

Cauchy problem with data on a characteristic cone for the Einstein-Vlasov equations

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Dedicated to Mario Novello

1 Introduction.

In recent papers [2, 3] (henceforth denoted by I and II) we have proved existence and uniqueness theorems for solutions of the Cauchy problem for the Einstein equations in *vacuum* with data on a characteristic cone C_O , generalising a construction of [7, 5]. We have used the tensorial splitting of the Ricci tensor of a Lorentzian metric g on a manifold V as the sum of a quasidiagonal hyperbolic system acting on g and a linear first order operator acting on a vector H , called the wave-gauge vector. The vector H vanishes if g is in wave gauge; that is, if the identity map is a wave map from (V, g) onto (V, \hat{g}) , with \hat{g} some given metric, which we have chosen to be Minkowski. The data needed for the reduced PDEs is the trace \bar{g} of g on C_O , but the geometric initial data is the degenerate quadratic form \tilde{g} induced by g on C_O , the missing part of \bar{g} is determined by a hierarchical system of ordinary differential equations¹, called the wave-map-gauge constraints, along the rays of C_O , deduced from the contraction of the Einstein tensor with a tangent to the rays (it is also possible to prescribe \tilde{g} up to a conformal factor and impose a coordinate condition).

The above generalises in a straightforward manner to Einstein equations coupled to matter fields whose energy-momentum tensor satisfies specific

¹For previous writing of these equations in the case of two intersecting surfaces in four-dimensional spacetime see Rendall [7] and Damour-Schmidt [5].

structure conditions. This is the case for Maxwell fields and scalar fields. However, it is not immediately clear that kinematic matter sources described by a Vlasov distribution field f fit into this scheme. In fact, there is a basic issue arising, related to the fact that the support of the distribution function f has to be contained within the subset of the tangent bundle where the momentum of the particles is timelike future pointing; outside of this region the Vlasov particles are *tachyons*. But a *no-tachyons condition* requires *a priori* knowledge of the whole metric on the initial data hypersurface, while in the wave-map scheme above only part of the metric is known before the constraints are solved, the remaining components being determined by ODEs along the generators of the hypersurface. It has been shown in [4] how to modify the scheme so that the whole metric can be prescribed on the initial characteristic hypersurface, which solves that problem, and allows e.g. the treatment of Vlasov particles with a given mass. However, the formulation in I and II has the clear advantage, that only geometrically significant objects are prescribed as free data. This is not the case with [4], as many components of the initial metric have a gauge character. It is therefore of interest to see whether kinetic matter fits into the original scheme of [7], as generalised in I and II. In this article we show that this is indeed the case. More precisely, we prove that the Cauchy problem on a characteristic cone for the *Einstein-Vlasov* system can also be split into a hierarchical system of ordinary differential equations as constraints and an evolution problem, in the important case of astrophysical studies where the masses of “particles” take a continuous set of positive values. We show how to construct physically relevant initial values in this case.

2 Einstein-Vlasov system.

2.1 Distribution function.

In kinetic theory the matter is composed of a collection of particles whose size is negligible at the considered scale: rarefied gases in the laboratory, galaxies or even clusters of galaxies at the cosmological scale. The number of particles is so great and their motion so chaotic that it is impossible to observe their individual motions. It is assumed that the state of the matter

in a spacetime (V, g) is represented² by a “one particle distribution function”:

Definition 1 A *distribution function* is a scalar function $f \geq 0$ on the tangent bundle TV to the spacetime V ,

$$f : TV \rightarrow R \quad \text{by} \quad (m, p) \mapsto f(m, p_m), \quad \text{with} \quad m \in V, \quad p \in T_m V.$$

The physical meaning of the distribution function is that it gives a mean³ number of particles with momentum $p \in T_m V$ at a point $m \in V$.

In astrophysics the particles are stars or even galaxies, they do not take a finite collection of a priori given masses, but rather a continuous family of positive masses.

One denotes by ω_g and ω_p respectively the volume forms on V and $T_m V$:

$$\theta = \omega_g \wedge \omega_p$$

where, in local coordinates

$$\omega_g = (\det g)^{\frac{1}{2}} dx^0 \wedge dx^1 \wedge \dots \wedge dx^n, \quad \omega_p := (\det g)^{\frac{1}{2}} dp^0 \wedge dp^1 \dots \wedge dp^n.$$

Definition 2 The *moment of order k of the distribution function f* is the *symmetric k -tensor*

$$T^{\alpha_1 \dots \alpha_p}(m) := \int_{T_m} f(m, p) p^{\alpha_1} \dots p^{\alpha_p} \omega_p.$$

The moment of order zero, integral on the fiber T_m , $m \in V$ of the distribution function:

$$M(m) := \int_{T_m} f \omega_p$$

gives the density of presence of particles at a point $m \in V$. The second moment of f defines the stress energy tensor it generates. It is a symmetric 2-tensor on V obtained at any point $m \in V$ by integrating on T_m the product of $f(m, p)$ by the tensor product $p \otimes p$.

²A mathematical justification of the oncoming of chaos in relativistic dynamics is a largely open problem.

³In the sense of Gibbs ensemble.

2.2 Vlasov equation.

In a Lorentzian spacetime (V, g) , in the absence of non gravitational forces and collisions, each particle follows a geodesic of the spacetime metric g , i.e. an orbit of the vector field $X = (p, P)$ on TV ,

$$p^\alpha := \frac{dx^\alpha}{d\lambda}, \quad \frac{dp^\alpha}{d\lambda} = P^\alpha, \quad \text{with} \quad P^\alpha := -\Gamma_{\lambda\mu}^\alpha p^\lambda p^\mu$$

with λ a canonical parameter and $\Gamma_{\lambda\mu}^\alpha$ the Christoffel symbols of the metric g . Recall that the scalar $g(p, p)$ is constant along an orbit of X in TV , recall also that (Liouville theorem) the volume form θ is invariant under the geodesic flow; that is, $\mathcal{L}_X \theta$ denoting the Lie derivative with respect to X , it holds that

$$\mathcal{L}_X \theta = 0.$$

In a collisionless model the physical law of conservation of particles imposes to the form $f\theta$ to be invariant under the vector field X , hence $f\theta$ has a zero Lie derivative with respect to X . Since we already know that $\mathcal{L}_X \theta = 0$ the invariance reduces to the **Vlasov** equation for f

$$\mathcal{L}_X f \equiv p^\alpha \frac{\partial f}{\partial x^\alpha} + P^\alpha \frac{\partial f}{\partial p^\alpha} = 0, \quad P^\alpha = -g_{\alpha\lambda} \Gamma_{\rho\sigma}^\alpha p^\rho p^\sigma,$$

which says that in the phase space TV the derivative of the distribution function in the direction of X vanishes

A fundamental theorem⁴ is

Theorem 3 *If the distribution function f satisfies the Vlasov equation, its moments have vanishing divergence. In particular*

$$\nabla_\alpha T^{\alpha\beta} = 0.$$

2.3 The Einstein-Vlasov system.

Given a smooth manifold V , the unknowns are the metric g on V and the distribution function f on TV . They must satisfy the Einstein-Vlasov system

$$S_{\alpha\beta} = T_{\alpha\beta}, \quad \text{on } V, \quad \mathcal{L}_X f = 0 \quad \text{on } TV,$$

⁴See for instance Marle [M].

where $S_{\alpha\beta}$ is the Einstein tensor of g , hence satisfies the contracted Bianchi identities, $\nabla_\alpha S^{\alpha\beta} \equiv 0$ and

$$T_{\alpha\beta} := \int_{R^{n+1}} p_\alpha p_\beta f(\cdot, p) \omega_p$$

is the second moment of the distribution function f , hence such that when the Vlasov equation holds

$$\nabla_\alpha T^{\alpha\beta} = 0.$$

The Einstein-Vlasov system is therefore coherent.

3 Cauchy problem on a characteristic cone

The proof of local existence for solutions of the Cauchy problem with initial data for g on a spacelike manifold S and for f on TS can be found in [1] and references therein. In what follows we consider the case where the initial manifold is a characteristic cone of g .

The future characteristic cone C_O of vertex O for a Lorentzian metric g is the set covered by future directed null geodesics issued from O . We choose as in I and II coordinates y^α such that the coordinates of O are $y^\alpha = 0$ and the components $g^{\lambda\mu}(0, 0)$ take the diagonal Minkowskian values, $(-1, 1, \dots, 1)$. If g is Lorentzian and C^2 in a neighbourhood U of O there is an eventually smaller neighbourhood of O , still denoted U , such that $C_O \cap U$ is an n dimensional manifold, differentiable except at O , and there exist in U coordinates $y := (y^\alpha) \equiv (y^0, y^i, i = 1, \dots, n)$ in which C_O is represented by the equation of a Minkowskian cone with vertex O ,

$$C_O := \{r - y^0 = 0\}, \quad r := \left\{ \sum (y^i)^2 \right\}^{\frac{1}{2}},$$

The null rays of C_O are represented by the generators of the Minkowskian cone with tangent vector ℓ . We overline as in I and II traces on C_O , and underline components in the y coordinates. The components of ℓ are $\overline{\ell}^0 = 1$, $\underline{\ell}^i = r^{-1}y^i$. We denote by Y_O the causal future of O , $y^0 \geq r$. We denote by TC_O the tangent bundle with base C_O , by $T_g^+ C_O$ the subbundle with fiber future timelike vectors for the metric g .

3.1 Cauchy problem for the Vlasov equation with data on C_O , given g .

Let the $n + 1$ dimensional spacetime (V, g) be a given Lorentzian manifold. The Vlasov equation, with $X = X(g)$, and unknown f

$$\mathcal{L}_X f = 0$$

is a first order linear partial differential equation for the distribution function f on the tangent bundle TV , differential equation on the geodesic flow which reads in the coordinates y, \underline{p} of TV

$$\frac{df(y(\lambda), \underline{p}(\lambda))}{d\lambda} = 0$$

where y and \underline{p} are solutions of the differential system

$$\frac{dy^\alpha}{d\lambda} = \underline{p}^\alpha, \quad \frac{d\underline{p}^\alpha}{d\lambda} = \underline{P}^\alpha \equiv -\Gamma_{\lambda\mu}^\alpha \underline{p}^\lambda \underline{p}^\mu.$$

This quasilinear first order differential system has, if its coefficients are Lipschitzian i.e. if the considered metric g is $C^{1,1}$, one and only one solution for $\lambda - \lambda_0$ small enough, $(y^i(\lambda), y^0(\lambda), \underline{p}^\alpha(\lambda))$, taking for $\lambda = \lambda_0$ given values

$$y^i(\lambda_0) = \bar{y}^i, \quad y^0(\lambda_0) = r \equiv \left\{ \sum (\bar{y}^i)^2 \right\}^{\frac{1}{2}}, \quad \underline{p}^\alpha(\lambda_0) = \underline{\pi}^\alpha,$$

$\underline{\pi}^\alpha$, a vector in R^{n+1} timelike for g . The image in TV of the obtained curves

$$\lambda \mapsto (y^i(\lambda), y^0(\lambda), \underline{p}^\alpha(\lambda)),$$

is the set of future timelike geodesics (we mean the curves and their tangents) issued from points of T^+C_O in a neighbourhood of O . Well known properties of timelike geodesics (in particular the conservation of $g(p, p)$ under the geodesic flow) show that conversely a past directed timelike geodesic issued from a point $(m, p) \in T^+Y_O$, with m in a small enough neighbourhood of O , will meet T^+C_O .

The initial data for f is a function \bar{f} on T^+C_O . The solution of the Cauchy problem for the Vlasov equation with this initial data is given in a neighborhood of T^+O in T^+Y_O by

$$\underline{f}(y(\lambda), \underline{p}(\lambda)) = \bar{f}(y^\alpha(\lambda_0), \underline{p}^\alpha(\lambda_0)).$$

3.2 Cauchy problem for g with data on C_O for the Einstein equations in wave gauge, given the stress energy tensor of f .

The Einstein tensor in wave gauge forms a quasilinear system of wave operators deduced from the identities

$$R_{\alpha\beta}^{(h)}(g) \equiv -\frac{1}{2}g^{\lambda\mu}\partial_{\lambda\mu}^2 g_{\alpha\beta} + q_{\alpha\beta}(g)(\partial g, \partial g).$$

The stress energy tensor of a given distribution function f_1 on a spacetime (V, g_1) is an integral operator on f_1 and g_1

$$T_{1,\alpha\beta} \equiv \int_{R^{n+1}} p_{1,\alpha} p_{1,\beta} f_1(\cdot, p) \omega_{g_1}.$$

One considers the system of quasilinear equations for a metric g_2 , with T_1 known,

$$\text{Einstein}^{(h)}(g_2) = T_1.$$

If T_1 is smooth there exists (Cagnac - Dossa theorem [6]) $t_0 > 0$ such that these equations have a solution g_2 with trace on $C_O \cap [0, t_0]$ the trace \bar{g} of a smooth spacetime metric.

3.3 Solution of the Einstein-Vlasov reduced system

One shows by iteration using known theorems that the Einstein-Vlasov system in wave gauge with unknowns g and f admits a solution with initial data \bar{g} and \bar{f} if \bar{g} is the trace of a smooth spacetime function, for instance $\bar{f} = 0$ and \bar{g} trace of the Minkowski metric in a neighbourhood of the vertex.

3.4 Solution of the original system

To show that a solution of the reduced system satisfies the original equations, if the initial data satisfy appropriate constraints, we proceed as in I where we have proved the decomposition $\ell^\alpha \bar{S}_{\alpha\beta} \equiv \mathcal{C}_\alpha + \mathcal{L}_\alpha$, where \mathcal{C}_α depends only on \bar{g} and \mathcal{L}_α is a linear homogeneous differential operator on the wave gauge vector. The Bianchi identities show, as in I, that a solution of the reduced system satisfies the original equations if the initial data satisfy the constraints, namely

$$\mathcal{C}_\alpha = \ell^\alpha \bar{T}_{\alpha\beta}.$$

We now show that in our astrophysical setting these constraints form again a hierarchical system of ordinary differential equations if we use coordinates x^α adapted to the null structure of C_O , defined by

$$x^0 = r - y^0, \quad x^1 = r \quad \text{and} \quad x^A = \mu^A(r^{-1}y^i),$$

$A = 2, \dots, n$, local coordinates on the sphere S^{n-1} , or angular polar pseudo coordinates, such that the curves $x^0 = 0$, $x^A = \text{constant}$ are the null geodesics issued from O with tangent $\ell := \frac{\partial}{\partial x^1}$. The trace \bar{g} on C_O of the, a priori general, Lorentzian metric g that we are going to construct is such that $\bar{g}_{11} = 0$ and $\bar{g}_{1A} = 0$; we use the notation

$$\bar{g} \equiv \bar{g}_{00}(dx^0)^2 + 2\nu_0 dx^0 dx^1 + 2\nu_A dx^0 dx^A + \bar{g}_{AB} dx^A dx^B.$$

In particular for the Minkowski metric $\eta \equiv \bar{\eta}$ it holds that

$$\eta_{00} = -1, \quad \nu_0 = \nu_A = 0, \quad \eta_{AB} = r^2 s_{AB},$$

where $s_{AB} dx^A dx^B$ is the metric of the round sphere S^{n-1} . The volume element in the tangent space to V at a point of C_O is in the x -coordinates

$$\bar{\omega}_p \equiv \nu_0 \det \tilde{g}.$$

4 Geometric initial data and gauge functions.

The geometric data is a degenerate quadratic form \tilde{g} induced on C_O by our unknown g ; it reads in coordinates x^1, x^A

$$\tilde{g} \equiv \tilde{g}_{AB} dx^A dx^B,$$

i.e. $\tilde{g}_{11} \equiv \tilde{g}_{1A} \equiv 0$ while $\tilde{g}_{AB} dx^A dx^B \equiv \bar{g}_{AB} dx^A dx^B$ is an x^1 -dependent Riemannian metric on S^{n-1} . While \tilde{g} is intrinsically defined, it is not so for \bar{g}_{00} , ν_0 , ν_A , they are gauge-dependent quantities, the unknowns of our constraints.

As in the spacelike case, the second fundamental form χ of C_O appears in the constraints, but here it depends only on the induced metric because the normal to C_O , $\ell \equiv \frac{\partial}{\partial x^1}$, is also tangent. We have defined χ in I as the tensor χ on C_O given by the Lie derivative $\mathcal{L}_\ell \tilde{g}$ with respect to ℓ of the degenerate quadratic form \tilde{g} , namely in the coordinates x^1, x^A :

$$\chi_{AB} \equiv \frac{1}{2} \partial_1 \bar{g}_{AB}, \quad \chi_{A1} \equiv \chi_{11} \equiv 0.$$

We denote by τ the mean extrinsic curvature $\tau := g^{AB} \chi_{AB}$.

5 The first constraint operator

We have found in I the identity

$$\bar{\ell}^\beta \bar{S}_{1\beta} \equiv \bar{R}_{11} \equiv -\partial_1 \tau + \nu^0 \partial_1 \nu_0 \tau - \frac{1}{2} \tau (\bar{\Gamma}_1 + \tau) - \chi_A^B \chi_B^A.$$

By definition of the wave-gauge vector H we have

$$\bar{\Gamma}_1 \equiv \bar{W}_1 + \bar{H}_1, \quad \bar{W}_1 = -\nu_0 x^1 \bar{g}^{AB} s_{AB},$$

and hence \bar{R}_{11} decomposes as

$$\bar{R}_{11} \equiv \mathcal{C}_1 + \mathcal{L}_1, \quad \text{where } \mathcal{L}_1 := -\frac{1}{2} \bar{H}_1 \tau \text{ vanishes in wave gauge}$$

while \mathcal{C}_1 involves only the values of the coefficients \bar{g}_{AB} and ν_0 on the light-cone; the first constraint reads

$$\mathcal{C}_1 := -\partial_1 \tau + \{\nu^0 \partial_1 \nu_0 - \frac{1}{2} (\bar{W}_1 + \tau)\} \tau - \chi_A^B \chi_B^A = \bar{T}_{11} \equiv (\nu_0)^2 \bar{T}^{00},$$

$$\bar{T}^{00} \equiv \det \tilde{g} \int_{R^{n+1}} (p^0)^2 \bar{f}(\cdot, p) d^{n+1} p, \quad d^{n+1} p = dp^0 dp^1 \dots dp^n.$$

The first constraint contains ν_0 as only unknown if \tilde{g} and \bar{f} are given. The resulting ODE is singular for $\tau = 0$ since τ multiplies $\partial_1 \nu_0$. However the alternative used by Rendall [R] and Damour and Schmidt [D-S] of prescribing only the conformal class of \tilde{g} , splitting χ into its conformally invariant traceless part σ and its unknown trace τ and imposing to ν_0 to annul the parenthesis in \mathcal{C}_1 does not work here because also \bar{T}_{11} contains ν_0 .

We look for a solution of the first constraint tending to 1 as r tends to zero, hence set $\nu_0 = 1 + u$ and write the first constraint as the first order differential equation (recall that $\nu^0 = (\nu_0)^{-1}$)

$$r \partial_1 u = r(a + b) + \beta + (a + 2b + 3\beta)u + (b + 3\beta)u^2 + \beta u^3,$$

where

$$a := \tau^{-1} \partial_1 \tau + \frac{1}{2} \tau + \tau^{-1} |\chi|^2, \quad b := -\frac{1}{2} \bar{g}^{AB} r s_{AB}, \quad \beta \equiv r \tau^{-1} \bar{T}^{00}.$$

To avoid analytical difficulties near the vertex, we assume that there exists a neighborhood $0 < r \leq r_0$ of the vertex O within C_O on which the initial data

\bar{f} vanishes and \tilde{g} coincides with $\tilde{\eta}$. In this neighbourhood the coefficients reduce to their Minkowskian values

$$a_0 = \frac{n-1}{2r}, \quad b_0 := -\frac{n-1}{2r}, \quad \beta_0 = 0,$$

the solution u tending to zero as r tends to zero is then $u = 0$ for $r \leq r_0$. For $r > r_0$ it is the solution u vanishing for $r = r_0$ of a smooth equation with smooth coefficients, equivalently of the integral equation

$$u(r, x^A) = \int_{r_0}^r \{(a+b) + (x^1)^{-1}[\beta + (a+2b+3\beta)u + (b+3\beta)u^2 + \beta u^3]\} dx^1$$

This solution can be computed by the Picard iteration method as long as the interval $[r_0, r]$ is small enough; its size, and the norm of u depend on the size of the various coefficients. We will always have $\nu_0 > 0$ in a small enough neighbourhood, as required for the Lorentzian character of \bar{g} .

6 The \mathcal{C}_A constraint

We have written in I the \mathcal{C}_A constraint operator. The \mathcal{C}_A constraint with kinetic source reads

$$\mathcal{C}_A - \bar{T}_{1A} \equiv -\frac{1}{2}(\partial_1 \xi_A + \tau \xi_A) + \tilde{\nabla}_B \chi_A^B - \frac{1}{2} \partial_A \tau + \partial_A \left(\frac{1}{2} \bar{W}_1 + \nu_0 \partial_1 \nu^0 \right) - \bar{T}_{1A} = 0$$

where ξ_A is defined as

$$\xi_A := -2\nu^0 \partial_1 \nu_A + 4\nu^0 \nu_C \chi_A^C + \left(\bar{W}^0 - \frac{2}{r} \nu^0 \right) \nu_A + \bar{g}_{AB} \bar{g}^{CD} (S_{CD}^B - \tilde{\Gamma}_{CD}^B)$$

while

$$\bar{T}_{1A}(x) \equiv \int_{R^{n+1}} p_1 p_A F(x, p) d^{m+1} p = \nu_0 \left\{ \int_{R^{n+1}} p^0 (\nu_A p^0 + g_{AB} p^B) d^{m+1} p \right.$$

When ν_0 has been determined the only unknown in $\mathcal{C}_A - \bar{T}_{1A}$ is ν_A . It satisfies a linear equation, its solution follows the same lines as in I.

7 The \mathcal{C}_0 constraint

When ν_0 and ν_A have been determined, the \mathcal{C}_0 constraint operator, $\ell^\beta \bar{S}_{0\beta} \equiv \bar{S}_{01}$, contains as only unknown $\bar{g}_{00} \equiv \underline{\bar{g}}_{00}$, we have found in *I* and *II* after long computations that the \mathcal{C}_0 constraint can be written, using the other constraints

$$\partial_1 \zeta + (\kappa + \tau) \zeta + \frac{1}{2} \{ \partial_1 \bar{W}^1 + (\kappa + \tau) \bar{W}^1 + \tilde{R} - \frac{1}{2} g^{AB} \xi_A \xi_B + \bar{g}^{AB} \tilde{\nabla}_A \xi_B \} + h_f = 0$$

$$h_f := \frac{1}{2} \bar{g}^{11} \bar{T}_{11} + \bar{g}^{1A} \bar{T}_{1A} + \bar{g}^{10} \bar{T}_{10},$$

$$\kappa \equiv \nu^0 \partial_1 \nu_0 - \frac{1}{2} (\bar{W}_1 + \tau), \quad \bar{W}_1 \equiv \nu_0 \bar{W}^0, \quad \bar{W}^0 \equiv \bar{W}^1 \equiv -r \bar{g}^{AB} s_{AB}.$$

The relation between ζ and \bar{g}_{00} is

$$\zeta := (\partial_1 + \kappa + \frac{1}{2} \tau) \bar{g}^{11} + \frac{1}{2} \bar{W}^1,$$

where \bar{g}^{11} and \bar{g}_{00} are linked by the linear relation

$$\underline{\bar{g}}_{00} \equiv \bar{g}_{00} \equiv -\bar{g}^{11} (\nu_0)^2 + \bar{g}^{AB} \nu_B \nu_A.$$

The component \bar{T}_{01} of the stress energy tensor of the distribution function reads

$$\bar{T}_{01}(\bar{x}) \equiv \int_{T_m} p_0 p_1 \bar{f}(\bar{x}, p) d^{n+1} p \equiv \nu_0 \int_{T_m} (\bar{g}_{00} p^0 + \nu_0 p^1 + \nu_A p^A) p^0 \bar{f}(\bar{x}, p) d^{n+1} p$$

The only unknown remaining in the \mathcal{C}_0 constraint is \bar{g}_{00} . It appears linearly. The solution equal to -1 in $[0, r_0]$ is smooth and negative in $[0, r_0 + \varepsilon]$.

8 The tachyon problem.

For a physical reason, namely the positivity of the masses of the considered ‘‘particles’’, the initial data \bar{f} of the distribution function must have its support in the subset

$$\bar{g}_{00}(p^0)^2 + 2\nu_0 p^0 p^1 + 2\nu_A p^0 p^A + \bar{g}_{AB} p^A p^B < 0, \quad (1)$$

since only \bar{g}_{AB} has been a priori given the above inequality gives an a posteriori restriction on the support of \bar{f} . It is natural to assume that this support in momentum space at the tip, where all components of the metric are known, is contained within the future Minkowski -cone. Then, by continuity of the data \bar{f} and of the solutions of the constraints, one obtains some neighborhood of the tip in C_O where this support will remain in the future cone of the constructed metric. This neighborhood will provide the relevant initial data. A precise statement to this effect can be obtained as follows:

Recall that $\underline{p}^0 \equiv p^0 + p^1$ is the time component of p in the y coordinates, the condition that we will give is that p is strictly timelike for a known metric on the support of \bar{f} . The following lemma is a result of the continuity properties found for \bar{g} :

Lemma 4 *Assume that there exists $r_0 \geq 0$ such that the support of the data \bar{f} in TC_O is contained in the subset*

$$(\underline{p}^0)^2 > k\{\bar{g}_{AB}p^A p^B + (p^1)^2\}, \quad k > 1 \quad \text{and} \quad p^0 > 0 \quad (2)$$

with $\bar{g}_{Ab} = \eta_{AB}$ for $r \leq r_0$ (possibly $r_0 = 0$). Then there is a neighbourhood $r_0 \leq r < R$ in C_O such that the support of \bar{f} in $TC_O \cap \{r_0 \leq r < R\}$ includes no tachyons for the metric \bar{g} .

Proof. The no-tachyon (positive masses) condition can be written with $\nu_0 = 1 + u$, $g^{00} = -1 + \alpha$ where u , α and ν_A tend to zero as r tends to r_0 (possibly zero)

$$(\underline{p}^0)^2 - (p^1)^2 - \alpha(p^0)^2 + 2p^0(u p^1 + \nu_A p^A) - \bar{g}_{AB}p^A p^B > 0. \quad (3)$$

Remark that $p^0 + p^1 \equiv \underline{p}^0$ implies

$$(p^0)^2 \leq 2\{(\underline{p}^0)^2 + (p^1)^2\}, \quad 2p^0 p^1 \leq (p^0)^2 + (p^1)^2 \leq 2(\underline{p}^0)^2 + 3(p^1)^2 \quad (4)$$

while setting $|\tilde{\nu}| := (\bar{g}^{AB}\nu_A\nu_B)^{\frac{1}{2}}$ we have

$$|\nu_A p^A| \leq (\bar{g}_{AB}p^A p^B)^{\frac{1}{2}} |\tilde{\nu}| \quad \text{hence} \quad 2p^0 \nu_A p^A \leq |\tilde{\nu}| \{ (p^0)^2 + \bar{g}_{AB}p^A p^B \} \quad (5)$$

The inequalities (4) and (5) show that the no tachyon condition (3) is implied by the following inequality

$$(\underline{p}^0)^2 - (2|\alpha| + 4|u| + |\tilde{\nu}|)(p^0)^2 > (p^1)^2 + \bar{g}_{AB}p^A p^B + 3|u|(p^1)^2 + |\tilde{\nu}| \{ (p^0)^2 + \bar{g}_{AB}p^A p^B \}. \quad (6)$$

The assumption (2) of the lemma (which depends only on the given data \tilde{g}) implies that there exist two numbers $k_1 < 1$ and $k_2 > 1$ with $k_1^{-1}k_2 = k$ such that

$$k_1(\underline{p}^0)^2 > k_2\{\bar{g}_{AB}p^A p^B + (p^1)^2\}, \quad k > 1 \quad \text{and} \quad p^0 > 0 \quad (7)$$

The properties of α, u and $\tilde{\nu}$, which tend to zero as r tends to r_0 , show that there exists $R > r_0$ such that for $r_0 \leq r \leq R$ it holds that

$$\begin{aligned} (\underline{p}^0)^2 - (2|\alpha| + 4|u| + |\tilde{\nu}|)(\underline{p}^0)^2 &> k_1(\underline{p}^0)^2, \\ (p^1)^2 + \bar{g}_{AB}p^A p^B + 3|u|(p^1)^2 + |\tilde{\nu}|\bar{g}_{AB}p^A p^B &< k_2\{\bar{g}_{AB}p^A p^B + (p^1)^2\} \end{aligned}$$

These inequalities complete the proof of the lemma. ■

9 Conclusion

From a smooth solution of the constraints in the x variables, Minkowskian for $0 \leq r < r_0$, one deduces that \bar{g} is the trace on C_O of a smooth spacetime function, which completes the proof of the existence of a solution of an Einstein-Vlasov spacetime in a neighbourhood of O taking the given data \tilde{g}, \bar{f} . Geometric uniqueness is also easy to prove. The existence up to the vertex is more involved: the technique of admissible series used in II could be used, but requires further work.

References

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