Some potentials for the curvature tensor on three dimensional manifolds

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Abstract

We study equations of Riemann-Lanczos type on three dimensional manifolds. Obstructions to global existence for global Lanczos potentials are pointed out. We check that the imposition of the original Lanczos symmetries on the potential leads to equations which do not have a determined type, leading to problems when trying to prove global existence. We show that elliptic equations can be obtained by relaxing those symmetry requirements in at least two different ways, leading to global existence of potentials under natural conditions. A second order potential for the Ricci tensor is introduced.

1 Introduction

When studying a set of field equations it is often convenient to study associated potentials rather than the fields themselves. A clear example is given

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by the Maxwell equations, where the usefulness of the scalar and vector potentials hardly needs advertising. Another one is provided by the hydrodynamics of irrotational flows. In general relativity a striking example is the Ernst potential, the introduction of which leads to solution-generating techniques, enables studies using inverse scattering methods, and so on. One can therefore hope that the introduction, in general relativity, of potentials that exist under fairly general circumstances can lead to a better understanding of the theory. For instance, four-dimensional potentials could perhaps be used to gain better understanding of the dynamics of the fields – this is indeed the case for Gowdy models, where transformations relating some metric functions with the Ernst potential give insight into singularity formation [17, 18]. Three-dimensional potentials could perhaps be used to study initial data sets — potentials for the extrinsic curvature tensor are in fact already used in the conformal method when generating solutions of the vector constraint equations.

In 1962 Lanczos [15] proposed a tensor potential for the Weyl tensor, and wrote what is referred to now as the Weyl-Lanczos equations [19, 22]. It was subsequently showed by Bampi and Caviglia [2] that the Weyl-Lanczos equations are always locally solvable for analytic metrics in four dimensions. The existence question for smooth, but not necessarily analytic, Lorentzian metrics has been eventually settled by Illge [12], who proved global existence of the Weyl-Lanczos potential on, say smooth, globally hyperbolic four-dimensional space-times. It has been pointed out in [1] that Illge's method works for analytic pseudo-Riemannian metrics of any signature. In the Riemannian case it should further lead to global existence for smooth metrics when the associated elliptic operators are invertible.

There exist several publications on the Weyl-Lanczos equations in 4 dimensions for particular space-times, we only mention two recent ones: in [16] O'Donnell finds solutions for Schwarzschild space-time, while in [8] Dolan and Muratori construct solutions for stationary, axi-symmetric, four-dimensional vacuum space-times. As far as other dimensions are concerned, Edgar and Höglund prove non-existence of a Lanczos potential for the Weyl tensor in $n \geq 7$ dimensions in [9]. In [10] another kind of potential — a double $(2,3)$-form $P^{ab}{}_{cde}$ — is introduced in arbitrary dimensions by Edgar and Senovilla.

In 1977 Udeschini Brinis [21] proposed the name "Riemann-Lanczos equations" for the generalisation[1] of the Weyl-Lanczos equations to the full

[1] Equations (1.1) were implicitly introduced by Lanczos in [14].

Riemann tensor:

$$R_{abcd} = L_{abc;d} - L_{abd;c} + L_{cda;b} - L_{cdb;a} \; . \tag{1.1}$$

Here one assumes at the outset that L is anti-symmetric in its first two indices. The existence question for the Riemann-Lanczos equations in the analytic category was considered by Bampi and Caviglia [2, 3].

The Weyl tensor vanishes identically in dimensions two and three, but we can still study the Riemann-Lanczos equations there. The local existence question has been settled in [7, 11] for all smooth metrics in dimension two, and for analytic metrics in dimension three. The purpose of this note is twofold: first, we present some no-go results for the Riemann-Lanczos equations. Indeed, we present some simple obstructions for the global existence of such potentials in Section 2, and we show non-ellipticity of a class of natural reformulations of those equations in Section 4. Next, we present three classes of related potentials for the Ricci tensor in dimension three: In Section 5 we construct two classes of first-order potentials for the Ricci tensor. Those potentials differ from the Riemann-Lanczos ones by relaxing the original hypothesis of anti-symmetry of L_{abc} in the first two indices. In Section 6 we present a second-order potential for the Ricci tensor; this involves an equation that arises naturally in the problem at hand.

2 Obstructions to existence of a global Riemann-Lanczos potential

We start with a few straightforward non-existence results for global curvature potentials. From (1.1) and from anti-symmetry of L in the first two indices one has the following equation for the Ricci scalar

$$R_g := \mathrm{tr}_g \mathrm{Ric} = 4 L^{ab}{}_{a;b} \; .$$

If M is compact, integration gives

$$\int_M R_g = 0 \; .$$

This implies:

(i) For compact two-dimensional manifolds a global Riemann-Lanczos potential cannot exist if M is not diffeomorphic to a torus.

(ii) In dimensions larger than or equal to three, if g is Einstein the existence of a global Riemann-Lanczos potential forces g to be Ricci-flat.

(iii) In particular it follows that three dimensional non-flat compact space-forms do not admit global Riemann-Lanczos potentials (the flat ones admit of course a trivial one).

3 The Ricci-Lanczos equation

Contracting (1.1) in b and d we obtain

$$\begin{aligned} R_{ac} &= 2(L_{(a}{}^{n}{}_{c);n} + L_{n(a}{}^{n}{}_{;c)}) \\ &= L_{anc}{}^{;n} + L_{cna}{}^{;n} + L^{n}{}_{an;c} + L^{n}{}_{cn;a} \ , \end{aligned} \tag{3.1}$$

We will refer to (3.1) as the *Ricci-Lanczos* equation.

Since in three dimensions the Riemann tensor is algebraically determined by the Ricci one, one expects that a solution of the Ricci-Lanczos equation provides a solution of the Riemann-Lanczos one. Let us show that this is indeed the case:

PROPOSITION 3.1 *Let (M, g) be a three-dimensional pseudo-Riemannian manifold. Every solution L_{abc} of the Ricci-Lanczos equations which is anti-symmetric in the first two indices provides a solution of the Riemann-Lanczos equations.*

PROOF: Recall the three dimensional identity

$$R_{abcd} = g_{ac}R_{bd} + g_{bd}R_{ac} - g_{ad}R_{bc} - g_{bc}R_{ad} - \frac{1}{2}R(g_{ac}g_{bd} - g_{ad}g_{bc}) \ . \tag{3.2}$$

In dimension three, pseudo-Riemannian metrics are, up to an overall sign, either Riemannian or Lorentzian. In an ON frame (so that $g_{00} = \epsilon = \pm 1$, $g_{11} = g_{22} = 1$) from (3.2) we have

$$\begin{aligned} R_{0101} &= R_{00} + \epsilon\left(R_{11} - \frac{1}{2}R\right) \ , \\ R_{0202} &= R_{00} + \epsilon\left(R_{22} - \frac{1}{2}R\right) \ , \\ R_{1212} &= R_{11} + R_{22} - \frac{1}{2}R \, , \\ R_{0102} &= \epsilon R_{12} \ , \\ R_{0112} &= -R_{02} \ , \\ R_{0212} &= R_{01} \ . \end{aligned} \tag{3.3}$$

We see that the first two equations are of the same type, similarly for equations four and five. For the components of the Ricci tensor and for the Ricci scalar we obtain

$$
\begin{aligned}
R_{00} &= 2(L_{010;1} + L_{020;2} - L_{011;0} - L_{022;0})\,, \\
R_{11} &= 2(-\epsilon L_{011;0} + L_{121;2} + \epsilon L_{010;1} - L_{122;1})\,, \\
R_{22} &= 2(-\epsilon L_{022;0} + L_{121;2} - L_{122;1} + \epsilon L_{020;2})\,, \\
R_{01} &= L_{021;2} + L_{120;2} - L_{022;1} - L_{122;0}\,, \\
R_{02} &= L_{012;1} + L_{121;0} - L_{120;1} - L_{011;2}\,, \\
R_{12} &= \epsilon(-L_{012;0} - L_{021;0} + L_{010;2} + L_{020;1})\,, \\
R &= 4(\epsilon(-L_{011;0} + L_{010;1} - L_{022;0} + L_{020;2}) + L_{121;2} - L_{122;1})\,.
\end{aligned}
\tag{3.4}
$$

It is straightforward to check that the insertion of (3.4) into (3.3) reproduces (1.1). □

4 Lack of ellipticity

In [12] Illge obtains a well posed system of equations by adding certain "gauge equations" to the four-dimensional Weyl-Lanczos equations. For Lorentzian metrics Illge's system of equations (4) is symmetric-hyperbolic, whence global existence and uniqueness of solutions of the associated Cauchy problem on globally hyperbolic manifolds immediately follows.[2] In Riemannian signature Illge's equation (4) [12, p. 554] is elliptic. In dimension three it is natural to ask the question whether the Ricci-Lanczos equation (3.1) can be completed, by adding some further equations, to an elliptic system. The answer is no, because the symbol[3] associated to the partial differential operator (3.1) is not surjective. Indeed, let $r_{ac}(k)$ denote the symbol of (3.1), so that

$$r_{ac}(k)(L) = L_{anc}k^n + L^n{}_{an}k_c + L_{cna}k^n + L^n{}_{cn}k_a\,.$$

[2]In particular Illge's Theorem 1 in [12] follows immediately from the symmetric hyperbolic character of this equation. In order to prove existence and uniqueness of equation (4), one does not need to use his second-order equation (8) in [9].

[3]By symbol we always mean the symbol associated with the highest order derivatives; this is sometimes called the principal symbol.

An analysis of the map adjoint to $r_{ac}(k)$ shows that the image of $r_{ac}(k)$ is the orthogonal of

$$\{\alpha(2k_a k_c - |k|^2 g_{ac}) \,, \alpha \in \mathbb{R}\} \,,$$

hence $r_{ac}(k)$ is never surjective. It follows that the addition of new equations will never lead to a surjective symbol, in particular it will never lead to an elliptic system (for which the symbol is bijective). This proves that a first order approach *à la Illge* cannot succeed for the three-dimensional Ricci-Lanczos equations.

An alternative approach to Illge's has been proposed by Andersson and Edgar in [1], by introducing "potentials for the potential". In Riemannian signature this leads to second order elliptic equations, and likewise one can ask whether the introduction of some new fields, say φ, so that L in (3.1) is a first order partial differential operator acting on φ, might lead to an elliptic second order system for φ. Again the answer is no: since the symbol of (3.1) is not surjective, composition from the right with another symbol will never lead to a bijective symbol. It follows that a second order approach *à la Andersson–Edgar* cannot succeed either.

Recall, now, that partial differential equations arising in geometry which are not of definite type can sometimes be replaced by better behaved equations by exploiting the identities satisfied by the objects at hand: a flagship example is provided by the Lorentzian Einstein equations, which are not hyperbolic, but their solutions can be constructed by considering an auxiliary hyperbolic system in wave coordinates. We have made various attempts to obtain a construction of a Riemann-Lanczos potential along those lines, exploiting the Bianchi identities, without success. In Appendix A we present a method which came closest to providing a solution, without however yielding one, except perhaps in some special cases. We describe one of our unsuccessful approaches there as some elements of the method there be used in the next section to obtain a Lanczos-type potential for the Ricci tensor. (Some readers interested in generalising the Andersson-Edgar-Illge approaches might moreover find it useful to know that this approach, as well as several similar ones, will not work.)

5 Ricci-Lanczos potentials

Our results were not very encouraging so far, but one can look for other methods for solving the Ricci-Lanczos equation. Now a consistency requirement for the Riemann-Lanczos equation was anti-symmetry of L in its first two indices, but there does not seem any special reason to impose such a restriction in the Ricci-Lanczos equation. (A solution of this equation will then provide a potential for the Ricci tensor, but will not necessarily lead to a solution of the Riemann-Lanczos equation.)

5.1 Potentials with no symmetry conditions

A simple ansatz which does not assume any symmetry for L, is the following:

$$L_{abc} = A_{ca;b} + \lambda A_{cb;a} + \sigma A^d{}_{d;c} g_{ab} \ , \tag{5.1}$$

where A is symmetric while λ and σ are some real constants; obviously we do not assume anymore that the g-trace of A vanishes. We set

$$W_a = A_a{}^b{}_{;b} \ , \tag{5.2}$$

so that

$$L_{nb}{}^n = (1+\sigma) A^c{}_{c;b} + \lambda W_b \ .$$

Inserting all this into (3.1) leads to

$$R_{ac} = 2A_{ac}{}^{;b}{}_b + 2\lambda(W_{a;c} + W_{c;a}) + 2(1+2\sigma)A^b{}_{b;ac} + 2\lambda \left(R_{(a}{}^{bd}{}_{c)} A_{bd} + R_{(a}{}^b A_{c)b} \right) \ . \tag{5.3}$$

The choice $\lambda = 0$ and $\sigma = -1/2$ immediately gives an elliptic equation for A,

$$R_{ac} = 2A_{ac}{}^{;b}{}_b \ . \tag{5.4}$$

It then follows that there exists an open set of (λ, σ)'s around $(0, -1/2)$ for which (5.3) is elliptic. We note the following:

(i) On a compact Riemannian manifold with boundary (5.4) can always be solved with the supplementary condition $A_{ab} = 0$ on the boundary;

(ii) Integration by parts together with the usual elliptic theory (*c.f., e.g.* [4, Theorem 4.1]) shows that on a compact Riemannian manifold without boundary (5.4) can be solved if and only if for all covariantly constant symmetric tensors σ_{ab} one has

$$\int_M \sigma^{ab} R_{ab} = 0 \ .$$

In this context we note the following result, pointed out to us by J. Jezierski [13]:

PROPOSITION 5.1 *Let $r_i(p)$, $i = 1, 2, 3$ denote the eigenvalues of the Ricci tensor at the point $p \in M$. Assume that either* *(a) there exists $p \in M$ such that*

$$\sqrt{r_1(p)} + \eta\sqrt{r_2(p)} + \epsilon\sqrt{r_3(p)} \neq 0 \ ,$$

for $\eta = \pm 1$ and $\epsilon = \pm 1$ (for negative r_i's the square root is understood in $\mathbb{C}$), or

(b) there exists $p \in M$ such that $R(p) = 0$, with the Ricci tensor at p being non-zero.

Then all symmetric covariantly constant tensors on M are proportional to the metric, with a constant proportionality factor.

REMARK 5.2 One can use the results in [20] to show that the set of metrics satisfying *(a)* is open and dense in the set of all metrics in a C^2 topology on a compact set. Note that (b) fails within the class of scalar flat metrics if and only if g is flat.

PROOF: Differentiating the equation $A_{bc;d} = 0$ and commuting derivatives one obtains

$$R_{abcd}A^a{}_e + R_{aecd}A^a{}_b = 0 \ .$$

Let B_{ab} be the trace-free part of A_{ab}, the last equation implies

$$R_{abcd}B^a{}_e + R_{aecd}B^a{}_b = 0 \ .$$

Contracting with g^{bc} and using (3.2) it follows that

$$B_e{}^b R_{bd} - 4R_e{}^b B_{bd} + \frac{R}{2}B_{ed} + B^{ab}R_{ab}g_{ed} = 0 \ .$$

Writing $\mathscr{B}$ for the matrix $B^a{}_b$ and $\mathscr{R}$ for $R^a{}_b$ we obtain the matrix equation

$$\mathscr{B}\mathscr{R} - 4\mathscr{R}\mathscr{B} + \frac{\mathrm{tr}_g\mathscr{R}}{2}\mathscr{B} + \mathrm{tr}_g(\mathscr{B}\mathscr{R})\mathrm{id} = 0 \ . \tag{5.5}$$

Taking the transpose and comparing the resulting equation with (5.5) we are led to

$$\mathscr{B}\mathscr{R} = \mathscr{R}\mathscr{B} \ , \tag{5.6}$$

so that (5.5) can be rewritten as

$$3\mathscr{B}\mathscr{R} = \frac{\mathrm{tr}_g\mathscr{R}}{2}\mathscr{B} + \mathrm{tr}_g(\mathscr{B}\mathscr{R})\mathrm{id} \ . \tag{5.7}$$

Now, $\mathscr{B}$ and $\mathscr{R}$ are symmetric, they commute by (5.6), and can therefore be simultaneously diagonalised. Writing r_i for the eigenvalues of $\mathscr{R}$ and b_i for those of $\mathscr{B}$, and noting that $b_3 = -b_1 - b_2$, (5.7) reduces to the system

$$\begin{pmatrix} 3r_1 - r_2 + r_3 & 2(r_3 - r_2) \\ 2(r_3 - r_1) & 3r_2 - r_1 + r_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0 \ .$$

The determinant of the matrix above is

$$-3(r_1^2 + r_2^2 + r_3^2 - 2r_1r_2 - 2r_2r_3 - 2r_1r_3) \ , \tag{5.8}$$

which equals

$$-3(\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3})(\sqrt{r_1}-\sqrt{r_2}+\sqrt{r_3})(\sqrt{r_1}+\sqrt{r_2}-\sqrt{r_3})(\sqrt{r_1}-\sqrt{r_2}-\sqrt{r_3}) \ . \tag{5.9}$$

Thus, if the product above is non-zero at p, then there exists a neighborhood $\mathscr{O}$ of p on which this product is non-zero, and B vanishes on $\mathscr{O}$. Then $A_{ab} = \sigma g_{ab}$ for some function σ on $\mathscr{O}$, and it follows from $A_{ab;c} = 0$ that σ is constant on $\mathscr{O}$. Uniqueness of solutions of ODE's implies that $A_{ab} = \sigma g_{ab}$ throughout M (M is connected by hypothesis).

To prove point (*b*), note that if $R(p) = 0$ but $R_{ab}(p) \neq 0$, then one eigenvalue of R_{ab}, say r_1, is strictly positive and one eigenvalue, say r_3, is strictly negative, which implies that no factor in (5.9) vanishes. □

Proposition 5.1 and Remark 5.2 show that generically the only symmetric trace-free covariantly constant tensors σ are constant multiples of g and therefore for generic metrics a necessary and sufficient condition for existence of a potential of the form (5.1) is

$$\int_M R = 0 \ . \tag{5.10}$$

In particular if g is scalar flat, $R \equiv 0$, then (5.1) is always solvable (if $R_{ab} \equiv 0$ then $A_{ab} = 0$ is a solution).

(iii) For Riemannian metrics on $\mathbb{R}^3$ which are asymptotically flat in the sense of [5] Equation (5.4) can always be solved;

(iv) For Lorentzian globally-hyperbolic three-dimensional space-times $(\mathscr{M}, g)$ Equation (5.4) can always be solved, with any arbitrarily prescribed smooth $(A_{ab}, A_{ab;c})|_{\mathscr{S}}$, where $\mathscr{S}$ is any Cauchy hypersurface in $\mathscr{M}$.

5.2 L_{abc}'s symmetric in the first two indices

It is of interest to enquire about ansatzes for the Ricci-Lanczos potential which satisfy further properties. For instance, instead of requiring L to be anti-symmetric in the first two-indices one could ask for symmetry in those. An integration by parts argument as at the beginning of Appendix A leads then to the natural ansatz

$$L_{abc} = A_{ac;b} + A_{bc;a} + W_a g_{bc} + W_b g_{ac} \; , \tag{5.11}$$

with W as in (5.2), and with again a symmetric A. This gives

$$L_{ab}{}^{a} = A_a{}^a{}_{;b} + 5W_b \; ,$$

and

$$R_{ac} = 2A_{ac}{}^{;b}{}_{b} + 7(W_{a;c} + W_{c;a}) + 2A^b{}_{b;ac} + 2W^b{}_{;b} g_{ac} + 2\left(R_{(a}{}^{bd}{}_{c)} A_{bd} + R_{(a}{}^{b} A_{c)b}\right) \; . \tag{5.12}$$

Let $\sigma(p)$ denote the symbol of the operator at the right-hand-side of (5.12), we have

$$\sigma(p)_{ac} = 2A_{ac}p^b p_b + 7(A_{ab}p^b p_c + A_{cb}p^b p_a) + 2A^b{}_b p_a p_c + 2A_{bd}p^b p^d g_{ac} \; . \tag{5.13}$$

To check injectivity we assume $\sigma(p) = 0$; contracting with g^{ac} gives

$$0 = A^a{}_a p^b p_b + 5A_{ab}p^a p^b \; . \tag{5.14}$$

Multiplying (5.13) with $p^a p^c$ leads to

$$0 = A^a{}_a p^b p_b + 9A_{ab}p^a p^b \; . \tag{5.15}$$

It clearly follows for $p \neq 0$ that the last two terms in (5.13) vanish. Multiplying (5.13) with p^a similarly leads now to $A_{ab}p^a = 0$, and using this result back in (5.13) injectivity of the symbol for $p \neq 0$ follows. Since the dimensions match, ellipticity ensues.

We note the following

(i) The operator, say P, defined by the right-hand-side of (5.12) is formally self-adjoint.

(ii) The kernel of P (and thus of P^*) consists of tensors such that the left-hand-side of (5.11) vanishes.

(iii) On a compact manifold with boundary we expect that there are no non-trivial elements of the kernel with zero-boundary conditions; if so, the usual coercitivity arguments should lead to existence of solutions of the associated boundary value problem for all R_{ab}. In particular there should be no obstruction for local existence of potentials for smooth, or C^3 metrics.

(iv) Again assuming that there are no elements in the kernel of A which go to zero at infinity[4], the usual theory [5] would give existence of solutions on $\mathbb{R}^3$ with asymptotically flat metrics.

(v) On compact manifolds without boundary, for those metrics for which the kernel of P is spanned by constant multiples of the metric tensor one obtains existence of the potential if and only if (5.10) holds. We expect such metrics to be generic.

(vi) However, in the Lorentzian case, it is not clear whether the associated evolution problem on globally hyperbolic manifolds is well posed.

6 A "non-Lanczos" potential for the Ricci tensor

In Appendix A we show that the equation (A.6) considered there fails to produce a Riemann-Lanczos potential in general. However, as shown in Proposition A.1, for any fixed k we obtain a class of elliptic operators acting on A, which in some situations can be solved to obtain potentials for the Ricci tensor. More precisely, for definiteness we consider (A.1) with $k_a = 0$,

$$L_{abc} = 2(A_{c[a;b]} - \beta A_{s[a}{}^{;s} g_{b]c}), \tag{6.1}$$

where β is a constant different from $1/2$, while

$$A_{ab} = A_{(ab)}$$

is a symmetric tensor field. The rationale for looking for such a form of A_{ab} is given at the beginning of Appendix A.

We restrict attention to scalar flat metrics so that (A.5) with $\alpha = -1$ and $\gamma = \delta = k = 0$ reads

$$P(A)_{ac} := 2(L_{(a}{}^{n}{}_{c);n} - L_{n(a}{}^{n}{}_{;c)}) = R_{ac}\ , \tag{6.2}$$

[4]This is the case for metrics which are flat outside of a compact set, and standard manipulations should give the result for general asymptotically flat metrics.

(As in (A.4), consistency of this equation requires that the Ricci tensor be trace-free.) Proposition A.1 shows that (6.2) is an elliptic equation for A. It should be emphasised that this does not lead to a solution of the Ricci-Lanczos equation (3.1), as the signs in (6.2) do not coincide with those in (3.1).

We have the following:

THEOREM 6.1 *Let* (M, g) *be a compact scalar flat Riemannian manifold without boundary, suppose that there exists* $\beta \neq 1/2$ *such that the formal adjoint* P^* *of* P *has trivial kernel.*[5] *Then there exists a globally defined solution of* (6.2) *on* M*.*

PROOF: Let S^2_0 denote the bundle of symmetric trace-free two-covariant tensors. Given a bundle B we denote by $\Gamma_{C^\infty}(B)$ the space of smooth sections of B. Ellipticity of P (*cf.* Proposition A.1) implies that we have the Berger-Ebin splitting [4, Theorem 4.1]

$$\Gamma_{C^\infty}(S^2_0 \oplus T^*M) = \mathrm{Im}\ P \oplus \mathrm{Ker}\ P^* \ .$$

It follows that if P^* has trivial kernel, then there exists at least one (and perhaps more than one) smooth solution (A, k) of the equation

$$P(A) = \mathrm{Ric} \ .$$

$\square$

If $\beta = 1$ then P is formally self-adjoint, and the integration by parts argument at the beginning of Appendix A shows that the kernel of P consists of trace-free symmetric tensors satisfying

$$A_{c[a;b]} = A_{s[a}{}^{;s} g_{b]c} \ . \tag{6.3}$$

One expects that generically there will be no such tensors, so that for generic Riemannian manifolds as in Theorem 6.1 solutions of (6.2) will exist, and will be unique in the class of solutions of the form (6.1) with $\beta = 1$. A similar result should be true for asymptotically flat Riemannian metrics.

[5]Throughout this work elements of a kernel are non-trivial twice differentiable solutions of the PDE; they can be assumed to be smooth if the metric is smooth.

A An ansatz for the Riemann-Lanczos potential that does not work

We seek L in the form

$$L_{abc} = 2(A_{c[a;b]} - \beta A_{s[a}{}^{;s} g_{b]c}) + (k_b g_{ac} - k_a g_{bc}) \ , \qquad \text{(A.1)}$$

where β is a constant, k_a is a vector field, while

$$A_{ab} = A_{(ab)}$$

is a symmetric tensor field. The rationale for looking for such a form of A_{ab} is the following: suppose that R_{ac} is of the form (3.1), and let A_{ab} be a symmetric tensor; multiplying (3.1) by A^{ac} and integrating over M (assuming that either M is compact without boundary, or that the integration by parts without boundary terms is justified one way or another) one obtains

$$\int_M L^{abc}(A_{c[a;b]} - A_{s[a}{}^{;s} g_{b]c}) = 0 \ .$$

So if L is of the form (A.1) with $\beta = 1$ and with $k_a = 0$ one obtains $L_{abc} = 0$, hence uniqueness of the resulting L's. The justification for the addition of the k part will follow shortly; in any case we will see that this addition will not solve the problems we are faced with, and that the ansatz (A.1) does not seem to lead to solutions of the Riemann-Lanczos problem except perhaps in special cases.

For any $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ consider the second-order linear differential operator

$$\mathscr{P}_{\alpha,\beta,\gamma,\delta}(A,k) = (P(A,k), S(A,k)) \ ,$$

where $L = L(A,k)$ is given by (A.1), and

$$\begin{aligned} P(A,k)_{ac} &:= 2(L_{(a}{}^{n}{}_{c);n} + \alpha L_{n(a}{}^{n}{}_{;c)}) \ , & \text{(A.2)} \\ S(A,k)_a &:= k_{a;n}{}^{n} - k^{n}{}_{;na} + (2\gamma - 1) R_a{}^n k_n + 2\delta k_a \\ &\quad + (1 - 2\beta)\Big(- \gamma R_{ab} A^{bn}{}_{;n} - \delta A_a{}^{n}{}_{;n} \\ &\qquad + R^{sn}(3A_{as;n} - 2A_{sn;a}) - R_{as;n} A^{sn}\Big) \ . & \text{(A.3)} \end{aligned}$$

We note that (A.2) with $\alpha = 1$ would be equal to the Ricci tensor of g when (1.1) holds. On the other hand if $\alpha = -1$ then

$$g^{ac} P(A,k)_{ac} = 2g^{ac}(L_a{}^{n}{}_{c;n} - L_{na}{}^{n}{}_{;c}) = 2(L^{cn}{}_{c;n} - L^{nc}{}_{n;c}) = 0 \qquad \text{(A.4)}$$

identically, whatever A and k.

It turns out that the operator (A.2) has good properties if $\alpha = -1$. However, we would like to solve the equation

$$P(A,k)_{ab} = R_{ab} \tag{A.5}$$

when $\alpha = 1$. Now, the value of α will not matter if $L_{n(a}{}^{n}{}_{;c)} = 0$. The question then arises, whether it is possible to choose k to enforce this last equation; this is the origin of (A.3). Indeed, suppose that the following system of equations for A and k holds:

$$\begin{aligned} R_{ac} &= 2(L_{(a}{}^{n}{}_{c);n} - L_{n(a}{}^{n}{}_{;c)}) \,, && \text{(A.6)} \\ 0 &:= k_{a;n}{}^{n} - k^{n}{}_{;na} + (2\gamma-1)R_{a}{}^{n}k_{n} + 2\delta k_{a} \\ &\quad +(1-2\beta)\Big(-\gamma R_{ab}A^{bn}{}_{;n} - \delta A_{a}{}^{n}{}_{;n} + R^{sn}(3A_{as;n} - 2A_{sn;a}) - R_{as;n}A^{sn}\Big) \,. && \text{(A.7)} \end{aligned}$$

In (A.2) L has to be expressed in terms of A and k as in (A.1). The constants β, γ and δ are left unspecified, and one would hope to be able to adjust them to get properties which are needed for the operators at hand. Since the trace of the left-hand-side of (A.6) vanishes automatically, (A.6) is consistent only if g is scalar flat (*i.e.*, $R_g = 0$), and therefore one would have to restrict one's attention to such metrics. Using the Bianchi identities one can derive a system of equations for the trace $L_{na}{}^{n}$ (see Proposition A.2 below). When the resulting equation (A.13) has no non-trivial solutions one will have produced a solution of the *Ricci-Lanczos* equations. It would finally follow from Proposition 3.1 that each solution of the Ricci-Lanczos equations provides also a solution of the Riemann-Lanczos equations.

This program can unfortunately not work for general Riemannian metrics: while (A.2) gives an elliptic equation for A except for some special values of β, ellipticity is spoiled by (A.3). Similarly for Lorentzian metrics (A.3) does not provide a good evolution equation for k. We note that this could be cured if one could find a means of ensuring *e.g.* that $k^{a}{}_{;a}$ vanishes, or to modify the ansatz (A.1) to obtain equations in which $k^{a}{}_{;a}$ comes with a factor different from minus one in (A.3), but we have not managed to do that.

Let us pass now to the details of what has been discussed above, as this will be useful for the purposes of Section 6:

PROPOSITION A.1 *Suppose that g is scalar flat, and assume that A is trace-free. Let L be defined by* (A.1). *If g is Riemannian, $\alpha = -1$, and $\beta \neq$*

1/2, then P is elliptic if viewed as acting on A at fixed k. However, S is not elliptic if viewed as acting on k at fixed A whatever the values of the parameters, and therefore $\mathscr{P}_{-1,\beta,\gamma,\delta}$ is not elliptic.

PROOF: Because the principal part of $\mathscr{P}_{-1,\beta,\gamma,\delta}$ is block-diagonal, to establish ellipticity one needs to show that for $0 \neq p \in TM$ the symbols $\sigma(P,p)$ of (A.2) and $\sigma(S,p)$ of (A.3) are linear bijections — the first from symmetric trace-free tensors to symmetric trace-free tensors, the second from vectors to vectors. We choose an ON frame for the metric g in which p is proportional to e_3. Rescaling p we can then assume that $p = (0, 0, p_3 = 1)$. By an abuse of notation we write $\sigma(P,p)$ for $\sigma(P,p)A$, similarly for other symbols. One finds

$$\begin{aligned}
\sigma(P,p)_{11} &= 2A_{11} + 2\beta A_{33} \\
\sigma(P,p)_{22} &= 2A_{22} + 2\beta A_{33} \\
\sigma(P,p)_{33} &= 2\alpha(A_{11} + A_{22} + 2\beta A_{33}) \\
\sigma(P,p)_{12} &= 2A_{12} \\
\sigma(P,p)_{13} &= ((1-\alpha)(1-\beta) + \alpha\beta)A_{13} \\
\sigma(P,p)_{23} &= ((1-\alpha)(1-\beta) + \alpha\beta)A_{23} \ .
\end{aligned} \tag{A.8}$$

We find that for any choice of α, β we always obtain

$$R_{33} = \alpha(R_{11} + R_{22}) \tag{A.9}$$

so that the operator (A.2) is not elliptic when mapping symmetric tensors into symmetric tensors. However, if we assume that the tensor A_{ab} is trace-free

$$A^n{}_n = 0 \ , \tag{A.10}$$

we can replace A_{33} by $-A_{11} - A_{22}$, and rewrite the first three of the symbol equations (A.8) as

$$\begin{aligned}
\sigma(P,p)_{11} &= 2((1-\beta)A_{11} - \beta A_{22}) \\
\sigma(P,p)_{22} &= 2((1-\beta)A_{22} - \beta A_{11}) \\
\sigma(P,p)_{33} &= 2\alpha(1-2\beta)(A_{11} + A_{22}) \ .
\end{aligned} \tag{A.11}$$

We see that for A_{ab} trace-free as assumed and for the choice $\alpha = -1$ we obtain that the symbol equations satisfy

$$\sigma(P,p)_{11} + \sigma(P,p)_{22} + \sigma(P,p)_{33} = 0 \ .$$

If $\beta \neq 1/2$ the linear map $(A_{11}, A_{22}) \to (\sigma(P,p)_{11}, \sigma(P,p)_{22})$ is bijective, which easily implies that the symbol $\sigma(P,p)$ has now rank five . As we have five independent components of A_{ab}, we obtain ellipticity of P for the choice $A^n{}_n = 0$ and $\alpha = -1$.

To finish the proof, we note that the symbol $\sigma(S,p)$ of (A.3), again for the choice of e_a so that $p = (0, 0, p_3 = 1)$, is given by

$$\sigma(S,p) = (0, k_2, k_3) \ ,$$

which is not elliptic. □

We set

$$Z_a = L_a{}^n{}_n \ , \quad W_a = A_a{}^n{}_{;n} \ ,$$

so that for trace-free tensors A_{ab} we have

$$Z_a = (1-2\beta)W_a - 2k_a \ . \tag{A.12}$$

Proposition A.2 *Suppose that g is scalar flat and that A is trace-free. Assume further that $S(A,k) = 0$ and that* (A.5) *holds. Then the vector field Z satisfies the system of equations*

$$(2\alpha\beta-\alpha-\beta+1)Z_a{}^{;c}{}_c+(2\alpha\beta-\alpha+\beta-1)Z_{c;a}{}^c+(1-2\beta+\gamma)R_{as}Z^s+\delta Z_a = 0, \tag{A.13}$$

which is elliptic if $\alpha \neq 0$, $\beta \neq 1/2$, and $2\alpha\beta - \alpha - \beta + 1 \neq 0$.

Proof: Since g is scalar flat we have $R_{ac}{}^{;c} = 0$. This leads to the three equations

$$\begin{aligned} 0 &= R_{ac}{}^{;c} \\ &= (2\alpha\beta-\alpha-\beta)(A_{an}{}^{;nc}{}_c + A^{cn}{}_{;nac}) + 2\beta A^{ns}{}_{;sna} + 2A_{ac;n}{}^{nc} \\ &\quad -A_{cn;a}{}^{nc} - A_{an;c}{}^{nc} + (2\alpha-1)(k_{a;c}{}^c + k_{c;a}{}^c) + 2k^n{}_{;na} \ , \end{aligned} \tag{A.14}$$

which can be rewritten using the Ricci identities and the expression for the Riemann tensor in terms of the Ricci tensor valid in 3 dimensions. Using the substitution $W_a = A_a{}^n{}_{;n}$, equations (A.14) turn into

$$\begin{aligned} 0 \ = \ & (2\alpha\beta-\alpha-\beta+1)W_{a;c}{}^c + (2\alpha\beta-\alpha+\beta-1)W_{c;a}{}^c \\ & +(1-2\beta+\gamma)R_{ac}W^c + \delta W_a - \delta A_a{}^n{}_{;n} - \gamma R_{ac}A^{cn}{}_{;n} + R^{sn}(3A_{as;n} - 2A_{sn;a}) \\ & -R_{as;n}A^{sn} + 2k^n{}_{;na} + (2\alpha-1)(k_{a;c}{}^c + k_{c;a}{}^c) \ . \end{aligned} \tag{A.15}$$

Here we have added and removed a term $\gamma R_{ac}W^c = \gamma R_{ac}A^{cn}{}_{;n}$, as well as a term $\delta W_a = \delta A_a{}^n{}_{;n}$. Expressing W in terms of Z using (A.12) one obtains

$$\begin{aligned}
0 &= (2\alpha\beta - \alpha - \beta + 1)Z_{a;c}{}^c + (2\alpha\beta - \alpha + \beta - 1)Z_{c;a}{}^c \\
&\quad +(1 - 2\beta + \gamma)R_{ac}Z^c + \delta Z_a \\
&\quad +2(2\alpha\beta - \alpha - \beta + 1)k_{a;c}{}^c + 2(2\alpha\beta - \alpha + \beta - 1)k_{c;a}{}^c \\
&\quad +2(1 - 2\beta + \gamma)R_{ac}k^c + 2\delta k_a \\
&\quad +(1 - 2\beta)\Big(- \delta A_a{}^n{}_{;n} - \gamma R_{ac}A^{cn}{}_{;n} + R^{sn}(3A_{as;n} - 2A_{sn;a}) \\
&\quad -R_{as;n}A^{sn} + 2k^n{}_{;na} + (2\alpha - 1)(k_{a;c}{}^c + k_{c;a}{}^c)\Big) \,. \qquad \text{(A.16)}
\end{aligned}$$

Equation (A.3) is equivalent to the vanishing of the sum of the last four lines above, leading to (A.13). To check ellipticity we calculate, choosing e_a as before and rescaling p so that $p = (0, 0, p_3 = 1)$, the symbol $\sigma(p)$ and obtain

$$\begin{aligned}
\sigma(p)_1 &= (2\alpha\beta - \alpha - \beta + 1)Z_1 \\
\sigma(p)_2 &= (2\alpha\beta - \alpha - \beta + 1)Z_2 \\
\sigma(p)_3 &= 2\alpha(2\beta - 1)Z_3 \qquad \text{(A.17)}
\end{aligned}$$

and ellipticity of (A.13) follows. □

A simple way of ensuring that (A.13) implies $Z = 0$ is the following; the restrictions on the constants are far from being optimal:

PROPOSITION A.3 *Suppose that* $\alpha = -1$, $\beta \in [0, 2/3]$, $\gamma = -1+\beta$ *and* $\delta < 0$. *If* M *is compact without boundary, then the only vector field* Z *satisfying* (A.13) *is* $Z = 0$.

PROOF: Let Z be a solution of (A.13) with $\alpha = -1$. Rewriting the $R_{ab}Z^b$ term as a commutator of covariant derivatives, this is equivalent to

$$(2 - 3\beta)Z_a{}^{;c}{}_c - \beta Z_{c;a}{}^c - (1 - 2\beta + \gamma)(Z_c{}^{;c}{}_a - Z_{c;a}{}^c) + \delta Z_a = 0 \,. \qquad \text{(A.18)}$$

Multiplying by Z^a and integrating by parts one obtains

$$\int_M \big((3\beta - 2)Z_{a;c}Z^{a;c} + (1 - \beta + \gamma)Z_{a;c}Z^{c;a} + (-2\beta + 1 + \gamma)(Z_c{}^{;c})^2 + \delta Z_a Z^a\big) = 0. \qquad \text{(A.19)}$$

The simplest choice $\gamma = -1 + \beta$ leads to

$$\int_M \big((3\beta - 2)Z_{a;c}Z^{a;c} - \beta(Z_c{}^{;c})^2 + \delta Z_a Z^a\big) = 0 \,. \qquad \text{(A.20)}$$

With our choice of δ and β all terms have the same signs, leading to $Z = 0$, as desired. $\square$

In the Lorentzian case we choose the signature $(-,+,+)$ and we use the index range $0, 1, 2$ to label local orthonormal frames e_a, with e_0 — timelike. We examine first whether our system of equations leading to (A.2), again for the ansatz (A.1), consists of a strictly hyperbolic system of equations (*cf.*, *e.g.* [6, p. 590] for definitions). Using an ON frame again, assuming $A^n{}_n = 0$, and that g is scalar flat, the five independent second-order equations (A.2) for A_{ab} read

$$\begin{aligned}
P(A,k)_{00} &= 2[((\alpha - 2\alpha\beta)\partial_{00} + \partial_{11} + (1-\beta)\partial_{22})A_{00} + \beta(\partial_{22} - \partial_{11})A_{11} \\
&\quad +(\beta + 2\alpha\beta - 1 - \alpha)(\partial_{01}A_{01} + \partial_{02}A_{02}) - 2\beta\partial_{12}A_{12}] + l.o. \\
P(A,k)_{11} &= 2[\beta(\partial_{00} + \partial_{22})A_{00} + (-\partial_{00} + (2\alpha\beta - \alpha)\partial_{11} + (1-\beta)\partial_{22})A_{11} \\
&\quad +(1 + \alpha - \beta - 2\alpha\beta)(\partial_{01}A_{01} - \partial_{12}A_{12}) - 2\beta\partial_{02}A_{02}] + l.o. \\
P(A,k)_{01} &= (1 + \beta + \alpha - 2\alpha\beta)\partial_{01}(A_{00} - A_{11}) + ((\alpha - 1 + \beta - 2\alpha\beta) \\
&\quad (\partial_{00} - \partial_{11}) + 2\partial_{22})A_{01} + (2\alpha\beta - \beta - 1 - \alpha)(\partial_{12}A_{02} + \partial_{02}A_{12}) + l.o. \\
P(A,k)_{02} &= (\alpha + 1 + \beta - 2\alpha\beta)\partial_{02}A_{11} + (2\alpha\beta - 1 - \beta - \alpha)(\partial_{12}A_{01} + \partial_{01}A_{12}) \\
&\quad +((\alpha - 1 + \beta - 2\alpha\beta)(\partial_{00} - \partial_{22}) + 2\partial_{11})A_{02} + l.o. \\
P(A,k)_{12} &= (2\alpha\beta - \beta - \alpha - 1)\partial_{12}A_{00} + (1 + \alpha + \beta - 2\alpha\beta)(\partial_{02}A_{01} + \partial_{01}A_{02}) \\
&\quad +(-2\partial_{00} + (1 - \beta + 2\alpha\beta - \alpha)(\partial_{11} + \partial_{22}))A_{12} + l.o.\,, \qquad \text{(A.21)}
\end{aligned}$$

where *l.o.* stands for all remaining lower-order terms in each equation. As before, consistency of zero-traces of A and Ric forces us to impose $\alpha = -1$; this will be assumed from now on.

In order to check whether the equations (A.21) form a hyperbolic second-order system, we must examine the *determinant* Q of the coefficient matrix of the A_{ab} in (A.21), where each ∂_i is replaced by a vector component ξ_i. This determinant can be written as

$$Q = \left| \sum Q^{ij}\xi_i\xi_j \right| \,,$$

where the Q^{ij} are the coefficients composing the coefficient matrix.

We wish to calculate Q for our system for $\alpha = -1$ and for a vector ξ with components $(1, k, 0)$. Computer algebra gives

$$Q = 8(4 - 20\beta + 33\beta^2 - 18\beta^3)(k-1)^5(1+k)^5 \qquad \text{(A.22)}$$

which clearly always has the roots $k = 1$ and $k = -1$, with multiplicities five each unless the prefactor vanishes. Thus the characteristic directions are the light-cone directions. Since we obtain multiple eigen-directions, we conclude that the system (A.21) is *not* strictly hyperbolic.

This, in itself, does not prove that the evolution problem for (A.6) considered as an equation for A is ill-posed, but shows that the results from the theory of strictly hyperbolic systems do not apply here.

For further reference we write P in full details,

$$\begin{aligned}
P(A,k)_{ab} &= 2A_{ab}{}^{;c}{}_{c} - A_a{}^c{}_{;bc} - A_b{}^c{}_{;ac} + (1-3\beta)(A_a{}^c{}_{;cb} + A_b{}^c{}_{;ca}) + 2\beta A^{cd}{}_{;cd}g_{ab} \\
&\quad -3(k_{a;b}+k_{b;a}) + 2k^c{}_{;c}g_{ab} \\
&= 2A_{ab}{}^{;c}{}_{c} - 2R_{(a}{}^{cd}{}_{b)}A_{cd} - 2R_{d(a}A_{b)}{}^{d} - 3\beta(A_a{}^c{}_{;cb} + A_b{}^c{}_{;ca}) + 2\beta A^{cd}{}_{;cd}g_{ab} \\
&\quad -3(k_{a;b}+k_{b;a}) + 2k^c{}_{;c}g_{ab} \\
&= 2A_{ab}{}^{;c}{}_{c} + ST\left\{-2R_{(a}{}^{cd}{}_{b)}A_{cd} - 2R_{d(a}A_{b)}{}^{d} - 6\beta A_a{}^c{}_{;cb} - 6k_{a;b}\right\} \ , \qquad \text{(A.23)}
\end{aligned}$$

where we use the symbol $ST\{\cdot\}$ to denote the symmetric traceless part.

When studying mapping properties of $\mathscr{P}_{-1,\beta,\gamma,\delta}$ one needs to consider the formal adjoint $\mathscr{P}^*_{-1,\beta,\gamma,\delta}$ of $\mathscr{P}_{-1,\beta,\gamma,\delta}$. Writing $\mathscr{P}^*_{-1,\beta,\gamma,\delta}$ as

$$\mathscr{P}^*_{-1,\beta,\gamma,\delta} = (P^*(A,k), S^*(A,k)) \ ,$$

after several integrations by parts one obtains

$$\begin{aligned}
P^*(A,k)_{ab} &= 2A_{ab}{}^{;c}{}_{c} + ST\Big\{-2R_{(a}{}^{cd}{}_{b)}A_{cd} - 2R_{d(a}A_{b)}{}^{d} - 6\beta A_a{}^c{}_{;cb} \\
&\qquad + (1-2\beta)\left(\gamma(k^cR_{ca})_{;b} + \delta k_{a;b} - 3(R_a{}^ck_b)_{;c} + 2(k^cR_{ab})_{;c} - k^cR_{ca;b}\right)\Big\} \ , \\
&= 2A_{ab}{}^{;c}{}_{c} + ST\Big\{ - 2A_a{}^c{}_{;bc} + (2-6\beta)A_a{}^c{}_{;cb} \\
&\qquad + (1-2\beta)\left(\gamma(k^cR_{ca})_{;b} + \delta k_{a;b} - 3(R_a{}^ck_b)_{;c} + 2(k^cR_{ab})_{;c} - k^cR_{ca;b}\right)\Big\} \ ,
\end{aligned} \qquad \text{(A.24)}$$

$$S^*(A,k)_a = k_{a;n}{}^{n} - k^n{}_{;na} + (2\gamma-1)R_a{}^nk_n + 2\delta k_a + 6A_a{}^b{}_{;b} \ . \qquad \text{(A.25)}$$

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