BOUNDARY REGULARITY OF CONFORMALLY COMPACT EINSTEIN METRICS

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Abstract. We show that C^2 conformally compact Riemannian Einstein metrics have conformal compactifications that are smooth up to the boundary in dimension 3 and all even dimensions, and polyhomogeneous in odd dimensions greater than 3.

1. Introduction

Suppose $\overline{M}$ is a smooth, compact manifold with boundary, and let M denote its interior and ∂M its boundary. (By "smooth," we always mean C^∞.) A Riemannian metric g on M is said to be *conformally compact* if for some smooth defining function ρ for ∂M in $\overline{M}$, $\rho^2 g$ extends by continuity to a Riemannian metric (of class at least C^0) on $\overline{M}$. The rescaled metric $\overline{g} = \rho^2 g$ is called a *conformal compactification* of g. If for some (hence any) smooth defining function ρ, $\overline{g}$ is in $C^k(\overline{M})$ or $C^{k,\lambda}(\overline{M})$, then we say g is conformally compact of class C^k or $C^{k,\lambda}$, respectively.

If g is conformally compact, the restriction of $\overline{g} = \rho^2 g$ to ∂M is a Riemannian metric on ∂M, whose conformal class is determined by g, independently of the choice of defining function ρ. This conformal class is called the *conformal infinity* of g.

Several important existence and uniqueness results [1, 2, 5, 9, 13] concerning conformally compact Riemannian Einstein metrics have been established recently. For many applications to physics and geometry, it turns out to be of great importance to understand the asymptotic behaviour of the resulting metrics near the boundary. This question has been addressed by Michael Anderson [2], who proved that if g is a 4-dimensional conformally compact Einstein metric with smooth conformal infinity, then the conformal compactification of g is smooth up

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to the boundary in suitable coordinates. It has long been conjectured that in higher dimensions, conformally compact Einstein metrics with smooth conformal infinities should have infinite-order asymptotic expansions in terms of ρ and $\log \rho$. The purpose of this paper is to confirm that conjecture.

The choice of special coordinates in Anderson's result cannot be dispensed with. Because the Einstein equation is invariant under diffeomorphisms, we cannot expect that the conformal compactification of an arbitrary conformally compact Einstein metric will necessarily have optimal regularity for all C^∞ structures on $\overline{M}$. For example, suppose g is an Einstein metric on M with a smooth conformal compactification, and let $\Psi\colon \overline{M} \to \overline{M}$ be a homeomorphism that restricts to the identity map of ∂M and to a diffeomorphism from M to itself. Then $\Psi^* g$ will still be Einstein with the same conformal infinity, but its conformal compactification $\rho^2 \Psi^* g$ may no longer be smooth. Thus the best one might hope for is that an arbitrary conformally compact Einstein metric can be made smoothly conformally compact after pulling back by an appropriate diffeomorphism. Even this is not true in general, because Fefferman and Graham showed in [7] that there is an obstruction to smoothness in odd dimensions.

Since Einstein metrics are always smooth (in fact, real-analytic) in suitable coordinates in the interior, only regularity at the boundary is at issue. For that reason, instead of assuming that $\overline{M}$ is compact, we will assume only that it has a compact boundary component Y, and restrict our attention to a collar neighborhood of Y in $\overline{M}$, which we may assume without loss of generality is diffeomorphic to $Y \times [0, 1)$. Throughout this paper, then, Y will be an arbitrary smooth, connected, compact, n-dimensional manifold without boundary, and we make the following identifications:

$$\overline{M} = Y \times [0, 1), \quad M = Y \times (0, 1), \quad \partial M = Y \times \{0\}.$$

Let $\rho\colon \overline{M} \to [0, 1)$ denote the projection onto the $[0, 1)$ factor; it is a smooth defining function for ∂M in $\overline{M}$. For $0 < R < 1$, we define

$$M_R = Y \times (0, R], \quad \overline{M}_R = Y \times [0, R].$$

In this context, we extend the definition of conformally compact metrics by saying that a smooth Riemannian metric g on M or M_R is conformally compact if $\rho^2 g$ extends to a continuous metric on $\overline{M}$ or $\overline{M}_R$, respectively. A continuous map $\Psi\colon \overline{M}_R \to \overline{M}$ (for some R) that restricts to the identity map of ∂M and to a (C^∞) diffeomorphism from M_R to its image will be called a *collar diffeomorphism*. If Ψ and its inverse are of class C^k (or $C^{k,\lambda}$) up to the boundary, we will call it a C^k (resp., $C^{k,\lambda}$) collar diffeomorphism.

The object of this paper is to prove that the following regularity holds:

Theorem A. *Let g be a smooth Riemannian metric on M. Suppose that $\dim M = n+1 \geq 3$; g is Einstein with $\mathrm{Ric}(g) = -ng$; g is conformally compact of class C^2; and the representative $\gamma = \rho^2 g|_{\partial M}$ of the conformal infinity of g is smooth. Let $\widetilde{\gamma}$ be any smooth representative of the conformal class $[\gamma]$. Then for any $0 < \lambda < 1$, there exists $R > 0$ and a $C^{1,\lambda}$ collar diffeomorphism $\Phi\colon \overline{M}_R \to \overline{M}$ such that $\Phi^* g$ can be written in the form*

$$\Phi^* g = \rho^{-2}(d\rho^2 + G(\rho)), \tag{1.1}$$

where $\{G(\rho) : 0 < \rho \leq R\}$ is a one-parameter family of smooth Riemannian metrics on Y, which has a continuous extension to $\overline{M}_R$ with $G(0) = \widetilde{\gamma}$, and has the following regularity:

(a) *If $\dim M$ is even or equal to 3, then G extends smoothly to $\overline{M}_R$, so $\Phi^* g$ is conformally compact of class C^∞.*
(b) *If $\dim M$ is odd and greater than 3, then G can be written in the form*
$$G(\rho) = \varphi(\rho, \rho^n \log \rho),$$
with $\varphi(\rho, z)$ a two-parameter family of Riemannian metrics on Y that is smooth in all of its arguments as a function on $Y \times [0, R] \times [R^n \log R, 0]$. Furthermore, $\Phi^* g$ is smoothly conformally compact if and only if $\partial_z \varphi(0,0)$ vanishes identically on ∂M.

Remark. The symmetric 2-tensor field $\partial_z \varphi(0,0)$ along ∂M can be determined in principle from local computations involving the conformal class $[\gamma]$ (cf. [7,8]), and in fact the vanishing of this tensor field is a necessary condition for the existence of a smoothly conformally compact Einstein metric on M with $[\gamma]$ as conformal infinity. Explicit formulae in low dimensions can be found in [10].

The main idea of the proof is to use the harmonic map equation to put g into a gauge in which it satisfies an elliptic equation, and then apply the polyhomogeneity results of [3]. The proof consists of four steps. First, we construct a preliminary collar diffeomorphism that makes $\rho^2 g$ coincide to second order along ∂M with a smooth product metric $\overline{h}$. Second, applying the inverse function theorem to the harmonic map equation, we show that there exists a collar diffeomorphism $H\colon \overline{M}_R \to \overline{M}_R$ that is harmonic in M_R, thought of as a map from (M_R, g) to (M_R, h), where $h = \rho^{-2}\overline{h}$. It follows that the metric $\widetilde{g} = (H^{-1})^* g$

satisfies the following "gauge-broken Einstein equation" near ∂M:

$$Q(\widetilde{g}, h) := \operatorname{Ric}(\widetilde{g}) + n\widetilde{g} - \delta^*_{\widetilde{g}}(\Delta_{\widetilde{g}h}(\mathrm{Id})) = 0 \tag{1.2}$$

(see, e.g., [13]), where $\Delta_{\widetilde{g}h}$ is the harmonic map Laplacian. The third step is to show that solutions to (1.2) satisfy the hypotheses of [3, Theorem 5.1.1] and therefore are polyhomogeneous (i.e., have asymptotic expansions in powers of ρ and $\log \rho$). The last step is to use a special defining function and Fermi coordinates near the boundary to put the metric into the form (1.1).

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2. Weighted Hölder spaces

Throughout most of this paper, we will use the notations and conventions of [13]. We define M_R and $\overline{M}_R$ as in the introduction. We assume throughout that $\dim M = n+1 \geq 3$. Any smooth local coordinates $\theta = (\theta^1, \ldots, \theta^n)$ on an open set $U \subset Y$ yield smooth coordinates $(\theta, \rho) = (\theta^1, \ldots, \theta^n, \rho)$ on the open subset $\Omega = U \times [0,1) \subset \overline{M}$. Choose finitely many such charts (U_i) to cover Y, with each set U_i chosen so that the coordinate functions extend smoothly to a neighborhood of $\overline{U}_i$ in Y. The resulting coordinates on $\Omega_i = U_i \times [0,1) \subset \overline{M}$ will be called *background coordinates* for $\overline{M}$.

Let B_1, B_2 be fixed open coordinate balls in the upper half-space $\mathbb{H}^{n+1} = \{(x,y) = (x^1, \ldots, x^n, y) : y > 0\}$, with $(0, \ldots, 0, 1) \in B_1 \subset \overline{B}_1 \subset B_2$. We will use the summation convention, with Greek indices generally understood to run from 1 to n, and Roman indices to run from 1 to $n+1$; sometimes it will be convenient to denote ρ by θ^{n+1} and y by x^{n+1}. Suppose $p \in M_R$, and let (θ_0, ρ_0) be the coordinate representation of p in some fixed background chart. If p is sufficiently close to ∂M, we can define a diffeomorphism $\Phi_p \colon \overline{B}_2 \to M$ by

$$(\theta, \rho) = \Phi_p(x, y) = (\theta_0 + \rho_0 x, \rho_0 y).$$

As is shown in [13], for R sufficiently small, there exists a countable set of points $\{p_i\} \subset M_R$ such that the sets $\{\Phi_{p_i}(\overline{B}_2)\}$ form a uniformly locally finite covering of M_R, and the sets $\{\Phi_{p_i}(B_1)\}$ still cover M_R. For each such map, set $\Phi_i = \Phi_{p_i}$, $V_1(p_i) = \Phi_i(B_1)$ and $V_2(p_i) = \Phi_i(B_2)$. Then for each i, $\left(V_2(p_i), \Phi_i^{-1}\right)$ is a coordinate chart on M_R, called a *Möbius chart*; the corresponding coordinates (x,y) will be called *Möbius coordinates.* It is shown in [13, Lemma 2.1] that if g is any $C^{k,\lambda}$ conformally compact metric on M, the pulled-back metrics $\Phi_i^* g$ are all uniformly $C^{k,\lambda}$ equivalent to the hyperbolic metric $y^{-2}\sum_j (dx^j)^2$ on $\overline{B}_2$. By compactness, they are also uniformly $C^{k,\lambda}$ equivalent on $\overline{B}_2$ to the Euclidean metric $\sum_j (dx^j)^2$.

We will be working in weighted Hölder spaces whose norms reflect the intrinsic geometry of a conformally compact metric. Before introducing them, let us record some elementary facts about Hölder spaces on subsets of $\mathbb{R}^m$. If $U \subset \mathbb{R}^m$ is a precompact open subset, k, p are nonnegative integers, and $\lambda \in [0,1)$, we denote by $C^{k,\lambda}(\overline{U};\mathbb{R}^p)$ the standard Hölder space of functions from $\overline{U}$ to $\mathbb{R}^p$, and we denote the usual Hölder norm on this space by $\|\cdot\|_{k,\lambda;\overline{U}}$. If $V \subset \mathbb{R}^p$ is any open set, $C^{k,\lambda}(\overline{U};V)$ denotes the open subset of $C^{k,\lambda}(\overline{U};\mathbb{R}^p)$ consisting of maps that take their values in V.

Lemma 2.1. *Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^p$ be convex, precompact open subsets, let k be a nonnegative integer, and let $\lambda \in [0,1)$.*

(a) *Each coordinate derivative defines a continuous linear map from $C^{k+1,\lambda}(\overline{U};\mathbb{R}^p)$ to $C^{k,\lambda}(\overline{U};\mathbb{R}^p)$.*

(b) *Pointwise multiplication defines a continuous bilinear map from $C^{k,\lambda}(\overline{U};\mathbb{R}) \times C^{k,\lambda}(\overline{U};\mathbb{R})$ to $C^{k,\lambda}(\overline{U};\mathbb{R})$.*

(c) *If $u \in C^{k,\lambda}(\overline{U};V)$ and $f \in C^{k,\lambda}(\overline{V};\mathbb{R})$ with $k \geq 1$, then $f \circ u \in C^{k,\lambda}(\overline{U};\mathbb{R})$, and there exists a constant C depending only on k, λ, U, and V such that*

$$\|f \circ u\|_{k,\lambda;\overline{U}} \leq C\|f\|_{k,\lambda;\overline{V}} \left(1 + \|u\|_{k,\lambda;\overline{U}}^{k+\lambda}\right).$$

(d) *If $f \in C^{k+2}(V;\mathbb{R})$ with $k \geq 1$, then composition $u \mapsto f \circ u$ defines a C^1 map from $C^{k,\lambda}(\overline{U};V)$ to $C^{k,\lambda}(\overline{U};\mathbb{R})$.*

(e) *Given $f \in C^{k+2}(\overline{V};\mathbb{R})$ and $u_0 \in C^{k,\lambda}(\overline{U};V)$ with $k \geq 1$, there exists $\delta > 0$ and a constant $C = C(U, V, k, \lambda, f, u_0, \delta)$ such that the following estimate holds for all $u \in C^{k,\lambda}(\overline{U};V)$ with $\|u - u_0\|_{k,\lambda;\overline{U}} \leq \delta$:*

$$\|f \circ u - f \circ u_0\|_{k,\lambda;\overline{U}} \leq C\|u - u_0\|_{k,\lambda;\overline{U}}.$$

Proof. The first two assertions are elementary and are left to the reader, and (e) follows easily from (d). Proofs of (c) and (d) can be found in [14]. q.e.d.

Now we proceed to define our weighted Hölder spaces on M_R. Let $E \to M$ be any tensor bundle, and let $R > 0$ be chosen so that M_R is covered by Möbius charts as above. For an integer $k \geq 0$ and $\lambda \in (0,1)$, we define the intrinsic Hölder space $C^{k,\lambda}(M_R;E)$ to be the set of locally $C^{k,\lambda}$ sections of E over M_R whose component functions in

Möbius coordinates satisfy a uniform $C^{k,\lambda}$ bound, with norm

$$\|u\|_{k,\lambda} := \sup_i \|\Phi_i^* u\|_{k,\lambda;\overline{B}_2},$$

where the supremum is over the countable collection of Möbius charts described above. Weighted versions of these Hölder spaces are defined by setting $C^{k,\lambda}_\delta(M_R;E) = \rho^\delta C^{k,\lambda}(M_R;E)$, with the norm $\|u\|_{k,\lambda,\delta} := \|\rho^{-\delta}u\|_{k,\lambda}$. It is shown in [13, Lemma 3.5] that this norm is equivalent to

$$\|u\|_{k,\lambda,\delta} \sim \sup_i \rho(p_i)^{-\delta}\|\Phi_i^* u\|_{k,\lambda;\overline{B}_j}$$

for either $j = 1$ or $j = 2$.

3. A preliminary normalization

Suppose that g satisfies the hypotheses of Theorem A. Let $\widetilde{\gamma}$ be an arbitrary smooth representative of the conformal class $[\gamma]$ on Y. Define a smooth product metric $\overline{h}$ on $\overline{M} = Y \times [0,1)$ by

$$\overline{h} = d\rho^2 + \widetilde{\gamma}.$$

Let $h = \rho^{-2}\overline{h}$, which is smoothly conformally compact and has $[\widetilde{\gamma}]$ as conformal infinity.

The goal of this section is to show that we can modify g by a collar diffeomorphism so that it agrees with h to second order along ∂M. We begin with the simple observation that they can be made to agree to first order by rescaling ρ.

Lemma 3.1. *Let g and h be as above. If $R > 0$ is sufficiently small, there exists a C^∞ collar diffeomorphism $G_0\colon \overline{M}_R \to \overline{M}$ that satisfies $\rho^2 G_0^* g = \rho^2 h + O(\rho)$ in any background coordinates.*

Proof. We can write $\widetilde{\gamma} = f\gamma$ for some smooth positive function $f\colon Y \to \mathbb{R}$. It is easy to check that the conclusion of the lemma is satisfied by $G_0(x,\rho) = (x, \rho/f(x))$, as long as R is chosen smaller than the infimum of f so that G_0 maps $\overline{M}_R$ into $\overline{M}$. q.e.d.

Next we show that we can make g agree with h to one higher order along ∂M. This lemma requires somewhat more work than might be expected because we are only assuming that g has a C^2 conformal compactification, and we need to make this normalization without losing any smoothness.

Lemma 3.2. *Let g and h be as above. For any sufficiently small $R > 0$, there exists a C^3 collar diffeomorphism $G\colon \overline{M}_R \to \overline{M}$ that satisfies $\rho^2 G^* g = \rho^2 h + O(\rho^2)$ in any background coordinates.*

Proof. After replacing g by $G_0^* g$, where G_0 is given by the preceding lemma, we may as well assume that $\rho^2 g|_{T\partial M} = \widetilde{\gamma}$. We will begin by showing that there exists a C^3 defining function r satisfying $|dr|^2_{r^2 g} = 1 + O(\rho^2)$. Let $\overline{g} = \rho^2 g$, which is a C^2 Riemannian metric on $\overline{M}_R$. Because $\operatorname{Ric}(g) = -ng$, it follows that $|d\rho|^2_{\overline{g}} = 1$ along ∂M (cf. [9, p. 192]). By Taylor's theorem, therefore, there is a function $b \in C^1(\overline{M}_R)$ such that

$$|d\rho|^2_{\overline{g}} = 1 + b\rho.$$

By Corollary 3.3.2 of [3], there exists a real-valued function $r \in C^3(\overline{M}_R) \cap C^\infty(M_R)$ such that

$$\begin{aligned} r|_{\partial M} &= 0, \\ \partial_\rho r|_{\partial M} &= 1, \\ \partial_\rho^2 r|_{\partial M} &= -b|_{\partial M}. \end{aligned}$$

Then r is a C^3 defining function for ∂M, which satisfies $r = \rho - \frac{1}{2} b\rho^2 + O(\rho^3)$, $dr = (1 - b\rho)d\rho + O(\rho^2)$. Therefore,

$$\begin{aligned} |dr|^2_{r^2 g} &= |dr|^2_{(r/\rho)^2 \overline{g}} \\ &= (r/\rho)^{-2} |dr|^2_{\overline{g}} \\ &= (1 - \tfrac{1}{2} b\rho)^{-2} (1 - b\rho)^2 |d\rho|^2_{\overline{g}} + O(\rho^2) \\ &= 1 + O(\rho^2). \end{aligned} \tag{3.1}$$

Let P denote the gradient of r with respect to the metric $r^2 g$; thus P is a C^2 vector field on $\overline{M}_R$. Because $P\rho = \langle dr, d\rho \rangle_{r^2 g} = \langle dr, dr + O(\rho)\rangle_{r^2 g} = 1 + O(\rho)$, we can write $P = \partial/\partial\rho + Q$, where Q is a C^2 vector field on $\overline{M}_R$ that is tangent to ∂M. Choose a smooth embedding $X = (X^1, \dots, X^N) \colon Y \hookrightarrow \mathbb{R}^N$ into some Euclidean space, and denote by the same symbol the extension of each coordinate function X^A to $\overline{M} = Y \times [0,1)$, chosen to be constant along the $[0,1)$ factor. By [3, Cor. 3.3.2] again, for each $A = 1, \dots, N$, there is a C^3 function $\widetilde{X}^A \colon \overline{M} \to \mathbb{R}$, smooth in M, satisfying

$$\begin{aligned} \widetilde{X}^A|_{\partial M} &= X^A|_{\partial M}, \\ \partial_\rho \widetilde{X}^A|_{\partial M} &= -QX^A|_{\partial M}, \\ \partial_\rho^2 \widetilde{X}^A|_{\partial M} &= -\partial_\rho Q X^A|_{\partial M}. \end{aligned}$$

It follows that $P\widetilde{X}^A = O(\rho^2)$. The C^3 map $\widetilde{X} \colon \overline{M} \to \mathbb{R}^N$ whose coordinate functions are $(\widetilde{X}^1, \dots, \widetilde{X}^N)$ thus satisfies $\widetilde{X}_* P = O(\rho^2)$.

We wish to use $\widetilde{X}$ and r to construct a collar diffeomorphism of $\overline{M}_R$. However, $\widetilde{X}$ might not map into $X(Y)$. To correct this, let $U \subset \mathbb{R}^N$ be a tubular neighborhood of $X(Y)$, and let $\Pi\colon U \to X(Y)$ be a smooth retraction. Define a C^3 map $Z\colon \overline{M}_R \to \overline{M}$ by

$$Z(x,\rho) = (X^{-1} \circ \Pi \circ \widetilde{X}(x,\rho), r(x,\rho)).$$

The restriction of Z to ∂M is the identity, and for some small $\varepsilon > 0$, Z is an embedding of $\overline{M}_\varepsilon$ into M.

Let $p \in M_R$ be arbitrary, and let $q = Z(p)$. Writing $Z_* g = (Z^{-1})^* g$, we conclude from (3.1) that

$$\begin{aligned}
|d\rho(q)|^2_{\rho^2 Z_* g} &= \rho(q)^{-2} |d\rho(q)|^2_{Z_* g} \\
&= \rho(Z(p))^{-2} |d(\rho \circ Z)(p)|^2_g \\
&= r(p)^{-2} |dr(p)|^2_g \\
&= |dr(p)|^2_{r^2 g} = 1 + O(\rho(p)^2) = 1 + O(\rho(q)^2).
\end{aligned}$$

To check the mixed tangential/normal components of $\rho^2 Z_* g$, let (θ^α, ρ) be any background coordinates on $\overline{M}_R$, and let $Z^\alpha = \theta^\alpha \circ Z$ denote the tangential component functions of Z in these coordinates. Observe that $d\theta^\alpha(Z_* P) = d\theta^\alpha(X_*^{-1} \circ \Pi_* \circ \widetilde{X}_* P)$. Because the component functions of Π_* (as a map from $\mathbb{R}^N$ to itself) are uniformly bounded, as are those of X_*^{-1} in background coordinates, it follows that the tangential components PZ^α of $Z_* P$ in background coordinates are $O(\rho^2)$. Thus

$$\begin{aligned}
\langle d\rho(q), d\theta^\alpha(q) \rangle_{\rho^2 Z_* g} &= \rho(q)^{-2} \langle d\rho(q), d\theta^\alpha(q) \rangle_{Z_* g} \\
&= \rho(Z(p))^{-2} \langle d(\rho \circ Z)(p), d(\theta^\alpha \circ Z)(p) \rangle_g \\
&= \langle dr(p), dZ^\alpha(p) \rangle_{r^2 g} \\
&= P_p Z^\alpha = O(\rho(p)^2) = O(\rho(q))^2.
\end{aligned}$$

We define our collar diffeomorphism by $G = Z^{-1}|_{M_{R_0}}$ for R_0 sufficiently small, and let $\widehat{g} = \rho^2 G^* g = \rho^2 Z_* g$. By construction, in any background coordinates,

$$|d\rho|^2_{\widehat{g}} = 1 + O(\rho^2) \tag{3.2}$$

$$\langle d\rho, d\theta^\alpha \rangle_{\widehat{g}} = O(\rho^2). \tag{3.3}$$

Inverting the coordinate matrix of $\widehat{g}$, therefore, we find that

$$\widehat{g} = d\rho^2 + \widehat{g}_{\alpha\beta}(\theta,\rho) d\theta^\alpha d\theta^\beta + O(\rho^2)$$

for some functions $\widehat{g}_{\alpha\beta}$ that are C^2 up to ∂M. Moreover, because the restriction of G to ∂M is the identity and $G^*\rho = \rho + O(\rho^2)$, $\widehat{g}_{\alpha\beta} = \widetilde{\gamma}_{\alpha\beta} = \overline{h}_{\alpha\beta}$ at points of ∂M.

To conclude the proof, we will use the Einstein equation to show that $\partial_\rho \widehat{g}_{\alpha\beta} = 0$ along ∂M, which implies $\widehat{g} = \overline{h} + O(\rho^2)$ as desired. In terms of $\widehat{g}$, the Einstein equation for G^*g translates to

$$-n\rho^{-2}\widehat{g}_{jk} = \widehat{R}_{jk} + (n-1)\rho^{-1}\rho_{;jk} + \rho^{-1}\rho_{;l}{}^{l}\widehat{g}_{jk} - n\rho^{-2}\rho_{;l}\rho_{;}{}^{l}\widehat{g}_{jk},$$

where the semicolons indicate covariant derivatives, all taken with respect to $\widehat{g}$ (cf. [12, p. 266]). Multiplying by ρ, using (3.2), and evaluating at $\rho = 0$, we obtain $(n-1)\rho_{;jk} + \rho_{;l}{}^{l}\widehat{g}_{jk} = 0$ along ∂M. Taking the trace with respect to $\widehat{g}$, we find that $\rho_{;l}{}^{l} = 0$ and therefore $\rho_{;jk} = 0$ along ∂M. Expanding this equation in terms of the Christoffel symbols of $\widehat{g}$ in background coordinates, we conclude that $\partial_\rho \widehat{g}_{\alpha\beta} = 0$ along ∂M as claimed. q.e.d.

4. The harmonic map normalization

In this section, we will show that g can be modified by a collar diffeomorphism so that it satisfies the elliptic equation (1.2) near the boundary. We seek a collar diffeomorphism that is harmonic from (M_R, g) to (M_R, h), where h is the smoothly conformally compact metric defined in the preceding section. In order to find one, we will parameterise the diffeomorphisms near the identity by small vector fields using the Riemannian exponential map of h.

For any small $R > 0$, let $\partial_R M_R = Y \times \{R\}$ denote the "inner boundary" of $\overline{M}_R$, and let $\mathring{C}^{k,\lambda}_{\delta}(M_R; TM)$ denote the set of vector fields in $C^{k,\lambda}_{\delta}(M_R; TM)$ that vanish on $\partial_R M_R$. If $v \in \mathring{C}^{k,\lambda}_{\delta}(M_R; TM)$, define a map $H_v \colon M_R \to M$ by

$$H_v(p) = \exp_p(v(p)),$$

where exp denotes the Riemannian exponential map of h. Since conformally compact metrics are complete at infinity [15], H_v is well-defined as a map from M_R into M as long as both R and v are sufficiently small.

Let us call a map $H \colon M_R \to M_R$ *admissible* if for each Möbius chart, H maps $\overline{V_1(p_i)}$ into $V_2(p_i)$. Because h is uniformly equivalent to the Euclidean metric in Möbius coordinates, for any admissible map H, the Riemannian distance $d_h(p, H(p))$ is uniformly bounded for $p \in M_R$. This implies that $d_{\overline{h}}(p, H(p)) \to 0$ uniformly as $p \to \partial M$, which in turn implies that any admissible map has a continuous extension to ∂M that fixes ∂M pointwise.

Lemma 4.1. *If $\delta \geq 0$ and v is sufficiently small in $\mathring{C}^{1,0}_{\delta}(M_R; TM)$, then H_v is an admissible map from M_R to itself.*

Proof. Note that

$$d_h(p, H_v(p)) = d_h(p, \exp_p(v(p))) \le |v(p)|_h. \tag{4.1}$$

Because d_h is uniformly equivalent to Euclidean distance in Möbius coordinates, it follows that H_v will be an admissible map if $\|v\|_{0,0,\delta}$ is sufficiently small (actually, for fixed $\delta > 0$, it suffices to choose R small enough), provided that H_v maps M_R to itself.

To establish that $H_v(M_R) \subset M_R$, assume that $\|v\|_{1,0,\delta}$ is small enough that $|\nabla v|_h \le \frac{1}{2}$ on M_R. Let $p \in M_R$ be arbitrary, and let $\gamma\colon [0,b] \to M$ be a unit-speed distance-minimizing h-geodesic from $\partial_R M_R$ to p. At any time t such that $v(\gamma(t)) \ne 0$, $|v(\gamma(t))|_h$ is a differentiable function of t and

$$\begin{aligned}\frac{d}{dt}|v(\gamma(t))|_h &= \frac{2\left\langle \nabla_{\gamma'(t)} v, v(\gamma(t))\right\rangle_h}{2|v(\gamma(t))|_h}\\ &\le |\nabla_{\gamma'(t)} v|_h \le \tfrac{1}{2}.\end{aligned}$$

Setting $f(t) = |v(\gamma(t))| - \frac{1}{2}t$, we see that f is continuous on $[0,b]$ and satisfies $f(0) = 0$ (recall that $v = 0$ on $\partial_R M_R$). At any t such that $|v(\gamma(t))| \ne 0$, f is differentiable and satisfies $f'(t) \le 0$ by the computation above, so that $f(t) \le 0$ for all $t \in [0,b]$. In particular, taking $t = b = d_h(p, \partial_R M_R)$, this implies that $|v(p)|_h \le \frac{1}{2} d_h(p, \partial_R M_R)$ for all p, which together with (4.1) implies that H_v maps M_R to itself. q.e.d.

Let Σ^2 denote the bundle of symmetric covariant 2-tensors over M. For any section w of Σ^2, write $g_w = h + w$. For any $0 < \lambda < 1$, define a map

$$\Theta\colon \mathring{C}^{2,\lambda}_{1+\lambda}(M_R; TM) \times C^{1,\lambda}_{1+\lambda}(M_R; \Sigma^2) \to C^{0,\lambda}_{1+\lambda}(M_R; TM) \times C^{1,\lambda}_{1+\lambda}(M_R; \Sigma^2)$$

by

$$\Theta(v, w) = \left((H_v)_*^{-1}(\Delta_{g_w h} H_v), w\right),$$

where $\Delta_{g_w h} H_v$ denotes the harmonic map Laplacian of H_v, viewed as a map from (M_R, g_w) to (M_R, h).

Lemma 4.2. *The map Θ is well-defined and of class C^1 in a neighborhood of $(0,0)$ in $\mathring{C}^{2,\lambda}_{1+\lambda}(M_R; TM) \times C^{1,\lambda}_{1+\lambda}(M_R; \Sigma^2)$.*

Proof. When v and w are understood, let us write $g = g_w$ and $H = H_v$ for brevity. Recall that g is uniformly $C^{1,\lambda}$ equivalent to the Euclidean metric in Möbius coordinates; a similar statement applies to h, but in that case we have uniform C^m equivalence for every m.

At any point $p \in M_R$, the value of the harmonic map Laplacian $\Delta_{gh} H$ at p is an element of $T_{H(p)}M$, so $H_*^{-1}(\Delta_{gh} H|_p) \in T_p M$. Thus the first component of $\Theta(v, w)$ does in fact define a section of TM as claimed.

For this proof, we will denote Möbius coordinates generically by x or $(x^j) = (x^1, \dots, x^{n+1})$, and the associated standard fiber coordinates on TM by v or $(v^j) = (v^1, \dots, v^{n+1})$. Letting $E^j(x, v)$ denote the (smooth) component functions of the h-exponential map in Möbius coordinates, we see that H has component functions given by

$$H^j(x) = E^j(x, v(x)).$$

Because $E^j(x, 0) = x^j$, it follows from Lemma 2.1(e) with $f = E^j$ and $u_0(x) = (x, 0)$ that the functions $A^j(x) = H^j(x) - x^j$ satisfy the following uniform bound for sufficiently small $v \in \mathring{C}^{2,\lambda}_{1+\lambda}(M_R; TM)$:

$$\|A^j\|_{2,\lambda;\overline{B}_1} \le C\|\Phi_i^* v\|_{2,\lambda;\overline{B}_1} \le C\rho(p_i)^{1+\lambda}\|v\|_{2,\lambda,1+\lambda}.$$

Calculating in Möbius coordinates, we have

$$(\Delta_{gh} H)^j = g^{kl}\left(-\partial_k\partial_l H^j + \Gamma^m_{kl}\partial_m H^j - (\Pi^j_{mq}\circ H)\partial_k H^m \partial_l H^q\right), \tag{4.2}$$

where Γ^m_{kl} are the Christoffel symbols of g and Π^j_{mq} are those of h. Note that we can write the difference $\Gamma^m_{kl} - \Pi^m_{kl}$ as follows:

$$\begin{aligned}
\Gamma^m_{kl} - \Pi^m_{kl} &= \tfrac{1}{2} g^{mj}\left(\partial_k g_{lj} + \partial_l g_{kj} - \partial_j g_{kl}\right)\\
&\qquad - \tfrac{1}{2} h^{mj}\left(\partial_k h_{lj} + \partial_l h_{kj} - \partial_j h_{kl}\right)\\
&= \tfrac{1}{2} g^{mj}\left(\partial_k w_{lj} + \partial_l w_{kj} - \partial_j w_{kl}\right)\\
&\qquad + \tfrac{1}{2}(g^{mj} - h^{mj})\left(\partial_k h_{lj} + \partial_l h_{kj} - \partial_j h_{kl}\right).
\end{aligned}$$

By virtue of Lemma 2.1 again, this time with f equal to the mj-component of the map taking an $(n+1)\times(n+1)$ matrix to its inverse, this last expression is in $C^{0,\lambda}(\overline{B}_1)$, with C^1 dependence on w, and satisfies an estimate of the form

$$\|\Gamma^m_{kl} - \Pi^m_{kl}\|_{0,\lambda;\overline{B}_1} \le C\rho(p_i)^{1+\lambda}\|w\|_{1,\lambda,1+\lambda}.$$

Now rewrite (4.2) as follows:

$$\begin{aligned}
(\Delta_{gh} H)^j = g^{kl}\Big(&-\partial_k\partial_l A^j + (\Gamma^m_{kl} - \Pi^m_{kl})\partial_m H^j + \Pi^m_{kl}\partial_m A^j - \Pi^j_{ml}\partial_k A^m\\
&- \Pi^j_{mq}\partial_k H^m \partial_l A^q + (\Pi^j_{mq}\circ \mathrm{Id} - \Pi^j_{mq}\circ H)\partial_k H^m \partial_l H^q\Big).
\end{aligned}$$

Another application of Lemma 2.1 (with $f = \Pi^j_{mq}$ for the last term) shows that this expression is in $C^{0,\lambda}(\overline{B}_1)$, with C^1 dependence on v and w, and with $C^{0,\lambda}$ norm bounded by a multiple of $\rho(p_i)^{1+\lambda}\left(\|v\|_{2,\lambda,1+\lambda} + \|w\|_{1,\lambda,1+\lambda}\right)$. Finally, since the pushforward map $H_*^{-1}\colon T_{H(p)}M \to T_p M$ is represented by the inverse of the matrix

$\partial H^j/\partial x^k(p) = \delta^j_k + \partial A^j/\partial x^k(p)$, one last application of Lemma 2.1 shows that Θ is a C^1 map as claimed. q.e.d.

Lemma 4.3. *If R is sufficiently small, the differential $D\Theta_{(0,0)}$ is a Banach space isomorphism from $\mathring{C}^{2,\lambda}_{1+\lambda}(M_R;TM) \times C^{1,\lambda}_{1+\lambda}(M_R;\Sigma^2)$ to $C^{0,\lambda}_{1+\lambda}(M_R;TM) \times C^{1,\lambda}_{1+\lambda}(M_R;\Sigma^2)$.*

Proof. At $(v,w) = (0,0)$, the differential of Θ can be computed as follows:

$$\begin{aligned} D\Theta_{(0,0)}(v,w) &= \left(\frac{\partial}{\partial t}\bigg|_{t=0} \big((H_{tv})_*^{-1}(\Delta_{hh} H_{tv})\big) + \frac{\partial}{\partial t}\bigg|_{t=0} (\Delta_{g_{tw}h}\,\mathrm{Id}), w\right) \\ &= (Lv + Aw, w), \end{aligned}$$

where L is the linearisation of the harmonic map Laplacian Δ_{hh} about the identity map, and A is some first-order linear differential operator that is bounded from $C^{1,\lambda}_{1+\lambda}(M_R;\Sigma^2)$ to $C^{0,\lambda}_{1+\lambda}(M_R;TM)$. Clearly this is invertible if and only if $L\colon \mathring{C}^{2,\lambda}_{1+\lambda}(M_R;TM) \to C^{0,\lambda}_{1+\lambda}(M_R;TM)$ is invertible.

A computation shows that $L = \nabla_h^*\nabla_h - \mathrm{Ric}(h)$. Because $\mathrm{Ric}(h)$ approaches $-nh$ at ∂M, it is straightforward to check that (in the terminology of [13]) the characteristic exponents of L are

$$s = 0,\ n+2,\ \frac{n+2 \pm \sqrt{n^2+8n}}{2}.$$

It follows that L has indicial radius $R = (n+2)/2$, and therefore by [13, Theorem C and Section 7], it is Fredholm as an operator from $C^{k+2,\lambda}_\delta(M;TM)$ to $C^{k,\lambda}_\delta(M;TM)$ for all $k \geq 0$, $0 < \lambda < 1$, and $-1 < \delta < n+1$. Moreover, [13, Lemma 7.12] shows that for a 1-form u supported in M_R,

$$(u, \nabla_h^*\nabla_h u) \geq \left(\frac{n^2}{4} + 1 - \varepsilon\right) \|u\|^2_{0,2},$$

where ε can be made as small as desired by taking R small. Since the operator $\nabla_h^*\nabla_h$ commutes with the index-raising isomorphism between 1-forms and vector fields, the same result holds for vector fields. It follows that $L \sim \nabla_h^*\nabla_h + n$ satisfies an a priori L^2 estimate of the form

$$\|v\|_{L^2} \leq C\|Lv\|_{L^2} \quad \text{for all } v \in C^\infty_c(M_R;TM)$$

when R is sufficiently small. Then the same argument as in the proof of Theorem C of [13] implies that $L\colon \mathring{C}^{k+2,\lambda}_\delta(M_R;TM) \to C^{k,\lambda}_\delta(M_R;TM)$ is an isomorphism for $-1 < \delta < n+1$; the only modification that needs to be made is to handle the Dirichlet boundary condition on the inner boundary $\partial_R M_R$, but as L is uniformly elliptic there, the required estimates follow easily from the standard theory of elliptic boundary value problems. q.e.d.

Now suppose g satisfies the hypotheses of Theorem A, and let $w = G^*g - h$, where G is given by Lemma 3.2. Let $\psi\colon \mathbb{R} \to [0,1]$ be a smooth cutoff function such that $\psi(t) \equiv 1$ for $t \le \frac{1}{2}$ and $\psi(t) \equiv 0$ for $t \ge 1$. For any small $s > 0$, define $\psi_s \in C^\infty(\overline{M}_R)$ by

$$\psi_s(p) = \psi\left(\frac{\rho(p)}{s}\right),$$

Then we define $w_s = \psi_s w$. Observe that $g_{w_s} = G^*g$ on the subset $M_{s/2}$ where $\psi_s \equiv 1$.

Lemma 4.4. *For any fixed small $R > 0$ and any $0 < \lambda < 1$, $w_s \to 0$ in $C^{1,\lambda}_{1+\lambda}(M_R; \Sigma^2)$ as $s \to 0$.*

Proof. It is shown in [13, Lemma 3.8] that ψ_s is uniformly bounded in $C^{1,\lambda}(M_R)$, with norm independent of s. Because ψ_s is supported in M_s, it follows from Lemma 2.1(b) that

$$\begin{aligned} \|w_s\|_{1,\lambda,1+\lambda} &\le C \sup_i \rho(p_i)^{-1-\lambda} \|\Phi_i^* w_s\|_{1,\lambda;\overline{B}_2} \\ &\le C \sup_{\{i:\rho(p_i)\le s\}} \rho(p_i)^{-1-\lambda} \|\Phi_i^* w\|_{1,\lambda;\overline{B}_2}. \end{aligned} \tag{4.3}$$

Writing w in background coordinates, we have

$$w = \rho^{-2}\overline{w}_{jk}(\theta,\rho) d\theta^j\, d\theta^k,$$

with $\overline{w}_{jk}(\theta,\rho) = O(\rho^2)$ and $\partial_l \overline{w}_{jk}(\theta,\rho) = O(\rho)$. Pulling back in Möbius coordinates, we obtain

$$\Phi_i^* w = y^{-2}\overline{w}_{jk}\big(\rho(p_i)x + \theta(p_i), \rho(p_i)y\big) dx^j\, dx^k.$$

It follows easily that the component functions of $\Phi_i^* w$ together with their first and second derivatives are bounded by a constant multiple of $\rho(p_i)^2$, and so $\|\Phi_i^* w\|_{1,\lambda;\overline{B}_2} \le C\rho(p_i)^2$. Inserting this into (4.3) completes the proof. q.e.d.

Theorem 4.5. *With g and h as above, for any $0 < \lambda < 1$, there exists a $C^{2,\lambda}$ collar diffeomorphism $\Psi\colon \overline{M}_R \to \overline{M}$ such that $\Psi^* g - h \in C^{1,0}_{1+\lambda}(M_R; \Sigma^2)$ and $\widetilde{g} = \Psi^* g$ satisfies (1.2) on M_{R_0} for some $0 < R_0 < R$.*

Proof. Let Θ and w_s be defined as above. It follows from Lemma 4.3 and the inverse function theorem that Θ is a bijection from a neighborhood of $(0,0)$ in $\mathring{C}^{2,\lambda}_{1+\lambda}(M_R; TM) \times C^{1,\lambda}_{1+\lambda}(M_R; \Sigma^2)$ to a neighborhood of $(0,0)$ in $C^{0,\lambda}_{1+\lambda}(M_R;TM) \times C^{1,\lambda}_{1+\lambda}(M_R;\Sigma^2)$. By Lemma 4.4, therefore, we can choose s small enough that $(0,w_s) = \Theta(v,w)$ for some $(v,w) \in \mathring{C}^{2,\lambda}_{1+\lambda}(M_R;TM) \times C^{1,\lambda}_{1+\lambda}(M_R;\Sigma^2)$. This is equivalent to the assertion that $w = w_s$ and H_v is harmonic from (M_R, g_{w_s}) to (M_R, h). Because $g_{w_s} = G^*g$ on $M_{s/2}$, H_v is also harmonic from $(M_{s/2}, G^*g)$ to $(M_{s/2}, h)$.

Because locally C^1 harmonic maps are smooth by elliptic regularity, $H_v\colon M_R \to M_R$ is smooth. Because its component functions $H^j(x)$ in Möbius coordinates differ from x^j by functions $A^j(x)$ that can be made as small as desired in $C^{2,\lambda}(\overline{B}_2)$ (by taking s sufficiently small), it follows that H_v is a diffeomorphism from M_R to itself, and by Lemma 4.1 it is an admissible map and therefore extends to a homeomorphism of $\overline{M}_R$ fixing ∂M pointwise, i.e., a collar diffeomorphism.

Define $\Psi = G \circ H_v^{-1}\colon M_R \to M$, and let $\widetilde{g} = \Psi^*g$. By the diffeomorphism invariance of the Einstein equation, $\widetilde{g}$ is Einstein; and by the diffeomorphism invariance of the harmonic map equation, the identity map is harmonic on $M_{s/2}$ from $\widetilde{g} = \left(H_v^{-1}\right)^* (G^*g)$ to h. Thus $\widetilde{g}$ satisfies (1.2) on $M_{s/2}$.

Next we will show that $\Psi^*g - h \in C^{1,0}_{1+\lambda}(M_R;\Sigma^2)$. Since $G^*g - h = w \in C^{1,\lambda}_{1+\lambda}(M_R;\Sigma^2)$ by the proof of Lemma 4.4, it suffices to show that $(H_v^{-1})^*G^*g - G^*g \in C^{1,\lambda}_{1+\lambda}(M_R;\Sigma^2)$. Let us abbreviate G^*g by $\widehat{g}$ and H_v^{-1} by $K\colon M_R \to M_R$. Again, by taking s (and thus also A^j) small enough, we can ensure that $H_v(V_2(p_i))$ contains $\overline{V}_1(p_i)$ for each Möbius chart, and therefore K is an admissible map. We will write the component functions of H_v as $H^j(x) = x^j + A^j(x)$ as above, with

$$\|A^j\|_{2,\lambda;\overline{B}_2} \le C\rho(p_i)^{1+\lambda}.$$

If s is chosen small enough, then the functions A^j will be uniformly small in $C^{2,\lambda}(\overline{B}_2)$, so the Jacobian matrix $\partial_k H^j(x)$ will be uniformly invertible on $\overline{B}_2$, independent of the choice of Möbius chart. Let us write $(I^j_k(x))$ for the components of the inverse matrix of $(\partial_k H^j(x))$. Because matrix inversion is continuous in the $C^{1,\lambda}$ norm by Lemma 2.1(d), the functions I^j_k are uniformly bounded in $C^{1,\lambda}(\overline{B}_2)$.

The chain rule shows that

$$\partial_k K^j(x) = I^j_k(K(x)), \tag{4.4}$$

which is continuous and uniformly bounded on $\overline{B}_1$. But this implies that K^j is uniformly C^1, and using (4.4) we conclude successively that $\partial_k K^j(x)$ is uniformly C^1, K^j is uniformly C^2, $\partial_k K^j(x)$ is uniformly $C^{1,\lambda}$ (by Lemma 2.1(c)), and thus K^j is uniformly $C^{2,\lambda}$ on $\overline{B}_1$.

Let us write $B^j(x) = K^j(x) - x^j$. The fact that $H_v \circ K = \mathrm{Id}$ translates in Möbius coordinates to

$$K^j(x) + A^j(K(x)) = x^j,$$

so $B^j(x) = -A^j(K(x))$, and applying Lemma 2.1(c) one more time, we conclude that

$$\|B^j\|_{2,\lambda;\overline{B}_1} = \|A^j \circ K\|_{2,\lambda;\overline{B}_1} \le C\|A^j\|_{2,\lambda;\overline{B}_2} \le C'\rho(p_i)^{1+\lambda}. \tag{4.5}$$

Now to show that $K^*\widehat{g} - \widehat{g} \in C^{1,0}_{1+\lambda}(M_R; \Sigma^2)$, we just compute in Möbius coordinates:

$$\begin{aligned}
K^*\widehat{g} - \widehat{g} &= \widehat{g}_{jk}(K(x))(dx^j + dB^j)(dx^k + dB^k) - \widehat{g}_{jk}(x)dx^j\,dx^k \\
&= \Big(\widehat{g}_{jk}(K(x)) - \widehat{g}_{jk}(x)\Big)dx^j\,dx^k \\
&\qquad + 2\widehat{g}_{jk}(K(x))\frac{\partial B^k}{\partial x^q}dx^j\,dx^q + \widehat{g}_{jk}(K(x))\frac{\partial B^j}{\partial x^m}\frac{\partial B^k}{\partial x^q}dx^m\,dx^q.
\end{aligned}$$

Because $\rho^2\widehat{g} \in C^2(\overline{M})$, the component functions $\widehat{g}_{jk}$ are uniformly bounded in $C^2(\overline{B}_2)$ by the properties of Möbius coordinates. The same is true of $\widehat{g}_{jk} \circ K$ by composition. The last two terms above thus have C^1 norms uniformly bounded by a constant multiple of $\rho(p_i)^{1+\lambda}$. Differentiating the first term, we obtain

$$\begin{aligned}
\partial_l\left(\widehat{g}_{jk} \circ K - \widehat{g}_{jk}\right) &= \left(\partial_m\widehat{g}_{jk} \circ K\right)\frac{\partial K^m}{\partial x^l} - \partial_l\widehat{g}_{jk} \\
&= \left(\partial_m\widehat{g}_{jk} \circ K\right)\frac{\partial B^m}{\partial x^l} + \left(\partial_l\widehat{g}_{jk} \circ K - \partial_l\widehat{g}_{jk}\right).
\end{aligned}$$

The first term is uniformly bounded by a multiple of $\rho(p_i)^{1+\lambda}$ thanks to (4.5). The same is true of the second term by a simple application of the mean value theorem and the fact that $K^j(x) - x^j = B^j(x)$. This completes the proof that $\Psi^*g - h \in C^{1,0}_{1+\lambda}(M_R; \Sigma^2)$.

It remains only to show that Ψ has a $C^{2,\lambda}$ extension to $\overline{M}_R$. Because G is C^3 on $\overline{M}_R$ by construction, it suffices to consider $H_v^{-1} = K$. Choose a fixed $p_i \in M_R$ and corresponding Möbius chart Φ_i. We can write the map K either in Möbius coordinates, with coordinate functions denoted by (K^j) as above:

$$K(x^j) = \left(K^1(x^j), \ldots, K^{n+1}(x^j)\right),$$

or in background coordinates $(\theta^j) = (\theta^1, \ldots, \theta^n, \rho)$, with coordinate functions that we will denote by $(\overline{K}^j)$:

$$K(\theta^j) = \left(\overline{K}^1(\theta^j), \ldots, \overline{K}^{n+1}(\theta^j)\right).$$

The two coordinate representations are related by

$$\overline{K}^m(\theta^j) = c^m + \rho(p_i) K^m\left(\frac{\theta^j - c^j}{\rho(p_i)}\right),$$

where c^j are constants defined by $(c^1, \ldots, c^{n+1}) = (\theta^1(p_i), \ldots, \theta^n(p_i), 0)$. Using once again the fact that $K^j(x) = x^j + B^j(x)$ with B^j satisfying (4.5), we compute

$$\frac{\partial \overline{K}^m}{\partial \theta^k} = \delta^m_k + \frac{\partial B^m}{\partial x^k}\left(\frac{\theta^j - c^j}{\rho(p_i)}\right),$$

so both $\overline{K}^m$ and $\partial \overline{K}^m/\partial x^k$ are uniformly bounded. Differentiating once more, we find

$$\frac{\partial^2 \overline{K}^m}{\partial \theta^k \partial \theta^l} = \rho(p_i)^{-1} \frac{\partial^2 B^m}{\partial x^k \partial x^l}\left(\frac{\theta^j - c^j}{\rho(p_i)}\right),$$

and therefore,

$$\begin{aligned}
&\left|\frac{\partial^2 \overline{K}^m}{\partial \theta^k \partial \theta^l}(\theta^j) - \frac{\partial^2 \overline{K}^m}{\partial \theta^k \partial \theta^l}(\widetilde{\theta}^j)\right| \\
&\qquad = \rho(p_i)^{-1}\left|\frac{\partial^2 B^m}{\partial x^k \partial x^l}\left(\frac{\theta^j - c^j}{\rho(p_i)}\right) - \frac{\partial^2 B^m}{\partial x^k \partial x^l}\left(\frac{\widetilde{\theta}^j - c^j}{\rho(p_i)}\right)\right| \\
&\qquad \le \rho(p_i)^{-1}\|B^m\|_{2,\lambda;\overline{B}_2}\left|\frac{\theta^j - \widetilde{\theta}^j}{\rho(p_i)}\right|^\lambda \\
&\qquad \le C\left|\theta^j - \widetilde{\theta}^j\right|^\lambda,
\end{aligned}$$

which shows that $\overline{K}^m$ is uniformly $C^{2,\lambda}$ up to the boundary as claimed. q.e.d.

5. Polyhomogeneity

Let $U_0 \subset \mathbb{R}^n$ be an open set, and let $U = U_0 \times (0, \varepsilon) \subset \mathbb{H}^{n+1}$. For any $\delta \in \mathbb{R}$, we denote by $\mathscr{C}^\delta$ the space of functions $f \in C^\infty(U)$ that satisfy, on any subset $K \times (0, \varepsilon_0)$ with $K \subset U_0$ compact and $0 < \varepsilon_0 < \varepsilon$, estimates of the following form for all integers $r \ge 0$ and all multi-indices α:

$$|(y\partial_y)^r \partial_x^\alpha f(x,y)| \le C_{r,\alpha} y^\delta.$$

(We use the multi-index notations $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\partial_x^\alpha = (\partial_{x^1})^{\alpha_1} \ldots (\partial_{x^n})^{\alpha_n}$.)

A smooth function $f\colon U \to \mathbb{R}$ is said to be *polyhomogeneous* (cf. [3, 16]) if there exists a sequence of real numbers $s_i \to +\infty$, a sequence of nonnegative integers $\{q_i\}$, and functions $f_{ij} \in C^\infty(U_0)$ such that

$$f(x,y) \sim \sum_{i=1}^{\infty} \sum_{j=0}^{q_i} y^{s_i} (\log y)^j f_{ij}(x) \tag{5.1}$$

in the sense that for any $\delta > 0$, there exists a positive integer N such that

$$f(x,y) - \sum_{i=1}^{N} \sum_{j=0}^{q_i} y^{s_i} (\log y)^j f_{ij}(x) \in \mathscr{C}^\delta.$$

A function or tensor field on M_R is said to be polyhomogeneous if its coordinate representation in every background chart is polyhomogeneous. (Note that the definition of polyhomogeneity is phrased somewhat differently in [3], but it is easy to verify that the two definitions are equivalent. Note also that there is a misprint in the first displayed inequality of [3, Section 3.1.5]: ∂_y^α in that inequality should be replaced by ∂_v^α.)

In this section, we will apply the theory of [3] to conclude that solutions to (1.2) are polyhomogeneous. A key step in the proof will be a regularity result for the linearised operator $D_1 Q_{(h,h)} = \frac{1}{2}(\Delta_L + 2n)$ from [13]. Following [3], we say that an interval $(\delta_-, \delta_+) \subset \mathbb{R}$ is a (*weak*) *regularity interval* for a second-order linear operator P on the spaces $C^{k,\lambda}_\delta(M_R; \Sigma^2)$ if whenever u is a locally C^2 section of Σ^2 such that $u \in C^{0,0}_{\delta_0}(M_R; \Sigma^2)$ and $Pu \in C^{0,\lambda}_\delta(M_R; \Sigma^2)$ with $\lambda \in (0,1)$ and $\delta_- < \delta_0 < \delta < \delta_+$, it follows that $u \in C^{2,\lambda}_\delta(M_R; \Sigma^2)$. (We caution the reader that the notations for weighted Hölder spaces used in [3] are different from those of [13] that we are using here, in terms of index positions and normalization of weights. The space of sections of Σ^2 that is denoted by $C^\delta_{k+\lambda}(M)$ in [3] is equal to the space that we would call $C^{k,\lambda}_{\delta+2}(M_R; \Sigma^2)$. The difference between the weight factors δ and $\delta + 2$ arises because we measure the size of the component functions in Möbius coordinates, while [3] measures them in background coordinates. To avoid confusion, we will use the notation $AC^\delta_{k+\lambda}(M)$ for the space denoted by $C^\delta_{k+\lambda}(M)$ in [3], so that $AC^\delta_{k+\lambda}(M) = C^{k,\lambda}_{\delta+2}(M_R; \Sigma^2)$. The condition that we have defined here would be expressed in [3] by saying that $(\delta_- - 2, \delta_+ - 2)$ is a regularity interval for P on the spaces $AC^\delta_{k+\lambda}(M)$.)

Theorem 5.1. *With $\widetilde{g} = \Psi^* g$ as in the preceding section, $\widetilde{g}$ is polyhomogeneous.*

Proof. For any small symmetric 2-tensor φ on M_R, define

$$F[\varphi] := \rho^2 Q(h + \rho^{-2}\varphi, h),$$

with Q as in (1.2). Then $\varphi = \rho^2(\widetilde{g} - h)$ satisfies $F[\varphi] = 0$. We wish to apply [3, Theorem 5.1.1] to F, and thereby conclude that φ is polyhomogeneous. To do so, we will consider $\varphi_0 \equiv 0$ as an approximate solution to $F[\varphi] = 0$, and check that F, φ, and φ_0 satisfy each of the hypotheses of Theorem 5.1.1.

(i) *F is a geometric operator in the sense of [3]*: This follows easily from the fact that F is an invariant operator on tensor fields. (Note that the notion of geometricity defined in [3] is considerably weaker than that of [13]; in particular, the definition in [3] does not require the coordinate expression of $F[\varphi]$ to depend only on the coefficients of a single metric and the covariant derivatives of its curvature.)

(ii) *F is quasilinear, and $F[\varphi]$ can be written in background coordinates as a smooth function of $\left(\rho, \theta^\alpha, \varphi_{ij}, \rho\partial_k\varphi_{ij}, \rho^2\partial_k\partial_l\varphi_{ij}\right)$*: This is easily seen by expanding $Q(\widetilde{g}, h)$ in background coordinates.

(iii) *$F[\varphi_0]$ is a smooth tensor field on $\overline{M}$ and an element of $AC_0^{\delta_0}(M) = C^{0,0}_{\delta_0+2}(M; \Sigma^2)$ for some $\delta_0 > 1$*: Observe that $F[\varphi_0] = F[0] = \rho^2 Q(h, h) = \rho^2(\operatorname{Ric}(h) + nh)$. Using the formula for the transformation of the Ricci tensor under a conformal change of metric (cf. [4, p. 59]) this can be written in background coordinates as

$$\begin{aligned} F[\varphi_0] &= \rho^2\left(R_{ij} + nh_{ij}\right) \\ &= \rho^2\overline{R}_{ij} + (n-1)\rho\rho_{;ij} + \rho\overline{h}^{kl}\rho_{;kl}\overline{h}_{ij} - n\overline{h}^{kl}\rho_{;k}\rho_{;l}\overline{h}_{ij} + n\overline{h}_{ij}, \end{aligned}$$

where the curvature and covariant derivatives are computed with respect to the smooth metric $\overline{h} = \rho^2 h$. Because of the way we constructed $\overline{h}$, $|d\rho|^2_{\overline{h}}$ is identically equal to 1 and $\rho_{;ij}$ is identically zero, so all terms after the first one cancel, showing that $F[\varphi_0] = \rho^2\overline{R}_{ij}$, which is smooth on $\overline{M}$ and $O(\rho^2)$ in background coordinates, and thus an element of $AC_0^2(M)$.

(iv) *$\varphi - \varphi_0 \in AC_1^\delta(M_R) = C^{1,0}_{\delta+2}(M_R; \Sigma^2)$ for some $\delta > 1$ and $R > 0$*: Since $\varphi - \varphi_0 = \varphi = \rho^2(\widetilde{g} - h)$, this is equivalent to the assertion that $\widetilde{g} - h \in C^{1,0}_\delta(M_R; \Sigma^2)$ for some $\delta > 1$, which is guaranteed by Theorem 4.5.

(v) *The linearised operator $F'[\varphi_0]$ is a geometric elliptic operator satisfying conditions* (4.2.1)–(4.2.4) *of [3]*: Actually, this is not true as stated, but something just as good is true. Note that

$F'[\varphi_0] = \rho^2 \circ D_1 Q_{(h,h)} \circ \rho^{-2} = \rho^2 \circ (\Delta_L + 2n) \circ \rho^{-2}$, which is certainly a geometric elliptic operator.

Define subbundles of Σ^2 as follows:

$$V_0 = \operatorname{span}(g);$$
$$V_1 = \left\{ q \in \Sigma^2 : \operatorname{Tr}_g q = 0,\ q(\operatorname{grad}\rho, \cdot) = 0 \right\};$$
$$V_2 = \operatorname{span}\left(g - \frac{n+1}{|d\rho|_g^2} d\rho \otimes d\rho \right);$$
$$V_3 = \left\{ d\rho \otimes \omega + \omega \otimes d\rho : \omega \in T^*M,\ \langle \omega, d\rho \rangle = 0 \right\}.$$

It is easy to check that Σ^2 admits an orthogonal decomposition $\Sigma^2 = V_0 \oplus V_1 \oplus V_2 \oplus V_3$. For $i = 0, \dots, 3$, let $\pi_i : \Sigma^2 \to V_i$ denote the orthogonal projection. The arguments of [9, pp. 199–202] show that $\Delta_L + 2n$ can be written in the form

$$\Delta_L + 2n = \sum_{i=0}^{3} P_i + \widetilde{P},$$

where in background coordinates $\widetilde{P}$ is of the form [3, (4.2.3)] and each P_i is an operator on sections of V_i that can be written

$$P_i = -\left(\rho^2 \partial_\rho^2 + (5-n)\rho\partial_\rho + B_i \right) \otimes \pi_i,$$

with $B_0 = B_2 = 4 - 2n$, $B_1 = 4 - 4n$, and $B_3 = 3 - 3n$. Thus

$$F'[\varphi_0] = \sum_{i=0}^{3} L_i + \widetilde{L},$$

where $\widetilde{L} = \rho^2 \circ \widetilde{P} \circ \rho^{-2}$ is again of the form [3, (4.2.3)], and

$$L_i = \rho^2 \circ P_i \circ \rho^{-2} = -\left(\rho^2 \partial_\rho^2 + (1-n)\rho\partial_\rho + b_i \right) \otimes \pi_i,$$

with $b_i = B_i - 4 < 0$. All of the arguments in [3, Sections 4 and 5] go through with only trivial changes if the ordinary differential operator denoted there by L_{ab} is replaced by the block-diagonal operator $L_0 \oplus L_1 \oplus L_2 \oplus L_3$.

(vi) *The interval* $(0,n)$ *is a regularity interval for the operator* $F'[\varphi_0]$ *on the spaces* $AC^\delta_{k+\lambda}(M_R)$: It is an immediate consequence of Lemma 6.4(b) of [13] that $(0,n)$ is a regularity interval for $\Delta_L + 2n$ on the spaces $C^{k,\lambda}_\delta(M;\Sigma_R)$, and it follows immediately from this that $(-2, n-2)$ is a regularity interval for $F'[\varphi_0] = \rho^2 \circ (\Delta_L + 2n) \circ \rho^{-2}$ on the same spaces. In the terminology of [3], this means that $(0,n)$ is a regularity interval on $AC^\delta_{k+\lambda}(M_R)$.

¿From [3, Theorem 5.1.1], therefore, we conclude that φ (and hence also $\widetilde{g}$) is polyhomogeneous. q.e.d.

6. The asymptotic expansion

To obtain the asymptotic expansion announced in Theorem A, we need to subject g to one more collar diffeomorphism. First, a preliminary lemma.

Lemma 6.1. *Suppose $\widetilde{g}$ is a Riemannian metric on M_R that is polyhomogeneous and conformally compact of class $C^{1,\lambda}$ for some $0<\lambda<1$, and satisfies $|d\rho|_{\rho^2\widetilde{g}} \to 1$ at ∂M. Then there exists a polyhomogeneous $C^{1,\lambda}$ defining function r such that $|dr|_{r^2\widetilde{g}} \equiv 1$ in a neighborhood of ∂M and $r/\rho \to 1$ at ∂M.*

Proof. Writing $\overline{g} = \rho^2\widetilde{g}$ and $r = \rho e^u$, we see that the conclusion is equivalent to

$$|d\rho|^2_{\overline{g}} + 2\rho\langle d\rho, du\rangle_{\overline{g}} + \rho^2|du|^2_{\overline{g}} = 1, \qquad u|_{\partial M} = 0.$$

It is shown in [12, Lemma 5.1] that this has a solution $u \in C^{2,\lambda}(\overline{M}_R;\mathbb{R})$ if $\overline{g}$ is $C^{3,\lambda}$ up to the boundary, by reducing it to finding the flow of the Hamiltonian vector field X_F, where $F\colon T^*\overline{M}_R \to \mathbb{R}$ is the function defined by

$$F(\theta,\xi) = 2\langle d\rho,\xi\rangle_{\overline{g}} + \rho|\xi|^2_{\overline{g}} - \frac{1-|d\rho|^2_{\overline{g}}}{\rho}.$$

(Here $(\theta,\xi) = (\theta^1,\dots,\theta^{n+1},\xi_1,\dots,\xi_{n+1})$ are standard coordinates on $T^*\overline{M}_R$ associated with background coordinates.) In the present situation, F is only of class $C^{0,\lambda}$ on $T^*\overline{M}_R$; but because it is polyhomogeneous, each of the following quantities is also $C^{0,\lambda}$ in each background chart:

$$\frac{\partial F}{\partial \xi_j},\ \frac{\partial F}{\partial \theta^\alpha},\ \rho\frac{\partial F}{\partial \rho},$$
$$\frac{\partial^2 F}{\partial \xi_j\partial\xi_k},\ \frac{\partial^2 F}{\partial \theta^\alpha\partial\theta^\beta},\ \frac{\partial^2 F}{\partial \xi_j\partial\theta^\alpha},\ \rho\frac{\partial^2 F}{\partial \theta^\alpha\partial\rho},\ \rho\frac{\partial^2 F}{\partial \xi_j\partial\rho},\ \rho^2\frac{\partial^2 F}{\partial \rho^2}.$$

Since the normal component of X_F satisfies

$$d\rho(X_F)|_{\partial T^*\overline{M}_R} = \left.\frac{\partial F}{\partial \xi_{n+1}}\right|_{\partial T^*\overline{M}_R} = 2\overline{g}^{n+1,n+1}|_{\partial T^*\overline{M}_R} = 2,$$

it follows that $V = \frac{1}{2}X_F$ satisfies the hypotheses of Lemma 6.3 below, so the flow-out by X_F from the boundary of $T^*\overline{M}_R$ exists and is polyhomogeneous and of class $C^{0,\lambda}$. The rest of the argument in [12, Lemma 5.1] then goes through to prove the existence of r as claimed. Moreover, the solution r so obtained is itself polyhomogeneous; matching lowest-order terms in the expansion of r with those in the expansion of $\overline{g}$, we find that $r \in C^{1,\lambda}(\overline{M}_R)$ as claimed. q.e.d.

It is clear from the proof of the preceding lemma that near any boundary point, one can choose new coordinates in which r is one of the coordinate functions and the metric is polyhomogeneous. In fact, we can do much better, as the next lemma shows.

Lemma 6.2. *If $\widetilde{g}$ satisfies the hypotheses of the preceding lemma, then for R sufficiently small, there exists a polyhomogeneous $C^{1,\lambda}$ collar diffeomorphism $\Gamma\colon \overline{M}_R \to \overline{M}$ such that $\Gamma^*\widetilde{g}$ has the form* (1.1).

Proof. Let r be the polyhomogeneous $C^{1,\lambda}$ defining function given by Lemma 6.1. As in the proof of Lemma 3.2, let P be the $r^2\widetilde{g}$-gradient of r, which is a polyhomogeneous $C^{0,\lambda}$ vector field on $\overline{M}_R$ whose normal component satisfies $d\rho(P) = 1$ along ∂M. Using Lemma 6.3 again, in each background coordinate chart Ω we obtain a uniquely determined $C^{0,\lambda}$ polyhomogeneous flow $(x,t) \mapsto \gamma_x(t)$, where for each $x \in \Omega \cap \partial M$, $\gamma_x\colon [0,\varepsilon] \to \overline{M}_R$ is the integral curve of P starting at x. Comparing lowest-order terms in the expansions of $\gamma_x(t)$ and P, we see that the flow is in fact $C^{1,\lambda}$ up to the boundary. The various maps thus obtained in different coordinate charts all agree where they overlap, so they patch together to define a global map $\Gamma\colon \partial M \times [0,\varepsilon] \to \overline{M}_R$. The inverse function theorem shows that Γ is a diffeomorphism in a neighborhood of $\partial M \times \{0\}$. Identifying $\partial M \times [0,\varepsilon]$ with $\overline{M}_\varepsilon$, we can view Γ as a collar diffeomorphism. It is easy to check that $\Gamma^*\widetilde{g}$ has the form (1.1) with $G(\rho)$ polyhomogeneous. q.e.d.

Here is the ODE lemma used in the proofs of Lemmas 6.1 and 6.2. It is adapted from Proposition B.1 in [6]. In this lemma, $\mathbb{H}^{m+1}$ denotes the upper half-space in $\mathbb{R}^{m+1}$, with coordinates $(x^1,\dots,x^m,y)$. In our application of this lemma in the proof of Lemma 6.1, the x^i-coordinates correspond to $(\theta^1,\dots,\theta^n,\xi_1,\dots,\xi_{n+1})$, and y corresponds to $\rho = \theta^{n+1}$.

Lemma 6.3. *Let U_0 be an open subset of $\mathbb{R}^m$, and let V be a C^1 vector field on $U = U_0 \times (0,\varepsilon) \subset \mathbb{H}^{m+1}$ of the form*

$$V = A^i(x,y)\frac{\partial}{\partial x^i} + (1 + B(x,y))\frac{\partial}{\partial y}.$$

Suppose that A^i and B satisfy the following estimates for some constants $C_0 > 0$ and $0 < \lambda < 1$:

$$\begin{aligned} &|B|,\ |\partial_{x^j}B|, |y\partial_y B| \le C_0 y^{\lambda}, \\ &|A^i|,\ |\partial_{x^j}A^i|,\ |y\partial_y A^i| \le C_0 y^{\lambda-1}. \end{aligned} \tag{6.1}$$

(a) *If K is any compact subset of U_0, there exists $\varepsilon > 0$ such that, for each $x_0 \in K$, there is a unique continuous solution $\gamma = \gamma_{x_0}$ on $[0,\varepsilon]$ to the initial value problem*

$$\begin{aligned} \gamma'(t) &= V(\gamma(t)), \\ \gamma(0) &= (x_0, 0). \end{aligned} \tag{6.2}$$

(b) *If the coefficient functions A^i and B are polyhomogeneous, then the map $(x,t) \mapsto \gamma_x(t)$ is polyhomogeneous and $C^{0,\lambda}$ up to the boundary.*

Proof. Let $x_0 \in K$ be arbitrary. Choose constants $C_1 > 0$ and $0 < \alpha < \lambda$, and let ε be a positive constant to be determined later. Let $\mathscr{X}$ denote the set of continuous maps $\gamma\colon [0,\varepsilon] \to \mathbb{R}^{m+1}$ of the form

$$\gamma(t) = (x_0 + a(t), t + b(t)) \tag{6.3}$$

satisfying

$$|a^i(t)| \le C_1 t^\alpha, \tag{6.4}$$

$$|b(t)| \le C_1 t^{\alpha+1}. \tag{6.5}$$

If ε is sufficiently small, all such maps take their values in U for $t \in (0,\varepsilon]$, and $\mathscr{X}$ is a complete metric space when endowed with the metric

$$d(\gamma,\widetilde{\gamma}) = \sup_{t\in[0,\varepsilon]} \left(t^{-\alpha}|a(t) - \widetilde{a}(t)| + t^{-\alpha-1}|b(t) - \widetilde{b}(t)| \right).$$

Note that for any $\gamma \in \mathscr{X}$, we have $|y(\gamma(t)) - t| = |b(t)| \le C_1 t^{1+\alpha} \le C_1 \varepsilon^\alpha t$, so choosing ε small enough that $C_1\varepsilon^\alpha < \frac{1}{2}$ implies that

$$\tfrac{1}{2} t \le y(\gamma(t)) \le \tfrac{3}{2} t. \tag{6.6}$$

Define a map $T\colon \mathscr{X} \to \mathscr{X}$ by

$$T\gamma(t) = \left(x_0^i + \int_0^t A^i(\gamma(\tau))\,d\tau,\ t + \int_0^t B(\gamma(\tau))\,d\tau \right).$$

It is straightforward to check that

$$\begin{aligned} \int_0^t |A^i(\gamma(\tau))|\,d\tau &\le \int_0^t C_0 y(\gamma(t))^{\lambda-1}\,d\tau \\ &\le \int_0^t C_0 C \tau^{\lambda-1}\,d\tau \le C' t^\lambda \le C'\varepsilon^{\lambda-\alpha} t^\alpha; \\ \int_0^t |B(\gamma(\tau))|\,d\tau &\le \int_0^t C_0 y(\gamma(t))^{\lambda}\,d\tau \\ &\le \int_0^t C_0 C \tau^{\lambda}\,d\tau \le C' t^{\lambda+1} \le C'\varepsilon^{\lambda-\alpha} t^{\alpha+1}. \end{aligned}$$

Thus if we choose ε small enough, it follows that T maps $\mathscr{X}$ into $\mathscr{X}$.

We will show that, after choosing ε even smaller if necessary, T is a contraction. Suppose $\gamma, \widetilde{\gamma} \in \mathscr{X}$. If (x^*, y^*) is any point along the line segment between $\gamma(t)$ and $\widetilde{\gamma}(t)$, (6.6) implies that $\frac{1}{2}t \le y^* \le \frac{3}{2}t$. To estimate $d(T\gamma, T\widetilde{\gamma})$, we use the mean-value theorem to obtain

$$\begin{aligned}
t^{-\alpha} &\int_0^t \big|A^i(\gamma(\tau)) - A^i(\widetilde{\gamma}(\tau))\big|\, d\tau \\
&\le t^{-\alpha} \sum_j \int_0^t |\partial_{x^j} A^i(x^*, y^*)|\, |a^j(\tau) - \widetilde{a}^j(\tau)|\, d\tau \\
&\qquad + t^{-\alpha} \int_0^t |\partial_y A^i(x^*, y^*)|\, |b(\tau) - \widetilde{b}(\tau)|\, d\tau \\
&\le t^{-\alpha} \int_0^t m C_0 (y^*)^{\lambda-1} \tau^{\alpha} d(\gamma, \widetilde{\gamma})\, d\tau \\
&\qquad + t^{-\alpha} \int_0^t C_0 (y^*)^{\lambda-2} \tau^{\alpha+1} d(\gamma, \widetilde{\gamma})\, d\tau \\
&\le t^{-\alpha} \int_0^t C_0 C \tau^{\lambda+\alpha-1} d(\gamma, \widetilde{\gamma})\, d\tau \\
&\le C' \varepsilon^{\lambda} d(\gamma, \widetilde{\gamma}),
\end{aligned}$$

for some (x^*, y^*) on the line segment between $\gamma(\tau)$ and $\widetilde{\gamma}(\tau)$. The analogous estimate for the y-component is similar. Thus T is a contraction if we choose ε sufficiently small. It therefore has a unique fixed point in $\mathscr{X}$, which is a solution to (6.2). By compactness, it is clear that for any given C_1 we can choose ε uniformly for $x_0 \in K$.

To see that the solution is unique, suppose γ is any continuous solution to (6.2). If we write γ in the form (6.3), the equation $T\gamma = \gamma$ together with (6.1) implies successively that $|b(t)| \le Ct$, then $|b(t)| \le Ct^{\lambda+1}$, and finally $|a^i(t)| \le Ct^{\lambda}$. It follows that (6.4) and (6.5) hold on $[0, \varepsilon]$ if C_1 and ε are chosen appropriately. Thus the restriction of γ to $[0, \varepsilon]$ is in $\mathscr{X}$, and so is equal to the unique fixed point of T.

Finally, we will prove the polyhomogeneity of the solution when V is polyhomogeneous. Recall the spaces $\mathscr{C}^{\delta}$ defined at the beginning of Section 5. We will need the following fact about these spaces, which is proved by a straightforward analysis of the Taylor expansion of $F(u + f, v + g)$ about $(u, v) = (u^1, \dots, u^m, v)$:

$$\begin{aligned}
&F\big(u(x,y) + f(x,y), v(x,y) + g(x,y)\big) - F\big(u(x,y), v(x,y)\big) \\
&\qquad \in \mathscr{C}^{\delta+\gamma} \qquad (6.7)
\end{aligned}$$

$$\text{when } F \in \mathscr{C}^{\delta},\ u^i \in \mathscr{C}^0,\ f^i \in \mathscr{C}^{\gamma},\ v \in \mathscr{C}^1,\ g \in \mathscr{C}^{\gamma+1},\ \gamma \ge 0.$$

For this proof, let $\mathscr{A}$ denote the space of polyhomogeneous functions on U, and for any $\delta \in \mathbb{R}$, define $\mathscr{A}^\delta = \mathscr{A} \cap \mathscr{C}^\delta$. Thus a polyhomogeneous function f is in $\mathscr{A}^\delta$ if and only if its leading term in the expansion (5.1) satisfies $s_1 \geq \delta$ and, if $s_1 = \delta$, $q_1 = 0$. Our hypotheses imply that $A^i \in \mathscr{A}^{\lambda-1}$ and $B \in \mathscr{A}^\lambda$.

For each $x_0 \in K$, let $a(x_0, t)$ and $b(x_0, t)$ denote the functions $a(t)$ and $b(t)$ obtained above with initial condition $(x_0, 0)$. The standard argument showing that solutions of ODEs depend smoothly upon initial values can be used to obtain estimates of the following form for all multi-indices α:

$$|\partial_x^\alpha a^i(x,t)| \leq C_\alpha t^\lambda, \quad |\partial_x^\alpha b(x,t)| \leq C_\alpha t^{\lambda+1}.$$

It is then straightforward to use the differential equation to obtain estimates on $(t\partial_t)^r \partial_x^\alpha a^i(x,t)$ and $(t\partial_t)^r \partial_x^\alpha b(x,t)$, showing that $a^i \in \mathscr{C}^\lambda$ and $b \in \mathscr{C}^{\lambda+1}$.

Suppose that for some integer $m \geq 1$, we have a "partial polyhomogeneous expansion" of the form

$$\begin{aligned} a^i(x,t) = p^i(x,t) + r^i(x,t), \qquad b(x,t) = q(x,t) + s(x,t), \\ \text{with } p^i \in \mathscr{A}^\lambda,\ r^i \in \mathscr{C}^{m\lambda},\ q \in \mathscr{A}^{\lambda+1},\ s \in \mathscr{C}^{m\lambda+1}. \end{aligned} \tag{6.8}$$

The discussion in the preceding paragraph shows that (6.8) holds with $m = 1$ and $p^i = q = 0$. Inserting (6.8) into $T\gamma = \gamma$, we obtain $r^i = r_0^i + r_1^i$, where

$$\begin{aligned} r_0^i &= \int_0^t A^i\big(x + p(x,\tau), \tau + q(x,\tau)\big)\, d\tau - p^i(x,t), \\ r_1^i &= \int_0^t \Big(A^i\big(x + p(x,\tau) + r(x,\tau), \tau + q(x,\tau) + s(x,\tau)\big) \\ &\qquad - A^i\big(x + p(x,\tau), \tau + q(x,\tau)\big)\Big)\, d\tau. \end{aligned}$$

¿From (6.7) we conclude that $r_1^i \in \mathscr{C}^{(m+1)\lambda}$. On the other hand, it is easy to check that r_0^i is polyhomogeneous, and since $r_0^i = r^i - r_1^i \in \mathscr{C}^{m\lambda}$, we have $r_0^i \in \mathscr{A}^{m\lambda}$. A similar argument shows that $s = s_0 + s_1$, with $s_0 \in \mathscr{A}^{m\lambda+1}$ and $s_1 \in \mathscr{C}^{(m+1)\lambda+1}$. We let P^i denote the sum of the (finitely many) terms in the expansion of r_0^i that are in $\mathscr{A}^{m\lambda} \setminus \mathscr{C}^{(m+1)\lambda}$, and Q the sum of the terms in s_0 that are in $\mathscr{A}^{m\lambda+1} \setminus \mathscr{C}^{(m+1)\lambda+1}$. Replacing p^i by $p^i + P^i$ and q by $q + Q$, we obtain (6.8) with $m + 1$ in place of m. Continuing by induction, we conclude that a and b are polyhomogeneous. Since both are in $\mathscr{A}^\lambda$, it follows that $(x,t) \mapsto \gamma_x(t)$ is a $C^{0,\lambda}$ map. q.e.d.

We are finally ready to prove the main theorem.

Proof of Theorem A. Suppose g satisfies the hypotheses of the theorem. Let Ψ be the collar diffeomorphism given by Theorem 4.5, and let $\widetilde{g} = \Psi^* g$, which is polyhomogeneous and conformally compact of class $C^{1,\lambda}$, $0 < \lambda < 1$. Theorem 4.5 shows that $\rho^2\Psi^* g - \rho^2 h \in C^{1,0}_{3+\lambda}(M_R; \Sigma^2)$, which implies that $\rho^2\Psi^* g - \rho^2 h = O(\rho^{1+\lambda})$ in background coordinates, so $\Psi^* g$ and h have the same conformal infinity $\widetilde{\gamma}$.

Then let Γ be the collar diffeomorphism given by Lemma 6.2, so that $\Gamma^*\widetilde{g}$ has the form (1.1). Because $r/\rho \to 1$ at ∂M, it follows that $\Gamma^*\widetilde{g}$ also has $\widetilde{\gamma}$ as conformal infinity. Because $\rho^2\Gamma^*\widetilde{g}$ is continuous up to ∂M and has a smooth restriction to ∂M, it follows that the log terms in the asymptotic expansion for $G(\rho)$ all occur with positive powers of ρ. Once polyhomogeneity is known, the detailed form of the expansion is established by matching powers and coefficients of various terms appearing in the equations. Such a study has been done by Robin Graham and Charles Fefferman [8] (see also [11]), and the results there imply that the expansions are of the form described in Theorem A. In the special case when $\dim M = 3$, it is possible to choose the conformal infinity $\widetilde{\gamma}$ to have constant Gaussian curvature, and an easy computation shows that the first log term vanishes in this case (cf. [7, 8, 11]), so $\rho^2\Gamma^*\widetilde{g}$ is always smooth.

The proof is completed by letting Φ be the collar diffeomorphism $\Psi \circ \Gamma$. q.e.d.

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