Generalisation to data with finite differentiability

In this section we continue to consider the problem (4.2.7), but with weaker assumptions, namely we take $h$, $H$ of class $C^{m-1}$, and $\varphi_+$, $\varphi_-$ of class $H^m$, with $m > n/2+2$ (we will see that it's the minimum required to get existence in our argument).

By density, we know that there exist $(h_p)$, $(H_p)$, $(\varphi_{+p})$, $(\varphi_{-p})$ of class $C^\infty$ with compact support such that

$$\lim_{p\to\infty} \| h_p - h \|_{C^{m-1}} = 0 \qquad \lim_{p\to\infty} \| H_p - H \|_{C^{m-1}} = 0$$
$$\lim_{p\to\infty} \| \varphi_{+p} - \varphi_+ \|_{H^m} = 0 \qquad \lim_{p\to\infty} \| \varphi_{-p} - \varphi_- \|_{H^m} = 0.$$

Now if we replace $h$, $H$, $\varphi_+$, $\varphi_-$ in the problem (4.2.7) by respectively $h_p$, $H_p$, $\varphi_{+p}$, $\varphi_{-p}$, we will get solutions $\varphi_p$. We can easily check that the argument used for existence and uniqueness of $\tilde{\varphi}$ (namely the solution for the problem with vanishing initial value) could be directly applied to a problem with non-vanishing initial data (the advantage of keeping the initial data is that the value of $u_{**p}$ obtained in lemma 4.4.2 won't depend on the norm $C^{m-1}$ of $\varphi_{+p}$, $\varphi_{-p}$, which, if it was the case, would lead us to assume more regularity than $H^m$ for $\varphi_+$, $\varphi_-$). Indeed for the proof of the lemma 4.4.1 , at the first use of the theorem of Stokes, the components of the integral on $N^+$ and on $N^-$ won't vanish but will give the norm $H^1$ of $\varphi_+$ on $N^+$, and the norm $H^1$ of $\varphi_-$ on $N^-$. At the further uses of the theorem of Stokes we will get the norm $H^m$ of $\varphi_-$ on $N^-$, and terms which are the norm $H^1$ on $N^+$ of $l^{\alpha_1} \circ q^{\alpha_2}(\varphi)$. The problem here is that we must estimate transverse derivatives of $\varphi$ on $N^+$. But by using the propagation equation we can estimate them by a function depending on the norm $H^m$ of $\varphi_+$ and $\rho$. On another hand, to get the bound of $w(\varphi^k)$ we need the norm $C^0(N^+)$ of $\varphi_+$ (which is not a problem as $m > n/2+2$), then all works similarly. For the proof of lemma 4.4.2, the components of the integral on $N^+$ and on $N^-$ vanish because we repeat the theorem of Stokes on a difference of functions with the same initial value. If we proceed in this way the values of $u_{*p}$, $u_{**p}$ depend on $\rho$, the upper bound of the norm $C^{m-1}$ of $h_p$, $H_p$ and of the norm $H^m$ of $\varphi_{+p}$, $\varphi_{-p}$. Hence as we can find $N \in I\!N$, $C_1$, $C_2$, $C_3$, $C_4$ such that for all $p \geq N$,

$$\| h_p \|_{C^{m-1}} \leq C_1, \qquad \| H_p \|_{C^{m-1}} \leq C_2, \qquad \| \varphi_{+p} \|_{H^m} \leq C_3, \qquad \| \varphi_{-p} \|_{H^m} \leq C_4,$$

there exists $u_{**}$ such for all $p \geq N$, we have a solution $\varphi_p$ defined on $[0; u_{**}] \times [0; V] \times Y$. Moreover $\varphi_p$ is in $C^\infty([0; u_{**}] \times [0; V] \times Y)$ and is unique. Now we can show the following lemma.

**Lemma 4.6.1** *For all $p, r \geq N$,*

$$\| \varphi_p(u) - \varphi_r(u) \|_{H^1([0;V]\times Y)} \leq \tilde{c_1} \int_0^u \| \varphi_p(s) - \varphi_r(s) \|_{H^1([0;V]\times Y)} \, ds$$
$$+ \tilde{c_2} \big[ \| \varphi_{+p} - \varphi_{+r} \|_{H^1(N^+)} + \| \varphi_{-p} - \varphi_{-r} \|_{H^1(N^-)} \big].$$

**Proof :**
We use an inequality of type (4.5.10) with $\varphi_p - \varphi_r$ instead of $\tilde{\varphi}^k$, as $\varphi_p$ and $\varphi_r$ have not the same initial values on $N^+$ and $N^-$, we get

$$\begin{aligned}\tilde{c_3} \| \varphi_p(u) - \varphi_r(u) \|_{H^1([0;V]\times Y)} &\leq \tilde{c_4} \| \varphi_{+p} - \varphi_{+r} \|_{H^1(N^+)} + \tilde{c_5} \| \varphi_{-p} - \varphi_{-r} \|_{H^1(N^-)} \\ &+ \int_\Omega \Big[ \big( - \Box(\varphi_p - \varphi_r) + (\varphi_p - \varphi_r) \big) \big( w(\varphi_p - \varphi_r) + l(\varphi_p - \varphi_r) \big) \\ &+ (-\lambda \hat{c} + (3n+1)\check{c}) \big( w(\varphi_p - \varphi_r) \big)^2 \Big] dV' \\ &+ \tilde{c_6} \int_0^u \| \varphi_p(s) - \varphi_r(s) \|_{H^1([0;V]\times Y)} \, ds. \end{aligned} \tag{4.6.1}$$

Recall that

$$\begin{aligned}\Box(\varphi_p - \varphi_r)(s,v,y) = &\big( h_p(\varphi_p, l(\varphi_p), D\varphi_p, s, v, y) w(\varphi_p) - h_r(\varphi_r, l(\varphi_r), D\varphi_r, s, v, y) w(\varphi_r) \big) \\ &+ \big( H_p(\varphi_p, l(\varphi_p), D\varphi_p, s, v, y) - H_r(\varphi_r, l(\varphi_r), D\varphi_r, s, v, y) \big). \end{aligned} \tag{4.6.2}$$

For the first part of the right member of (4.6.2), we write

$$\begin{aligned} &h_p(\varphi_p, l(\varphi_p), D\varphi_p, s, v, y) w(\varphi_p) - h_r(\varphi_r, l(\varphi_r), D\varphi_r, s, v, y) w(\varphi_r) \\ &\quad = h_p(\varphi_p, l(\varphi_p), D\varphi_p, s, v, y) \big( w(\varphi_p) - w(\varphi_r) \big) \\ &\quad\quad + \big( h_p(\varphi_p, l(\varphi_p), D\varphi_p, s, v, y) - h_r(\varphi_r, l(\varphi_r), D\varphi_r, s, v, y) \big) w(\varphi_r). \end{aligned}$$

As we can bound $h_p(\varphi_p, l(\varphi_p), D\varphi_p, s, v, y)$ by a constant depending on $\rho$, when we will multiply $h_p(\varphi_p, l(\varphi_p), D\varphi_p, s, v, y)\big(w(\varphi_p) - w(\varphi_r)\big)$ by $w(\varphi_p - \varphi_r) + l(\varphi_p - \varphi_r)$ under the integral on $\Omega$ in (4.6.1), we obtain (with use of $ab \leq 1/2(a^2 + b^2)$)

$$\begin{aligned} &\int_\Omega h_p(\varphi_p, l(\varphi_p), D\varphi_p, s, v, y) \big( w(\varphi_p) - w(\varphi_r) \big) \big( w(\varphi_p - \varphi_r) + l(\varphi_p - \varphi_r) \big) dV' \\ &\quad \leq \int_\Omega \Big[ c(\rho) \frac{3}{2} \big( w(\varphi_p - \varphi_r) \big)^2 + c(\rho) \frac{1}{2} \big( l(\varphi_p - \varphi_r) \big)^2 \Big] dV'. \end{aligned}$$

Then we will absorb the term with $w(\varphi_p - \varphi_r)$ by choosing a $\lambda$ large enough. For $\big(h_p(\varphi_p, l(\varphi_p), D\varphi_p, s, v, y) - h_r(\varphi_r, l(\varphi_r), D\varphi_r, s, v, y)\big) w(\varphi_r)$ we proceed as we have done in (4.5.45), hence

$$\begin{aligned} &|h_p(\varphi_p, l(\varphi_p), D\varphi_p, s, v, y) - h_r(\varphi_r, l(\varphi_r), D\varphi_r, s, v, y)| \\ &\quad \leq \tilde{c'}_1(\rho) |\varphi_p - \varphi_r| + \tilde{c'}_2(\rho) |l(\varphi_p - \varphi_r)| + \tilde{c'}_3(\rho) |D\varphi_p - \varphi_r|. \end{aligned}$$

and as we can bound $w(\varphi_r)$ by a constant depending on $\rho$, we get (with use of $ab \leq$

$1/2(a^2+b^2))$

$$\int_\Omega \big(h_p(\varphi_p,l(\varphi_p),D\varphi_p,s,v,y)-h_r(\varphi_r,l(\varphi_r),D\varphi_r,s,v,y)\big)w(\varphi_r)\big(w(\varphi_p-\varphi_r)+l(\varphi_p-\varphi_r)\big)dV'$$
$$\leq \int_\Omega \overline{c}(\rho)\big[\tilde{c'}_1(\rho)|\varphi_p-\varphi_r|^2+\frac{\tilde{c'}_1+3\tilde{c'}_2+\tilde{c'}_3}{2}\big(l(\varphi_p-\varphi_r)\big)^2+\tilde{c'}_3\big(D(\varphi_p-\varphi_r)\big)^2$$
$$+\frac{\tilde{c'}_1+\tilde{c'}_2+\tilde{c'}_3}{2}\big(w(\varphi_p-\varphi_r)\big)^2\big]dV'.$$

Here also we absorb the term containing $w(\varphi_p-\varphi_r)$ by choosing a $\lambda$ large enough. For the second part of the right member of (4.6.2), we treat the term with $H_p$ and $H_r$ similarly as above. And for $(\varphi_p-\varphi_r)\big(w(\varphi_p-\varphi_r)+l(\varphi_p-\varphi_r)\big)$ under the integral on $\Omega$ in (4.6.1), we just use $ab\leq 1/2(a^2+b^2)$. Finally we have

$$\|\varphi_p(u)-\varphi_r(u)\|_{H^1([0;V]\times Y)}\leq \tilde{c}_1\int_0^u \|\varphi_p(s)-\varphi_r(s)\|_{H^1([0;V]\times Y)}\,ds$$
$$+\tilde{c}_2\big[\,\|\varphi_{+p}-\varphi_{+r}\|_{H^1(N^+)}+\|\varphi_{-p}-\varphi_{-r}\|_{H^1(N^-)}\,\big].$$

Now by applying the linear Gronwall lemma, we obtain

$$\|\varphi_p(u)-\varphi_r(u)\|_{H^1([0;V]\times Y)}\leq \tilde{c}\big[\,\|\varphi_{+p}-\varphi_{+r}\|_{H^1(N^+)}+\|\varphi_{-p}-\varphi_{-r}\|_{H^1(N^-)}\,\big].$$

Hence we get that $(\varphi_p)$ is a Cauchy sequence in $H^1([0;V]\times Y)$ and so converges to $\varphi$ in $H^1([0;V]\times Y)$. As the norm $H^m([0;V]\times Y)$ of $\varphi_p$ is uniformly bounded, we can show by using interpolation inequalities that $(\varphi_p)$ converges to $\varphi$ in $H^{m'}([0;V]\times Y)$ for all $1<m'<m$.

Then by proceeding as we have done in section 4.4 we show that $\varphi$ is a solution of (4.2.7) and is in $C^0\big([0;u_{**}]\times[0;V]\times Y\big)$ (recall that $m>n/2+2$ hence $\varphi_+$ and $\varphi_-$ are in $C^2$). By taking the limit in lemma 4.4.1 we can show that for all $u\in[0;u_{**}]$, $\varphi(u)$ is in $H^{m'}([0;V]\times Y)$ for any $0<m'<m$.

Moreover if $\varphi_+$ and $\varphi_-$ are in $C^{m-1}\cap H^m$, we can use again $\tilde{\varphi}$ (the solution of the problem with vanishing initial values) and show its regularity as we have done in the case $C^\infty$ thus if $m>n/2+3+j$, we get $\tilde{\varphi}$, and then $\varphi$, in $C^j([0;u_{**}]\times[0;V]\times Y)$.

**Proposition 4.6.1** *Let $0<V<\min_{y\in Y}(v_{\max}(y))$, $m>n/2+2$, if $h$ and $H$ are of class $C^{m-1}$, $\varphi_+$ and $\varphi_-$ are of class $H^m$, there exists $u_{**}>0$ and a unique solution $\varphi$ of the problem (4.2.7) in $C^0\big([0;u_{**}]\times[0;V]\times Y\big)$, moreover $u\mapsto\|\varphi(u)\|_{H^{m'}([0;V]\times Y)}$ is uniformly bounded on $[0;u_{**}]$ for all $0<m'<m$.*
*Moreover if $\varphi_+$ and $\varphi_-$ are in $C^{m-1}\cap H^m$ with $m>n/2+3+j$, then $\varphi$ is in $C^j([0;u_{**}]\times[0;V]\times Y)$.*

If $v_{\max}$ is not constant, to obtain a neighborhood of the whole $N^-$, we must proceed as in the smooth case, and show that we can solve the same problem with a prescribed function on $\underset{u,v}{\cup}\partial Y_{uv}$ (compatible with the given functions on $N^+\cup N^-$). As we do not want $u_{**}$ to depend on the norm $C^{m-1}$ of the given function on $\underset{u,v}{\cup}\partial Y_{uv}$, we try to obtain the existence of a solution with non-zero data on $\underset{u,v}{\cup}\partial Y_{uv}$. In order to achieve this we must change the vector contracted with the energy momentum tensor (indeed when we apply the theorem of Stokes, we get terms under an integral on a hypersurface $\partial\Omega^T = \underset{0\le u\le\tau,\ 0\le v\le V}{\cup} \partial Y_{uv}$ with transverse derivatives to this hypersurface which we don't control, the new vector field will permit to absorb this terms). This kind of energy estimates is done in L. Hörmander [8] chapter XXIV paragraph 24.1.

So let us consider the problem (4.2.7) with the weaker assumptions described at the beginning of this section, with one more boundary condition, namely a given function $\varphi_Y$ on $\underset{u,v}{\cup} \partial Y_{uv}$ of class $H^m$ compatible with the given functions on $N^+ \cup N^-$ as precised in the smooth case (see end of section Global process). As we have done before, we first smooth the functions (we denote $\left(\varphi_{Yp}\right)$ a sequence of $C^\infty$ functions with compact support which converges to $\varphi_Y$ in $H^m$). We restart the argument (once more without vanishing values on $N^+ \cup N^-$) with $X = e^{-\lambda t}(e_0 - \frac{\delta}{\sqrt{n-1}} f_1)$ where $\delta > 0$ chosen below, $f_1$ is the normal to $\underset{u,v}{\cup} \partial Y_{uv}$ (in the sense of $f_1$ and the tangent vector space of $\underset{u,v}{\cup} \partial Y_{uv}$ generate $TM$ and $f_1$ has no component along $e_0$ and $e_1$), directed to the exterior of $\Omega$ (we assume that $n > 1$, otherwise we are in dimension $1+1$, there's no $y$). When we apply the theorem of Stokes, for the integral on $\partial\Omega$, we will get more terms than before as precised below. In all the following, to be less heavy, we don't write the subscript $p$ on $\varphi^k$ but we keep in mind that we work with the smooth functions $a^0_p$, etc...

First we examine the integral on $\Omega \cap N^-_\tau$. Recall that in any local orthonormal basis with (4.4.1),

$$\int_{\Omega\cap N^-_\tau} \sum_\mu Y^\mu dS_\mu \;=\; \int_{\Omega\cap N^-_\tau} (Y^0 - Y^1) dS.$$

As $e_0$, $e_1$ are always tangent to $\partial Y$, locally we have $f_1 = \sum_{i=2}^n a_i e_i$, and for all $i \ge 2$,

$$T^0_{\ i} - T^1_{\ i} = -\nabla_0\varphi^k\nabla_i\varphi^k - \nabla_1\varphi^k\nabla_i\varphi^k = -(\nabla_0\varphi^k + \nabla_1\varphi^k)\nabla_i\varphi^k = -2l(\varphi^k)\nabla_i\varphi^k,$$

we get

$$\begin{aligned}
-&\int_{\Omega\cap N^-_\tau} \sum_{\mu,\nu} T^\mu_{\ \nu} X^\nu dS_\mu \\
&= \int_{\Omega\cap N^-_\tau} \frac{1}{2}\big(4(l(\varphi^k))^2 + |D\varphi^k|^2 + (\varphi^k)^2 - 4\sum_{i=2}^n \frac{\delta}{\sqrt{n-1}} a_i w(\varphi^k)\nabla_i\varphi^k\big) e^{-\lambda t} dS \\
&\ge \int_{\Omega\cap N^-_\tau} \frac{1}{2}\big(3(l(\varphi^k))^2 + \frac{1}{2}|D\varphi^k|^2 + (\varphi^k)^2\big) e^{-\lambda t} dS
\end{aligned}$$

by using $4ab \ge -\frac{1}{2}a^2 - 8b^2$ with $a = \nabla_i\varphi^k$, $b = \frac{\delta}{\sqrt{n-1}} a_i w(\varphi^k)$, and assuming that $\delta^2 \le 1/(8\max_i(\max_{Z'} |a_i|^2))$ with $Z' = [0; u'_1] \times [0; V] \times Y$ ($\max_i(\max_{Z'} |a_i|^2$ doesn't vanish because $f_1$ doesn't vanish), as we work on compact sets, we can choose a finite number of local basis hence $\delta$ can be chosen globally). We will obtain similar inequalities when we do it again with $l^{\alpha_1} \circ q^{\alpha_2}(\varphi^k)$ instead of $\varphi^k$. Similarly for the integral on $\Omega \cap N^-$, we will still recover the norm $H^m$ of $\varphi_-$. We see that for the moment this will just change the constants in the proof.

Now we look at the integral on $\Omega \cap N_V^+$. As

$$\int_{\Omega\cap N_V^+} \sum_\mu Y^\mu dS_\mu = \int_{\Omega\cap N_V^+} (Y^0 + Y^1) dS$$

and

$$T^0_{\ i} + T^1_{\ i} = -(\nabla_0\varphi^k - \nabla_1\varphi^k)\nabla_i\varphi^k = -2w(\varphi^k)\nabla_i\varphi^k,$$

we obtain

$$\begin{aligned}
&-\int_{\Omega\cap N_V^+} \sum_{\mu,\nu} T^\mu_{\ \nu} X^\nu dS_\mu \\
&\quad = \int_{\Omega\cap N_V^+} \frac{1}{2}\big(4(w(\varphi^k))^2 + |D\varphi^k|^2 + (\varphi^k)^2 - 4\sum_{i=2}^n \frac{\delta}{\sqrt{n-1}} a_i w(\varphi^k)\nabla_i\varphi^k\big) e^{-\lambda t} dS \\
&\quad \geq \int_{\Omega\cap N_V^+} \frac{1}{2}\big(3(w(\varphi^k))^2 + \frac{1}{2}|D\varphi^k|^2 + (\varphi^k)^2\big) e^{-\lambda t} dS \geq 0
\end{aligned}$$

and the analogous when we replace $\varphi^k$ by $l^{\alpha_1} \circ q^{\alpha_2}(\varphi^k)$. For the integral on $\Omega \cap N^+$, it gives terms which are the norm $H^1$ on $N^+$ of $l^{\alpha_1} \circ q^{\alpha_2}(\varphi^k)$, we still estimate them by a function depending on the norm $H^m$ of $\varphi_{+p}$ and $\rho$.

Therefore we take care of the integral on $\partial\Omega^T$. To estimate it, we complete $(e_0, f_1, )$ in a local orthonormal basis $(e_0, f_1, ..., f_n)$ (it is possible as $e_0$ is tangent to $\partial Y$). Then, as

$$\int_{\partial\Omega^T} \sum_\mu Y^\mu dS_\mu = \int_{\partial\Omega^T} g(Y, f_1) f_1 \rfloor dV$$

($f_1 \rfloor dV$ defined in appendix 4.7 **2)**) and in $(e_0, f_1, ..., f_n)$,

$$\begin{aligned}
T^1_{\ 0} &= \nabla_{f_1}\varphi^k \nabla_0\varphi^k \\
T^1_{\ 1} &= \frac{1}{2}\big((\nabla_0\varphi^k)^2 + (\nabla_{f_1}\varphi^k)^2 - (\nabla_{f_2}\varphi^k)^2 - ... - (\nabla_{f_n}\varphi^k)^2 - (\varphi^k)^2\big)
\end{aligned}$$

we get

$$\begin{aligned}
&-\int_{\partial\Omega^T} \sum_{\mu,\nu} T^\mu_{\ \nu} X^\nu dS_\mu \\
&= \int_{\partial\Omega^T} \big[-\nabla_{f_1}\varphi^k\nabla_0\varphi^k + \frac{\delta}{2\sqrt{n-1}}\big((\nabla_0\varphi^k)^2 + (\nabla_{f_1}\varphi^k)^2 - \sum_{j=2}^n (\nabla_{f_j}\varphi^k)^2 - (\varphi^k)^2\big] e^{-\lambda t} dS \\
&\geq \int_{\partial\Omega^T} \big[\big(\frac{\delta}{2\sqrt{n-1}} - \frac{1}{2\varepsilon}\big)(\nabla_0\varphi^k)^2 + \big(\frac{\delta}{2\sqrt{n-1}} - \frac{\varepsilon}{2}\big)\nabla_{f_1}\varphi^k)^2 - \frac{\delta}{2\sqrt{n-1}}\sum_{j=2}^n (\nabla_{f_j}\varphi^k)^2 \\
&\qquad - \frac{\delta}{2\sqrt{n-1}}(\varphi^k)^2\big] e^{-\lambda t} dS \\
&\geq - \parallel \varphi_{Yp} \parallel_{H^1}
\end{aligned}$$

by using $ab \geq -\frac{1}{2\varepsilon}a^2 - \frac{\varepsilon}{2}b^2$ with $a = \nabla_0\varphi^k$, $b = \nabla_{f_1}\varphi^k$ and choosing $\varepsilon > 0$ small enough (we need to absorb terms which contain $\nabla_{f_1}\varphi^k$ to be allowed to estimate the expression above by the norm $H^1$ of $\varphi_{Yp}$). When we do it again with $l^{\alpha_1} \circ q^{\alpha_2}(\varphi^k)$ instead of $\varphi^k$, we proceed similarly, but we need to control more the term $-\frac{\delta}{2\sqrt{n-1}} l^{\alpha_1} \circ q^{\alpha_2}(\varphi^k)$. We can absorb it by multiplying the previous estimations by suitables coefficients after writing $l$ and $q$ as linear combination of $(e_0, f_1, ..., f_n)$ (when we do the estimation with $\alpha + 1$-derivatives of $\varphi^k$, we must for example multiply the estimation with $\alpha$-derivatives of $\varphi^k$ by two).

Now we examine the second member of the equality in the theorem of Stokes, namely the integral on $\Omega$. The new vector fields $X$ will produce more terms than the previous one. Recall that

$$\sum_{\mu,\nu} \nabla_\mu (T^\mu_{\ \nu} X^\nu) = \sum_{\mu,\nu} (\nabla_\mu T^\mu_{\ \nu}) X^\nu + \sum_{\mu,\nu} T^\mu_{\ \nu} (\nabla_\mu X^\nu)$$

and that

$$\sum_\mu \nabla_\mu T^\mu_{\ \nu} \ = \ (\Box\varphi^k - \varphi^k) \nabla_\nu \varphi^k.$$

Therefore if we write $X$ in $(w, l, e_2, ..., e_n)$, we obtain

$$\sum_{\mu,\nu} (\nabla_\mu T^\mu_{\ \nu}) X^\nu \ = \ e^{-\lambda t} (\Box\varphi^k - \varphi^k) \big( l(\varphi^k) + w(\varphi^k) - \frac{\delta}{\sqrt{n-1}} \sum_{i=2}^n a_i \nabla_i \varphi^k \big).$$

As we can uniformly bound the $a_i$'s, the factor with $\nabla_i \varphi^k$ $(i \geq 2)$ will just produce more terms with $D\varphi^k$ which can be estimated as we have done for the factor $l(\varphi^k)$ or $w(\varphi^k)$ in the proof. As we can do the analogous with $l^{\alpha_1} \circ q^{\alpha_2}(\varphi^k)$ instead of $\varphi^k$, it won't change the principle of the argument. On the other hand for $\sum_{\mu,\nu} T^\mu_{\ \nu} (\nabla_\mu X^\nu)$, we have in $(e_0, e_1, e_2, ..., e_n)$, $X^0 = e^{-\lambda t}$, $X^1 = 0$, $X^i = -\frac{\delta}{\sqrt{n-1}} a_i e^{-\lambda t}$ for all $i \geq 2$. Then when we detail $\nabla_\mu X^\nu$ as we have done in (4.5.4), we will obtain in $\sum_{\mu,\nu} T^\mu_{\ \nu} (\nabla_\mu X^\nu)$ a term $\lambda {e_0}^{\,0} e^{-\lambda t} \big( - T^0_{\ 0} + \frac{\delta}{\sqrt{n-1}} \sum_{i=2}^n a_i T^0_{\ i} \big)$, and other terms which can be uniformly bounded by $T^0_{\ 0}$ multiplyed by a constant (independent of $\lambda$) as we control the $a_i$'s and their partial derivatives of the first order, and as for all $0 \leq i, j \leq n$, $|T^i_{\ j}|$ can be bounded by $|T^0_{\ 0}|$. We need to keep a term of the form $-c\lambda |T^0_{\ 0}|$ when we take the opposite of $\sum_{\mu,\nu} T^\mu_{\ \nu} (\nabla_\mu X^\nu)$ to get our $H^m$-estimations of $\varphi^k$. This can be realised by noticing that $-\frac{\delta}{\sqrt{n-1}} a_i T^0_{\ i} \ \leq \ |T^0_{\ 0}| (\frac{1}{2(n-1)} + \frac{1}{2} \delta^2 \max_i (\max_{Z'} |a_i|^2))$, adding on $i \geq 2$ and choosing $\delta^2 \leq \frac{1}{2(n-1) \max_i (\max_{Z'} |a_i|^2)}$.

Finally by choosing $\delta > 0$ such that $\delta^2 \leq \frac{1}{2 \max_i (\max_{Z'} |a_i|^2)} \min \big( \frac{1}{n-1}, \frac{1}{4} \big)$ we obtain estimations as in lemma 4.4.1 and lemma 4.4.2. The remainder of the argument works similarly. Thus this gives $u_{*p}$, $u_{**p}$ depend on $\rho$, the upper bound of the norm $C^{m-1}$ of $h_p$, $H_p$ and of the norm $H^m$ of $\varphi_{+p}$, $\varphi_{-p}$, $\varphi_{Yp}$, and then $u_{**}$ such for all $p \geq N$, we have a solution $\varphi_p$ of the problem with a prescribed function on $\bigcup_{u,v} \partial Y_{uv}$, with $\varphi_p$ defined on

$[0; u_{**}] \times [0; V] \times Y$.
Moreover $\varphi_p$ is in $C^\infty([0; u_{**}] \times [0; V] \times Y)$ and is unique.

Furthermore, as we have done at the beginning of the section, but by using the vector field $X$ defined above, we can show the analogous lemma of lemma 4.6.1, namely

**Lemma 4.6.2** *For all $p, r \geq N$,*

$$\| \varphi_p(u) - \varphi_r(u) \|_{H^1([0;V]\times Y)} \leq \tilde{c_1} \int_0^u \| \varphi_p(s) - \varphi_r(s) \|_{H^1([0;V]\times Y)} \, ds$$
$$+\tilde{c_2}\big[ \| \varphi_{+p} - \varphi_{+r} \|_{H^1(N^+)} + \| \varphi_{-p} - \varphi_{-r} \|_{H^1(N^-)} + \| \varphi_{Yp} - \varphi_{Yr} \|_{H^1(\partial\mathcal{Y})} \big].$$

From which we deduce, by the same way as before, the convergence of $(\varphi_p)$ to $\varphi$ solution of the problem (4.2.7) with prescribed data on $\underset{u,v}{\cup} \partial Y_{uv}$, with the same regularity as in proposition 4.6.1.

Now we repeat the process described in the smooth case at the end of section 4.4, it leads to the following theorem.

**Theorem 4.6.1** *Let $m > n/2 + 2$, if $h$ and $H$ are of class $C^{m-1}$, $\varphi_+$ and $\varphi_-$ are of class $H^m$, there exists a unique solution $\varphi$ of the problem (4.2.7) in $C^0\big(\underset{y\in Y}{\cup} \underset{0<V<v_{\max}(y)}{\cup} \{P(u,v,y); (u,v,y) \in [0; u_{**}(V, \tilde{Y}(V))] \times [0;V] \times \tilde{Y}(V)\big)$, moreover $u \mapsto \| \varphi(u) \|_{H^{m'}([0;V]\times Y)}$ is uniformly bounded on $[0; u_{**}(V, \tilde{Y}(V))]$ for all $0 < m' < m$. Moreover if $\varphi_+$ and $\varphi_-$ are in $C^{m-1} \cap H^m$ with $m > n/2 + 3 + j$, then $\varphi$ is in $C^j\big(\underset{y\in Y}{\cup} \underset{0<V<v_{\max}(y)}{\cup} \{P(u,v,y); (u,v,y) \in [0; u_{**}(V, \tilde{Y}(V))] \times [0;V] \times \tilde{Y}(V)\}\big)$.*

This gives a neighborhood of whole $N^-$ even in the case of $v_{\max}$ is not constant. Notice that this neighborhood becomes thiner and thiner when we increase the value of $v$.

## 4.7 Appendix A

### 4.7.1 Proof of lemma 4.5.1

We calculate $\sum_\mu \nabla_\mu T^\mu{}_\nu$.

$$\begin{aligned}
&\sum_\mu \nabla_\mu T^\mu{}_\nu \\
&= \sum_\mu \nabla_\mu \big(\nabla^\mu \tilde{\varphi}^k \nabla_\nu \tilde{\varphi}^k - \frac{1}{2}\big((\sum_\alpha \nabla^\alpha \tilde{\varphi}^k \nabla_\alpha \tilde{\varphi}^k) + (\tilde{\varphi}^k)^2\big)\delta^\mu{}_\nu\big) \\
&= \sum_\mu \big((\nabla_\mu \nabla^\mu \tilde{\varphi}^k)\nabla_\nu \tilde{\varphi}^k + \nabla^\mu \tilde{\varphi}^k (\nabla_\mu \nabla_\nu \tilde{\varphi}^k) - \frac{1}{2}\delta^\mu{}_\nu \nabla_\mu \big((\sum_\alpha \nabla^\alpha \tilde{\varphi}^k \nabla_\alpha \tilde{\varphi}^k) + (\tilde{\varphi}^k)^2\big)\big) \\
&= \sum_\mu (\nabla_\mu \nabla^\mu \tilde{\varphi}^k)\nabla_\nu \tilde{\varphi}^k + \sum_\mu \nabla^\mu \tilde{\varphi}^k (\nabla_\mu \nabla_\nu \tilde{\varphi}^k) - \frac{1}{2}\nabla_\nu \big((\sum_\alpha \nabla^\alpha \tilde{\varphi}^k \nabla_\alpha \tilde{\varphi}^k)\big) - \tilde{\varphi}^k \nabla_\nu \tilde{\varphi}^k.
\end{aligned}$$

For the first term of the right member of the equality above, we can notice that

$$\big(\sum_\mu \nabla_\mu \nabla^\mu \tilde{\varphi}^k\big)\nabla_\nu \tilde{\varphi}^k = \big(\sum_{\mu,\alpha} ((\nabla_\mu g^{\mu\alpha})\nabla_\alpha \tilde{\varphi}^k + g^{\mu\alpha}\nabla_\mu \nabla_\alpha \tilde{\varphi}^k)\big)\nabla_\nu \tilde{\varphi}^k.$$

But as $\nabla$ is the connexion of Levi Civita, we have $\nabla g = 0$ so in particular $\nabla_\mu g = 0$, hence

$$\big(\sum_\mu \nabla_\mu \nabla^\mu \tilde{\varphi}^k\big)\nabla_\nu \tilde{\varphi}^k = \Box\tilde{\varphi}^k \nabla_\nu \tilde{\varphi}^k.$$

For the second and third one, we see that if $\tilde{\varphi}^k$ is of class $C^2$, the second one vanishes the third one. Indeed we have

$$\sum_\mu \nabla^\mu \tilde{\varphi}^k \big(\nabla_\mu \nabla_\nu \tilde{\varphi}^k\big) = \sum_{\mu,\alpha} g^{\mu\alpha}\nabla_\alpha \tilde{\varphi}^k \big(\nabla_\mu \nabla_\nu \tilde{\varphi}^k\big) \tag{4.7.1}$$

and

$$\begin{aligned}
&-\frac{1}{2}\nabla_\nu\big(\sum_\alpha \nabla^\alpha \tilde{\varphi}^k \nabla_\alpha \tilde{\varphi}^k\big) \\
&\quad= -\frac{1}{2}\sum_{\mu,\alpha}\Big(\big(\nabla_\nu g^{\mu\alpha}\big)\nabla_\mu \tilde{\varphi}^k \nabla_\alpha \tilde{\varphi}^k + g^{\mu\alpha}\big(\nabla_\nu \nabla_\mu \tilde{\varphi}^k\big)\nabla_\alpha \tilde{\varphi}^k + g^{\mu\alpha}\nabla_\mu \tilde{\varphi}^k\big(\nabla_\nu \nabla_\alpha \tilde{\varphi}^k\big)\Big) \\
&\quad= -\frac{1}{2}\sum_{\mu,\alpha} g^{\mu\alpha}\big(\nabla_\nu \nabla_\mu \tilde{\varphi}^k\big)\nabla_\alpha \tilde{\varphi}^k - \frac{1}{2}\sum_{\mu,\alpha} g^{\mu\alpha}\nabla_\mu \tilde{\varphi}^k\big(\nabla_\nu \nabla_\alpha \tilde{\varphi}^k\big).
\end{aligned} \tag{4.7.2}$$

But if we consider the one-form $d\tilde{\varphi}^k = \sum_\mu (\partial_\mu \tilde{\varphi}^k)dx^\mu$, as we know that for any one-form $\omega = \sum_\beta \omega_\beta dx^\beta$,

$$\nabla_\nu \omega_\beta = \partial_\nu \omega_\beta - \sum_\lambda \Gamma^\lambda{}_{\nu\beta}\omega_\lambda$$

where $\nabla_\mu \partial_\nu = \sum_\lambda \Gamma^\lambda{}_{\mu\nu}\partial_\lambda$, we get

$$\begin{aligned}
\nabla_\nu \nabla_\mu \tilde{\varphi}^k &= \nabla_\nu \partial_\mu \tilde{\varphi}^k = \partial_\nu \partial_\mu \tilde{\varphi}^k - \sum_\lambda \Gamma^\lambda{}_{\nu\mu}\partial_\lambda \tilde{\varphi}^k \\
\nabla_\mu \nabla_\nu \tilde{\varphi}^k &= \nabla_\mu \partial_\nu \tilde{\varphi}^k = \partial_\mu \partial_\nu \tilde{\varphi}^k - \sum_\lambda \Gamma^\lambda{}_{\mu\nu}\partial_\lambda \tilde{\varphi}^k.
\end{aligned}$$

For the connexion of levi Civita the torsion is null hence $\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$. So we get if $\tilde{\varphi^k}$ is $C^2$,

$$\nabla_\nu \nabla_\mu \tilde{\varphi}^k = \nabla_\mu \nabla_\nu \tilde{\varphi}^k.$$

Therefore as $g$ is symetric, we can exchange $\alpha$ and $\mu$ in the second sum of (4.7.2) and so (4.7.1) vanishes (4.7.2). Then as $\tilde{\varphi}^k$ is of class $C^2$, it remains

$$\sum_\mu \nabla_\mu T^\mu{}_\nu = \big(\Box\tilde{\varphi}^k - \tilde{\varphi}^k\big)\nabla_\nu \tilde{\varphi}^k.$$

### 4.7.2 Proof of lemma 4.5.2

Let $Z$ be any vector field on $M$, we recall that $(e_i)$ is any local orthonormal basis with (4.4.1). By definition we know that the infinitesimal element of volume on $M$ is

$$dV = \theta^0 \wedge ... \wedge \theta^n$$

where $(\theta^i)$ is the dual basis associated to $(e_i)$, and that

$$dS_\mu = e_\mu \rfloor dV$$

where $\rfloor$ is a contraction, i.e. if $X$ is a vector field and $\alpha$ is a p-form then $X \rfloor \alpha$ is the (p-1)-form such that $(X \rfloor \alpha)(W_1, ..., W_{p-1}) = \alpha(X, W_1, ..., W_{p-1})$, here $(X \rfloor dV)(W_1, ..., W_{p-1}) = det(X, W_1, ..., W_{p-1})$.
In particular if $X \rfloor dV$ acts on a hypersurface where $X$ is tangent, it vanishes because we obtain a determinant with a vector $X$ linearly dependent to the other vectors.
As we have $l = \dfrac{e_0 + e_1}{2}$ and $w = \dfrac{e_0 - e_1}{2}$, $(l, w, e_2, ..., e_n)$ is also a local basis of $TM$. Hence when we write $Z$ in this base, we get

$$\sum_\mu Z^\mu dS_\mu = (\sum_\mu Z^\mu e_\mu) \rfloor dV = (Z^l l + Z^w w + \sum_{i=2}^n Z^i e_i) \rfloor dV \tag{4.7.3}$$

and as $l, e_2, ..., e_n$ are tangent to $N_\tau^-$, when this $n$-form acts on vectors of $TN_\tau^-$ it remains,

$$\sum_\mu Z^\mu dS_\mu|_{TN_\tau^-} = Z^w w \rfloor dV.$$

But we notice that (as $l$ is isotropic and orthogonal to $(e_i)$ for $2 \leq i \leq n$)

$$g(Z, l) = g(Z^l l + Z^w w + \sum_{i=2}^n Z^i e_i, l) = Z^w g(w, l) = Z^w g(\frac{e_0 - e_1}{2}, \frac{e_0 + e_1}{2}) = -\frac{1}{2} Z^w$$

Thus

$$\int_{N_\tau^-} \sum_\mu Z^\mu dS_\mu = \int_{N_\tau^-} -2g(Z, l) w \rfloor dV.$$

Now as

$$g(Z, l) = g(Z^0 e_0 + Z^1 e_1 + ... + Z^n e_n, \frac{e_0 + e_1}{2}) = -\frac{1}{2} Z^0 + \frac{1}{2} Z^1$$

we obtain

$$\int_{N_\tau^-} \sum_\mu Z^\mu dS_\mu = \int_{N_\tau^-} (Z^0 - Z^1) w \rfloor dV.$$

Similarly as $w, e_2, ..., e_n$ are tangent to $N_V^+$, when the $n$-form (4.7.3) acts on vectors of $TN_V^+$ it remains,

$$\sum_\mu Z^\mu dS_\mu|_{TN_V^+} = Z^l l \rfloor dV.$$

As $w$ is isotropic and orthogonal to $(e_i)$ for $2 \leq i \leq n$, we have

$$g(Z,w) = g(Z^l l + Z^w w + \sum_{i=2}^{n} Z^i e_i, w) = Z^l g(l,w) = Z^l g(\frac{e_0+e_1}{2}, \frac{e_0-e_1}{2}) = -\frac{1}{2}Z^l$$

Thus

$$\int_{N_V^+} \sum_\mu Z^\mu dS_\mu = \int_{N_V^+} -2g(Z,w)l \rfloor dV.$$

Now as

$$g(Z,w) = g(Z^0 e_0 + Z^1 e_1 + ... + Z^n e_n, \frac{e_0-e_1}{2}) = -\frac{1}{2}Z^0 - \frac{1}{2}Z^1$$

we get

$$\int_{N_V^+} \sum_\mu Z^\mu dS_\mu \ = \ \int_{N_V^+} (Z^0 + Z^1) l \rfloor dV.$$

### 4.7.3 Proof of lemma 4.4.3

Recall that we assume that $\mathcal{S}$ is a compact Riemannian manifold (with or without boundary), $m$ in $I\!N$, and that $f$ is in $H^1(\mathcal{S}) \cap H^m(\mathcal{S})$. Let $1 < m' < m$, $m' \in I\!N$, and $0 < \sigma < 1$ such that $m' = \sigma + (1-\sigma)m$, i.e. $\sigma = (m-m')/(m-1)$ ($m-1$ is an integer greater than or equal to 1). First notice that as $m - 1 \geq 1$, there exists $c$ such that

$$\| f \|^{m-1}_{H^{m'}(\mathcal{S})} \ \leq \ c \left( \| f \|^{m-1}_{L^2(\mathcal{S})} + ... + \| \nabla^{m'} f \|^{m-1}_{L^2(\mathcal{S})} \right)$$

(in all the following $c$ will denote a constant which can be able to change at each lines). Now we use one of the Gagliardo-Nirenberg-Ni inequalities (see T. Aubin [1] chapter 3 paragraph 7.6 Theorem 3.70 with $p = q = r = 2$ and $\nabla f$ instead of $f$), namely if $j < m$

$$\| \nabla^j f \|_{L^2} \leq c \| \nabla^m f \|_{L^2}^{\frac{j-1}{m-1}} \| \nabla f \|_{L^2}^{\frac{m-j}{m-1}} .$$

We obtain

$$\begin{aligned}
&\| f \|^{m-1}_{H^{m'}(\mathcal{S})} \\
&\leq c\Big( \| f \|^{m-1}_{L^2} + \| \nabla f \|^{m-1}_{L^2} + \| \nabla^m f \|_{L^2} \| \nabla f \|^{m-2}_{L^2} + ... + \| \nabla^m f \|^{m'-1}_{L^2} \| \nabla f \|^{m-m'}_{L^2} \Big) \\
&= c\Big( \| f \|^{m-1}_{L^2} + \Big( \| \nabla f \|^{m'-1}_{L^2} + \| \nabla^m f \|_{L^2} \| \nabla f \|^{m'-2}_{L^2} + ... + \| \nabla^m f \|^{m'-1}_{L^2} \Big) \| \nabla f \|^{m-m'}_{L^2} \Big) \\
&\leq c\Big( \| f \|^{m'-1}_{L^2} \| f \|^{m-m'}_{L^2} + \Big( \| \nabla f \|_{L^2} + \| \nabla^m f \|_{L^2} \Big)^{m'-1} \| \nabla f \|^{m-m'}_{L^2} \Big) \\
&\leq c\Big( \| f \|^{m'-1}_{H^m} \| f \|^{m-m'}_{L^2} + \| f \|^{m'-1}_{H^m} \| \nabla f \|^{m-m'}_{L^2} \Big) \\
&\leq c \ \ \| f \|^{m'-1}_{H^m} \| f \|^{m-m'}_{H^1} .
\end{aligned}$$

Hence f is in $H^{m'}(\mathcal{S})$ and there exists $c > 0$ such that

$$\| f \|_{H^{m'}(\mathcal{S})} \ \leq \ c \ \| f \|^{\sigma}_{H^1(\mathcal{S})} \| f \|^{1-\sigma}_{H^m(\mathcal{S})} .$$

## 4.8 Appendix B

Simpler example of equation with nonlinearity which looks like "dissipation" type : Let, in dimension two, $c' > 0$ and

$$\begin{cases} \dfrac{\partial^2 \varphi}{\partial u \partial v} = -\dfrac{1}{2}\Big(\dfrac{\partial \varphi}{\partial u}\Big)^3 \\ \varphi|_{N^+} = \varphi_+(u) = c'u \\ \varphi|_{N^-} = \varphi_-(v) = c'v \end{cases} \tag{4.8.1}$$

We get

$$\frac{1}{(\partial_u \varphi)^2}(u,v) = v + \frac{1}{c'^2}$$

then

$$\varphi(u,v) = \sqrt{\frac{1}{v + \frac{1}{c'^2}}}\; u + c'v.$$

$\varphi$ is a smooth solution of (4.8.1) on the future of $N^+ \cup N^-$ entirely.
In our estimation we will get in (4.5.10) (the flat metric case is a little simpler) :

$$\begin{aligned} &\frac{1}{2}\int_0^V \big((\varphi^k)^2(\tau,v) + (\partial_v \varphi^k)^2(\tau,v)\big)dv \\ &\quad \leq \int_0^\tau \int_0^V \Big[\big(-(\partial_u \varphi^{k-1})^2 \partial_u \varphi^k(u,v) + \varphi^k(u,v)\big)\big(\partial_u \varphi^k(u,v) + \partial_v \varphi^k(u,v)\big) \\ &\qquad -\frac{\lambda}{2}\big(\partial_u \varphi^k(u,v)\big)^2\Big]dvdu + c(c'). \end{aligned}$$

This can be bound "independently" of $\partial_u \varphi^{k-1}$, $\partial_u \varphi^k$ if we choose a $\lambda$ large enough and if we have a bound of $|\partial_u \varphi^{k-1}|$. As we have done to get (4.5.39) this is achieved if we can obtain an estimation of the norm $H^m([0;V])$ with $m > n/2 + 2$, i.e. $m > 3$, of $\varphi^{k-1}(u)$. But when we differentiate with respect to $v$ to get the norm $H^2([0;V])$, there's still a problem. We obtain

$$\begin{aligned} &\frac{1}{2}\int_0^V \big((\partial_v \varphi^k)^2(\tau,v) + (\partial_v \partial_v \varphi^k)^2(\tau,v)\big)dv \\ &\quad \leq \int_0^\tau \int_0^V \Big[\big(-2\partial_u \varphi^{k-1} \partial_u \partial_v \varphi^{k-1}(u,v) - (\partial_u \varphi^{k-1})^2 \partial_u \partial_v \varphi^k(u,v) + \partial_v \varphi^k(u,v)\big) \\ &\qquad \times \big(\partial_u \partial_v \varphi^k(u,v) + \partial_v \partial_v \varphi^k(u,v)\big) - \frac{\lambda}{2}\big(\partial_u \partial_v \varphi^k(u,v)\big)^2\Big]dvdu + c(c'). \end{aligned}$$

And then we need a bound of $|\partial_u \partial_v \varphi^{k-1}|$, this could be achieved if we can estimate the norm $H^m([0;V])$ with $m > n/2 + 3$, i.e. $m > 4$, of $\varphi^{k-1}(u)$.Therefore, an $H^m$ estimate for $\varphi^k$ requires an $H^{m'}$ estimate with $m' > m + 2$ for $\varphi^{k-1}$. And so our approach here can't be directly generalised to cover such cases.

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# Chapitre 5

# A quasilinear symmetric hyperbolic system

**Abstract**

We consider a characteristic initial value problem for a class of symmetric hyperbolic systems with initial data given on two smooth null intersecting characteristic surfaces. We prove existence and uniqueness of solutions on a (one-sided) neighborhood of one of the initial surfaces.

## Sommaire



## 5.1 Introduction

The characteristic initial value problem for the vacuum Einstein equations with initial data given on two smooth null intersecting surfaces has been studied by H.-J. Seifert and H. Müller zum Hagen [14] and by A. D. Rendall [15]. In those papers existence of a solution is established in a neighborhood of the intersection of the initial data hypersurfaces. The initial aim of this work was to prove existence of the solution in a whole neighborhood of the initial data surfaces, rather than of their intersection. This is why we consider a particular class of symmetric hyperbolic systems whose form comes from the

Newman-Penrose decomposition of Einstein's equations, see the book of S. Klainerman and F. Nicolo [11] (cf also thesis of O. Lengard [12], chapter 4). The principal part of the symmetric hyperbolic systems studied here is of a form which seems to apply to Einstein's equations. But, because of certain lower order terms, the application of our results on the systems associated to Einstein's equations (in harmonic formulation, or formulation of Klainerman-Nicolo) is not obvious. We are studying this problem actually.

## 5.2 Presentation of the problem

Let $Y$ be a $(n-1)$-dimensional compact manifold without boundary, and let $U, v_{\max}$ be two strictly positive real numbers, set

$$\mathscr{M} := \{u \in [0,U], v \in [0, v_{\max}[, y \in Y\} \tag{5.2.1}$$

We let $e_-$ be a vector field of the form

$$e_- := e_-{}^u \partial_u \ , \tag{5.2.2}$$

similarly $e_+$ will be a vector field on $\mathscr{M}$ of the form

$$e_+ = e_+{}^v \partial_v \ , \tag{5.2.3}$$

where the smooth functions $e_\pm{}^i$ are required to be strictly positive. Results with weaker differentiability conditions will be derived in Section 5.7, and more general $e_\pm$ will be considered in Section 5.8. We let

$$N^- := \{u = 0\} \ , \quad N^+ := \{v = 0\} \ ,$$

and we start by considering the quasilinear problem

$$\begin{aligned} e_-(\varphi) + L\psi \quad &= a^0(\varphi, \cdot) + a^1(\varphi, \cdot)\psi && (5.2.4)\\ -L^*\varphi + e_+(\psi) \quad &= b^0(\varphi, \cdot) + b^1(\varphi, \cdot)\psi && (5.2.5)\\ \varphi|_{N^-} = \varphi_{0-} & && (5.2.6)\\ \psi|_{N^+} = \psi_{0+} & && (5.2.7) \end{aligned}$$

where $\cdot$ denotes a possible dependence upon the coordinates and where $e_-{}^u$, $e_+{}^v$, $L$ can depend on $\varphi$ (in this case $e_-(\varphi) = e_-{}^u(\varphi, \cdot)\partial_u\varphi$ and so on). The fields $\varphi$ and $\psi$ are sections of vector bundles over $\mathscr{M}$, each equipped with a scalar product. $L$ is a differential operator (mapping sections of the $\psi$-bundle to those of the $\varphi$-bundle) such that

$$L = \sum_{j=1}^{r} A^j q_j$$

where the $A^j$'s are smooth, and the $q_j$'s $(1 \le j \le r)$ are tangent to the

$$Y_{uv} = N_u^- \cap N_v^+ = \{u\} \times \{v\} \times Y \ ,$$

where the $N_u^-$ are the level sets of $u$, and the $N_v^+$ are the level sets of $v$. In other words, $L$ is a differential operator in the $y$-variables, with coefficients depending possibly upon all the variables $(u, v, y)$, and upon the field $\varphi$. Finally, $L^*$ is the formal adjoint of $L$, obtained by integrating by parts over $Y$ (making use of the scalar product $< \cdot, \cdot >$ on the $\psi$-bundle), more precisely we assume that

$$L^* = -\sum_{j=1}^{r}(A^j)^* q_j - \sum_{j=1}^{r} q_j\left((A^j)^*\right) ,$$

with $\sum_{j=1}^{r} q_j\left((A^j)^*\right)$ just depending on $(\varphi, u, v, y)$ (not depending on the gradient of $\varphi$).

We assume that there exists a real number $\hat{R} > 0$ such that $a^0$, $a^1$, $b^0$, $b^1$ and ${e_-}^u$, ${e_+}^v$, $A^j$'s are defined in their first variable on an open set which contains the union over $p$ in $\varphi_{0-}(N^-)$ of closed balls $\overline{B(p, \hat{R})}$ of radius $\hat{R}$ centred at $p$ :

$$\Xi := \cup_{p\in\varphi_{0-}(N^-)}\overline{B(p, \hat{R})} \, . \tag{5.2.8}$$

Here the balls are taken with respect to some Riemannian metric on the relevant bundles. We first require $a^0$, $a^1$, $b^0$, $b^1$, $\varphi_{0-}$, $\psi_{0+}$ to be smooth, we will see further that existence and uniqueness will hold under less stringent differentiability conditions.

We note that $\psi$ on $N^-$ can be obtained by integrating (5.2.5), considered as a linear ODE for $\psi|_{N^-}$, as $\varphi$ is already known there by (5.2.6). Similarly we can integrate (5.2.4) to obtain $\varphi$ on $N^+$ in a neighborhood of $N^+ \cap N^-$ ; however, because this equation is non-linear in $\varphi$, a global solution might sometimes fail to exist on $N^+$ ; see chapter 4 (4.2.2) for an explicit example where this happens. This indicates in particular that for systems with non-linearities more general than in (5.2.4)-(5.2.7) the results proved below will not be true anymore.

The purpose of this work is to sketch the proof of existence of solutions of (5.2.4)-(5.2.7), using a method inspired by that of A. Majda [13].

## 5.3 Iterative scheme

One starts by constructing a sequence of approximate solutions $\left(\varphi^k, \psi^k\right)_{k\in\mathbb{N}}$ of the problem (5.2.4)-(5.2.7) by induction. We set

$$\varphi^0 = \varphi_{0+} + v(\partial_v\varphi_0) + \cdots + \frac{v^{s-1}}{(s-1)!}(\partial_v^{s-1}\varphi_0) \, , \tag{5.3.1}$$

where $\varphi_{0+}$ is the value of $\varphi$ on $N^+$, obtained as the unique smooth solution of (in order to simplify we denote here $A^j(\varphi_{0+})$ for $A^j(\varphi_{0+}, u, 0, y)$)

$$\begin{aligned} &{e_-}^u(\varphi_{0+})\partial_u\varphi_{0+}(u, y) = \\ &-\sum_j A^j(\varphi_{0+})q_j(\psi_{0+})(u, y) + a^0(\varphi_{0+}(u, y), u, 0, y) + a^1(\varphi_{0+}(u, y), u, 0, y)\psi_{0+}(u, y) \end{aligned}$$

such that $\varphi_{0+}(0, y) = \varphi_{0-}(0, y)$ (note that $\varphi_{0+}$ can be just defined in a neighborhood of $N^+ \cap N^-$ but this is enough for our purpose). For $i \in \mathbb{N}^*$ the higher derivatives $\partial_v^i\varphi_0$ are defined by induction as the unique smooth solution of the equation of the form

$$e_-\left((\partial_v^i\varphi_0)(u, y)\right) = B(u, y)(\partial_v^i\varphi_0)(u, y) + C(u, y)$$

obtained by taking $\partial_v^i$ of (5.2.4) at $(u,0,y)$ and then commuting $e_-$ and $\partial_v^i$, with the initial condition $(\partial_v^i \varphi_0)(0,y) = \partial_v^i(\varphi_{0-})(0,y)$, (for more details, see the appendix 5.9, just after (5.9.1)). This choice of $\varphi^0$ is necessary to obtain estimations in our argument, and is one of the elements which reduce the time $u_*$ in Lemma 5.4.1 as $\varphi_{0+}$ is not necessary defined on all $N^+$. Further we let $\psi^0 = \psi_{0+}$, and for $k \geq 0$ we define the $(\varphi^{k+1}, \psi^{k+1})$'s as solutions of the problem

$$\begin{cases} e_-(\varphi^{k+1}) + L\psi^{k+1} \quad = a^0(\varphi^k, \cdot) + a^1(\varphi^k, \cdot)\psi^{k+1} \\ -L^*\varphi^{k+1} + e_+(\psi^{k+1}) = b^0(\varphi^k, \cdot) + b^1(\varphi^k, \cdot)\psi^{k+1} \\ \varphi^{k+1}|_{N^-} = \varphi_{0-} \\ \psi^{k+1}|_{N^+} = \psi_{0+} \end{cases} \tag{5.3.2}$$

Here $e_-{}^u$, $e_+{}^v$ and the $A^j$'s are evaluated at $\varphi^k$ in the case when they depend on $\varphi$. This can be written as a linear symmetric hyperbolic system :

$$\begin{pmatrix} e_- & L \\ -L^* & e_+ \end{pmatrix} \begin{pmatrix} \varphi^{k+1} \\ \psi^{k+1} \end{pmatrix} = \begin{pmatrix} a^0(\varphi^k, \cdot) \\ b^0(\varphi^k, \cdot) \end{pmatrix} + \begin{pmatrix} 0 & a^1(\varphi^k, \cdot) \\ 0 & b^1(\varphi^k, \cdot) \end{pmatrix} \begin{pmatrix} \varphi^{k+1} \\ \psi^{k+1} \end{pmatrix}.$$

The existence of a solution $(\varphi^{k+1}, \psi^{k+1})$ follows from Theorem 1 of A. D. Rendall's article [15], which proves existence and uniqueness of a solution of a quasilinear equation with prescribed data on two transverse characteristic hypersurfaces in a neighborhood $\mathscr{U}$ of the intersection of these hypersurfaces (see in particular his paragraph on symmetric hyperbolic equations). Rendall's method, as applied to (5.3.2), can be summarised as follows : First, one solves the equations for $\varphi^{k+1}$, $\psi^{k+1}$, and all their partial derivatives along $N^+$ and $N^-$, where the first two equations in (5.3.2) are understood as propagation equations on the hypersurfaces. As equations (5.3.2) are linear, this can be done globally on $N^+$ and $N^-$. In a second step $\mathscr{U}$ is determined by the application of the standard theorems for the spacelike Cauchy problem. More precisely, a suitable set of functions, constructed using the solution of the propagation equations above, defines initial data on a spacelike Cauchy surface $\Sigma$ containing the intersection of the hypersurfaces $N^- \cap N^+$. Since (5.3.2) is linear we obtain a solution $(\varphi^{k+1}, \psi^{k+1})$ on the domain of dependence $\mathscr{U}$ of the auxiliary spacelike surface $\Sigma$. This solution is $C^\infty$ if all the remaining functions are, hence we can iterate (5.3.2), obtaining a sequence of solutions of $C^\infty$ class.

As we do not wish to address the issues of definition of domains of dependence for general symmetric hyperbolic systems, it suffices for our purposes to note that the set $\mathscr{U}$ above can be taken to be $k$-independent, and to contain a whole neighborhood of $N^+ \cup N^-$.

The reader will note that we are using $\psi^{k+1}$ rather than $\psi^k$ in the right-hand-sides of the first two equations of (5.3.2), this appears to be necessary to obtain $k$-independent *a priori* estimates with our method and to get the convergence to the solution (indeed we will see further that the convergence of $(\psi^k)$ is assured by extracting a subsequence, which wouldn't work if we had both $\psi^k$ and $\psi^{k+1}$ in the iterative system).

**Remark 5.3.1** : The fact that our iterative system is linear guarantees the existence of a global solution, and not only existence in a neighborhood of the intersection of the null hypersurfaces. This is one of the reasons why one cannot allow a dependence on $\psi$ in the coefficients of $e_-{}^u$, $e_+{}^v$ and $L$. The scheme we are using would have forced us to take $e_-{}^u$, $e_+{}^v$, $A^j$'s at $\psi^{k+1}$, and linearity would have been lost. The example discussed in chapter 4

(4.2.2) shows in any case that such non-linearities are an obstruction to prove the result we are aiming at.

The fact that $a^i(\varphi^k,\cdot)$, $b^i(\varphi^k,\cdot)$ $(i=0,1)$ and $e_-{}^u(\varphi^k,\cdot)$, $e_+{}^v(\varphi^k,\cdot)$, $A^j(\varphi^k,\cdot)$'s are well defined in (5.3.2) is assured by Lemma 5.4.1, which implies that all the $|\varphi^k|$'s are contained in the compact set $\Xi$ defined in (5.2.8).

Now we describe the global process to get the convergence of $\left(\varphi^k,\psi^k\right)$ to $(\varphi,\psi)$ solution of (5.2.4)-(5.2.7). Let $0<V<v_{\max}$. Let $R=\hat{R}/c$ where $c$ is the multiplicative constant in front of the norm $H^s([0,V]\times Y)$ when we bound the $L^\infty$ norm by the $H^s$ norm (with $s>n/2$), and $\hat{R}$ has been defined in (5.2.8). Thus $a^0$, $a^1$, $b^0$ and $b^1$ can be composed with $\varphi^k$ in the relevant variable as long as $\parallel \varphi^k(u)-\varphi_{0-} \parallel_{H^s([0,V]\times Y)}\leq R$ (we denote $\varphi^k(u)$, $\psi^k(u)$ the restrictions of $\varphi^k$, $\psi^k$ to $\{u\}\times[0,V]\times Y$, the norm $H^s$ considered here is the same as the one used in chapter 4, defined in subsection 4.4.1). Therefore the iterative scheme is well defined for $u\in[0,u'_k]$, with $u'_k$ defined as

$$u'_k \text{ is the largest number such that } \sup_{0\leq u\leq u'_k} \parallel \varphi^k(u)-\varphi_{0-} \parallel_{H^s([0,V]\times Y)}\leq R. \qquad (5.3.3)$$

The fact that there exists $u_*$ such that for all $k$ in $I\!N$ we have $u'_k>u_*$ is established in the following lemma.

## 5.4 Estimations for the approximating sequence of solutions

**Lemma 5.4.1** *For any integer $s>n/2+1$ there exists $R'>0$, $u_*>0$ such that for all $k$ in $I\!N$,*

$$\begin{aligned}
\max_{0\leq u\leq u_*} \parallel \varphi^k(u)-\varphi_{0-} \parallel_{H^s([0,V]\times Y)} &\leq R\,,\\
\int_0^{u_*} \parallel \psi^k(u) \parallel^2_{H^s([0,V]\times Y)}\, du &\leq R'\\
\max_{0\leq u\leq u_*} \parallel \psi^k(u) \parallel_{H^{s-1}([0,V]\times Y)} &\leq R''\,.
\end{aligned}$$

**Remark 5.4.1** The assumption $s>n/2+1$ is necessary to control the norm $L^\infty$ and $C^1$ of $\varphi^k$ (the norm $C^1$ to control $\partial_v\frac{e_+{}^v}{e_-{}^u}$ in the case of $e_+$, $e_-$ depend on $\varphi$, otherwise we just need $s>n/2$).

**Remark 5.4.2** In the third inequality, we obtain a bound of the norm $H^{s-1}$ of $\psi^k$ instead of the norm $H^s$ because to get it we need to control the Sobolev norm of $\varphi^k$ of one degree more than those of $\psi^k$.

PROOF : The proof is an energy inequality, with a non-standard weight function which allows us to control some terms which do not occur in the energy integrand.

For the first inequality we proceed by induction. Let us assume that

$$\max_{0\leq u\leq u_*} \parallel \varphi^{k-1}(u)-\varphi_{0-} \parallel_{H^s([0,V]\times Y)} \leq R\,.$$

We will estimate the energy of $\varphi^k$ on $\{u\}\times[0,V]\times Y$ by integrating $\partial_u$ of this energy with respect to $u$. Recall that $e_-(\varphi^k)=e_-^{\ u}(\varphi^{k-1},\cdot)\partial_u\varphi^k$, with $e_-^{\ u}>0$, the lower bound being uniform on compact sets. First, as $\varphi_{0-}$ just depends on variable $(v,y)$, we have

$$\begin{aligned}
&\partial_u\Big(\int_{\{u\}\times[0,V]\times Y}|\varphi^k-\varphi_{0-}|^2e^{-\lambda(u+v)}dS\Big) \\
&=\int_{\{u\}\times[0,V]\times Y}\big[\frac{2}{e_-^{\ u}}<\varphi^k-\varphi_{0-},e_-(\varphi^k)>+(-\lambda+\tilde{c_1})|\varphi^k-\varphi_{0-}|^2\big]e^{-\lambda(u+v)}dS\ ,
\end{aligned}\tag{5.4.1}$$

where $\tilde{c_1}\ dS:=\partial_u(dS)$. By using the first equation in (5.3.2) we get

$$\begin{aligned}
&\int_{\{u\}\times[0,V]\times Y}\frac{2}{e_-^{\ u}}<\varphi^k-\varphi_{0-},e_-(\varphi^k)>e^{-\lambda(u+v)}dS \\
&\qquad=\int_{\{u\}\times[0,V]\times Y}\frac{2}{e_-^{\ u}}<\varphi^k-\varphi_{0-},-L\psi^k+a^0(\varphi^{k-1},u,v,y) \\
&\qquad\qquad+a^1(\varphi^{k-1},u,v,y)\psi^k>e^{-\lambda(u+v)}dS\ ,
\end{aligned}$$

(in order to simplify we write $e_-^{\ u}$ for $e_-^{\ u}(\varphi^{k-1},u,v,y)$ and $L$ for $L(\varphi^{k-1},u,v,y)$).

The weight $e^{-\lambda(u+v)}$ will allow us to bound the energy independently of $\psi^k$ by taking $\lambda$ large enough. Now by using the adjoint of $L$ and the second equation of (5.3.2), we obtain (the sum over $j$ is implicit),

$$\begin{aligned}
&\int_{\{u\}\times[0,V]\times Y}\frac{2}{e_-^{\ u}}<\varphi^k-\varphi_{0-},-L\psi^k>e^{-\lambda(u+v)}dS \\
&=\int_{\{u\}\times[0,V]\times Y}q_j\big(\frac{2}{e_-^{\ u}}\big)<(A^j)^*(\varphi^k-\varphi_{0-}),\psi^k>e^{-\lambda(u+v)}dS \\
&\quad+\int_{\{u\}\times[0,V]\times Y}\frac{2}{e_-^{\ u}}\big[<e_+(-\psi^k)+b^0(\varphi^{k-1},u,v,y)+b^1(\varphi^{k-1},u,v,y)\psi^k,\psi^k> \\
&\qquad\qquad+<L^*\varphi_{0-},\psi^k>\big]e^{-\lambda(u+v)}dS \\
&=\int_{\{u\}\times[0,V]\times Y}q_j\big(\frac{2}{e_-^{\ u}}\big)<(A^j)^*(\varphi^k-\varphi_{0-}),\psi^k>e^{-\lambda(u+v)}dS \\
&\quad+\int_{\{u\}\times[0,V]\times Y}\big[\frac{1}{e_-^{\ u}}e_+\big(-|\psi^k|^2\big) \\
&\qquad+\frac{2}{e_-^{\ u}}\big(<b^0(\varphi^{k-1},u,v,y)+b^1(\varphi^{k-1},u,v,y)\psi^k,\psi^k> \\
&\qquad\quad+<L^*\varphi_{0-},\psi^k>\big)\big]e^{-\lambda(u+v)}dS\ .
\end{aligned}$$

We can write

$$dS=\hat{c}(u,v,y)dv\ d\mu_Y \tag{5.4.2}$$

with $\hat{c}$ positive, where $d\mu_Y$ is some $u$- and $v$-independent volume form on $Y$. Now as

$e_+ = e_+^{\ v}\partial_v$ with $e_+^{\ v} > 0$, by integrating by parts with respect to $v$, we get

$$\begin{aligned}
&\int_{\{u\}\times[0,V]\times Y} \frac{e_+^{\ v}}{e_-^{\ u}}\partial_v\big(-|\psi^k|^2\big)e^{-\lambda(u+v)}\hat{c}dv\, d\mu_Y \\
&\le \int_{\{u\}\times\{0\}\times Y} \frac{e_+^{\ v}}{e_-^{\ u}}|\psi_{0+}|^2 e^{-\lambda u}\hat{c}(u,0,y)d\mu_Y \\
&\quad + \int_{\{u\}\times[0,V]\times Y} |\psi^k|^2\partial_v\big(\frac{e_+^{\ v}}{e_-^{\ u}}e^{-\lambda(u+v)}\hat{c}\big)dvd\mu_Y \\
&\le \tilde{c}(R) \parallel \psi_{0+}(u) \parallel^2_{L^2(Y)} \\
&\quad + \int_{\{u\}\times[0,V]\times Y} |\psi^k|^2\big[\partial_v\left(\frac{e_+^{\ v}}{e_-^{\ u}}\right) + \frac{e_+^{\ v}}{e_-^{\ u}}\frac{\partial_v\hat{c}}{\hat{c}} - \lambda\frac{e_+^{\ v}}{e_-^{\ u}}\big]e^{-\lambda(u+v)}dS\ ,
\end{aligned} \tag{5.4.3}$$

because $s > n/2$ implies that $H^s \hookrightarrow L^\infty$ holds and so with the inductive assumption on the norm $H^s$ of $\varphi^{k-1}$, we control (and we bound away from zero) $\frac{e_+^{\ v}}{e_-^{\ u}}(\varphi^{k-1}, u, v, y)$. (Throughout this paper we will use symbols $\tilde{c}, c_i, \tilde{c}_i, \overline{c}, \overline{c}_i$, *etc.*, to denote constants which might change from lemma to lemma.) This leads to

$$\begin{aligned}
&\partial_u\big(\int_{\{u\}\times[0,V]\times Y} |\varphi^k - \varphi_{0-}|^2 e^{-\lambda(u+v)}dS\big) \\
&\le \tilde{c}(R) \parallel \psi_{0+}(u) \parallel^2_{L^2(Y)} \\
&+ \int_{\{u\}\times[0,V]\times Y} \Big[\left(\left(\partial_v\left(\frac{e_+^{\ v}}{e_-^{\ u}}\right) + \left(\frac{\partial_v\hat{c}}{\hat{c}} - \lambda\right)\frac{e_+^{\ v}}{e_-^{\ u}} + \big(q_j(\frac{1}{e_-^{\ u}})\big)^2\right)|\psi^k|^2\right. \\
&+ \frac{2}{e_-^{\ u}}\big( < b^0(\varphi^{k-1}, u, v, y) + b^1(\varphi^{k-1}, u, v, y)\psi^k, \psi^k > \\
&+ < \varphi^k - \varphi_{0-}, a^0(\varphi^{k-1}, u, v, y) + a^1(\varphi^{k-1}, u, v, y)\psi^k > \\
&+ < L^*\varphi_{0-}, \psi^k > \big) + r|(A^j)^*(\varphi^k - \varphi_{0-})|^2 + (-\lambda + \tilde{c}_1)|\varphi^k - \varphi_{0-}|^2\Big]e^{-\lambda(u+v)}dS\ .
\end{aligned} \tag{5.4.4}$$

Let us denote by

$$\tilde{c}_2(R) = \max_{(\theta,u,v,y)\in Z,\ h\in\{a^0,a^1,b^0,b^1\}} |h(\theta, u, v, y)|\ ,$$

where

$$Z = \Xi \times [0, u_1'] \times [0, V] \times Y \tag{5.4.5}$$

(recall that $\Xi$ has been defined in (5.2.8)). Since $s > n/2$ we have the embedding $H^s \hookrightarrow L^\infty$, which guarantees that for $\|\varphi^{k-1} - \varphi_{0-}\|_{H^s}$ small enough the values of the fields $\varphi^{k-1}$ will belong to $\Xi$). It follows that

$$\begin{aligned}
&\partial_u\big(\int_{\{u\}\times[0,V]\times Y} |\varphi^k - \varphi_{0-}|^2 e^{-\lambda(u+v)}dS\big) \\
&\le \tilde{c}(R) \parallel \psi_{0+}(u) \parallel^2_{L^2(Y)} + \tilde{c}_3(R) \parallel \varphi_{0-} \parallel^2_{H^1([0,V]\times Y)} \\
&+ \int_{\{u\}\times[0,V]\times Y} \big[\frac{2}{e_-^{\ u}}\tilde{c}_2(R)^2 + r|(A^j)^*(\varphi^k - \varphi_{0-})|^2 + \big(\frac{2}{e_-^{\ u}} - \lambda + \tilde{c}_1\big)|\varphi^k - \varphi_{0-}|^2 \\
&+ \big(\frac{3}{e_-^{\ u}}\tilde{c}_2(R)^2 + \frac{2}{e_-^{\ u}} + \partial_v\frac{e_+^{\ v}}{e_-^{\ u}} + (\frac{\partial_v\hat{c}}{\hat{c}} - \lambda)\frac{e_+^{\ v}}{e_-^{\ u}} + \big(q_j(\frac{1}{e_-^{\ u}})\big)^2\big)|\psi^k|^2\big]e^{-\lambda(u+v)}dS\ .
\end{aligned}$$

**Remark 5.4.3** The above equation clearly shows that when we will work with spaces with finite differentiability we will have to assume that $\varphi_{0-}$ is in a Sobolev space of one degree more than $\psi_{0+}$.

If $s > n/2+1$, $H^s \hookrightarrow C^1$ holds hence we can also control $\partial_v \left(\frac{e_+^v}{e_-^u}\right)(\varphi^{k-1}, u, v, y)$. Notice also that if we denote

$$||| (A^j)^* |||_{C^0(Z)} := \max_{|X|\leq 1} \| (A^j)^* X \|_{C^0(Z)} \tag{5.4.6}$$

we will have

$$|(A^j)^*(\varphi^k - \varphi_{0-})| \leq ||| (A^j)^* |||_{C^0(Z)} |\varphi^k - \varphi_{0-}|.$$

Finally by choosing $\lambda > 0$ large enough and integrating with respect to $u$, we will get

$$\int_{\{\tau\}\times[0,V]\times Y} |\varphi^k - \varphi_{0-}|^2 dS' \leq \int_0^\tau \left[\tilde{c}_4(R) + \int_{\{u\}\times[0,V]\times Y} |\varphi^k - \varphi_{0-}|^2 dS'\right] du ,$$

where

$$dS' := e^{-\lambda(u+v)} dS .$$

The above calculations are the heart of our proof. The usual argument, in which one examines the equations satisfied by the derivatives of $\varphi^k$, leads then to

$$\| \varphi^k(\tau) - \varphi_{0-} \|^2_{H^s([0,V]\times Y)} \leq \int_0^\tau \left[\tilde{c}_{15}(R) + \tilde{c}_{16} \| \varphi^k(u) - \varphi_{0-} \|^2_{H^s([0,V]\times Y)}\right] du , \tag{5.4.7}$$

with some constants $\tilde{c}_{15}(R) > 0$, $\tilde{c}_{16} > 0$. (In any case details of the proof of (5.4.7) can be found in 5.9.1.)

Gronwall's lemma gives that for all $0 \leq u \leq \tau$,

$$\| \varphi^k(u) - \varphi_{0-} \|^2_{H^s([0,V]\times Y)} \leq u_1' \tilde{c}_{15}(R) e^{\tilde{c}_{16} u} ,$$

where $u_1'$ has been defined in (5.3.3). Then by using (5.4.7) , we get

$$\| \varphi^k(u) - \varphi_{0-} \|^2_{H^s([0,V]\times Y)} \leq \tau(\tilde{c}_{15}(R) + \tilde{c}_{16} u_1' \tilde{c}_{15}(R) e^{\tilde{c}_{16} u_1'} ,$$

Choosing

$$u_* \leq \frac{R^2}{\tilde{c}_{15}(R) + \tilde{c}_{16} u_1' \tilde{c}_{15}(R) e^{\tilde{c}_{16} u_1'}}$$

yields

$$\max_{0\leq u\leq u_*} \| \varphi^k(u) - \varphi_{0-} \|_{H^s([0,V]\times Y)} \leq R . \tag{5.4.8}$$

Let us pass now to the second inequality of Lemma 5.4.1. For that we consider again the inequality (5.9.2) with $|\gamma| = s$ and we choose $\lambda > 0$ large enough so that

$$-\lambda \min_Z\left(\frac{e_+^v}{e_-^u}\right) + \tilde{c}_{14}(R) < -1 \quad \text{and} \quad -\lambda + \tilde{c}_9(R) < 1 ,$$

thus we get (recall that $\hat{q}^\gamma = \partial_v^{\gamma_1} \circ q^{\gamma_2}$)

$$\begin{aligned}\| \psi^k(u) \|^2_{H^s([0,V]\times Y)} &+ \frac{d}{du}\Big(\int_{\{u\}\times[0,V]\times Y} \big|\hat{q}^\gamma(\varphi^k - \varphi_{0-})\big|^2 e^{-\lambda(u+v)} dS\Big) \\ &\leq \tilde{c}_{13}(R) + \| \varphi^k(u) - \varphi_{0-} \|^2_{H^{|\gamma|}([0,V]\times Y)} \ .\end{aligned}$$

Then by integrating with respect to $u$, we obtain by using (5.4.8), that

$$\int_0^{u_*} \| \psi^k(u) \|^2_{H^s([0,V]\times Y)} \, du \ \leq \ \tilde{c}_{17}(R) =: R' \ .$$

It remains to show the third inequality of Lemma 5.4.1. On one hand we have for all $0 \leq u \leq u_*$, $0 \leq \tilde{v} \leq V$,

$$\begin{aligned}&\int_{\{u\}\times[0,\tilde{v}]\times Y} \partial_v\big(|\psi^k(u,v,y)|^2 \hat{c}(u,v,y)\big) \, dv \, d\mu_Y \\ &= \int_Y |\psi^k(u,\tilde{v},y)|^2 \hat{c}(u,\tilde{v},y) \, d\mu_Y - \int_Y |\psi_{0+}(u,y)|^2 \hat{c}(u,0,y) \, d\mu_Y \ .\end{aligned}$$

On another hand, by using the second equation of (5.3.2), we obtain

$$\begin{aligned}&\int_{\{u\}\times[0,\tilde{v}]\times Y} \partial_v\big(|\psi^k(u,v,y)|^2 \hat{c}(u,v,y)\big) dv \, d\mu_Y \\ &= \int_{\{u\}\times[0,\tilde{v}]\times Y} \frac{2}{e_+^{\ v}} < e_+(\psi^k), \psi^k > dS + \int_{\{u\}\times[0,\tilde{v}]\times Y} |\psi^k|^2 \frac{\partial_v \hat{c}}{\hat{c}} dS \\ &= \int_{\{u\}\times[0,\tilde{v}]\times Y} \frac{2}{e_+^{\ v}} < L^*\varphi^k + b^0(\varphi^{k-1},u,v,y) + b^1(\varphi^{k-1},u,v,y)\psi^k, \psi^k > dS \\ &\quad + \int_{\{u\}\times[0,\tilde{v}]\times Y} |\psi^k|^2 \frac{\partial_v \hat{c}}{\hat{c}} dS \\ &\leq \int_{\{u\}\times[0,\tilde{v}]\times Y} \frac{1}{e_+^{\ v}} \big(|L^*\varphi^k|^2 + |b^0(\varphi^{k-1},u,v,y)|^2\big) \frac{e^{-\lambda(u+v)}}{e^{-\lambda(u_*+V)}} dS \\ &\quad + \int_{\{u\}\times[0,\tilde{v}]\times Y} \Big(\frac{2}{e_+^{\ v}}(1 + \| b^1 \|_{C^0(Z)}) + \frac{\partial_v \hat{c}}{\hat{c}}\Big) |\psi^k|^2 \frac{e^{-\lambda(u+v)}}{e^{-\lambda(u_*+V)}} dS \end{aligned} \tag{5.4.9}$$

(recall that here $e_+{}^v = e_+{}^v(\varphi^{k-1},u,v,y)$ and $L^* = L^*(\varphi^{k-1},u,v,y)$).

Thus, by using again the first inequality of Lemma 5.4.1, we get

$$\begin{aligned}&\int_Y |\psi^k(u,\tilde{v},y)|^2 \hat{c}(u,\tilde{v},y) e^{-\lambda(u+v)} d\mu_Y \\ &\quad \leq \tilde{c}_{18} \| \psi_{0+}(u) \|^2_{L^2(Y)} + \tilde{c}_{19}(R) + \tilde{c}_{20}(R) \int_0^{\tilde{v}} \int_Y |\psi^k(u)|^2 e^{-\lambda(u+v)} dS.\end{aligned}$$

By proceeding similarly with $\hat{q}^\gamma(\psi^k)$ instead of $\psi^k$, $1 \leq |\gamma| \leq s-1$, (where $\hat{q}^\gamma = \partial_v^{\gamma_1} \circ q^{\gamma_2}$) we obtain following inequalities

$$\begin{aligned}&\int_Y |\hat{q}^\gamma(\psi^k(u,\tilde{v},y))|^2 \hat{c}(u,\tilde{v},y) e^{-\lambda(u+v)} d\mu_Y \\ &\quad \leq \tilde{c}_{23}(R) + \sum_{0\leq|\gamma_j|\leq|\gamma|} \tilde{c}_{\gamma_j}(R) \int_0^{\tilde{v}} \int_Y |\hat{q}^{\gamma_j}(\psi^k)|^2 e^{-\lambda(u+v)} \hat{c}(u,\tilde{v},y) d\mu_Y \ dv.\end{aligned}$$

(as for the proof of the first inequality of Lemma 5.4.1, we don't detail them, it can be found in 5.9.2). Adding these inequalities for all $1 \leq |\gamma| \leq s-1$, Now applying the linear Gronwall lemma, and then integrating with respect to $v$, we can write

$$\int_{\{u\}\times[0,V]\times Y} \sum_{1\leq|\gamma|\leq s-1} |\hat{q}^{\gamma}(\psi^k)(u,v,y)|^2 e^{-\lambda(u+v)} dS \leq V\tilde{c}_{24}(R)e^{\tilde{c}_{25}(R)V}.$$

As this inequality is available for all $0 \leq u \leq u_*$, it gives

$$\max_{0\leq u\leq u_*} \parallel \psi^k(u) \parallel_{H^{s-1}([0,V]\times Y)} \leq R''.$$

$\square$

**Lemma 5.4.2** *If $s > n/2+2$ in the previous lemma, there exists $0 < u_{**} \leq u_*$, $\alpha < 1$, such that for all $k$ in $\mathbb{N}$,*

$$\max_{0\leq u\leq u_{**}} \parallel \varphi^{k+1}(u) - \varphi^k(u) \parallel_{H^0([0,V]\times Y)} \leq \alpha \max_{0\leq u\leq u_{**}} \parallel \varphi^k(u) - \varphi^{k-1}(u) \parallel_{H^0([0,V]\times Y)} .$$

**Remark 5.4.4** The assumption $s > n/2+2$ is necessary to control the norm $C^1$ of $\psi^k$, but if $e_-$ doesn't depend on $\varphi$, we just need $s > n/2+1$ (notably to control the norm $L^\infty$ of $\psi^k$).

Proof of Lemma 5.4.2 : The proof is similar to that of the previous lemma : If we differentiate with respect to $u$ the square of the norm $H^0([0,V]\times Y)$ of $\varphi^{k+1}(u) - \varphi^k(u)$, we obtain

$$\begin{aligned}
&\partial_u\Big(\int_{\{u\}\times[0,V]\times Y} |\varphi^{k+1} - \varphi^k|^2 e^{-\lambda(u+v)} dS\Big) \\
&= \int_{\{u\}\times[0,V]\times Y} \Big[2 < \varphi^{k+1} - \varphi^k, \frac{1}{e_-^{\,u}(\varphi^k)} e_-(\varphi^{k+1}) - \frac{1}{e_-^{\,u}(\varphi^{k-1})} e_-(\varphi^k) > \\
&\quad + (-\lambda + \tilde{c}_1)|\varphi^{k+1} - \varphi^k|^2\Big] e^{-\lambda(u+v)} dS \ . \qquad (5.4.10)
\end{aligned}$$

The first equation in (5.3.2) gives (recall that the $A^j$'s depend on $\varphi$, that is why we write $L(\varphi^k)$ or $L(\varphi^{k-1})$),

$$\begin{aligned}
&\int_{\{u\}\times[0,V]\times Y} 2 < \varphi^{k+1} - \varphi^k, \frac{1}{e_-^{\,u}(\varphi^k)} e_-(\varphi^{k+1}) - \frac{1}{e_-^{\,u}(\varphi^{k-1})} e_-(\varphi^k) > e^{-\lambda(u+v)} dS \\
&= \int_{\{u\}\times[0,V]\times Y} 2 < \varphi^{k+1} - \varphi^k, -\frac{1}{e_-^{\,u}(\varphi^k)} L(\varphi^k)\psi^{k+1} + \frac{1}{e_-^{\,u}(\varphi^{k-1})} L(\varphi^{k-1})\psi^k \\
&\quad + \frac{a^0}{e_-^{\,u}}(\varphi^k, u, v, y) - \frac{a^0}{e_-^{\,u}}(\varphi^{k-1}, u, v, y) \qquad (5.4.11) \\
&\quad + \frac{a^1}{e_-^{\,u}}(\varphi^k, u, v, y)\psi^{k+1} - \frac{a^1}{e_-^{\,u}}(\varphi^{k-1}, u, v, y)\psi^k > e^{-\lambda(u+v)} dS \ .
\end{aligned}$$

On one hand for the terms containing function $a^0$, we proceed as follows : notice that

$$
\begin{aligned}
&|\frac{a^0}{e\_^u}(\varphi^k, u, v, y) - \frac{a^0}{e\_^u}(\varphi^{k-1}, u, v, y)| \\
&\qquad = |\int_0^1 \frac{\partial}{\partial q}\frac{a^0}{e\_^u}(q\varphi^k + (1-q)\varphi^{k-1}, u, v, y)dq| \\
&\qquad = |\int_0^1 (\frac{\partial}{\partial \theta}\frac{a^0}{e\_^u})(q\varphi^k + (1-q)\varphi^{k-1}, u, v, y)(\varphi^k - \varphi^{k-1})dq| \\
&\qquad \leq \| \frac{a^0}{e\_^u} \|_{C^1(Z)} |\varphi^k(u) - \varphi^{k-1}(u)|
\end{aligned}
$$

by using the first inequality of Lemma 5.4.1 ($Z$ is defined in (5.4.5)). We obtain that

$$
\begin{aligned}
&\int_{\{u\}\times[0,V]\times Y} 2 < \varphi^{k+1} - \varphi^k, \frac{a^0}{e\_^u}(\varphi^k, u, v, y) \\
&\quad - \frac{a^0}{e\_^u}(\varphi^{k-1}, u, v, y) > e^{-\lambda(u+v)}dS \\
&\leq \int_{\{u\}\times[0,V]\times Y} \big(|\varphi^{k+1} - \varphi^k|^2 + \tilde{c}_2|\varphi^k - \varphi^{k-1}|^2\big)e^{-\lambda(u+v)}dS \ .
\end{aligned}
$$

with

$$
\tilde{c}_2 = \| \frac{a^0}{e\_^u} \|^2_{C^1(Z)} \ .
$$

For the terms containing the function $a^1$, we can proceed similarly and write

$$
\begin{aligned}
&\frac{a^1}{e\_^u}(\varphi^k, u, v, y)\psi^{k+1} - \frac{a^1}{e\_^u}(\varphi^{k-1}, u, v, y)\psi^k \\
&\quad = \frac{a^1}{e\_^u}(\varphi^k, u, v, y)(\psi^{k+1} - \psi^k) + \big(\frac{a^1}{e\_^u}(\varphi^k, u, v, y) - \frac{a^1}{e\_^u}(\varphi^{k-1}, u, v, y)\big)\psi^k.
\end{aligned}
$$

By using both inequalities of Lemma 5.4.1, as $s - 1 > n/2$ the embedding $H^{s-1} \hookrightarrow L^\infty$ implies a bound $\tilde{c}(R'')$ of the norm $L^\infty([0, V] \times Y)$ of $\psi^k$ uniformly on $0 \leq u \leq u_*$, we get

$$
\begin{aligned}
&\int_{\{u\}\times[0,V]\times Y} 2 < \varphi^{k+1} - \varphi^k, \frac{a^1}{e\_^u}(\varphi^k, u, v, y)\psi^{k+1} \\
&\quad - \frac{a^1}{e\_^u}(\varphi^{k-1}, u, v, y)\psi^k > e^{-\lambda(u+v)}dS \\
&\leq \int_{\{u\}\times[0,V]\times Y} \big[2|\varphi^{k+1} - \varphi^k|^2 + \| \frac{a^1}{e\_^u} \|^2_{C^0(Z)} |\psi^{k+1} - \psi^k|^2 \\
&\quad + \tilde{c}_3|\varphi^k - \varphi^{k-1}|^2\big]e^{-\lambda(u+v)}dS \ .
\end{aligned}
$$

with

$$
\tilde{c}_3 = \| \frac{a^1}{e\_^u} \|^2_{C^1(Z)} \tilde{c}(R'')^2
$$

On another hand for the terms containing $L\psi^{k+1}$ , $L\psi^k$, we proceed as follows. First recall that in the estimations to get rid of the terms with $|\psi^{k+1} - \psi^k|^2$, we need to use the second equation of (5.3.2) and then integrate by parts with respect to $v$ to make appear a term $-\lambda|\psi^{k+1}-\psi^k|^2$ under the integral. We must not forget that $L$ can depend on $\varphi$, so we have to take care of it when we use the adjoint of $L$, and then the second equation of (5.3.2). We have

$$\begin{aligned}
&2 < \varphi^{k+1}-\varphi^k, -\frac{1}{e_^u(\varphi^k)}L(\varphi^k)\psi^{k+1} + \frac{1}{e_^u(\varphi^{k-1})}L(\varphi^{k-1})\psi^k > \\
&= \frac{-2}{e_^u(\varphi^k)} < \varphi^{k+1}-\varphi^k, L(\varphi^k)\psi^{k+1} - L(\varphi^{k-1})\psi^k > \\
&\quad - \left(\frac{2}{e_^u(\varphi^k)} - \frac{2}{e_^u(\varphi^{k-1})}\right) < \varphi^{k+1}-\varphi^k, L(\varphi^{k-1})\psi^k > \ .
\end{aligned}$$

When we integrate the second term of the right-hand-side above on $\{u\}\times[0,V]\times Y$, we can bound the result by

$$\begin{aligned}
&\int_{\{u\}\times[0,V]\times Y} [\ \| \frac{1}{e_^u} \|^2_{C^1(Z)} |\varphi^k-\varphi^{k-1}|^2 \tilde{c}_4 \| \varphi^{k-1} \|^2_{C^0([0,V]\times Y)} \| \psi^k \|^2_{C^1([0,V]\times Y)} \\
&\quad + |\varphi^{k+1}-\varphi^k|^2 ] e^{-\lambda(u+v)} dS
\end{aligned}$$

where $\tilde{c}_4 = c' \max_j \| A^j \|^2_{C^1(Z)}$. Now notice that

$$\begin{aligned}
&\int_{\{u\}\times[0,V]\times Y} \frac{-2}{e_^u(\varphi^k)} < \varphi^{k+1}-\varphi^k, L(\varphi^k)\psi^{k+1} - L(\varphi^{k-1})\psi^k > e^{-\lambda(u+v)} dS \\
&= \int_{\{u\}\times[0,V]\times Y} \frac{-2}{e_^u(\varphi^k)} [ < \varphi^{k+1}-\varphi^k, L(\varphi^k)(\psi^{k+1}-\psi^k) > \\
&\quad + < \varphi^{k+1}-\varphi^k, \big(L(\varphi^k)-L(\varphi^{k-1})\big)\psi^k > ] e^{-\lambda(u+v)} dS \\
&= \int_{\{u\}\times[0,V]\times Y} [\sum_{j=1}^r q_j \big(\frac{-2}{e_^u(\varphi^k)}\big) < -(A^j)^*(\varphi^{k+1}-\varphi^k), \psi^{k+1}-\psi^k > \\
&\quad + \frac{-2}{e_^u(\varphi^k)} \big( < L^*(\varphi^k)\varphi^{k+1} - L^*(\varphi^{k-1})\varphi^k, \psi^{k+1}-\psi^k > \\
&\quad - < \big(L^*(\varphi^k) - L^*(\varphi^{k-1})\big)\varphi^k, \psi^{k+1}-\psi^k > \\
&\quad + < \varphi^{k+1}-\varphi^k, \big(L(\varphi^k)-L(\varphi^{k-1})\big)\psi^k > \big) ] e^{-\lambda(u+v)} dS \ .
\end{aligned}$$

Thus we will use the second equation of (5.3.2) for the term with $< L^*(\varphi^k)\varphi^{k+1} - L^*(\varphi^{k-1})\varphi^k, \psi^{k+1}-\psi^k >$ of the right-hand-side above. For the remainder we can bound the expression by

$$\begin{aligned}
&\int_{\{u\}\times[0,V]\times Y} [\ \| \frac{1}{e_^u} \|_{C^1(Z)} \tilde{c}_4 \big(|\varphi^k-\varphi^{k-1}|^2 + |\psi^{k+1}-\psi^k|^2\big) \\
&\quad + \| \frac{1}{e_^u} \|_{C^0(Z)} \big(\tilde{c}_4^2 |\varphi^k-\varphi^{k-1}|^2 \| \varphi^k \|^2_{C^1([0,V]\times Y)} + |\psi^{k+1}-\psi^k|^2 \\
&\quad + |\varphi^{k+1}-\varphi^k|^2 + \tilde{c}_4^2 |\varphi^k-\varphi^{k-1}|^2 \| \psi^k \|^2_{C^1([0,V]\times Y)} \big) ] e^{-\lambda(u+v)} dS \ .
\end{aligned}$$

So for the moment we have

$$\begin{aligned}
&\partial_u\Big(\int_{\{u\}\times[0,V]\times Y}|\varphi^{k+1}-\varphi^k|^2 e^{-\lambda(u+v)}dS\Big)\\
&\leq \int_{\{u\}\times[0,V]\times Y}\Big[\frac{-2}{{e_-}^u(\varphi^k)} < L^*(\varphi^k)\varphi^{k+1}-L^*(\varphi^{k-1})\varphi^k, \psi^{k+1}-\psi^k >\\
&+\tilde{c}_5(R)|\varphi^k-\varphi^{k-1}|^2+(\tilde{c}_6(R)-\lambda)|\varphi^{k+1}-\varphi^k|^2+\tilde{c}_7|\psi^{k+1}-\psi^k|^2\Big]e^{-\lambda(u+v)}dS\ .
\end{aligned}$$

Then for the term with $< L^*(\varphi^k)\varphi^{k+1}-L^*(\varphi^{k-1})\varphi^k, \psi^{k+1}-\psi^k >$, we use the second equation of (5.3.2), we get

$$\begin{aligned}
&\int_{\{u\}\times[0,V]\times Y}\frac{-2}{{e_-}^u(\varphi^k)} < L^*(\varphi^k)\varphi^{k+1}-L^*(\varphi^{k-1})\varphi^k, \psi^{k+1}-\psi^k > e^{-\lambda(u+v)}dS\\
&=\int_{\{u\}\times[0,V]\times Y}\frac{2}{{e_-}^u(\varphi^k)} < -{e_+}^v(\varphi^k)\partial_v\psi^{k+1}+{e_+}^v(\varphi^{k-1})\partial_v\psi^k\\
&\quad + b^0(\varphi^k,u,v,y)-b^0(\varphi^{k-1},u,v,y)+b^1(\varphi^k,u,v,y)\psi^{k+1}\\
&\qquad - b^1(\varphi^{k-1},u,v,y)\psi^k, \psi^{k+1}-\psi^k > \big]e^{-\lambda(u+v)}dS\ . \qquad (5.4.12)
\end{aligned}$$

We treat terms containing $b^0$, $b^1$ as we have done for $a^0$, $a^1$, for the terms with $\partial_v$, we write

$$\begin{aligned}
&-{e_+}^v(\varphi^k)\partial_v\psi^{k+1}+{e_+}^v(\varphi^{k-1})\partial_v\psi^k\\
&\quad = -{e_+}^v(\varphi^k)\partial_v(\psi^{k+1}-\psi^k)-\big({e_+}^v(\varphi^k)-{e_+}^v(\varphi^{k-1})\big)\partial_v\psi^k\ .
\end{aligned}$$

When we integrate it on $\{u\}\times[0,V]\times Y$ after multiplying it by $\frac{2}{{e_-}^u}e^{-\lambda(u+v)}$, the second term above can be bounded similarly as before and for the first term, we integrate it by parts with respect to $v$ as we have done in (5.4.3) but with $\psi^{k+1}-\psi^k$ instead of $\psi^k$. As $\psi^{k+1}(u,0,y)=\psi^k(u,0,y)=\psi_{0+}$ the integral on $\{u\}\times\{0\}\times Y$ disappears, it leads to

$$\begin{aligned}
&\int_{\{u\}\times[0,V]\times Y}\frac{2}{{e_-}^u} < -{e_+}^v(\varphi^k)\partial_v\psi^{k+1}+{e_+}^v(\varphi^{k-1})\partial_v\psi^k, \psi^{k+1}-\psi^k >\\
&\quad e^{-\lambda(u+v)}\hat{c}dv\ d\mu_Y\\
&\leq \int_{\{u\}\times[0,V]\times Y}\big[|\psi^{k+1}-\psi^k|^2\Big(\partial_v\left(\frac{{e_+}^v}{{e_-}^u}\right)+\frac{{e_+}^v}{{e_-}^u}\frac{\partial_v\hat{c}}{\hat{c}}-\lambda\frac{{e_+}^v}{{e_-}^u}\Big)\\
&+\parallel\frac{1}{{e_-}^u}\parallel_{C^0(Z)}\big(\tilde{c}_8|\varphi^k-\varphi^{k-1}|^2\parallel\psi^k\parallel^2_{C^1([0,V]\times Y)}+|\psi^{k+1}-\psi^k|^2\big)\big]e^{-\lambda(u+v)}dS\ ,
\end{aligned}$$

with $\tilde{c}_8=\parallel{e_+}^v\parallel^2_{C^1([0,V]\times Y)}$. Hence

$$\begin{aligned}
&\partial_u\Big(\int_{\{u\}\times[0,V]\times Y}|\varphi^{k+1}-\varphi^k|^2 e^{-\lambda(u+v)}dS\Big)\\
&\quad\leq \int_{\{u\}\times[0,V]\times Y}\big[\tilde{c}_9(R)|\varphi^k-\varphi^{k-1}|^2+(\tilde{c}_6(R)-\lambda)|\varphi^{k+1}-\varphi^k|^2\\
&\quad + (\tilde{c}_{10}(R)-\lambda)|\psi^{k+1}-\psi^k|^2\big]e^{-\lambda(u+v)}dS\ .
\end{aligned}$$

Furthermore by choosing $\lambda$ large enough, we obtain

$$\begin{aligned}\partial_u\Big(\int_{\{u\}\times[0,V]\times Y} & |\varphi^{k+1}-\varphi^k|^2 e^{-\lambda(u+v)}dS\Big)\\ &\leq \int_{\{u\}\times[0,V]\times Y} \tilde{c}_9(R)|\varphi^k-\varphi^{k-1}|^2 e^{-\lambda(u+v)}dS\ .\end{aligned} \tag{5.4.13}$$

Now if we choose $0<u_{**}\leq u_*$ such that

$$\alpha^2 := u_{**}\tilde{c}_9(R) < 1$$

and if we integrate (5.4.13) with respect to $u$, as $\varphi^{k+1}(0,v,y)=\varphi^k(0,v,y)=\varphi_{0-}$, it leads to

$$\| \varphi^{k+1}(u)-\varphi^k(u)\|_{H^0([0,V]\times Y)} \leq \alpha \max_{0\leq u\leq u_{**}} \| \varphi^k(u)-\varphi^{k-1}(u)\|_{H^0([0,V]\times Y)}\ .$$

As it holds for all $0\leq u\leq u_{**}$, the proof of Lemma 5.4.2 is complete.

□

## 5.5 Convergence of the approximating sequence

Lemma 5.4.2 gives us the estimate

$$\begin{aligned}\sum_{k=1}^{N} & \max_{0\leq u\leq u_{**}} \| \varphi^{k+1}(u)-\varphi^k(u)\|_{H^0([0,V]\times Y)}\\ &\leq \sum_{k=1}^{N}\alpha^k \max_{0\leq u\leq u_{**}} \| \varphi^1(u)-\varphi^0(u)\|_{H^0([0,V]\times Y)}\ .\end{aligned}$$

As $\alpha<1$, by taking the limit $N\to\infty$, we see that the series converges, hence there exists $\varphi$ in $L^\infty\big([0,u_{**}],H^0([0,V]\times Y)\big)$ such that

$$\lim_{k\to\infty}\max_{0\leq u\leq u_{**}} \| \varphi^k(u)-\varphi(u)\|_{H^0([0,V]\times Y)} = 0. \tag{5.5.1}$$

Now by interpolation and the bounds in Lemma 5.4.1 we can show that $\big(\varphi^k(u)\big)$ converges to $\varphi(u)$ in all spaces $H^{s'}$ with $1<s'<s$. Indeed, we have for any $k,l$ in $I\!N$,

$$\begin{aligned}\max_{0\leq u\leq u_{**}} & \| \varphi^k(u)-\varphi^l(u)\|_{H^{s'}}\\ &\leq c \max_{0\leq u\leq u_{**}} \| \varphi^k(u)-\varphi^l(u)\|_{H^0}^{1-\frac{s'}{s}} \max_{0\leq u\leq u_{**}} \| \varphi^k(u)-\varphi^l(u)\|_{H^s}^{\frac{s'}{s}}\\ &\leq c\ (2R)^{\frac{s'}{s}} \max_{0\leq u\leq u_{**}} \| \tilde{\varphi}^k(u)-\tilde{\varphi}^l(u)\|_{H^0}^{1-\frac{s'}{s}}\ .\end{aligned}$$

From this and (5.5.1) we deduce that $\big(\varphi^k(u)\big)$ is a Cauchy sequence in $H^{s'}([0,V]\times Y)$, which is a complete space, so $\big(\varphi^k(u)\big)$ converges to a function $f(u)\in H^{s'}([0,V]\times Y)$. As convergence in $H^{s'}$ implies also convergence in $H^0$, by the uniqueness of the limit in $H^0$, we obtain that $f(u)=\varphi(u)$. If we choose $\frac{n}{2}+1<s'<s$ we get by the embedding

$H^{s'}([0,V]\times Y) \hookrightarrow C^1([0,V]\times Y)$ and so for all $0 \le u \le u_{**}$, $\big(\varphi^k(u)\big)$ converges to $\varphi(u)$ in $C^1([0,V]\times Y)$. Moreover by taking the limit in Lemma 5.4.1, we get

$$\max_{0\le u\le u_{**}} \parallel \varphi - \varphi_{0-} \parallel_{H^{s'}([0,V]\times Y)} \le R. \tag{5.5.2}$$

Now we take care of the convergence of $\big(\psi^k\big)$. By Lemma 5.4.1 we know that for all $0 \le u \le u_{**}$, $\big(\psi^k(u)\big)$ is bounded in norm $H^{s-1}([0,V]\times Y)$. As $H^{s-1}$ is reflexive, we can extract a subsequence $\big(\psi^{k'}(u)\big)$ weakly converging to $\psi(u)$ in $H^{s-1}([0,V]\times Y)$, moreover

$$\parallel \psi(u) \parallel_{H^{s-1}([0,V]\times Y)} \le \liminf \parallel \psi^{k'}(u) \parallel_{H^{s-1}([0,V]\times Y)} \le R''.$$

By compactness of embedding $H^{s-1}([0,V]\times Y) \hookrightarrow H^{s''}([0,V]\times Y)$ for all $0 < s'' < s-1$, we get that $\big(\psi^{k'}(u)\big)$ strongly converges to $\psi(u)$ in $H^{s''}([0,V]\times Y)$. If we take $n/2+1 < s'' < s-1$, the embedding $H^{s''}([0,V]\times Y) \hookrightarrow C^1([0,V]\times Y)$ implies that for all $0 \le u \le u_{**}$, $\big(\psi^{k'}(u)\big)$ converges to $\psi(u)$ in $C^1([0,V]\times Y)$.

Furthermore, one can easily check that $(\varphi,\psi)$ is a solution of the initial problem. Indeed, for all $0 \le u \le u_{**}$ fixed, by the continuity of $b^0$, $b^1$, as $(\varphi^{k'},\psi^{k'})$ converges in $C^1([0,V]\times Y)$, and as operators $L^*$, $e_+$ are tangent to $N_u^-$, we directly obtain that $(\varphi,\psi)$ satisfy (5.2.5). For (5.2.4), it suffices to notice that

$$\varphi^{k'}(u) - \varphi^{k'}(0) = \int_0^u \frac{1}{e_-{}^u(\varphi^{k'-1})} e_-(\varphi^{k'})(s)ds\ ,$$

thus by applying the dominated convergence theorem of Lebesgue, and then differentiate with respect to $u$, $(\varphi,\psi)$ satisfy (5.2.4). (5.2.6) and (5.2.7) are automatically satisfied as they hold for all the iterates.

**Remark 5.5.1** : Notice that in our argument we must require that $s > n/2+2$ to obtain a suitable convergence to $(\varphi,\psi)$ for our problem even when $e_-$ doesn't depend on $\varphi$.

## 5.6 Uniqueness and regularity of the solution

The question of uniqueness is easily taken care of. Let $(\varphi_1,\psi_1)$, $(\varphi_2,\psi_2)$ be two solutions of our problem, we first show that $\varphi_1 = \varphi_2$ by proceeding as in Lemma 5.4.2 with $\varphi_1 - \varphi_2$ instead of $\varphi^{k+1} - \varphi^k$. This gives, when we integrate with respect to $u$, for all $0 \le u \le u_{**}$

$$\parallel \varphi_1(u) - \varphi_2(u) \parallel_{H^0([0,V]\times Y)} \le \int_0^u \tilde{C} \parallel \varphi_1(s) - \varphi_2(s) \parallel_{H^0([0,V]\times Y)} ds.$$

By applying the linear Gronwall lemma, we get $\parallel \varphi_1(u) - \varphi_2(u) \parallel_{H^0([0,V]\times Y)} = 0$. Then as $\varphi_1 = \varphi_2$, by (5.2.5) we get

$$e_+{}^v(\varphi_1)\partial_v(\psi_1 - \psi_2) \;=\; b^1(\varphi_1)(\psi_1 - \psi_2)\ .$$

Thus $\psi_1 - \psi_2$ can be explicitly calculated, as initial values $(\psi_1 - \psi_2)(u,0,y)$ vanish, it implies that $\psi_1 = \psi_2$.

Now we consider the question of regularity of $(\varphi,\psi)$. We start by proving continuity on $[0,u_{**}]\times[0,V]\times Y$. From the first equation of (5.3.2) and Lemma 5.4.1 we have

$$|e_-(\varphi^k)(u,v,y)| \le \overline{C}(R)$$

hence by integrating in $u$ along the integral curve of $e_-$ between the points $P(u,v,y)$ and $Q(u+h,v,y)$, we get

$$|\varphi^k(u+h,v,y) - \varphi^k(u,v,y)| \leq \overline{C'}(R)|h|$$

now by taking the limit in $C^0([0,V]\times Y)$ when $k\to\infty$ we have

$$|\varphi(u+h,v,y) - \varphi(u,v,y)| \leq \overline{C'}(R)|h|.$$

Thus $\varphi$ is in $C^{0,1}\big([0,u_{**}], C^0([0,V]\times Y)\big) \subset C^0\big([0,u_{**}], C^0([0,V]\times Y)\big)$ hence $\varphi$ is in $C^0\big([0,u_{**}]\times[0,V]\times Y\big)$. In the same way, for all $\gamma\in I\!N^{r+1}$, such that $s-1>n/2+|\gamma|+1$, we can obtain (by induction on $|\gamma|$) that $\hat{q}^\gamma(\varphi)$ is in $C^{0,1}\big([0,u_{**}], C^0([0,V]\times Y)\big)$ (by commuting $e_-$ and $\hat{q}^\gamma$ in $e_-\circ\hat{q}^\gamma(\varphi^k)$ (recall that the $\hat{q}=(\partial_v, q_1,\cdots,q_r)$), using the first equation of (5.3.2), and Lemma 5.4.1, with embedding $H^{s-1}\hookrightarrow C^{|\gamma|}$). In particular $L^*(\varphi)$ is in $C^0\big([0,u_{**}]\times[0,V]\times Y\big)$. Furthermore as $\psi$ is solution of an elementary linear ODE of the form $\partial_v\psi = F\psi + G$ with $F$, $G$ continuous in $(u,v,y)$, we have

$$\psi(u,v,y) = [\psi_{0+}(u,y) + \int_0^v G(u,s,y)e^{-\int_0^s F(u,\sigma,y)d\sigma}ds]e^{\int_0^v F(u,s,y)ds}$$

and so $\psi$ is in $C^0\big([0,u_{**}]\times[0,V]\times Y\big)$.

Notice that by induction on $|\gamma|$, $s-1>n/2+|\gamma|+1$, we will show (by commuting $e_+$ and $\hat{q}^\gamma$ in $e_+\circ\hat{q}^\gamma(\psi)$, using (5.2.5) and previous result on regularity of $\varphi$) that $\hat{q}^\gamma(\psi)$ is solution of a similar elementary linear ODE with functions continuous in $(u,v,y)$, and so is in $C^0\big([0,u_{**}]\times[0,V]\times Y\big)$.

As $e_-$, $\partial_v$, $q_1,\ldots,$ $q_r$ generate $TM$, and $\hat{q} = (\partial_v, q_1,\ldots,q_r)$, to show that $\varphi$ is in $C^1\big([0,u_{**}]\times[0,V]\times Y\big)$, it remains to show that $e_-(\varphi)$ is in $C^0\big([0,u_{**}]\times[0,V]\times Y\big)$, but this comes from (5.2.4) and the previous result on $L\psi$.

Similarly to show that $\psi$ is in $C^1\big([0,u_{**}]\times[0,V]\times Y\big)$, it remains to show that $e_-(\psi)$ is in $C^0\big([0,u_{**}]\times[0,V]\times Y\big)$. This can be achieved by the same process as for $\hat{q}^\gamma(\psi)$ as we know now that $e_-(\varphi)$ is continuous in all its variables.

Hence if in our problem data are smooth, we can choose $s$ as large as we want, and induction shows that $(\varphi,\psi)$ is in $C^\infty\big([0,u_{**}]\times[0,V]\times Y\big)$.

Notice that we can repeat the argument with any $0<V<v_{\max}$ as much as necessary. Then we just have to verify the uniqueness of the solution on the union over $V$ of the $\big([0,u_{**}(V)]\times[0,V]\times Y\big)$'s. This is provided by taking the causal past $J_P^-$ of the points $P(u_{**},V,\tilde{y})$ ($\tilde{y}\in Y$). Thus we obtain existence and uniqueness of a smooth solution of our problem on a neighborhood of whole $N^-$. This leads to the following theorem.

**Theorem 5.6.1** *If $a^0$, $a^1$, $b^0$, $b^1$, $\varphi_{0-}$, $\psi_{0+}$ are $C^\infty$, there exists a unique $C^\infty$ solution $(\varphi,\psi)$ of the problem (5.2.4)-(5.2.7) on* $\bigcup\limits_{0<V<v_{\max}}[0,u_{**}(V))]\times[0,V]\times Y$.

## 5.7 Finite differentiability of data

Now we consider the same problem, but with weaker assumptions, namely we take $a^0$, $a^1$, $b^0$, $b^1$ of class $C^s$, and $\varphi_{0-}$ of class $H^{s+1}$, $\psi_{0+}$ of class $H^s$, with $s>n/2+2$ (we will see that this is the minimum required to get existence in our argument).

By density, we know that there exist $(a_p^0)$, $(a_p^1)$, $(b_p^0)$, $(b_p^1)$, $(\varphi_{0-p})$, $(\psi_{0+p})$ of class $C^\infty$ with compact support such that

$$\lim_{p\to\infty} \| a_p^0 - a^0 \|_{C^s} = 0 \qquad \lim_{p\to\infty} \| a_p^1 - a^1 \|_{C^s} = 0$$
$$\lim_{p\to\infty} \| b_p^0 - b^0 \|_{C^s} = 0 \qquad \lim_{p\to\infty} \| b_p^1 - b^1 \|_{C^s} = 0$$
$$\lim_{p\to\infty} \| \varphi_{0-p} - \varphi_{0-} \|_{H^{s+1}} = 0 \qquad \lim_{p\to\infty} \| \psi_{0+p} - \psi_{0+} \|_{H^s} = 0.$$

Now if we replace $a^0$, $a^1$, $b^0$, $b^1$, $\varphi_{0-}$, $\psi_{0+}$ in the problem (5.3.2) by respectively $a_p^0$, $a_p^1$, $b_p^0$, $b_p^1$, $\varphi_{0-p}$, $\psi_{0+p}$, we will get solutions $(\varphi_p, \psi_p)$ by proceeding as we have done in the smooth case. The values of $u_{*p}$, $u_{**p}$ depend on $R$, the upper bound of the norm $C^s$ of $a_p^0$, $a_p^1$, $b_p^0$, $b_p^1$, and of the norm $H^{s+1}$ of $\varphi_{0-p}$, $H^s$ of $\psi_{0+p}$. Hence as we can find $N \in I\!N$, $C$ such that for all $p \geq N$,

$$\| a_p^0 \|_{C^s} \leq C, \qquad \| a_p^1 \|_{C^s} \leq C, \qquad \| b_p^0 \|_{C^s} \leq C, \qquad \| b_p^1 \|_{C^s} \leq C,$$
$$\| \varphi_{0-p} \|_{H^{s+1}} \leq C, \qquad \| \psi_{0+p} \|_{H^s} \leq C,$$

there exists $u_{**}$ such for all $p \geq N$, $(\varphi_p, \psi_p)$ exists on $[0, u_{**}] \times [0, V] \times Y$.

As shown in Section 5.5, for all $0 < s' < s$, $(\varphi_p^k)$ converges to $\varphi_p$ in $H^{s'}$ (by interpolation), and $(\psi_p^k)$ converges to $\psi_p$ in $H^{s'-1}$ (after extracting a subsequence). Thus for all $0 < s' < s$, $(\varphi_p, \psi_p)$ satisfy the inequalities of Lemma 5.4.1 with $s'$ instead of $s$. Moreover $(\varphi_p, \psi_p)$ is in $C^\infty([0, u_{**}] \times [0, V] \times Y)$ and is unique. We shall need the following result :

**Lemma 5.7.1** *For all $r, p \geq N$,*

$$\begin{aligned}
&\max_{0\leq u\leq u_{**}} \| \varphi_r(u) - \varphi_p(u) \|^2_{H^0([0,V]\times Y)} \\
&\leq \| \varphi_{0-r} - \varphi_{0-p} \|^2_{H^0([0,V]\times Y)} + \tilde{c}\big( \max_{0\leq u\leq u_{**}} \| \psi_{0+r}(u) - \psi_{0+p}(u) \|^2_{L^2(Y)} \\
&\quad + \| a_r^0 - a_p^0 \|^2_{C^0(Z)} + \| b_r^0 - b_p^0 \|^2_{C^0(Z)} + \| a_r^1 - a_p^1 \|^2_{C^0(Z)} + \| b_r^1 - b_p^1 \|^2_{C^0(Z)} \big).
\end{aligned}$$

Proof : We proceed similarly as in proof of Lemma 5.4.2 with $\varphi_r - \varphi_p$ instead of $\varphi^{k+1} - \varphi^k$, by writing

$$\begin{aligned}
&\left|\frac{a_r^0}{e_^u}(\varphi_r, u, v, y) - \frac{a_p^0}{e_^u}(\varphi_p, u, v, y)\right| \\
&\qquad = \left|\frac{a_r^0}{e_^u}(\varphi_r, u, v, y) - \frac{a_r^0}{e_^u}(\varphi_p, u, v, y) + \frac{a_r^0}{e_^u}(\varphi_p, u, v, y) - \frac{a_p^0}{e_^u}(\varphi_p, u, v, y)\right| \\
&\qquad \leq \left\| \frac{a_r^0}{e_^u} \right\|_{C^1(Z)} |\varphi_r(u) - \varphi_p(u)| + \left\| \frac{1}{e_^u} \right\|_{C^0(Z)} \| a_r^0 - a_p^0 \|_{C^0(Z)}
\end{aligned}$$

and

$$\left|\frac{a_r^1}{e\_^u}(\varphi_r, u, v, y)\psi_r - \frac{a_p^1}{e\_^u}(\varphi_p, u, v, y)\psi_p\right| \tag{5.7.1}$$
$$\begin{aligned}
&= \left|\frac{a_r^1}{e_^u}(\varphi_r, u, v, y)(\psi_r - \psi_p) + \left(\frac{a_r^1}{e_^u}(\varphi_r, u, v, y) - \frac{a_p^1}{e_^u}(\varphi_p, u, v, y)\right)\psi_p\right| \\
&\leq \left\| \frac{a_r^1}{e_^u} \right\|_{C^0(Z)} |\psi_r(u) - \psi_p(u)| + \left\| \frac{a_r^1}{e_^u} \right\|_{C^1(Z)} |\varphi_r(u) - \varphi_p(u)||\psi_p| \\
&\quad + \left\| \frac{1}{e_^u} \right\|_{C^0(Z)} \| a_r^1 - a_p^1 \|_{C^0(Z)} |\psi_p|
\end{aligned}$$

(Z defined in (5.4.5)). Thus we obtain

$$\begin{aligned}
&\partial_u\Big(\int_{\{u\}\times[0,V]\times Y}|\varphi_r-\varphi_p|^2e^{-\lambda(u+v)}dS\Big)\\
&\le \int_{\{u\}\times[0,V]\times Y}\Big[2<\varphi_r-\varphi_p,-\frac{1}{e_-^{\ u}(\varphi_r)}L\psi_r+\frac{1}{e_-^{\ u}(\varphi_p)}L\psi_p)>+\parallel\frac{1}{e_-^{\ u}}\parallel^2_{C^0(Z)}\parallel a_r^0-a_p^0\parallel^2_{C^0(Z)}\\
&+\parallel\frac{1}{e_-^{\ u}}\parallel^2_{C^0(Z)}\parallel a_r^1-a_p^1\parallel^2_{C^0(Z)}\parallel\psi_p(u)\parallel^2_{C^0(Z'')}+(5+\parallel\frac{a_r^0}{e_-^{\ u}}\parallel^2_{C^1(Z)}\\
&+\parallel\frac{a_r^1}{e_-^{\ u}}\parallel^2_{C^1(Z)}\parallel\psi_p(u)\parallel^2_{C^0(Z'')}-\lambda+\tilde{c}_1)|\varphi_r-\varphi_p|^2\\
&+\parallel\frac{a_r^1}{e_-^{\ u}}\parallel^2_{C^0(Z)}|\psi_r-\psi_p|^2\Big]e^{-\lambda(u+v)}dS
\end{aligned}$$

where $Z''=[0,V]\times Y$. Proceeding as in the proof of Lemma 5.4.2, after using the adjoint of $L$, and analysing the terms involving $b^0$, $b^1$ as those involving $a^0$, $a^1$, we get (if $s-1>n/2+1$ to have the embedding $H^{s-1}\hookrightarrow C^1$ and control $\max\limits_{0\le u\le u_{**}}\parallel\psi_p(u)\parallel_{C^1(Z'')}$,

$$\begin{aligned}
&\partial_u\Big(\int_{\{u\}\times[0,V]\times Y}|\varphi_r-\varphi_p|^2e^{-\lambda(u+v)}dS\Big)\\
&\le\overline{c}\parallel\psi_{0+r}(u)-\psi_{0+p}(u)\parallel^2_{L^2(Y)}+\overline{c}_1\big(\parallel a_r^0-a_p^0\parallel^2_{C^0(Z)}+\parallel b_r^0-b_p^0\parallel^2_{C^0(Z)}\big)\\
&\quad+\overline{c}_2\big(\parallel a_r^1-a_p^1\parallel^2_{C^0(Z)}\parallel\psi_p(u)\parallel^2_{C^0(Z'')}+\parallel b_r^1-b_p^1\parallel^2_{C^0(Z)}\big)\\
&\quad+\int_{\{u\}\times[0,V]\times Y}\big[(\overline{c}_3-\lambda)|\varphi_r-\varphi_p|^2+(\overline{c}_4-\lambda\overline{c}_5)|\psi_r-\psi_p|^2\big]e^{-\lambda(u+v)}dS\ .
\end{aligned}$$

Hence by choosing $\lambda$ large enough, and integrating in $u$ we obtain the result of the lemma. □

This lemma implies that $\big(\varphi_p(u)\big)$ is a Cauchy sequence, and so is converging, in $H^0([0,V]\times Y)$. Then by interpolation we get that $\big(\varphi_p(u)\big)$ converges to $\varphi(u)$ in $H^{s'}([0,V]\times Y)$ for all $0<s'<s$. To get convergence of $\big(\psi_p\big)$, we proceed as we have done in the smooth case with $\psi_p$ instead of $\psi^k$. Thus, by following the argument of the Section 5.5 in the smooth case but with $\psi_p$ instead of $\psi^k$, $\varphi_p$ instead of $\varphi^k$, we get that $(\varphi,\psi)$ satisfy (5.2.4) and (5.2.5). Uniqueness and regularity are also obtained by the same method as in the smooth case. We simply note that we must assume $s>n/2+j+2$ to get $(\varphi,\psi)$ in $C^j([0,u_{**}]\times[0,V]\times Y)$.

We repeat the argument on $0<V<v_{\max}$ as much as necessary (the uniqueness holds by the same argument as in section 5.5).

**Theorem 5.7.1** *If $a^0$, $a^1$, $b^0$, $b^1$ are $C^s$, $\varphi_{0-}$ is $H^{s+1}$, $\psi_{0+}$ is $H^s$, with $s>n/2+2$, then there exists a unique solution $(\varphi,\psi)$ of the problem (5.2.4)-(5.2.7) in* $C^0\big(\bigcup\limits_{0<V<v_{\max}}[0,u_{**}(V)]\times[0,V]\times Y\big)$. *Moreover for all $V>0$, $u\mapsto\parallel\varphi(u)\parallel_{H^s([0,V]\times Y)}$, $u\mapsto\parallel\psi(u)\parallel_{H^{s-1}([0,V]\times Y)}$ are uniformly bounded on $[0,u_{**}(V)]$.*

*Moreover if $s>n/2+j+2$, then $(\varphi,\psi)$ are in* $C^j\big(\bigcup\limits_{0<V<v_{\max}}[0,u_{**}(V)]\times[0,V]\times Y\big)$.

## 5.8 $e_+$ and $e_-$ with components tangential to $N^+ \cap N^-$

Now we consider the problem (5.2.4)-(5.2.7) with

$$e_- := e_-{}^u \partial_u + \sum_j e_-{}^{q_j} q_j \ , \tag{5.8.1}$$

and

$$e_+ := e_+{}^v \partial_v + \sum_j e_+{}^{q_j} q_j \tag{5.8.2}$$

where $e_-{}^{q_j}$, $e_+{}^{q_j}$ can also depend on $\varphi$.

We sketch again the proof of Lemma 5.4.1 but with the new definitions of $e_+$, $e_-$. The equality (5.4.1) still holds if we add in its right-hand-side the term

$$\begin{aligned}
&-\frac{2}{e_-{}^u} < \varphi^k - \varphi_{0-}, e_-{}^{q_j} q_j(\varphi^k - \varphi_{0-}) > -\frac{2}{e_-{}^u} < \varphi^k - \varphi_{0-}, e_-{}^{q_j} q_j(\varphi_{0-}) > \\
&\quad =: I + II
\end{aligned}$$

under the weighted integral on $\{u\} \times [0, V] \times Y$. We integrate $I$ by parts, indeed notice that if $X$ is a vector field, $f$ a function, we have

$$\int_\Omega 2 < \phi, X(\phi) > f dVol = \int_{\partial\Omega} |\phi|^2 f X \cdot n dS - \int_\Omega |\phi|^2 \big(X(f) + f \mathrm{div} X\big) dVol$$

where $n$ is the exterior normal to $\partial\Omega$. Here as we apply this with $\Omega = Y$, and as $Y$ has no boundary, it will give (the sums on $j$ are implicit in all this section)

$$\begin{aligned}
&\int_{\{u\}\times[0,V]\times Y} \frac{2}{e_-{}^u} < \varphi^k - \varphi_{0-}, e_-{}^{q_j} q_j(\varphi^k - \varphi_{0-}) > e^{-\lambda(u+v)} \hat{c} dv \, d\mu_Y \\
&= \int_{\{u\}\times[0,V]\times Y} |\varphi^k - \varphi_{0-}|^2 \big[ q_j \big( \frac{2}{e_-{}^u} e_-{}^{q_j} e^{-\lambda(u+v)} \hat{c} \big) + \frac{2}{e_-{}^u} e_-{}^{q_j} e^{-\lambda(u+v)} \hat{c} \ \mathrm{div}(q_j) \big] dv \, d\mu_Y \\
&\leq \int_{\{u\}\times[0,V]\times Y} c_1(R) |\varphi^k - \varphi_{0-}|^2 e^{-\lambda(u+v)} dS \ .
\end{aligned}$$

For $II$ notice that

$$\begin{aligned}
&\int_{\{u\}\times[0,V]\times Y} -\frac{2}{e_-{}^u} < \varphi^k - \varphi_{0-}, e_-{}^{q_j} q_j(\varphi_{0-}) > e^{-\lambda(u+v)} dS \\
&\leq \int_{\{u\}\times[0,V]\times Y} c_2(R) |\varphi^k - \varphi_{0-}|^2 e^{-\lambda(u+v)} dS + c_3(R) \parallel \varphi_{0-} \parallel^2_{H^1([0,V]\times Y)} \ .
\end{aligned}$$

At this step we see that the $q_j$'s components of $e_-$ won't change the principle of the proof. Furthermore, as $e_+$ has $q_j$'s components, in (5.4.4) we will have more at the right side

$$\int_{\{u\}\times[0,V]\times Y} \frac{2}{e_-{}^u} < e_+{}^{q_j} q_j(\psi^k), \psi^k > e^{-\lambda(u+v)} dS$$

which can be bounded as above by

$$\int_{\{u\}\times[0,V]\times Y} c_5(R)|\psi^k|^2 e^{-\lambda(u+v)} dS \ .$$

For higher derivatives, it is similar. Hence the first inequality of Lemma 5.4.1 is satisfied.

Furthermore, as the first inequality of Lemma 5.4.1 involves directly the second one, the second inequality of Lemma 5.4.1 is also satisfied.

For the third one, (5.4.9) still holds if we replace $e_+$ by $e_+{}^v\partial_v$ at the second line and if we add at the third line a term $-\frac{2}{e_+v} < e_+{}^{q_j} q_j(\psi^k), \psi^k >$ under the integral. We can treat this term by integrating by parts as we have already done. We proceed similarly for higher derivatives and so the third inequality of Lemma 5.4.1 is satisfied.

Now we look at the proof of Lemma 5.4.2. We see that (5.4.10) still holds if we replace $e_-$ by $e_-{}^u\partial_u$, and that (5.4.11) is satisfied if we add in its right member under the weighted integral a term

$$< \varphi^{k+1} - \varphi^k, -\frac{2}{e_-{}^u(\varphi^k)} e_-{}^{q_j}(\varphi^k) q_j(\varphi^{k+1}) + \frac{2}{e_-{}^u(\varphi^{k-1})} e_-{}^{q_j}(\varphi^{k-1}) q_j(\varphi^k) > \ .$$

We write

$$\begin{aligned} &- \frac{2}{e_-{}^u(\varphi^k)} e_-{}^{q_j}(\varphi^k) q_j(\varphi^{k+1}) + \frac{2}{e_-{}^u(\varphi^{k-1})} e_-{}^{q_j}(\varphi^{k-1}) q_j(\varphi^k) \\ &= -\frac{2}{e_-{}^u(\varphi^k)} e_-{}^{q_j}(\varphi^k)(q_j(\varphi^{k+1}) - q_j(\varphi^k)) \\ &\quad - \Big(\frac{2}{e_-{}^u(\varphi^k)} e_-{}^{q_j}(\varphi^k) - \frac{2}{e_-{}^u(\varphi^{k-1})} e_-{}^{q_j}(\varphi^{k-1})\Big) q_j(\varphi^k) \ . \end{aligned}$$

Then for the term

$$\int_{\{u\}\times[0,V]\times Y} < \varphi^{k+1} - \varphi^k, -\frac{2}{e_-{}^u(\varphi^k)} e_-{}^{q_j}(\varphi^k)(q_j(\varphi^{k+1}) - q_j(\varphi^k)) > e^{-\lambda(u+v)} dS \ ,$$

we integrate it by parts as we have done above but with $\varphi^{k+1} - \varphi^k$ instead of $\varphi^k - \varphi_{0-}$, we can bound it by

$$\int_{\{u\}\times[0,V]\times Y} c_6(R)|\varphi^{k+1} - \varphi^k|^2 e^{-\lambda(u+v)} dS \ .$$

For the remainder, we have

$$\begin{aligned} &\int_{\{u\}\times[0,V]\times Y} < \varphi^{k+1} - \varphi^k, \\ &\quad - \Big(\frac{2}{e_-{}^u(\varphi^k)} e_-{}^{q_j}(\varphi^k) - \frac{2}{e_-{}^u(\varphi^{k-1})} e_-{}^{q_j}(\varphi^{k-1})\Big) q_j(\varphi^k) > e^{-\lambda(u+v)} dS \\ &\le \int_{\{u\}\times[0,V]\times Y} \big[|\varphi^{k+1} - \varphi^k|^2 + c_7(R)|\varphi^k - \varphi^{k-1}|^2\big] e^{-\lambda(u+v)} dS \ . \end{aligned}$$

Now to complete the proof of Lemma 5.4.2, it remains to add in the right member of (5.4.12) under the weighted integral a term

$$< -\frac{2}{e_-{}^u(\varphi^k)} e_+{}^{q_j}(\varphi^k) q_j(\psi^{k+1}) + \frac{2}{e_-{}^u(\varphi^{k-1})} e_+{}^{q_j}(\varphi^{k-1}) q_j(\psi^k), \psi^{k+1} - \psi^k > \ .$$

and to treat it in the same way as above.

Furthermore, as the $q_j$'s are tangent to $N_u^-$, the convergence of $(\varphi^k, \psi^k)$ is obtained as before.

The uniqueness of $\varphi$ holds by the same argument as previously. For the uniqueness of $\psi$, we can notice that if $(\varphi, \psi_1)$, $(\varphi, \psi_2)$ are two solutions of the problem, we have by (5.2.5)

$$e_+{}^v(\varphi)\partial_v(\psi_1 - \psi_2) + e_+{}^{q_j}(\varphi)q_j(\psi_1 - \psi_2) = b^1(\varphi)(\psi_1 - \psi_2)\,.$$

But we can always choose a local coordinate system $(w_1, \ldots, w_{n+1})$ such that $\partial_{w_1} = e_+{}^v(\varphi)\partial_v + e_+{}^{q_j}(\varphi)q_j = e_+$, thus $\psi_1 - \psi_2$ satisfies a linear ODE. As $N^+ = \{P(u, v, y); \tilde{f}(u, v, y) = 0\}$ with $\tilde{f}(u, v, y) = v$, by writing $\tilde{f}$ locally in variables $(w_i)$, we get that $N^+ = \{P(w_1, \ldots, w_{n+1}); \tilde{f}(w_1, \ldots, w_{n+1}) = 0\}$. The fact that $\frac{\partial \tilde{f}}{\partial w_1} \neq 0$ is assured by $e_+{}^v(\varphi) \neq 0$ which implies that $\partial_{w_1}$ has always a component in $\partial_v$ and so is always transverse to $N_+$. Now as $\tilde{f}$ is $C^1$ and $\frac{\partial \tilde{f}}{\partial w_1} \neq 0$, the implicit functions theorem gives that there exists a function $\hat{f}$ such that $w_1 = \hat{f}(w_2, \ldots, w_{n+1})$ hence $N^+ = \{P(w_1, \ldots, w_{n+1}); w_1 = \hat{f}(w_2, \ldots, w_{n+1})\}$. By solving the linear ODE with respect to $w_1$, we obtain that locally, in a neighborhood of $N_+$,

$$\begin{aligned}
&(\psi_1 - \psi_2)(w_1, \ldots, w_{n+1}) = \\
&\quad (\psi_1 - \psi_2)(\hat{f}(w_2, \ldots, w_{n+1}), w_2, \ldots, w_{n+1}) e^{\int_{\hat{f}(w_2,\ldots,w_{n+1})}^{w_1} b^1(\varphi)(s, w_2, \ldots, w_{n+1})ds}\,.
\end{aligned}$$

As $(\psi_1 - \psi_2)(\hat{f}(w_2, \ldots, w_{n+1}), w_2, \ldots, w_{n+1})$ is $\psi_1 - \psi_2$ on $N^+$ it vanishes, and so $\psi_1 - \psi_2$ vanishes in a neighborhood of $N^+$, in particular on $N_{\tilde{v}}^+$ for a suitable $\tilde{v} > 0$. As we work on compact sets, we can choose a finite number of local coordinate system $(w_1, \ldots, w_{n+1})$ and start again to solve the linear ODE but with initial values on $N_{\tilde{v}}^+$ instead of $N^+$. In this way we see that the uniqueness of $\psi$ still holds.

For the regularity the fact that $\varphi$ is in $C^0\big([0, u_{**}] \times [0, V] \times Y\big)$ is obtained as before. Then, as we have written it above, $\psi$ satisfies a local linear ODE with functions continuous in all their variables, thus $\psi$ is continuous in all its variables (we start with $\psi_+$ as initial values which is continuous in all its variables to obtain $\psi$ continuous in a neighborhood of $N^+$, then we do it again as much as necessary). Furthermore all works similarly and we can say that $(\varphi, \psi)$ is a smooth solution on $[0, u_{**}] \times [0, V] \times Y$ if the given functions are $C^\infty$.

The argument which permits to get the solution under less stringent differentiability conditions is analogous as previously. We repeat the argument on $0 < V < v_{\max}$. It leads to the following theorem.

**Theorem 5.8.1** *Let in (5.2.4)-(5.2.5),*

$$\begin{aligned}
e_- &= e_-{}^u\partial_u + \sum_j e_-{}^{q_j}q_j \\
e_+ &= e_+{}^v\partial_v + \sum_j e_+{}^{q_j}q_j
\end{aligned}$$

*with $e_-{}^{q_j}$, $e_+{}^{q_j}$ eventually depending on $\varphi$, Let $s > n/2 + 2$, if $a^0$, $a^1$, $b^0$, $b^1$ are $C^s$, $\varphi_{0-}$ is $H^{s+1}$, $\psi_{0+}$ is $H^s$, there exists a unique solution $(\varphi, \psi)$ of the problem (5.2.4)-(5.2.7)*

*on* $C^0\big(\underset{0<V<v_{\max}}{\cup}[0,u_{**}(V)]\times[0,V]\times Y\big)$, *moreover for all* $V>0$, $u\mapsto\|\varphi(u)\|_{H^s([0,V]\times Y)}$, $u\mapsto\|\psi(u)\|_{H^{s-1}([0,V]\times Y)}$ *are uniformly bounded on* $[0,u_{**}(V)]$.
*Moreover if* $s>n/2+j+2$, *then* $(\varphi,\psi)$ *are in* $C^j\big(\underset{0<V<v_{\max}}{\cup}[0,u_{**}(V)]\times[0,V]\times Y\big)$.

## 5.9 Appendix : Higher derivatives estimates in the proof of Lemma 5.4.1

### 5.9.1 First inequality :

To obtain the higher order energy estimate in Lemma 5.4.1 we will proceed similarly as with the $L^2$ estimate, with all the composition of $|\gamma|$ vector fields in $\{\partial_v,q_1,\dots,q_r\}$ of $\varphi^k$ replacing $\varphi^k$ $(1\le|\gamma|\le s)$. At the left side, we will always have a part of the norm $H^s([0,V]\times Y)$ of $\varphi^k(\tau)$, at the right side the role of $\lambda$ will be to absorb all which contains $\psi^k$ and its compositions with $\partial_v,q_1,\dots,q_r$ by choosing a $\lambda$ large enough. Thus it will remain at the right side a sum of a constant and of the integral with respect to $u$ on $[0,\tau]$ of a part of the norm $H^s([0,V]\times Y)$ of $\varphi^k(u)$. Then we will add all these inequalities and we will apply the linear Gronwall lemma.

So to get the whole energy, we must restart with $\partial_v^{\gamma_1}\circ q^{\gamma_2}(\varphi^k)$ (and all the possible commutations of $\partial_v$ and $q_i$ $(1\le i\le r)$) instead of $\varphi^k$ for all $0\le\gamma_1+|\gamma_2|\le s$. We notice that vector fields $\partial_v$, $e_-$, $q_1$, … , $q_r$ don't commute with each other. Here we just detail the estimate with $\partial_v^{\gamma_1}\circ q^{\gamma_2}(\varphi^k)$ (for any vector field $X$, any $\alpha$ in $\mathbb{N}$, $X^\alpha=X\circ\cdots\circ X$ $\alpha$-times), but it is exactly the same way for any commutation of $\partial_v$ and $q_i$ $(1\le i\le r)$ in $\partial_v^{\gamma_1}\circ q^{\gamma_2}(\varphi^k)$. By taking again (5.4.1), we obtain

$$\begin{aligned}
&\partial_u\Big(\int_{\{u\}\times[0,V]\times Y}\big|\partial_v^{\gamma_1}\circ q^{\gamma_2}(\varphi^k-\varphi_{0-})\big|^2e^{-\lambda(u+v)}dS\Big)\\
&\quad=\int_{\{u\}\times[0,V]\times Y}\Big[\frac{2}{e_-^{\;u}}<\partial_v^{\gamma_1}\circ q^{\gamma_2}(\varphi^k-\varphi_{0-}),e_-\circ\partial_v^{\gamma_1}\circ q^{\gamma_2}(\varphi^k)>\\
&\qquad+(-\lambda+\tilde{c_1})\big|\partial_v^{\gamma_1}\circ q^{\gamma_2}(\varphi^k-\varphi_{0-})\big|^2\Big]e^{-\lambda(u+v)}dS.
\end{aligned}$$

For more lightness in the following we denote $\hat{q}^\gamma=\partial_v^{\gamma_1}\circ q^{\gamma_2}$. As we want to use the first equation of (5.3.2), we need to commute $e_-$ with $\hat{q}^\gamma$. Notice that

$$e_-\circ\hat{q}^\gamma=\hat{q}^\gamma\circ e_-+\sum_{0\le|\gamma'|\le|\gamma|-1}\kappa_{\gamma'}\hat{q}^{\gamma'}\circ e_-+\sum_{1\le|j|\le|\gamma|}\kappa'_j\hat{q}^j\ ,$$

where $\kappa_{\gamma'}$'s, $\kappa'_j$'s are smooth functions depending on $\varphi^{k-1}$ multiplied by partial derivatives until order $|\gamma|$ of $\varphi^{k-1}$, as $e_-$ depends on $\varphi^{k-1}$. Thus we get

$$\int_{\{u\}\times[0,V]\times Y} \frac{2}{e_-^{\,u}} < \hat{q}^\gamma(\varphi^k - \varphi_{0-}), e_- \circ \hat{q}^\gamma(\varphi^k) > e^{-\lambda(u+v)} dS$$
$$\leq \int_{\{u\}\times[0,V]\times Y} \frac{2}{e_-^{\,u}} \Big[ < \hat{q}^\gamma(\varphi^k - \varphi_{0-}), \hat{q}^\gamma \circ e_-(\varphi^k) >$$
$$+ \sum_{0\leq|\gamma'|\leq|\gamma|-1} < \hat{q}^\gamma(\varphi^k - \varphi_{0-}), \kappa_{\gamma'} \hat{q}^{\gamma'} \circ e_-(\varphi^k) >$$
$$+ \sum_{1\leq|j|\leq|\gamma|} < \hat{q}^\gamma(\varphi^k - \varphi_{0-}), \kappa'_j \hat{q}^j(\varphi^k - \varphi_{0-}) >$$
$$+ \sum_{1\leq|j|\leq|\gamma|} < \hat{q}^\gamma(\varphi^k - \varphi_{0-}), \kappa'_j \hat{q}^j(\varphi_{0-}) > \Big] e^{-\lambda(u+v)} dS.$$

By using the following Gagliardo-Nirenberg-Moser inequality

$$\| f_1^{(\beta_1)} \dots f_\mu^{(\beta_\mu)} \|_{L^2} \leq c \sum_\nu \big( \prod_{i\neq\nu} \| f_i \|_{L^\infty} \big) \| f_\nu \|_{H^s},$$

where $|\beta_1| + \cdots + |\beta_\mu| = s$, we can see that ($s > n/2$ implies $H^s \hookrightarrow L^\infty$ so we control the norm $L^\infty$ and the norm $H^s$ of $\varphi^{k-1}$ by the induction's assumption)

$$\int_{\{u\}\times[0,V]\times Y} \frac{2}{e_-^{\,u}} \Big[ \sum_{1\leq|j|\leq|\gamma|} < \hat{q}^\gamma(\varphi^k - \varphi_{0-}), \kappa'_j \hat{q}^j(\varphi^k - \varphi_{0-}) >$$
$$+ \sum_{1\leq|j|\leq|\gamma|} < \hat{q}^\gamma(\varphi^k - \varphi_{0-}), \kappa'_j \hat{q}^j(\varphi_{0-}) > \Big] e^{-\lambda(u+v)} dS$$
$$\leq \tilde{c}_5(R) \big( \| \varphi^k(u) - \varphi_{0-} \|^2_{H^{|\gamma|}([0,V]\times Y)} + \| \varphi^k(u) - \varphi_{0-} \|^2_{L^\infty([0,V]\times Y)}$$
$$+ \| \varphi_{0-} \|^2_{H^{|\gamma|}([0,V]\times Y)} + \| \varphi_{0-} \|^2_{L^\infty([0,V]\times Y)} \big).$$

Now by the first equation of (5.3.2), we get

$$\int_{\{u\}\times[0,V]\times Y} \frac{2}{e_-^{\,u}} < \hat{q}^\gamma(\varphi^k - \varphi_{0-}), \kappa_{\gamma'} \hat{q}^{\gamma'} \circ e_-(\varphi^k - \varphi_{0-}) > e^{-\lambda(u+v)} dS$$
$$= \int_{\{u\}\times[0,V]\times Y} \frac{2}{e_-^{\,u}} < \hat{q}^\gamma(\varphi^k - \varphi_{0-}), \kappa_{\gamma'} \hat{q}^{\gamma'} \big( - L\psi^k + a^0(\varphi^{k-1}, u, v, y)$$
$$+ a^1(\varphi^{k-1}, u, v, y)\psi^k \big) > e^{-\lambda(u+v)} dS.$$

On one hand, by using the following Moser inequality

$$\| F(f) \|_{H^s} \leq c(\| F \|_{C^s}, \| f \|_{L^\infty})(1 + \| f \|_{H^s}),$$

we can write

$$\int_{\{u\}\times[0,V]\times Y} \frac{2}{e_-^{\,u}} < \hat{q}^\gamma(\varphi^k - \varphi_{0-}), \kappa_{\gamma'} \hat{q}^{\gamma'} \big( a^0(\varphi^{k-1}, u, v, y)$$
$$+ a^1(\varphi^{k-1}, u, v, y)\psi^k \big) > e^{-\lambda(u+v)} dS$$
$$\leq \overline{c}(R) \| \varphi^k(u) - \varphi_{0-} \|^2_{H^{|\gamma|}([0,V]\times Y)}$$
$$+ c(\| a^0 \|_{C^{|\gamma|}}, \| a^1 \|_{C^{|\gamma|}}, R) \big( 1 + \| \psi^k(u) \|^2_{H^{|\gamma|}([0,V]\times Y)} \big)$$

(we can assume, as we need it further, $c(\| a^0 \|_{C^{|\gamma|}}, \| a^1 \|_{C^{|\gamma|}}, R) > 0$). On another hand for the term with $L$, if we set $\hat{q}^{\gamma'}\big(-L\psi^k\big) = \tilde{\kappa}_{\gamma''}\hat{q}^{\gamma''}(\psi^k)$ with $|\gamma''| = |\gamma'| + 1$ and $\tilde{\kappa}_{\gamma''}$'s smooth functions, depending on $\varphi^{k-1}$, we have for any $0 \leq |\gamma'| \leq |\gamma| - 1$,

$$\begin{aligned}
&\int_{\{u\}\times[0,V]\times Y} \frac{2}{e_-^{\ u}} < \hat{q}^{\gamma}(\varphi^k - \varphi_{0-}), \kappa_{\gamma'}\hat{q}^{\gamma'}\big(-L\psi^k\big) > e^{-\lambda(u+v)}dS \\
&= \int_{\{u\}\times[0,V]\times Y} \frac{2}{e_-^{\ u}} < \kappa_{\gamma'}\tilde{\kappa}_{\gamma''}\hat{q}^{\gamma}(\varphi^k - \varphi_{0-}), \hat{q}^{\gamma''}\big(\psi^k\big) > e^{-\lambda(u+v)}dS \\
&\leq \tilde{c}_6(R)\big( \| \varphi^k(u) - \varphi_{0-} \|^2_{H^{|\gamma|}([0,V]\times Y)} + \| \varphi^k(u) - \varphi_{0-} \|^2_{L^\infty([0,V]\times Y)} \big) \\
&\quad + \overline{c}(R) \| \psi^k(u) \|^2_{H^{|\gamma|}([0,V]\times Y)} .
\end{aligned}$$

For the term
$< \hat{q}^{\gamma}(\varphi^k - \varphi_{0-}), \hat{q}^{\gamma}(-L\psi^k) >$ under the integral on $\{u\} \times [0,V] \times Y$, we want to use the second equation of (5.3.2), for that we notice that

$$\begin{aligned}
&<\hat{q}^{\gamma}(\varphi^k - \varphi_{0-}), \hat{q}^{\gamma}(-L\psi^k) > \\
&\quad = - < \hat{q}^{\gamma}(\varphi^k - \varphi_{0-}), L\, \hat{q}^{\gamma}(\psi^k) > - < \hat{q}^{\gamma}(\varphi^k - \varphi_{0-}), \sum_{\substack{|\mu|+|\nu|=|\gamma| \\ \mu \neq 0}} \breve{\kappa}_{\mu,\nu}\hat{q}^{\mu}(L)\hat{q}^{\nu}(\psi^k) >
\end{aligned}$$

where $\breve{\kappa}_{\mu,\nu}$'s are smooth functions, depending on $\varphi^{k-1}$. Hence (the sum over $j$ is implicit)

$$\begin{aligned}
&\int_{\{u\}\times[0,V]\times Y} \frac{2}{e_-^{\ u}} < \hat{q}^{\gamma}(\varphi^k - \varphi_{0-}), \hat{q}^{\gamma}(-L\psi^k) > e^{-\lambda(u+v)}dS \\
&= \int_{\{u\}\times[0,V]\times Y} q_j\Big(\frac{2}{e_-^{\ u}}\Big) < -(A^j)^*(\hat{q}^{\gamma}(\varphi^k - \varphi_{0-})), \hat{q}^{\gamma}(\psi^k) > e^{-\lambda(u+v)}dS \\
&+ \int_{\{u\}\times[0,V]\times Y} \frac{2}{e_-^{\ u}} \Big[ - < L^*\hat{q}^{\gamma}(\varphi^k - \varphi_{0-}), \hat{q}^{\gamma}(\psi^k) > \\
&\quad - \sum_{\substack{|\mu|+|\nu|=|\gamma| \\ \mu \neq 0}} < \breve{\kappa}_{\mu,\nu}\overline{\kappa}_{\mu}\hat{q}^{\gamma}(\varphi^k - \varphi_{0-}), \hat{q}\circ\hat{q}^{\nu}(\psi^k) > \Big] e^{-\lambda(u+v)}dS ,
\end{aligned}$$

where $\overline{\kappa}_{\mu}$'s are smooth functions, depending on $\varphi^{k-1}$, $\hat{q}^{\mu}(L) = \overline{\kappa}_{\mu}\hat{q}$. Now

$$\begin{aligned}
&- < L^*\hat{q}^{\gamma}(\varphi^k - \varphi_{0-}), \hat{q}^{\gamma}(\psi^k) > \\
&\quad = - < \hat{q}^{\gamma}(L^*\varphi^k - \varphi_{0-}), \hat{q}^{\gamma}(\psi^k) > + < \sum_{\substack{|\mu|+|\nu|=|\gamma| \\ \mu \neq 0}} \breve{\kappa}_{\mu,\nu}\overline{\kappa}_{\mu}\hat{q}\circ\hat{q}^{\nu}(\varphi^k - \varphi_{0-}), \hat{q}^{\gamma}(\psi^k) > .
\end{aligned}$$

But, as $\mu \neq 0$, $|\nu| \leq |\gamma| - 1$, and so $\hat{q}\circ\hat{q}^{\nu}$ is a derivative of degree less than or equal to $|\gamma|$, hence

$$\begin{aligned}
&\int_{\{u\}\times[0,V]\times Y} -\frac{2}{e_-^{\ u}} \sum_{\substack{|\mu|+|\nu|=|\gamma'| \\ \mu \neq 0}} < \breve{\kappa}_{\mu,\nu}\overline{\kappa}_{\mu}\hat{q}^{\gamma}(\varphi^k - \varphi_{0-}), \hat{q}\circ\hat{q}^{\nu}(\psi^k) > e^{-\lambda(u+v)}dS \\
&\leq \tilde{c}_7(R)\big( \| \varphi^k(u) - \varphi_{0-} \|^2_{H^{|\gamma|}([0,V]\times Y)} + \| \varphi^k(u) - \varphi_{0-} \|^2_{L^\infty([0,V]\times Y)} \big) \\
&\quad + \overline{c}(R) \| \psi^k(u) \|^2_{H^{|\gamma|}([0,V]\times Y)} .
\end{aligned}$$

Similarly,

$$\int_{\{u\}\times[0,V]\times Y} \frac{2}{e_-^u} < \sum_{\substack{|\mu|+|\nu|=|\gamma| \\ \mu\neq 0}} \breve{\kappa}_{\mu,\nu}\overline{\kappa}_\mu \hat{q}\circ\hat{q}^\nu(\varphi^k-\varphi_{0-}), \hat{q}^\gamma(\psi^k) > e^{-\lambda(u+v)}dS$$
$$\leq \tilde{c}_7(R)\big( \| \varphi^k(u)-\varphi_{0-} \|^2_{H^{|\gamma|}([0,V]\times Y)} + \| \varphi^k(u)-\varphi_{0-} \|^2_{L^\infty([0,V]\times Y)} \big)$$
$$+ \overline{c}(R) \| \psi^k(u) \|^2_{H^{|\gamma|}([0,V]\times Y)} .$$

Recall that with (5.4.6)

$$|(A^j)^*(\hat{q}^\gamma(\varphi^k-\varphi_{0-}))| \leq \||(A^j)^*\||_{C^0(Z)}|\hat{q}^\gamma(\varphi^k-\varphi_{0-})|.$$

Finally, for the moment, we have (as $s > n/2$ we can bound the norm $L^\infty([0,V]\times Y)$ by a constant multiplying the norm $H^s([0,V]\times Y)$)

$$\partial_u\big( \int_{\{u\}\times[0,V]\times Y} \big|\hat{q}^\gamma(\varphi^k-\varphi_{0-})\big|^2 e^{-\lambda(u+v)}dS\big)$$
$$\leq \int_{\{u\}\times[0,V]\times Y} \frac{2}{e_-^u} < \hat{q}^\gamma(-L^*\varphi^k), \hat{q}^\gamma(\psi^k) > e^{-\lambda(u+v)}dS + \tilde{c}_8(R) \| \varphi_{0-} \|^2_{H^{s+1}([0,V]\times Y)}$$
$$+ (-\lambda+\tilde{c}_9) \| \varphi^k(u)-\varphi_{0-} \|^2_{H^s([0,V]\times Y)} \tilde{c}_{10}(R) \| \psi^k(u) \|^2_{H^s([0,V]\times Y)} .$$

**Remark 5.9.1** The above inequality shows that when we will work with less stringent differentiability, we will have to assume that $\varphi_{0-}$ is in a Sobolev space of one degree more than $\psi_{0+}$.

Now by using the second equation of (5.3.2), and Moser inequality, we obtain

$$\int_{\{u\}\times[0,V]\times Y} \frac{2}{e_-^u} < \hat{q}^\gamma(-L^*\varphi^k), \hat{q}^\gamma(\psi^k) > e^{-\lambda(u+v)}dS$$
$$= \int_{\{u\}\times[0,V]\times Y} \frac{2}{e_-^u} < \hat{q}^\gamma\big( - e_+(\psi^k) + b^0(\varphi^{k-1},u,v,y) + b^1(\varphi^{k-1},u,v,y)\psi^k\big),$$
$$\hat{q}^\gamma(\psi^k) > e^{-\lambda(u+v)}dS$$
$$\leq \int_{\{u\}\times[0,V]\times Y} -\frac{2}{e_-^u} < \hat{q}^\gamma\circ e_+(\psi^k), \hat{q}^\gamma(\psi^k) > e^{-\lambda(u+v)}dS$$
$$+c(\| b^0 \|_{C^{|\gamma|}}, \| b^1 \|_{C^{|\gamma|}}, R)\big(1+ \| \psi^k(u) \|^2_{H^{|\gamma|}([0,V]\times Y)}\big) .$$

Furthermore, by writing

$$\hat{q}^\gamma\circ e_+ = e_+\circ\hat{q}^\gamma + \sum_{1\leq|j|\leq|\gamma|} \tilde{\kappa}_j\hat{q}^j$$

where $\tilde{\kappa}_j$'s are smooth functions depending on $\varphi^{k-1}$, and by integrating by parts with respect to $v$, as $-|\hat{q}^\gamma(\psi^k)|^2 \frac{e_+^{\ v}}{e_-^{\ u}} e^{-\lambda(u+v)}\hat{c}$ is negative ($\hat{c}$ defined in (5.4.2)), we get

$$\begin{aligned}
&\int_{\{u\}\times[0,V]\times Y} -\frac{2}{e_-^{\ u}} < \hat{q}^\gamma \circ e_+(\psi^k), \hat{q}^\gamma(\psi^k) > e^{-\lambda(u+v)}\hat{c})dv\, d\mu_Y \\
&\leq \int_Y |\hat{q}^\gamma(\psi^k)(u,0,y)|^2 \frac{e_+^{\ v}}{e_-^{\ u}} e^{-\lambda(u)}\hat{c}(u,0,y)d\mu_Y \\
&+ \int_{\{u\}\times[0,V]\times Y} |\hat{q}^\gamma(\psi^k)|^2 \partial_v\big(\frac{e_+^{\ v}}{e_-^{\ u}} e^{-\lambda(u+v)}\hat{c}\big)dvd\mu_Y \\
&\quad +\tilde{c}_{11}(R) \parallel \psi^k(u) \parallel^2_{H^{|\gamma|}([0,V]\times Y)} \\
&\leq \tilde{c}(R)\int_Y |\hat{q}^\gamma(\psi^k)(u,0,y)|^2 d\mu_Y \\
&+\big(\tilde{c}_{12}(R) - \lambda \min_Z(\frac{e_+^{\ v}}{e_-^{\ u}})\big) \parallel \psi^k(u) \parallel^2_{H^{|\gamma|}([0,V]\times Y)} .
\end{aligned} \tag{5.9.1}$$

We show that $\hat{q}^\gamma(\psi^k)(u,0,y)$ is independent of $k$. Indeed, on one hand, as vector fields $q_i$ are tangent to $N^+$, $q^{\gamma_j}(\psi^k)(u,0,y)$ $(1 \leq |\gamma_j| \leq s-1)$ is given by $\psi_{0+}$. On another hand notice that $(\partial_v\psi^k)(u,0,y)$ can be calculated with the second equation of (5.3.2) on $N^+$, namely

$$\begin{aligned}
&-L^*\varphi^k(u,0,y) + e_+^{\ v}(\varphi^{k-1})(\partial_v\psi^k)(u,0,y) \\
&\quad = b^0(\varphi^{k-1}(u,0,y),u,0,y) + b^1(\varphi^{k-1}(u,0,y),u,0,y)\psi^k(u,0,y).
\end{aligned}$$

But $\psi^k(u,0,y) = \psi_{0+}(u,y)$ and $\varphi^{k-1}(u,0,y) = \varphi_{0+}(u,y)$ because $\varphi^0$ at $(u,0,y)$ is equal to $\varphi_{0+}$ which is the unique smooth solution of

$$\begin{aligned}
&e_-^{\ u}(\varphi_{0+})\partial_u(\varphi_{0+})(u,y) + L(\varphi_{0+})\psi_{0+}(u,y) \\
&\quad = a^0(\varphi_{0+}(u,y),u,0,y) + a^1(\varphi_{0+}(u,y),u,0,y)\psi_{0+}(u,y)
\end{aligned}$$

such that $\varphi_{0+}(0,y) = \varphi_{0-}(0,y)$, and that $\varphi^k(u,0,y)$ is determined by solving

$$\begin{aligned}
&e_-^{\ u}(\varphi^{k-1})\partial_u(\varphi^k)(u,0,y) + L(\varphi^{k-1})\psi^k(u,0,y) \\
&\quad = a^0(\varphi^{k-1}(u,0,y),u,0,y) + a^1(\varphi^{k-1}(u,0,y),u,0,y)\psi^k(u,0,y).
\end{aligned}$$

Thus $(\partial_v\psi^k)(u,0,y)$ is independent of $k$. Now $q^{\gamma_j}(\partial_v\psi^k))(u,0,y)$ $(1 \leq |\gamma_j| \leq s-2)$ is given by $(\partial_v\psi^k)(u,0,y)$. So we just have to commute $q_j$ and $\partial_v$ in $\hat{q}^\gamma$ and verify that $\partial_v \circ \cdots \circ \partial_v(\psi^k)(u,0,y)$ is independent of $k$. This can be shown by induction, let us detail this for $\partial_v \circ \partial_v(\psi^k)(u,0,y)$, higher orders being similar. First we notice that doing it for $\partial_v \circ \partial_v(\psi^k)(u,0,y)$ is equivalent to doing it for $\partial_v \circ e_+(\psi^k)(u,0,y)$. From the second equation of (5.3.2) we deduce

$$\begin{aligned}
\partial_v \circ e_+(\psi^k) &= \partial_v \circ L^*(\varphi^{k-1})\varphi^k + (\partial_1 b^0)(\varphi^{k-1},\cdot)\partial_v\varphi^{k-1} + (\partial_3 b^0)(\varphi^{k-1},\cdot) \\
&+ [(\partial_1 b^1)(\varphi^{k-1},\cdot)\partial_v\varphi^{k-1} + (\partial_3 b^1)(\varphi^{k-1},\cdot)]\psi^k + b^1(\varphi^{k-1},\cdot)\partial_v\psi^k
\end{aligned}$$

(where $\partial_j$ means partial derivative with respect to the $j^{\text{th}}$ variable). If we commute $\partial_v$ and $L^*$, it will add some terms with $\partial_v\varphi^k$, $\partial_v\varphi^{k-1}$, $L^*\varphi^k$. As we know that $\psi^k(u,0,y)$, $\varphi^k(u,0,y)$, $e_+(\psi^k)(u,0,y)$, $L^*\varphi^k(u,0,y)$ are independent of k, it will be the same for

$\partial_v \circ \partial_v(\psi^k)(u,0,y)$ if $\partial_v(\varphi^k(u,0,y))$ is also independent of k. This follows from the first equation of (5.3.2), from which we deduce

$$\partial_v \circ e_-(\varphi^k) = -\partial_v \circ L(\varphi^{k-1}\psi^k + (\partial_1 a^0)(\varphi^{k-1},\cdot)\partial_v\varphi^{k-1} + (\partial_3 a^0)(\varphi^{k-1},\cdot) \\ + [(\partial_1 a^1)(\varphi^{k-1},\cdot)\partial_v\varphi^{k-1} + (\partial_3 a^1)(\varphi^{k-1},\cdot)]\psi^k + a^1(\varphi^{k-1},\cdot)\partial_v\psi^k.$$

By commuting $\partial_v$ and $e_-$, $\partial_v$ and $L$, and taking the expression at $(u,0,y)$ we see that $(\partial_v\varphi^k)(u,0,y)$ is solution of a linear ODE, namely

$$e_-(\partial_v\varphi^k)(u,0,y) = B(u,y)\partial_v\varphi^k(u,0,y) + C(u,y)\partial_v\varphi^{k-1}(u,0,y) + D(u,y).$$

But $\partial_v\varphi^0$ at $(u,0,y)$ is equal to $\partial_v\varphi_0(u,y)$ where $\partial_v\varphi_0$ is the solution of this ODE with $\partial_v\varphi_0$ instead of $\partial_v\varphi^{k-1}$, namely

$$e_-\big(\partial_v\varphi_0(u,y)\big) = B(u,y)(\partial_v\varphi_0)(u,y) + C(u,y)(\partial_v\varphi_0)(u,y) + D(u,y)$$

with $(\partial_v\varphi_0)(0,y) = (\partial_v\varphi_{0-})(0,y)$, in this way $(\partial_v\varphi^k)(u,0,y)$ is always equal to $(\partial_v\varphi_0)(u,y)$, and so is independent of $k$. By our choice of $\varphi^0$, it will be the same until $\partial_v^{s-1}(\varphi^k)(u,0,y)$ which implies $\partial_v^{\alpha}(\psi^k)(u,0,y)$ is independent of $k$ for all $0 \le \alpha \le s$. Hence $\hat{q}^{\gamma}(\psi^k)(u,0,y)$ is independent of $k$.

We come back to (5.9.1) and now we can write

$$\partial_u\Big(\int_{\{u\}\times[0,V]\times Y} \big(\hat{q}^{\gamma}(\varphi^k - \varphi_{0-})\big)^2 e^{-\lambda(u+v)} dS\Big) \tag{5.9.2}$$
$$\le \tilde{c}_{13}(R) + (-\lambda + \tilde{c}_9(R)) \parallel \varphi^k(u) - \varphi_{0-} \parallel^2_{H^{|\gamma|}([0,V]\times Y)} \\ + (-\lambda \min_Z(\frac{e_+^{\,v}}{e_-^{\,u}}) + \tilde{c}_{14}(R)) \parallel \psi^k(u) \parallel^2_{H^{|\gamma|}([0,V]\times Y)} \ .$$

By choosing $\lambda > 0$ large enough and integrating with respect to $u$, we have

$$\int_{\{\tau\}\times[0,V]\times Y} \big|\hat{q}^{\gamma}(\varphi^k - \varphi_{0-})\big|^2 dS' \le \int_0^{\tau} \big[\tilde{c}_{13}(R) + \parallel \varphi^k(u) - \varphi_{0-} \parallel^2_{H^{|\gamma|}([0,V]\times Y)}\,\big]\, du$$

with $\tilde{c}_{13}(R) > 0$. The sum on $0 \le |\gamma| \le s$ of these inequalities can be written as it follows

$$\parallel \varphi^k(\tau) - \varphi_{0-} \parallel^2_{H^s([0,V]\times Y)} \le \int_0^{\tau} \big[\tilde{c}_{15}(R) + \tilde{c}_{16} \parallel \varphi^k(u) - \varphi_{0-} \parallel^2_{H^s([0,V]\times Y)}\,\big] du \tag{5.9.3}$$

with $\tilde{c}_{15}(R) > 0$.

### 5.9.2 Second inequality :

We proceed similarly as we have done for the estimation of the norm $H^0$ of $\psi^k$, but this time with $\hat{q}^{\gamma}\psi^k$ instead of $\psi^k$ $(1 \le |\gamma| \le s-1)$. On one hand we have for all $0 \le u \le u_*$, $0 \le \tilde{v} \le V$,

$$\int_{\{u\}\times[0,\tilde{v}]\times Y} \partial_v\big(|\hat{q}^{\gamma}(\psi^k)(u,v,y)|^2\hat{c}(u,v,y)\big)\, dv\, d\mu_Y \\ = \int_Y |\hat{q}^{\gamma}(\psi^k)(u,\tilde{v},y)|^2\hat{c}(u,\tilde{v},y)\, d\mu_Y - \int_Y |\hat{q}^{\gamma}(\psi^k)(u,0,y)|^2\hat{c}(u,0,y)\, d\mu_Y \ .$$

On another hand,

$$\begin{aligned}
&\int_{\{u\}\times[0,\tilde{v}]\times Y} \partial_v\big(|\hat{q}^\gamma(\psi^k)(u,v,y)|^2\hat{c}(u,v,y)\big)dv\ d\mu_Y \\
&= \int_{\{u\}\times[0,\tilde{v}]\times Y} \frac{2}{e_+^v} < e_+(\hat{q}^\gamma(\psi^k)), \hat{q}^\gamma(\psi^k) > dS + \int_{\{u\}\times[0,\tilde{v}]\times Y} |\hat{q}^\gamma(\psi^k)|^2 \frac{\partial_v \hat{c}}{\hat{c}} dS.
\end{aligned}$$

To use the second equation of (5.3.2) we need to commute $\hat{q}$ and $e_+$. As we have done in the proof of the first inequality of Lemma 5.4.1, it will give more in the estimation a constant multiplied by the norm $H^{|\gamma|}$ of $\psi^k$. Now

$$\begin{aligned}
&\int_{\{u\}\times[0,\tilde{v}]\times Y} \frac{2}{e_+^v} < \hat{q}^\gamma(e_+(\psi^k)), \hat{q}^\gamma(\psi^k) > dS \\
&= \int_{\{u\}\times[0,\tilde{v}]\times Y} \frac{2}{e_+^v} < \hat{q}^\gamma(L^*\varphi^k + b^0(\varphi^{k-1},u,v,y) + b^1(\varphi^{k-1},u,v,y)\psi^k), \psi^k > dS \\
&\leq \int_{\{u\}\times[0,\tilde{v}]\times Y} \frac{1}{e_+^v}\big(|\hat{q}^\gamma(L^*\varphi^k)|^2 + (\hat{q}^\gamma|b^0(\varphi^{k-1},u,v,y))|^2\big)\frac{e^{-\lambda(u+v)}}{e^{-\lambda(u_*+V)}}dS \\
&\quad + \int_{\{u\}\times[0,\tilde{v}]\times Y} \frac{1}{e_+^v}\big(2|\hat{q}^\gamma(\psi^k)|^2 + < \hat{q}^\gamma(b^1(\varphi^{k-1},u,v,y)\psi^k), \hat{q}^\gamma(\psi^k) > \big)\frac{e^{-\lambda(u+v)}}{e^{-\lambda(u_*+V)}}dS
\end{aligned}$$

**Remark 5.9.2** The fact that we assume here that $|\gamma| \leq s-1$ (instead of $|\gamma| \leq s$) comes from the above inequality in which we must control the norm $L^2$ of $\hat{q}^\gamma(L^*\varphi^k)$.

Thus, as $\hat{q}^\gamma(\psi^k)(u,0,y)$ is independent of $k$ (see proof of first inequality of Lemma 5.4.1), by using again the first inequality of Lemma 5.4.1, and by using Moser inequalities as we have already done, we get

$$\begin{aligned}
&\int_Y |\hat{q}^\gamma(\psi^k(u,\tilde{v},y))|^2\hat{c}(u,\tilde{v},y)e^{-\lambda(u+v)}d\mu_Y \\
&\quad \leq \tilde{c}_{20}(R) + \sum_{0\leq|\gamma_j|\leq|\gamma|} \tilde{c}_{\gamma_j}(R)\int_0^{\tilde{v}}\int_Y |\hat{q}^{\gamma_j}(\psi^k)|^2 e^{-\lambda(u+v)}dS.
\end{aligned}$$

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**Résumé :** Nous considérons des équations d'onde semilinéaires avec données initiales sur deux hypersurfaces caractéristiques transverses. Nous montrons l'existence et l'unicité d'une solution dans un voisinage de la totalité de l'une des hypersurfaces. La première partie traite, dans une métrique plate, d'une équation dont le second membre ne contient pas de gradient. Nous reprenons la méthode de Galerkin avec une décomposition spectrale suivant l'une des directions isotropiques, et des estimations d'énergie dans des espaces de Sobolev avec un nombre de dérivées non homogène. Dans la deuxième partie, le second membre dépend du gradient. Nous travaillons dans une métrique Lorentzienne, avec une méthode itérative et des inégalités d'énergie, obtenues grâce au tenseur d'énergie impulsion, sur des tranches d'espace-temps parallèles à l'une des directions isotropiques, dans des espaces de Sobolev pondérés. La troisième partie présente le même genre de résultat pour un système symétrique hyperbolique quasilinéaire.

**Title :** Characteristic initial value problems for nonlinear wave equations

**Abstract :** We consider semilinear wave equations with initial values on two tranversely intersecting null hypersurfaces. We show existence and uniqueness of a solution in a neighborhood of one of the initial characteristic hypersurfaces, or of both. The first part treats, in a flat metric case, of an equation which has no gradient on its right-hand-side. We deal with the Galerkin's method with a spectral decomposition along one of the isotropic directions and energy estimates in Sobolev spaces which have different orders partial derivatives according to the variables. In the second part the right-hand-side of the equation depends on gradient. We work in a Lorentzian metric, with an iterative method. The energy estimates are obtained by using the energy momentum tensor, on slices of space-time tangential to one of the isotropic directions, in weighted Sobolev spaces. The third part presents similar results for a quasilinear symmetric hyperbolic system.

**Discipline :** Mathématiques

**Mots-clés :** Equations d'onde non linéaires, hypersurfaces caractéristiques, problème de Cauchy, systèmes symétriques hyperboliques quasilinéaires, métrique Lorentzienne, méthode de Galerkin, inégalités d'énergie, espaces de Sobolev, tenseur d'énergie impulsion, résultats d'existence et d'unicité.

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