On the dynamics of Gowdy space times

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Abstract

We study the behavior near the singularity t=0 of Gowdy metrics. We prove existence of an open dense set of boundary points near which the solution is smoothly "asymptotically velocity term dominated" (AVTD). We show that the set of solutions which are AVTD near the whole boundary and which satisfy a uniformity condition is open in the set of all solutions. We analyse in detail the asymptotic behavior of "power law" solutions at the (hitherto unchartered) points at which the asymptotic velocity equals zero or one. Several other related results are established.

Contents

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1 Introduction

The Gowdy family of space-times [8] constitutes an interesting toy model to study formation of singularities in general relativity. This family of metrics is sufficiently simple to hope to analyse the resulting singularities in an exhaustive way. It is sufficiently non-trivial so that the relevant dynamical behavior has not been understood so far. The questions of interest are the curvature blow-up – or lack thereof – at the boundary $t = 0$ of the associated space-time, as well as existence of Cauchy horizons. The reader is referred to [6] for a further discussion of this issue, we simply note that the relevant geometric information can be obtained by deriving a sharp asymptotic expansion of the solutions near the singular set $t = 0$. The main purpose of this work is to prove a stability result for the existence of such expansions.

In Gowdy space times the essential part of the Einstein equations reduces to a nonlinear wave-map-type system of equations [8] for a map x from $(M, g_{\alpha\beta})$ to the hyperbolic plane $(\mathscr{H}, h_{ab})$, where $M = [T, 0) \times \mathrm{S}^1$ with the flat metric $g = -dt^2 + d\theta^2$. The solutions are critical points of the Lagrangean

$$\mathscr{L}[x] = \frac{1}{2} \int_M t g^{\alpha\beta} h_{ab} \partial_\alpha x^a \partial_\beta x^b \, d\theta dt \, . \tag{1.1}$$

This differs from the usual wave-map Lagragean by a supplementary multiplicative factor t. It is sometimes convenient to use coordinates $P, Q \in \mathbb{R}$ on the hyperbolic plane in which the hyperbolic metric h_{ab} takes the form

$$h = dP^2 + e^{2P} dQ^2. \tag{1.2}$$

Let $X_t = \frac{\partial x}{\partial t}, X_\theta = \frac{\partial x}{\partial \theta}$, D denote the Levi-Civita connection of h_{ab}, and $D_\theta \equiv \frac{D}{D\theta} := D_{X_\theta}$, $D_t \equiv \frac{D}{Dt} := D_{X_t}$. The Euler-Lagrange equations for (1.1) take the form

$$\frac{DX_t}{Dt} - \frac{DX_\theta}{D\theta} = -\frac{X_t}{t} \tag{1.3}$$

or, in coordinates,

$$\Box x^a + \Gamma^a_{bc} \circ x \partial_\mu x^b \partial^\mu x^c = -\frac{\partial_t x^a}{t},$$

where the Γ's are the Christoffel symbols of h_{ab}, and $\Box = \partial_t^2 - \partial_\theta^2$. Global existence of smooth solutions on $(-\infty, 0)$ of the Cauchy problem for (1.3) has been established by V. Moncrief [13].

For further use we note the non-vanishing Christoffel symbols of h:

$$\Gamma^P_{QQ} = -e^{2P} \ , \quad \Gamma^Q_{PQ} = \Gamma^Q_{QP} = 1 \ . \tag{1.4}$$

In the (P, Q) coordinates one thus has

$$\begin{aligned}
\partial_t^2 P - \partial_\theta^2 P &= -\frac{\partial_t P}{t} + e^{2P}\left((\partial_t Q)^2 - (\partial_\theta Q)^2\right) \ , \\
\partial_t^2 Q - \partial_\theta^2 Q &= -\frac{\partial_t Q}{t} - 2\left(\partial_t P \partial_t Q - \partial_\theta P \partial_\theta Q\right) \ .
\end{aligned}$$

We consider solutions defined on sets $\Omega(a, b, t_0)$, where

$$t_0 < 0 \ , \ a \le b \ , \quad \Omega(a, b, t_0) := \{t_0 \le t < 0 \ , \ a + t \le \theta \le b - t\} \tag{1.5}$$

(see Figure 1). Thus our analysis is local, if the solution satisfies certain properties on an interval $[a, b] \subset S^1$, then the conclusions hold on that interval. Throughout this work we assume that the initial data for the map x at t_0 are smooth functions of θ. We prove the following theorem (the Geroch group is defined in Section 4; the position function φ_∞ is defined by Equation (3.24)):

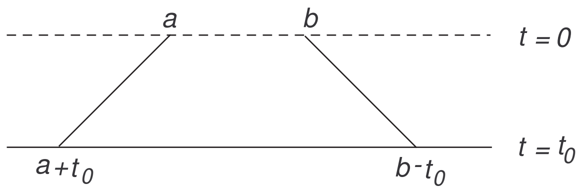

Figure 1: The set $\Omega(a,b,t_0)$.

THEOREM 1.1 *Let $(\mathring{x}(t_0,\cdot),\mathring{X}_t(t_0,\cdot))$ be Cauchy data for a solution of the Gowdy equations on $\Omega(a,b,t_0)$ such that the associated solution $\mathring{x}$ has* uniformly controlled blow-up*; by this we mean that*

$$\sup_{\theta\in[a-|t|,b+|t|]}\left(\sum_{k=0}^{\ell}|t^{k+1}D_\theta^k X_\theta|+\sum_{k=1}^{\ell}|t^{k+1}D_\theta^k X_t|\right)(t,\theta)\to_{t\to 0} 0\ , \qquad (1.6)$$

with $\ell=2$, and with X_t in (1.6) *equal to $\partial_t\mathring{x}$,* etc. *There exists $\eta>0$ such that for all initial data $(x(t_0,\cdot),X_t(t_0,\cdot))$ satisfying*[1]

$$\|(x(t_0,\cdot)-\mathring{x}(t_0,\cdot),X_t(t_0,\cdot)-\mathring{X}_t(t_0,\cdot))\|_{H^3\oplus H^2}<\eta$$

the associated solution x also satisfies (1.6)*, and further the following holds:*

(i) *For all $\theta\in[a,b]$ the function $|tX_t|(t,\theta)$ converges to a velocity function $v(\theta)$ as t tends to zero, uniformly in θ.*

(ii) *There exists an open dense set on which v is smooth.*

(iii) *$[a,b]$ can be covered by a finite number of intervals $[a_i,b_i]$ with the following property: for each i there exists an element G_i of the Geroch group such that G_ix has a smooth velocity function $0\le v<1$ and a smooth position function φ_∞, except perhaps on the boundary of the set $\{v(\theta)=0\}$. Further G_ix satisfies a power law blow-up, Equation (8.20).*

(iv) *In the associated space-time the curvature scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ blows up in finite proper time on every causal curve approaching*

$$\mathscr{B}:=\{0\}\times([a,b]\setminus\{v(\theta)=1\})\times S^1\times S^1\ .$$

[1] We equip $\Omega(a,b,t_0)$ with the Riemannian metric $dt^2+d\theta^2$, this together with the metric h on $\mathscr{H}_2$ induces Riemannian metrics on all the bundles involved. We use those metrics to measure the distance between points on those bundles.

In particular the associated Gowdy space-time is inextendible across $\mathscr{B}$.

REMARK 1.2 Actually it suffices to have a sequence $t_i \to 0$ along which (1.6) holds. A recent result of Ringström [18][2] can be used to lower to $\ell = 1$ the threshold ℓ in (1.6).

The proof of Theorem 1.1 can be found at the end of Section 12.

It is of interest to enquire whether the uniform blow-up condition (1.6) is necessary for AVTD behavior of the solutions. Consider, for example, an $\mathrm{AVTD}_3^{(P,Q)}$ solution, as defined in Section 3, for which the error terms in (3.5)-(3.6) and in their derivative counterparts (see (3.10)) are uniform in θ. If v_1 is strictly smaller than one (no negative lower bound assumed), then the solution satisfies (1.6)[3]. This shows in particular that the set of solutions satisfying the hypotheses of Theorem 1.1 is not empty, as existence of a large class of $\mathrm{AVTD}_\infty^{(P,Q)}$ solutions satisfying $v_1 < 1$ follows from the results in [14].

The second main result of this work is the proof that for every solution there exists an open dense set $\hat{\Omega} \subset S^1$ near which we have complete control of the solution:

THEOREM 1.3 *Consider a solution x defined on $\Omega(a, b, t_0)$. There exists an open dense set $\hat{\Omega} \subset [a, b]$ such that x is $AVTD_\infty^{(P,Q)}$ in a neighborhood of $\{0\} \times \hat{\Omega}$.*

The examples discussed in Section 3 show that the result is sharp, with the following proviso: the known examples have a velocity function defined everywhere, even at points where it is not continuous, while Theorems 1.3 and 12.1 leave open the possibility of existence of points where the velocity is not defined. Such points (if any) are characterised in point (i) of Proposition 12.3.

The third main result of this paper is an exhaustive analysis of the asymptotic behavior of power-law solutions at points at which v vanishes,

[2]The results presented in [18] have been made available as a preprint [17] while this paper was being prepared for publication.

[3]This follows immediately from Proposition 1.4 together with the calculations of the proof of Lemma 8.7. For points θ at which $v_1 \geq 1$ one expects Q_∞ to have vanishing derivatives, compare point (ii) of Proposition 3.1 and Theorem 10.2. Then, if $v_1(\theta_0) \geq 1$ and if $\partial_\theta^i Q_\infty(\theta_0) = 0$, $i = 1, 2, 3$, then (3.5)-(3.6) and their derivative counterparts give pointwise decay of the function appearing under the sup at the left-hand-side of (1.6), but uniformity is far from being clear.

or equals one. This last case is especially important for the discussion of strong cosmic censorship, we refer the reader to [6] for applications. We note that no results concerning those velocities were available so far in the non-polarised case.

The results discussed above are established through a series of auxiliary results which have some interest in their own. We say that a solution x satisfies *a power law blow-up*, or is of *power-law type*, or is a *power-law solution*, if the norm of the theta derivatives vector $|X_\theta|$ does not blow up faster than $|t|^{\epsilon-1}$, for some positive constant ϵ, when approaching the singularity $t = 0$. All solutions of the smooth Cauchy problem on T^3 analysed in detail so far satisfy[4] a power law decay.

It is simple to show that every solution with a power law decay has a continuous asymptotic velocity function v (see the proof of Theorem 8.3 below). The associated solutions of the vacuum Einstein equations have curvature blowing up uniformly, except perhaps at the set of points θ at which $v(\theta) = 1$. Consider the set of initial data for solutions satisfying a power law decay and for which $v < 1$, uniformly in θ. We show – see Theorem 11.1 below – that this set is open in the set of all initial data; this is one of the steps of the proof of Theorem 1.1. We further show that for those solutions v is smooth except perhaps at the boundary of the set of points at which v vanishes. Theorem 11.1 leads to a sharper version of the *stability of the singularity theorem* for $(2/3, 2/3, -1/3)$ Kasner metrics, see Theorem 9.1.

An important element of our analysis is the action of the Geroch group, as defined in Section 4. In fact, the key ingredients of our analysis are the results in [4] together with the following:

(i) The analysis of the action of the Geroch group in the work of Rendall and Weaver [15];

(ii) The reformulation of the wave-map equations as a first-order system of scalar equations by Christodoulou and Tahvildar-Zadeh [3];

(iii) The small-derivatives stability result of Ringström [16].

We finish this introduction by recalling some results from [4] which will be useful in the sequel:

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[4]More precisely, the examples known to us satisfy a power law decay on all sets $C^0_{t_0}(\psi)$ as defined in Equation (2.3) below. The constants are uniform in ψ away from the points at which the asymptotic velocity has spikes, or discontinuities, or crosses zero or one. At the end of Section 2 we give an explicit self-similar solution that does not satisfy the power law decay, but this solution does not fit into a Cauchy problem framework.

PROPOSITION 1.4 (Time-weighted pointwise estimates; Proposition 3.2.1 in [4]) *Let $x(t_0,\theta) \in C^k(S^1)$, $k \geq 1$, $X_t(t_0,\theta) \in C^{k-1}(S^1)$. For all $t \geq t_0$ we have*

(i) $(|X_t|^2+|X_\theta|^2)(t,\theta) \leq 2\left\{\sup_{\psi\in[\theta-t+t_0,\theta+t-t_0]}(|X_t|^2+|X_\theta|^2)(t_0,\psi)\right\}\left(\frac{t_0}{t}\right)^2$.

(ii) *If $k \geq 2$, then there exist constants C depending only upon the arguments listed such that, for all $1 \leq |\alpha| \leq k$,*

$$|D^\alpha x|(t,\theta) \leq C(|\alpha|, t_0, \|X_\theta(t_0)\|_{C^{|\alpha|-1}}, \|X_t(t_0)\|_{C^{|\alpha|-1}})|t|^{-|\alpha|}. \quad (1.7)$$

REMARK 1.5 It has been pointed out to us by H. Ringström that the proof of Proposition 3.2.1 in [4] (compare [4, Equation (3.2.5)] together with the argument leading to Equation (3.2.9) there) actually gives an inequality somewhat stronger than (i) above:

$$\left(\frac{t}{t_0}\right)^2(|X_t|^2+|X_\theta|^2)(t,\theta) \leq \frac{1}{2}\left\{\sup_{\psi\in[\theta-t+t_0,\theta+t-t_0]}|X_t-X_\theta|^2(t_0,\psi)+\sup_{\psi\in[\theta-t+t_0,\theta+t-t_0]}|X_t+X_\theta|^2(t_0,\psi)\right\}. \quad (1.8)$$

Equation (1.8) carries more information about the solution than the inequality in (i), which can be seen *e.g.* when the initial data have small θ-derivatives.

PROPOSITION 1.6 (Time-weighted Sobolev decay; Proposition 3.3.1 in [4]) *Let $x \in C^i([t_0,0)\times S^1)$ and let $X_\theta(t_0,\cdot)$, $X_t(t_0,\cdot) \in H_i(S^1)$, $i \geq 1$. Then there exist constants depending only upon the arguments listed such that*

(i) *For all $1 \leq |\alpha| \leq i+1$,*

$$g^{(\alpha)}(t) \equiv \oint d\theta |t|^{2|\alpha|}|D^\alpha x|^2 \leq C(|\alpha|, \|X_\theta(t_0)\|_{H_{|\alpha|-1}(S^1)}, \|X_t(t_0)\|_{H_{|\alpha|-1}(S^1)}, t_0).$$

(ii) *If at least one differentiation is a θ differentiation we have*

$$\lim_{t\to 0} g^{(\alpha)}(t) = 0.$$

(iii) *If at least one differentiation is a θ differentiation then $\frac{g^{(\alpha)}(t)}{|t|} \in L^1([t_0,0])$ and*

$$\int_{t_0}^0 \frac{g^{(\alpha)}(s)}{|s|}ds \leq C'(|\alpha|, t_0, \|X_\theta(t_0)\|_{H_{|\alpha|-1}(S^1)}, \|X_t(t_0)\|_{H_{|\alpha|-1}(S^1)}).$$

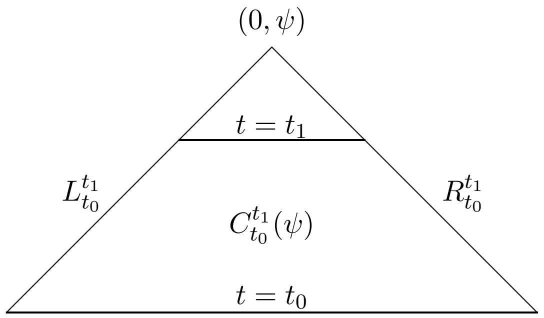

Figure 2: The truncated domains of dependence $C_{t_0}^{t}(\psi)$.

2 Problems with θ derivatives, self-similar solutions

As already mentioned in the introduction, all published solutions of the Gowdy equations known to us, and for which the asymptotic behavior is reasonably well understood [4, 5, 10, 14–16], have the property that

$$|tX_\theta| \le Ct^\epsilon \tag{2.1}$$

for some $\epsilon > 0$, with the bound being optimal[4]. The power law is very useful for the control of the analytic properties of the solutions, but it is not necessary for curvature blow-up. In any case the bound (2.1) certainly implies, for all $\psi \in S^1$,

$$\int_{C_{t_0}^0(\psi)} |X_\theta|^2 \, dt \, d\theta < \infty \;, \tag{2.2}$$

where for $t_0 < 0$ the set $C_{t_0}^0(\psi)$ is defined as (compare Figure 2)

$$C_{t_0}^0(\psi) = \{t_0 \le t < 0 \;,\; -|t| \le \theta - \psi \le |t|\} \;. \tag{2.3}$$

We shall say that $\lim_{C_{t_0}^0(\psi)} f = \alpha$ if

$$\lim_{t\to 0} \sup_{-|t|\le\theta-\psi\le|t|} |f(t,\theta) - \alpha| = 0 \;. \tag{2.4}$$

Such limits look a little awkward at first sight; however, they arise naturally when considering the behavior of the geometry along causal curves with endpoints on the boundary $t = 0$. Further, such limits appear naturally in our results below.

We have a partial converse to (2.1):

PROPOSITION 2.1 (Proposition 3.4.1 in [4]) *At every ψ at which* (2.2) *holds we have, for all multi-indices α,*

$$\lim_{C^0_{t_0}(\psi)} |t|^{|\alpha|+1} |D^\alpha X_\theta| = 0 \ .$$

REMARK 2.2 In Section 6 below we give further integral conditions which ensure pointwise convergence of $|tX_t|$ to a number $v(\psi)$. Yet another criterion for existence of $v(\psi)$ is given by Proposition 12.3.

Proposition 2.1 begs the question of existence of solutions for which (2.2) fails. An obvious candidate is given by self-similar solutions:

$$x(t,\theta) = y(\theta/t) \ , \tag{2.5}$$

for some map y from M to the hyperbolic space. It would be of interest to find all solutions satisfying (2.5). Here we note the following family of such solutions: let $\alpha, \beta \in \mathbb{R}$ and let $\Gamma : \mathbb{R} \to \mathscr{H}$ be an affinely parameterised unit-speed geodesic in $\mathscr{H}$, for $|\theta| < -t$ set

$$x(t,\theta) = \Gamma\Big(\alpha \arcsin\Big(\frac{\theta}{t}\Big) + \beta\Big) \ . \tag{2.6}$$

It is easily checked that (2.6) solves the Gowdy equation (1.3). Equation (2.2) fails for the solution (2.6) when $\alpha \neq 0$, as expected. It turns out that the solutions (2.6) do not fit into the Cauchy problem framework because they are singular on the whole light cone $|\theta| = -t$, while the solutions we are interested in are smooth at $t = t_0$. Singular solutions can sometimes be used to produce smooth examples of bad behavior, but we have not managed to exploit this solution to do that. In view of our stability results here it would be important to construct a solution of the Cauchy problem where (2.2) fails, or to prove that such solutions do not exist.

3 AVTD$_k$ behavior, spikes and discontinuities

In [5, 11, 14] a large class of solutions of (1.3) has been constructed with the following behavior:

$$P(t,\theta) = -v_1(\theta)\ln|t| + P_\infty(\theta) + o(1) \ , \quad 0 < v_1(\theta) < 1 \ , \tag{3.1}$$
$$Q(t,\theta) = Q_\infty(\theta) + |t|^{2v_1(\theta)}\Big(\psi_Q(\theta) + o(1)\Big) \ . \tag{3.2}$$

A generalisation of those formulae to arbitrary velocities requires a careful study of the field equations. For instance, an analysis of the indicial exponents of the linearised equations suggests the following behavior of Q at points where $v(\theta) = n \in \mathbb{N}^*$ (compare [6] for $n = 1$)[5]

$$Q(t,\theta) = Q_\infty(\theta) + \frac{t^{2n}}{(2n)!}\partial_\theta^{2n} Q_\infty(\theta) \ln|t| + \psi_Q(\theta) t^{2n} + o(t^{2n}) \; . \quad (3.3)$$

While there is no existence statement for solutions with a non-zero coefficient in the $\ln|t|$ term above, we expect that such solutions can actually be constructed. We note that if $v \in \mathbb{N}^*$ on an interval, then no log term will occur on that interval.

Applying the "solution-generating transformation" (3.19) described below to (3.3) at a point at which one further has $Q_\infty(\theta) = 0$ leads to a solution (P', Q') with a negative P-velocity $v_1'(\theta) = -n$ and with a logarithmically blowing-up Q function

$$Q'(t,\theta) = e^{2P_\infty(\theta)} \left(\frac{1}{(2n)!} \partial_\theta^{2n} Q_\infty(\theta) \ln|t| + \psi_Q(\theta) \right) + o(1) \; . \quad (3.4)$$

The above discussion suggests that the following will capture the asymptotic behavior of a large class of solutions of the Gowdy equations:

$$P(t,\theta) = -v_1(\theta)\ln|t| + P_\infty(\theta) + o(1) \; , \quad (3.5)$$

$$Q(t,\theta) = Q_\infty(\theta) + \begin{cases} |t|^{2v_1(\theta)}\Big(\psi_Q(\theta) + o(1)\Big) \; , & 0 < v_1(\theta) \notin \mathbb{N} \; ; \\ |t|^{2v_1(\theta)}\Big(Q_{\ln}(\theta)\ln|t| + \psi_Q(\theta) + o(1)\Big) \; , & 0 < v_1(\theta) \in \mathbb{N} \; ; \\ Q_{\ln}(\theta)\ln|t| + o(1) \; , & v_1(\theta) \in -\mathbb{N}^* \; ; \\ o(1) \; , & -\mathbb{N}^* \not\ni v_1(\theta) \le 0 \; . \end{cases} \quad (3.6)$$

The function

$$v := |v_1| \quad (3.7)$$

will be called the *velocity function*, while Q_∞ will be called the *Q-position function*. Those functions have the following geometric interpretation [9]: for $v > 0$ the path

$$\tau \to \Gamma_\theta(\tau) := (P(-e^{-\tau},\theta), Q(-e^{-\tau},\theta))$$

[5] The expected *a priori* estimate $|t^{i+1} D_\theta^i X_\theta| \to 0$ implies that $\partial_\theta^i Q_\infty(\theta) = 0$ for $i = 1, \ldots, 2n-1$, which is implicit in (3.3).

approaches – in a sense made precise by (3.5)-(3.6) – the affinely parameterised h-geodesic

$$\tau \to \mathring{\Gamma}_\theta(\tau) := (v_1(\theta)\tau - \varphi(\theta), Q_\infty(\theta)) \ ,$$

with v - the length of the velocity vector of $\mathring{\Gamma}_\theta$. The point $Q_\infty(\theta)$ is then the uniquely defined point on the conformal boundary of the hyperbolic space at which the geodesic $\mathring{\Gamma}_\theta$ accumulates. Clearly this interpretation breaks down at $v(\theta) = 0$, which suggests that solutions might display strange features, not necessarily compatible with (3.5)-(3.6), at the boundary of the set $\{v(\theta) = 0\}$.

We shall say that a map $x = (P, Q)$ is in the AVTD$^{(P,Q)}$ class if there exist functions v_1, Q_∞, and $Q_{\ln}$ such that

$$P(t,\theta) = -v_1(\theta)\ln|t| + O(1) \ , \tag{3.8}$$

$$Q(t,\theta) = \begin{cases} Q_\infty(\theta) + o(1) \ , & v_1 \notin -\mathbb{N}^* \ ; \\ Q_{\ln}(\theta)\ln|t| + Q_\infty(\theta) + o(1) \ , & v_1 \in -\mathbb{N}^* \ . \end{cases} \tag{3.9}$$

We shall say that a solution is in the AVTD$_k^{(P,Q)}$ class if (3.5)-(3.6) hold with functions v_1, P_∞, Q_∞, $Q_{\ln}$ and ψ_Q which are of C_k differentiability class (on closed intervals the derivatives are understood as one-sided ones at the end points). For the purposes of the AVTD$_k^{(P,Q)}$ definition the function $Q_{\ln}$ is assumed to be extended by 0 to the set $v_1(\theta) \notin -\mathbb{N}^*$; we emphasise that such an extension will *not* be assumed in Definition 3.4 below. For $k > 0$ we will assume that the behavior (3.5)-(3.6) is preserved under differentiation in the following way:

$$\forall\ 0 \le i + j \le k \qquad \partial_\theta^j (t\partial_t)^i \Big(P(t,\theta) + v_1(\theta)\ln|t| - P_\infty(\theta) \Big) = o(1) \ , \tag{3.10}$$

similarly for Q.

Note that the classes AVTD$^{(P,Q)}$ and AVTD$_0^{(P,Q)}$ do not coincide.

Unless explicitly stated otherwise the $o(1)$ symbol denotes *pointwise convergence to zero as t tends to zero at fixed θ*. Similarly $O(1)$ means *uniformly bounded in t at fixed θ*. An alternative meaning of $o(t)$, which will be sometimes used, is provided by uniform convergence to zero on the set $\Omega(a, b, t_0)$ as defined by (1.5). Yet another possibility is convergence to zero in a $\lim_{C^0_{t_0}(\psi)}$ sense, as defined by (2.4); in any case we will make precise statements when needed.

Using the above solutions, Rendall and Weaver [15] have constructed solutions of (1.3) which display "spiky features". They separate those solutions into two classes, one called "false spikes" and one called "true spikes". An instructive example of this behavior is provided by a family of Gowdy maps discovered by Moncrief [12], and analysed in detail in [4, Appendix B]. They are given there in terms of the polar coordinates on the hyperbolic space,

$$h = d\rho^2 + \sinh^2\rho \, d\varphi^2 \ , \tag{3.11}$$

which are related to the (P,Q) coordinates by the formulae[6]

$$\begin{aligned} e^P &= \cosh\rho + \sinh\rho \ \cos\varphi \\ &= \frac{1}{2}\left(e^{\rho}(1+\cos\varphi) + e^{-\rho}(1-\cos\varphi)\right) \ , \qquad (3.12)\\ e^P Q &= \sinh\rho \ \sin\varphi \ . \qquad (3.13) \end{aligned}$$

Moncrief's ansatz

$$\rho = \rho(t) \ , \quad \varphi = n\theta \ , \quad n \in \mathbb{N} \ , \tag{3.14}$$

leads to the following: every solution is uniquely determined by two numbers $v_\infty \in [0,1)$, $\rho_\infty \in \mathbb{R}$, such that

$$\rho = -v_\infty \ln|t| + \rho_\infty + o(1) \ . \tag{3.15}$$

Inserting (3.14)-(3.15) into (3.12) one finds

$$P(t,\theta) = \begin{cases} -v_\infty \ln|t| + \rho_\infty + \ln\left(\frac{1+\cos(n\theta)}{2}\right) + o(1) \ , & n\theta \neq \pi \mod 2\pi; \\ v_\infty \ln|t| - \rho_\infty + o(1) \ , & n\theta = \pi \mod 2\pi, \end{cases} \tag{3.16}$$

$$Q(t,\theta) = \begin{cases} \frac{\sin(n\theta)}{1+\cos(n\theta)} + o(1) \ , & n\theta \neq \pi \mod 2\pi; \\ 0 \ , & n\theta = \pi \mod 2\pi. \end{cases} \tag{3.17}$$

If we define $v_1 : S^1 \to \mathbb{R}$ by the equation

$$v_1(\theta) := \lim_{t\to 0} |t| P_t \ , \tag{3.18}$$

then $v_1 = v_\infty > 0$ except at n isolated *spike points* $\theta_m = (1+2m)\pi/n$, $m \in \mathbb{N} \cap [1,n]$, at which v_1 is equal to $-v_\infty < 0$. Equation (3.16) shows that the subleading term in P blows up logarithmically at the spike points, so that no uniformity in θ for that term can be expected near those points in norms which control pointwise behavior of P and Q. It is interesting that

[6] We are very grateful to Marsha Weaver for several enlightening discussions concerning the issues discussed in this section, and for providing us formulae (3.12)-(3.13).

even though Q_∞ blows up as one approaches the spike points, it is finite, actually vanishing, there. Next, even though at each fixed θ we have

$$\lim_{t\to 0} P_\theta(t,\theta) = 0 \ ,$$

there are timelike curves reaching the boundary along which P_θ does not go to zero: for example, for $n = 1$,

$$\lim_{t\to 0} P_\theta(t,\pi+\alpha e^{-\rho(t)}) = \infty \ , \quad \lim_{t\to 0} P_\theta(t,\pi+\alpha e^{-2\rho(t)}) = \frac{\alpha}{2} \ .$$

The above examples provide solutions with an arbitrary finite number of spikes. Solutions with a countably infinite number of spikes accumulating at some point $\theta_\infty \in S^1$ can be constructed as follows: Consider an $\mathrm{AVTD}_\infty^{(P,Q)}$ solution such that the function Q_∞ in (3.6) has an infinite number of isolated zeros θ_i accumulating at θ_∞, and such that v_1 avoids zero in a neighborhood of θ_∞; the existence of such solutions follows from [14]. Following [15], one then performs the following "inversion" of the hyperbolic plane:

$$e^{-P'} = \frac{e^{-P}}{Q^2 + e^{-2P}} \ , \quad Q' = \frac{Q}{Q^2 + e^{-2P}} \ . \tag{3.19}$$

This is an isometry of h and therefore maps solutions into solutions. It is easily seen that (P', Q') will have a spike at each of the points θ_i, yielding the desired spiky solution.

The discontinuities discussed so far consisted of isolated points at which v_1 changes sign. Solutions with jumps of v_1 can be constructed as follows: consider any $\mathrm{AVTD}_\infty^{(P,Q)}$ solution such that the zero set of the function Q_∞ in (3.6) is a closed interval $[a, b]$, with v_1 strictly positive near the end points. As before, the existence of such solutions follows from [14]. It is easily seen from (3.19) that the velocity function v_1' associated with the map $x' = (P', Q')$ will jump from $v_1(a)$ to $-v_1(a)$ at a, and will be continuous from the right there. Further, as θ increases from a to b the new velocity function v_1' will continuously attain the value $-v_1(b)$ when b is approached from the left, and jump to $v_1(b)$ immediately afterwards.

Clearly, the above behaviors can be combined to give infinite sequences of pointwise jumps and/or intervals on which v_1' is negative, with the set of discontinuities of v_1' accumulating at a given point.

In fact, let $F \subset S^1$ or $F \subset [a, b]$ be any non-empty closed set without interior, we claim that there exists a smooth function φ_F such that

$$\varphi_F^{-1}(\{0\}) = F \ .$$

In order to see this, let $x \notin F$, and let (x_-, x_+) be the largest open interval containing x which does not meet F (hence $x_\pm \in F$), we set $\sigma(x) = (x - x_-)(x_+ - x)$. Define

$$\varphi_F(x) = \begin{cases} 0, & x \in F; \\ e^{-1/\sigma(x)}, & \text{otherwise.} \end{cases}$$

Then φ_F has all the required properties. Using the function φ_F as Q_∞, with v_1 equal, *e.g.*, to the constant function $1/2$, after performing an inversion we obtain a new function v_1' which equals $-1/2$ on F, and $1/2$ on $S^1 \setminus F$. If F is a fat Cantor set one obtains a rather wild set of spikes, with measure as close as desired to that of S^1 or that of $[a, b]$ by choosing F suitably.

The spikes discussed so far are called *false spikes*, as they can be thought of as an artifact of the (P, Q) coordinate system above: no discontinuous behavior occurs in the (ρ, φ) representation of the solutions.[7] However, the (P, Q) coordinates are very useful when analysing the *Gowdy-to-Ernst transformation*, because that transformation has a very simple form precisely in the (P, Q) coordinates: given a solution $x = (P, Q)$ of the Gowdy equations, one defines a new solution $\hat{x}$ by performing the "Gowdy-to-Ernst" transformation [15]:

$$\hat{P} := -P - \ln|t| \; , \quad e^{\hat{P}} \partial_t \hat{Q} := -e^P \partial_\theta Q \; , \quad e^{\hat{P}} \partial_\theta \hat{Q} := -e^P \partial_t Q \; . \tag{3.20}$$

The new map satisfies again the Gowdy equation (1.3). As shown by Rendall and Weaver, this has significant consequences: By definition, a *true spike* is the image of a false spike after a Gowdy-to-Ernst transformation has been performed; equation (3.20) shows that any discontinuity in v_1 leads to a discontinuity in the velocity $\hat{v}_1$ associated with the map $(\hat{P}, \hat{Q})$. For AVTD solutions this typically leads to a discontinuity in the geometric velocity function $\hat{v} = |\hat{v}_1|$. For instance, for the solutions (3.14) the transformation

[7] Applying isometries of the hyperbolic plane to the solution has the effect of reshuffling Killing vectors, and can thus be considered as an irrelevant "coordinate transformation" from the space-time point of view. We note that the isometry (3.19) changes the orientation of the hyperbolic plane. The accompanying relabeling of the Killing vectors changes the space-time orientation, leading thus to a non-equivalent solution if a space-time orientation has been chosen. However, we can always perform a second inversion about a different point, regaining the original orientation. If the map $\theta \to Q_\infty(\theta)$ is surjective, this will always introduce at least one "false spike" in the transformed solution. In other words, in the surjective case there will be no (P, Q) representation of the solution without "false spikes". This gives some geometric meaning to those.

(3.20) leads to

$$\hat{P} = \begin{cases} -(1-v_\infty)\ln|t| - \rho_\infty - \ln\left(\frac{1+\cos(n\theta)}{2}\right) + o(1)\ , & n\theta \neq \pi \mod 2\pi; \\ -(1+v_\infty)\ln|t| + \rho_\infty + o(1)\ , & n\theta = \pi \mod 2\pi, \end{cases} \tag{3.21}$$

which clearly results in a $|\hat{v}|$ which is *not* continuous at $n\theta = \pi \mod 2\pi$.

We refer the reader to [15, Section 6] for a further discussion of iterations of the above.

It is interesting to enquire about independence of the conditions (3.8)-(3.9). It turns out that (3.8) is the key requirement, up to a Gowdy-to-Ernst transformation:

PROPOSITION 3.1 (i) *At each point θ at which* (3.8) *holds with $v_1 > 0$ we also have* (3.9). *If the term $O(1)$ in* (3.8) *is uniform in θ over some interval I then Q_∞ is continuous on I. If further v_1 is uniformly bounded away from zero, then the term $o(1)$ in* (3.8) *is uniform in $\theta \in I$.*

(ii) *If $v_1 > 1$ on an interval I, with the term $O(1)$ in* (3.8) *uniform in θ, then there exists a constant A such that*

$$\forall\, \theta \in I \quad Q(t,\theta) \to_{t\to 0} A$$

(we say that x is asymptotically polarised *on I).*

(iii) *At points θ or intervals I on which* (3.8) *holds with $v_1 < 1$ the conclusions of point (i) above hold for the Gowdy-to-Ernst transformed map $\hat{x}$. If $v_1 < 0$ on I then the conclusion of point (ii) holds for $\hat{Q}$. (In the case of an interval I we assume that the term $O(1)$ in* (3.8) *is uniform in θ.)*

PROOF: Point (i) of Proposition 1.4 shows that

$$|Q_t| + |Q_\theta| \le \frac{C(\theta)e^{-P}}{|t|} \le C(\theta)|t|^{v_1(\theta)-1}\ . \tag{3.22}$$

Integrating in t one obtains (3.9). If $C(\theta)$ can be made θ-independent, then Q_∞ is a uniform limit of continuous functions, and therefore continuous, which gives (i). If $v_1(\theta) > 1$, then (3.22) shows that $|Q_\theta|$ tends to zero as t goes to zero, which easily implies that Q_∞ is constant over I. This proves (ii). Applying the Gowdy-to-Ernst transformation (3.20) one finds that the hatted velocity function $\hat{v}_1$ associated with $\hat{P}$ equals $1 - v_1$, and point (iii) follows. $\square$

Let us summarize the properties of the (P,Q) coordinates which follow from the above:

(i) They provide a simple explicit formula for the Gowdy-to-Ernst transformation;

(ii) They describe faithfully the geometric behavior of the solutions except near those points at the conformal boundary with $\varphi = \pi \mod 2\pi$;

(iii) They describe faithfully the geometric behavior of those solutions on intervals of θ on which v vanishes.

The problem with $\varphi = \pi \mod 2\pi$ above is avoided by turning to the (ρ,φ)-description of the solutions. So, instead of (3.5)-(3.6) we write

$$\rho(t,\theta) = -v(\theta)\ln|t| + \rho_\infty(\theta) + o(1)\ , \qquad (3.23)$$

$$\varphi(t,\theta) = \varphi_\infty(\theta) + \begin{cases} |t|^{2v(\theta)}\Big(\psi_\varphi(\theta)+o(1)\Big)\ , & 0<v(\theta)\notin N^*\ ; \\ |t|^{2v(\theta)}\Big(\varphi_{\ln}(\theta)\ln|t| + \psi_\varphi(\theta)+o(1)\Big)\ , & v(\theta)\in N^*\ ; \\ o(1)\ , & v(\theta)=0\ . \end{cases} \qquad (3.24)$$

(We impose the usual restriction that $\rho \geq 0$ so v above is necessarily non-negative, though in some situations it might be convenient not to do this, allowing ρ to be negative, but then identifying the points (ρ,φ) with $(-\rho,\varphi-\pi)$.) It follows from (3.12) that the notation v for the ρ-velocity in (3.23) is comptatible with (3.5) and (3.7). A map $x=(\rho,\varphi)$ will be said to be AVTD$^{(\rho,\varphi)}$ on an interval $[a,b]$ if there exist real valued functions v and φ_∞ such that for $\theta\in[a,b]$ we have

$$\rho(t,\theta) = -v(\theta)\ln|t| + O(1)\ , \qquad (3.25)$$

$$\varphi(t,\theta) = \varphi_\infty(\theta) + o(1)\ . \qquad (3.26)$$

x will be said to be AVTD$_k^{(\rho,\varphi)}$ on an interval $[a,b]$ if (3.23)-(3.24) holds with functions v, ρ_∞, φ_∞, $\varphi_{\ln}$ and ψ_φ which are C^k on $[a,b]$. For $k>0$ the derivatives are assumed to behave as in (3.10).

Because of the usual polar coordinate singularity at $\rho=0$ the (ρ,φ) coordinates do not always reflect the geometric character of the dynamics for solutions on intervals on which $v(\theta)=0$.

We note the following result, which follows immediately from the calculations in the proof of Proposition 4.2 below. It shows that the only points θ at which the distinction between $\mathrm{AVTD}_k^{(\rho,\varphi)}$ and $\mathrm{AVTD}_k^{(P,Q)}$ behavior matters are those at which v vanishes (where (ρ,φ) might be singular) or at which $\varphi_\infty = \pi \mod 2\pi$ (where the restriction of Q to the conformal boundary of the hyperbolic space is singular):

Proposition 3.2 *Let $k \geq 0$.*

(i) *If the map x is $AVTD_k^{(P,Q)}$ on $[a_1, b_1]$ with v_1 avoiding zero on $[a_1, b_1]$, then it is $AVTD_k^{(\rho,\varphi)}$. Further $\varphi_\infty \mod 2\pi$ avoids π on $[a_1, b_1]$ if $v_1 > 0$, while $\varphi_\infty \equiv \pi \mod 2\pi$ if $v_1 < 0$.*

(ii) *If the map x is $AVTD_k^{(\rho,\varphi)}$ on $[a_1, b_1]$ with $\varphi_\infty \mod 2\pi$ avoiding π on $[a_1, b_1]$, then it is $AVTD_k^{(P,Q)}$.*

Remark 3.3 We have an obvious equivalent of points (i) and (ii) of Proposition 3.1 for the (ρ,φ) representation of the solutions, with identical proof, regardless of whether or not φ_∞ meets $\pi \mod 2\pi$.

What has been said so far in this section leads naturally to the following definition:

Definition 3.4 *A map $x : \Omega(a, b, t_0) \to \mathscr{H}_2$ will be said to be AVTD, respectively $AVTD_k$, on $[a_1, b_1] \subset [a, b]$ if there exists a function $v : [a_1, b_1] \to \mathbb{R}^+$ such that:*

(i) *For every interval $I \subset [a_1, b_1]$ on which v vanishes the map x is $AVTD^{(P,Q)}$, respectively $AVTD_k^{(P,Q)}$, near $\{0\} \times I$, with $|v_1| = v$.*

(ii) *For every interval $I \subset [a_1, b_1]$ on which v has no zeros the map x is $AVTD^{(\rho,\varphi)}$, respectively $AVTD_k^{(\rho,\varphi)}$, near $\{0\} \times I$.*

4 The Geroch group and its action

We consider solutions of the Gowdy equations defined on $\Omega(a, b, t_0)$ (see (1.5)), for some $a \leq b$, $t_0 < 0$. We fix[8] once and for all a (P, Q) coordinate system on $(\mathscr{H}_2, h)$. Following [7], we define the *Geroch group* $\mathscr{G}$ as the set of finite strings of the form $G = G_1 G_2 \cdots G_k$, where each of the G_i's

[8]We emphasise that there are several coordinate systems (P, Q) in which the metric h takes the form (1.2), differing from each other by an isometry of h.

is either an isometry of $(\mathscr{H}_2, h)$, denoted by I_i, or is a Gowdy-to-Ernst transformation (3.20), denoted by E. The Geroch group acts on solutions as follows: First, $G_1 \cdots G_n$ acts on x by first acting with G_n on x, then acting with G_{n-1} on $G_n x$, *etc.* Next, if x is a solution of (1.3), then we start by writing it in the (P, Q) coordinate system just chosen. Isometries act on solutions by composition. This implies that the $\mathscr{G}$-group product $I_1 I_2$ of two isometries I_1 and I_2 is the composition $I_1 \circ I_2$ of I_1 with I_2. The action of a Gowdy-to-Ernst transformation E on a solution is defined as follows: we integrate (3.20) with the integration constant chosen so that $\hat{Q}(t_0, a) = Q(t_0, a)$. This leads to the group product $E^2 = \mathrm{Id}_{\mathscr{H}_2}$, where $\mathrm{Id}_{\mathscr{H}_2}$ is the identity isometry of the hyperbolic plane. The $\mathscr{G}$–group products EI and IE, are defined by the above action on solutions.

Let $G = G_1 G_2 \cdots G_k$, then any two adjacent Gowdy-to-Ernst transformations can be canceled out, leading to a shorter presentation of G. Similarly any two adjacent isometries can be replaced by a single isometry. This leads eventually to a presentation of G such that isometries and Gowdy-to-Ernst transformations alternate. The number of Gowdy-to-Ernst transformations in the resulting presentation of G will be called the *order* of G. Thus, the order of I is zero, the order of E, or IE, or EI, is one, *etc.*

The behavior near $t = 0$ of all the known to us solutions of the Cauchy problem for the Gowdy equations is captured in the definition of the set $\mathscr{U}_1$ below; the set $\mathscr{U}_2$ is then a subset of $\mathscr{U}_1$ with a *genericity condition*; we expect $\mathscr{U}_2$ to be useful in the analysis of the strong cosmic censorship problem in the class of Gowdy space-times. The following comment is in order here: as emphasised in the previous section, the (P, Q) variables provide a parametrization of the hyperbolic space which does not always correctly reflect the geometric aspects of the asymptotic behavior of the solutions, so the reader might wonder why to invest so much effort to characterise the asymptotics of the Gowdy solutions in terms of those variables rather than, say, the (ρ, φ) variables of (3.11). The answer is that the Gowdy-to-Ernst transformation takes a simple form in the (P, Q) variables, compare (8.33), and this is what forces us to carry out the analysis below.

Definition 4.1 *1. Let $\mathscr{U}_1$ be the set of smooth solutions of the Gowdy equations defined on $\Omega(a, b, t_0)$ for which* (3.5)-(3.6) *holds with some functions v_1, P_∞, Q_∞ defined on $[a, b]$, a function ψ_Q defined on the set $\{v_1 > 0\}$, and a function $Q_{\ln}$ defined on the set $\{v_1 \in \mathbb{Z}^*\}$, satisfying the following:*

(i) *v_1 is uniformly bounded.*

(ii) *v_1 is continuous on an open dense subset of $[a, b]$.*

(iii) *The restrictions of* Q_∞ *to the sets* $\{\theta : v_1(\theta) \notin -\mathbb{N}\}$ *and* $\{\theta : v_1(\theta) \in -\mathbb{N}\}$ *are continuous functions on those sets.*

(iv) *The restriction of* P_∞ *to the set* $\{\theta : Q_\infty$ *and* v_1 *are continuous at* $\theta\}$ *is a continuous function on this set, similarly for* $Q_{\ln}$ *and* ψ_Q*.*

(v) *At points at which* $Q_{\ln}(\theta) \neq 0$ *the error terms* $o(1)$ *in* (3.5)-(3.6) *have the property that they remain* $o(1)$ *after multiplication by* $\ln|t|$.

2. We define $\mathscr{U}_2$ *to be the subset of* $\mathscr{U}_1$ *consisting of those solutions for which the sets of discontinuities and of critical points of* v_1 *and of* Q_∞ *are finite.*

It is an open question whether there exist solutions of the smooth Cauchy problem for the Gowdy equations which are not in $\mathscr{U}_1$.

In our analysis below we will need the following:

PROPOSITION 4.2 $\mathscr{U}_1$ *and* $\mathscr{U}_2$ *are invariant under the action of isometries of* $(\mathscr{H}_2, h)$.

REMARK 4.3 The arguments of the proof of Theorem 12.1 show that the set of solutions which are in $\mathscr{U}_1$ is stable under the action of the whole Geroch group. It is not completely clear what happens with the action of the Geroch group on solutions in $\mathscr{U}_2$, since the integration of the $\hat{Q}$ equations might introduce non-generic behavior.

PROOF: Let ϕ be an isometry of the hyperbolic space into itself. We will write

$$\hat{x} = \phi \circ x \ ,$$

and denote by $(\hat{P}, \hat{Q})$ the associated coordinate functions.

If $\theta_0 \in [a,b]$ is such that $v_1(\theta_0) = 0$, then continuity of ϕ shows that (3.5)-(3.6) does hold for $(\hat{P}(t,\theta), \hat{Q}(t,\theta))$ at θ_0, with the new velocity $\hat{v}_1(\theta_0) = 0 = v_1(\theta)$.

Let $\mathscr{V}$ be the open dense set on which v_1 is continuous, then $\mathscr{V}$ is a nonempty countable union of open intervals I_i, $\mathscr{V} = \cup_i I_i$. By definition the velocity function v_1 is continuous on each I_i. We rewrite the I_i's as

$$I_i = \underbrace{\{v_1 > 0\}}_{I_{i+}} \cup \underbrace{\{v_1 < 0\}}_{I_{i-}} \cup \underbrace{\{v_1 = 0\}}_{I_{i0}} \ .$$

The simplest set to analyse is I_{i0}: for $(t,\theta) \in [t_0, 0) \times I_{i0}$ the map $x(t,\theta)$ stays in a compact set, so does $\phi \circ x$ for any isometry ϕ, and the property that (P_∞, Q_∞) are continuous on I_{i0} is clearly preserved under the action of isometries.

Let us use the angle φ of (3.11) to parameterise the conformal boundary of the hyperbolic space, then to every point for which $v_1(\theta) > 0$ we can assign a unique $\varphi_\infty(\theta)$ such that the trajectory $t \to x(t,\theta)$ asymptotes to the point at infinity $\varphi_\infty(\theta)$. From (3.12)-(3.13) we have

$$v_1(\theta) > 0 \implies Q_\infty(\theta) = \frac{\sin(\varphi_\infty(\theta))}{1+\cos(\varphi_\infty(\theta))}\ , \tag{4.1}$$

except at

$$\varphi_\infty(\theta) = \pi \mod 2\pi \tag{4.2}$$

where a more careful analysis is required. We emphasise that a possible singularity arising here would only reflect the singular behavior of the Q-parameterisation of the conformal boundary, and not a singularity of $\varphi_\infty(\theta)$: continuity of φ_∞ on the set $\{v > 0\}$ can be established as in Proposition 3.1, working directly in the (ρ, φ) coordinates, regardless of whether or not $\cos(\theta) = -1$. In any case Q_∞ is continuous on I_{i+} by hypothesis, and so is therefore φ_∞.

Inverting (3.12)-(3.13) one finds

$$e^\rho = \frac{e^{-P} + e^P(1+Q^2)}{2} + \sqrt{\left(\frac{e^{-P} + e^P(1+Q^2)}{2}\right)^2 - 1}\ . \tag{4.3}$$

(The alternative solution

$$e^{-\rho} = \frac{e^{-P} + e^P(1+Q^2)}{2} + \sqrt{\left(\frac{e^{-P} + e^P(1+Q^2)}{2}\right)^2 - 1} \tag{4.4}$$

always leads to a negative ρ. Equations (4.3)-(4.4) reflect the fact that a point (P, Q) corresponds both to (ρ, φ) and $(-\rho, \varphi - \pi)$. In the current proof we follow the usual convention that $\rho \geq 0$.) It follows that

$$\rho(t,\theta) = -v_1(\theta)\ln|t| + \rho_\infty(\theta) + o(1)\ , \tag{4.5}$$

for some number $\rho_\infty(\theta)$. Inserting (4.5) into (3.13) one is then led to

$$\varphi(t,\theta) = \varphi_\infty(\theta) + \left\{ \begin{array}{ll} |t|^{2v_1(\theta)}\Big(\psi_\varphi(\theta) + o(1)\Big)\ , & v_1(\theta) \notin N^*\ , \\ |t|^{2v_1(\theta)}\Big(\varphi_{\ln}(\theta)\ln|t| + \psi_\varphi(\theta) + o(1)\Big)\ , & v_1(\theta) \in N^*\ , \end{array}\right. \tag{4.6}$$

for some numbers $\psi_\varphi(\theta)$, $\varphi_{\ln}(\theta)$. As φ_∞ is a continuous function of θ on I_{i+}, continuity in θ of $\rho_\infty(\theta)$ there follows. Continuity of $\psi_\varphi(\theta)$ and $\varphi_{\ln}(\theta)$ over the sets $I_{i+}\cap\{v_1\in\mathbb{N}^*\}$ and $I_{i+}\cap\{v_1\notin\mathbb{N}^*\}$ follows in an identical manner.

Recall, now, that any isometry ϕ of the hyperbolic space extends to a smooth diffeomorphism of the conformal boundary of $(\mathscr{H}_2,h)$, say χ. This shows that on the set $\{v(\theta)>0\}$ the solution $\hat{x}:=\phi\circ x$ will have a φ-position function

$$\hat{\varphi}_\infty(\theta)=\chi(\varphi_\infty(\theta)) \ .$$

The fact that isometries extend smoothly to the conformal boundary further shows that the asymptotic behavior (4.5)-(4.6) is preserved under the action of isometries of the hyperbolic space, so that the map $(\hat{\rho},\hat{\varphi})$ will satisfy the hatted version of (4.5)-(4.6) at points with positive v_1.

Let θ be any point such that $\chi(\varphi_\infty(\theta))\neq\pi \mod 2\pi$. A straightforward analysis of (3.12)-(3.13) shows that one will recover (3.5)-(3.6) for the map $(\hat{P},\hat{Q})$ at θ, with $\hat{v}_1(\theta)=v_1(\theta)>0$.

Set

$$\begin{aligned}\hat{I}_{i+} \ &:= \ \underbrace{\{\chi(\varphi_\infty(\theta))\neq\pi \mod 2\pi\}}_{\hat{I}_{i++}}\bigcup\\ &\quad\ \underbrace{\{\exists \text{ interval } J \text{ around } \theta \text{ such that } \chi\circ\varphi_\infty|_J\equiv\pi \mod 2\pi\}}_{\hat{I}_{i+-}}\subset I_{i+} \ .\end{aligned}$$

Then $\hat{I}_{i+}$ is clearly open in I_{i+}. Suppose that $\theta\in I_{i+}$ is such that there are no points of $\hat{I}_{i++}$ in a neighborhood of θ, then Q_∞ is constant and equal to $\pi \mod 2\pi$ on that neighborhood, hence $\theta\in\hat{I}_{i+-}$. It follows that $\hat{I}_{i+}$ is dense in I_{i+}.

On $\hat{I}_{i++}$ the function $\hat{\varphi}_\infty-\pi$ avoids integer multiples of 2π, and continuity of $\hat{v}_1(\theta)=v_1(\theta)>0$ on I_{i+} follows. Similarly one obtains a new continuous position function $\hat{Q}_\infty$ by using (4.1) with φ_∞ there replaced by $\hat{\varphi}_\infty$.

On the other hand, at points at which $\hat{\varphi}_\infty=\pi \mod 2\pi$ and $v_1\notin\mathbb{N}^*$ we have from (3.12)-(3.13) and (4.5)-(4.6)

$$\hat{P}(t,\theta) \ = \ v_1(\theta)\ln|t|+\hat{\rho}_\infty(\theta)+\ln\left(1+\frac{(\hat{\psi}_\varphi(\theta))^2}{4}\right)+o(1) \ , \tag{4.7}$$

$$\hat{Q}(t,\theta) \ = \ -\frac{1}{2}e^{\hat{\rho}(\infty)-\hat{P}(\infty)}\hat{\psi}_\varphi(\theta)+o(1) \ . \tag{4.8}$$

Analogous equations hold with supplementary $\ln|t|$ terms for $v_1\in\mathbb{N}^*$, compare (3.4). This shows that (3.5)-(3.6) hold again, with a negative v_1. Further v_1 is continuous on $\hat{I}_{i+-}$. It follows that I_{i+} contains an open dense set on which v_1 is continuous.

Consider, finally, points at which $v_1(\theta) < 0$. Equation (4.3) leads to

$$\begin{aligned}\rho(t,\theta) &= \left(-P + \frac{e^{2P}(1+3Q^2)}{4} + O(e^{4P})\right)(t,\theta)\\ &= -|v_1(\theta)|\ \ln|t| - P_\infty(\theta) + o(1)\ .\end{aligned}$$

Inserting this into (3.13) yields

$$\sin(\varphi) = \frac{e^P}{\sinh\rho}Q = 2e^{2P_\infty}(1+o(1))|t|^{2|v_1|(\theta)}Q\ ,$$

so that $\sin(\varphi(t,\theta))$ goes to zero as t does. Equation (3.12) shows that we must have $\varphi(t,\theta) \to_{t\to 0} \pi \mod 2\pi$ and one obtains, again modulo 2π,

$$\varphi(t,\theta) = \pi + \begin{cases} |t|^{2|v_1(\theta)|}2e^{2P_\infty(\theta)}\left(Q_\infty(\theta)+o(1)\right), & v_1 \notin -\mathbb{N}^*\ ;\\ |t|^{2|v_1|(\theta)}2e^{2P_\infty(\theta)}\left(Q_{\ln}(\theta)\ln|t| + Q_\infty(\theta)+o(1)\right), & v_1(\theta)\in -N^*\ .\end{cases}$$

(For $v_1(\theta) \in -N^*$ we have used the hypothesis that $\ln|t|\times o(1)$ remains $o(1)$.) The calculations done so far show that (3.5)-(3.6) hold for all $\theta\in[a,b]$ for the map $\phi\circ x$. A repetition of the arguments given on I_{i+} justifies continuity on an open dense subset of I_{i-}, and the proposition for $\mathscr{U}_1$ easily follows.

The result for $\mathscr{U}_2$ follows immediately from the calculations above, using the fact that for maps in $\mathscr{U}_2$ all level sets of Q_∞ form a finite collection of points. □

5 A symmetric hyperbolic system

Let us set

$$\begin{aligned} P_t = f_1, &\quad P_\theta = g_1,\\ e^P Q_t = f_2, &\quad e^P Q_\theta = g_2.\end{aligned}$$

Equation (1.3) takes then the form of the following first order symmetric hyperbolic system:

$$\partial_t\begin{pmatrix} f_1\\ f_2\\ g_1\\ g_2\end{pmatrix} = \begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0\end{pmatrix}\partial_\theta\begin{pmatrix} f_1\\ f_2\\ g_1\\ g_2\end{pmatrix} + \begin{pmatrix} f_2^2-g_2^2-\frac{f_1}{t}\\ -f_1f_2+g_1g_2-\frac{f_2}{t}\\ 0\\ f_1g_2-g_1f_2\end{pmatrix}, \tag{5.1}$$

The new unknowns f_a, g_a agree with the coefficient functions of $1-$forms, ψ^A_μ, of Christodoulou and Tahvildar-Zadeh's work on spherically symmetric wave maps [3], when an appropriate trivialisation of the bundle of vectors tangent to the hyperbolic space has been chosen: Indeed, if we set

$$e_1 = \partial_P \ , \qquad e_2 = e^{-P}\partial_Q \ , \tag{5.2}$$

then, in view of (1.2), the e_a's form a globally defined h–orthonormal frame with constant structure coefficients (and thus constant connection coefficients), and

$$f_a = h(e_a, X_t) \ , \qquad g_a = h(e_a, X_\theta) \ .$$

We consider solutions defined on domains of dependence $\Omega(a, b, t_0)$, defined in (1.5). By Proposition 1.4 we have

$$\sup_{\Omega(a,b,t_0)} (|tf_1| + |tf_2| + |tg_1| + |tg_2|) < \infty \ . \tag{5.3}$$

Throughout this work the value of various irrelevant constants may change from line to line.

Since $|X_\theta|^2 = g_1^2 + g_2^2$, Proposition 1.6 implies

$$\lim_{t\to 0} \int_{a+t}^{b-t} t^2(g_1^2 + g_2^2)d\theta = 0 \ , \tag{5.4}$$

$$\int_{a+t}^{b-t} t(g_1^2 + g_2^2)d\theta \in L^1([t_0, 0]) \ . \tag{5.5}$$

We further note that by (1.4) we have

$$\begin{aligned} D_\mu X_\nu &= (\partial_\mu\partial_\nu P - e^{2P}\partial_\mu Q \partial_\nu Q)\partial_P + (\partial_\mu\partial_\nu Q + Q_\mu P_\nu + Q_\nu P_\mu)\partial_Q \\ &= (\partial_\mu\partial_\nu P - e^{2P}\partial_\mu Q \partial_\nu Q)e_1 + e^P(\partial_\mu\partial_\nu Q + Q_\mu P_\nu + Q_\nu P_\mu)e_2 \ , \end{aligned} \tag{5.6}$$

so that

$$D_\theta X_\theta = (\partial_\theta g_1 - g_2^2)e_1 + (\partial_\theta g_2 + g_1 g_2)e_2 \ . \tag{5.7}$$

It then easily follows from (5.3)-(5.5) together with Proposition 1.6 that

$$\lim_{t\to 0} \int_{a+t}^{b-t} t^4 \left((\partial_\theta g_1)^2 + (\partial_\theta g_2)^2\right) d\theta = 0 \ , \tag{5.8}$$

$$\int_{a+t}^{b-t} t \left((\partial_\theta g_1)^2 + (\partial_\theta g_2)^2\right) d\theta \in L^1([t_0, 0]) \ . \tag{5.9}$$

It turns out that we also have

PROPOSITION 5.1

$$\lim_{t\to 0}\int_{a+t}^{b-t} t^2 f_2^2 d\theta = 0 \ , \tag{5.10}$$

$$\int_{a+t}^{b-t} t f_2^2 d\theta \in L^1([t_0,0]) \ . \tag{5.11}$$

PROOF: Let

$$F(t) = \int_{a+t}^{b-t} t f_1 d\theta \ ,$$

then

$$\frac{dF}{dt} \ = \ t(g_1 - f_1)(t, b-t) - t(g_1 + f_1)(t, a+t) + \int_{a+t}^{b-t} t(f_2^2 - g_2^2) \ .$$

Now, tf_1 is bounded, hence so if F, and by integration of the last equation we obtain

$$\int_{t_0}^{t}\int_{a+t}^{b-t} t f_2^2 \, d\theta dt \le \int_{t_0}^{t}\int_{a+t}^{b-t} t g_2^2 \, d\theta dt + C$$

for all $t_0 \le t < 0$. Equation (5.5) together with the monotone convergence theorem imply (5.11). In order to prove (5.10) we calculate

$$\begin{aligned}
\left|\frac{d}{dt}\int_{a+t}^{b-t} t^2 f_2^2\right| \ &= \ \Bigg| - t^2 f_2^2(t, b-t) - t^2 f_2^2(t, a+t) \\
&\qquad + \int_{a+t}^{b-t} f_2\left(\partial_\theta g_2 - f_1 f_2 + g_1 g_2\right)\Bigg| \\
&\le \ C\left(1 + \int_{a+t}^{b-t}\left(|t|^3(\partial_\theta g_2)^2 + |t|(f_2^2 + g_1^2 + g_2^2)\right)\right) \ .
\end{aligned} \tag{5.12}$$

The function on the right-hand side of the last line is in $L^1([t_0,0])$ by (5.5), (5.9) and (5.11). Integrating (5.12) one concludes that the limit at the left-hand side of (5.10) exists. If this limit were different from zero (5.11) couldn't hold, whence the result. □

We are ready to prove now:

PROPOSITION 5.2 *For $k \geq 0$ we have*

$$\lim_{t\to 0} \int_{a+t}^{b-t} |t|^{2(k+1)} \Big((\partial_\theta^k g_1)^2 + (\partial_\theta^k g_2)^2\Big)\, d\theta = 0 \,, \tag{5.13}$$

$$\int_{a+t}^{b-t} |t|^{2k+1} \Big((\partial_\theta^k g_1)^2 + (\partial_\theta^k g_2)^2\Big)\, d\theta \in L^1([t_0, 0]) \,, \tag{5.14}$$

$$\lim_{t\to 0} \int_{a+t}^{b-t} |t|^{2(k+1)} (\partial_\theta^k f_2)^2 d\theta = 0 \,, \tag{5.15}$$

$$\int_{a+t}^{b-t} |t|^{2k+1} (\partial_\theta^k f_2)^2 d\theta \in L^1([t_0, 0]) \,. \tag{5.16}$$

PROOF: The cases $k = 0$ have already been established, as well as (5.13) and (5.14) with $k = 1$. A simple induction argument, using the formulae (1.4) for the Christoffel symbols, shows that

$$D_\theta^k X_\theta = \Big(\partial_\theta^k g_1 + F_k(\partial_\theta^{k-1} g, \ldots, g)\Big) e_1 + \Big(\partial_\theta^k g_2 + G_k(\partial_\theta^{k-1} g, \ldots, g)\Big) e_2, \tag{5.17}$$

where g denotes both $g_1,\ g_2$, while the $F_k(\cdot)$'s and $G_k(\cdot)$'s are polynomials in the variables $\partial_\theta^m g,\ 0 \leq m \leq k-1$ with the number of derivatives and factors in each of the terms $\partial_\theta^{i_1} g \cdot \partial_\theta^{i_2} g \ldots \partial_\theta^{i_n} g$ satisfying

$$n \geq 2 \,, \quad \sum_{m=1}^{n} (i_m + 1) \leq k + 1 \,.$$

This, together with Proposition 1.6, proves (5.13) and (5.14).

Next, for $k \geq 1$ we compute

$$\partial_\theta^k f_2 = \sum_{i=0}^{k} C(i,k) \langle D_\theta^i X_t, D_\theta^{k-i} e_2 \rangle \tag{5.18}$$

since $f_2 = \langle X_t, e_2 \rangle$. By induction we obtain

$$D_\theta^k e_2 = \Big(\partial_\theta^{k-1} g + F'_{k-1}(\partial_\theta^{k-2} g, \ldots, g)\Big) e_1 + \Big(\partial_\theta^{k-1} g + G'_{k-1}(\partial_\theta^{k-2} g, \ldots, g)\Big) e_2, \tag{5.19}$$

where the $F'_{k-1}(\cdot)$'s and $G'_{k-1}(\cdot)$'s have the same property as described above. Thus

$$\begin{aligned} |t|^{k+1} |\partial_\theta^k f_2| &\leq \sum_{i=0}^{k} C(i,k) |t|^{i+1} |D_\theta^i X_t| |t|^{k-i} |D_\theta^{k-i} e_2| \\ &\leq C \sum_{i=1}^{k} |t|^i (|\partial_\theta^{i-1} g| + |F_{i-1}| + |G_{i-1}|) + |t|^{k+1} |D_\theta^k X_t|. \end{aligned}$$

The proof is completed by combining Proposition 1.6 with Equations (5.13)-(5.14). □

6 Existence of a velocity function

Numerical experiments (see [2] and references therein) suggest that the limit

$$v(\theta) := \lim_{t\to 0} |tX_t|(t,\theta) \qquad (6.1)$$

exists, and is a continuous function of θ except for a "small" exceptional set of θ's in S^1. We set

$$\Omega_{t-\mathrm{reg}} := \{\theta \in S^1 \ \text{ such that the limit (6.1) exists } \} , \qquad (6.2)$$

so that v is a well-defined function on $\Omega_{t-\mathrm{reg}}$. The existence of this limit is useful when analysing the geometry of the associated space-time. Now Proposition 5.1 shows that $t^2 f_2^2$ goes to zero in L^2 as t tends to zero, and since

$$|tX_t|^2 = t^2(f_1^2 + f_2^2) ,$$

the whole information about the limit (6.1) is contained in tf_1, except possibly for a negligible set.

6.1 Existence of a weak velocity function v_{weak}

We have the following:

PROPOSITION 6.1 *There exists $v_{\mathrm{weak}} \in L^\infty(S^1)$ such that for any $p \in (1,\infty)$*

$$|t| f_1(t,\cdot) \stackrel{L^p}{\rightharpoonup} v_{\mathrm{weak}} ,$$

where $\stackrel{L^p}{\rightharpoonup}$ denotes weak convergence in $L^p(S^1)$.

REMARK 6.2 In the proof of Theorem 1.3 we establish existence of an open dense set $\hat\Omega \subset S^1$ such that v_{weak} has a smooth representative v on $\hat\Omega$, with pointwise convergence to v on $\hat\Omega$.

PROOF: Let $v_i(\theta) = 2^{-i} f_1(-2^{-i},\theta)$, then the sequence v_i is bounded in L^2 and therefore there exists v_{weak} and a subsequence v_{i_j} which converges weakly to v_{weak}. Let ϕ be any smooth function on S^1, we have

$$\begin{aligned}
\partial_t \int_{S^1} t f_1 \phi &= \int_{S^1} \left(t\partial_\theta g_1 - t(f_2^2 - g_2^2)\right)\phi \\
&= \int_{S^1} \left(-t g_1 \partial_\theta \phi - t(f_2^2 - g_2^2)\phi\right) .
\end{aligned}$$

Integrating one finds

$$\begin{aligned}&\left|\int_{S^1} t_1 f_1(t_1,\theta)\phi(\theta)d\theta - \int_{S^1} t_2 f_1(t_2,\theta)\phi(\theta)d\theta\right| \\ &\qquad = \left|\int_{t_2}^{t_1}\int_{S^1}\left(-tg_1\partial_\theta\phi - t(f_2^2-g_2^2)\phi\right)d\theta dt\right| .\end{aligned}$$

Setting $t_1 = -2^{-i_j}$ and letting j go to infinity one obtains

$$\begin{aligned}&\left|\int_{S^1} v_{\mathrm{weak}}(\theta)\phi(\theta)d\theta - \int_{S^1} |t_2| f_1(t_2,\theta)\phi(\theta)d\theta\right| \\ &\qquad \le \int_{t_2}^{0}\int_{S^1}\left(|tg_1\partial_\theta\phi| + |t|(f_2^2+g_2^2)\phi\right)d\theta dt .\end{aligned}$$

Since the integrand in the last line is in $L^1([t_0,0]\times S^1)$ we obtain

$$\lim_{t\to 0}\int_{S^1} |t| f_1(t,\theta)\phi(\theta)d\theta = \int_{S^1} v_{\mathrm{weak}}(\theta)\phi(\theta)d\theta ,$$

so that $|t|f_1$ converges to v_{weak} in the sense of distributions. Weak convergence in L^p follows by elementary functional analysis, using the fact that smooth functions are dense in $L^{p'}$ for $p\in(1,\infty)$, with p' – the Hölder conjugate of p. Lower semi-continuity of the norm with respect to weak convergence implies

$$\|v_{\mathrm{weak}}\|_{L^p(S^1)} \le \lim_{j\to\infty}\|2^{-i_j} f_1(-2^{i_j},\cdot)\|_{L^p(S^1)} \le 2\pi \sup_{[t_0,0)\times S^1}|tf_1| ,$$

so that

$$\|v_{\mathrm{weak}}\|_{L^\infty(S^1)} \le \frac{1}{2\pi}\sup_{p\in[2,\infty)}\|v_{\mathrm{weak}}\|_{L^p(S^1)} \le \sup_{[t_0,0)\times S^1}|tf_1| .$$

□

The information contained in v_{weak} seems to be very poor. For instance, since v_{weak} is defined only almost everywhere, one can imagine situations in which v_{weak} has a smooth representative, but nevertheless the dynamics has very rough features at some points. This actually happens in the solutions with "spikes" discussed in Section 3. In those last examples one has pointwise convergence of $|tX_t|$ everywhere; this suggests that even an exhaustive understanding of the properties of the velocity function might not be enough to understand the dynamics of the Gowdy models.

6.2 From space-time integrals to pointwise velocity

We introduce

$$\delta g_1^\psi(t) = g_1(t, \psi - t) - g_1(t, \psi + t) , \tag{6.3}$$

$$\begin{aligned} \Omega_{\theta-\mathrm{reg}} \quad := \quad & \Big\{ \psi \in S^1 \;\text{ such that }\; \delta g_1^\psi \in L^1([t_0, 0)) \;\text{ and} \\ & \int_{C^0_{t_0}(\psi)} (f_2^2 + g_1^2 + g_2^2)\, dt\, d\theta < \infty \Big\} . \end{aligned} \tag{6.4}$$

Our aim in this section is to present an integral criterion for pointwise existence of a velocity function v on $\Omega_{\theta-\mathrm{reg}}$; this will be used later in this work.

Theorem 6.3 *For every $\psi \in \Omega_{\theta-\mathrm{reg}}$ there exists a number $v(\psi) \in \mathbb{R}$ such that*

$$\lim_{C^0_{t_0}(\psi)} |tX_t| = v(\psi) , \quad \textit{with} \;\; \lim_{C^0_{t_0}(\psi)} |tf_2| = 0 \tag{6.5}$$

(recall that $\lim_{C^0_{t_0}(\psi)}$ has been defined in (2.4)). This implies in particular

$$\Omega_{\theta-\mathrm{reg}} \subset \Omega_{t-\mathrm{reg}} .$$

Moreover for $\psi \in \Omega_{\theta-\mathrm{reg}}$ we have

$$\lim_{C^0_{t_0}(\psi)} |t^2 D_\theta X_\theta| = \lim_{C^0_{t_0}(\psi)} |t^2 D_\theta X_t| = \lim_{C^0_{t_0}(\psi)} |tX_\theta| = 0 . \tag{6.6}$$

.

Remark 6.4 We remark that for those points $\psi \in \Omega_{\theta-\mathrm{reg}}$ for which $v(\psi) \neq 1$ we have curvature blow-up in the associated space-time.

Proof: We have

$$\begin{aligned} \Box(tf_1) \;&=\; \partial_\theta g_1 - (1 + 2tf_1)(f_2^2 + g_2^2) + 2tf_2 \partial_\theta g_2 - 2tg_2 \partial_\theta f_2 \\ &\quad + 4tf_2 g_1 g_2 , \end{aligned} \tag{6.7}$$

$$\begin{aligned} \Box(tf_2) \;&=\; \partial_\theta g_2 + f_1 f_2 + g_1 g_2 - 2tf_2 \partial_\theta g_1 + 2tg_2 \partial_\theta f_1 \\ &\quad + tf_2 g_2^2 - tf_2 g_1^2 - tf_2^3 + tf_1^2 f_2 . \end{aligned} \tag{6.8}$$

In particular

$$\begin{aligned} \Box(tf_2)^2 \;&=\; 2tf_2 \Box(tf_2) + 2(\partial_t(tf_2))^2 - 2t^2(\partial_\theta f_2)^2 \\ &=\; 2tf_2 \Box(tf_2) - 2t^2(\partial_\theta f_2)^2 + 2t^2(\partial_\theta g_2 + f_2^2 - g_2^2)^2 \end{aligned} \tag{6.9}$$

using (5.1). It follows from (2.2), (6.10), Propositions 2.1 and [4, point b) of Lemma 3.4.1] that for $\psi \in \Omega_{\theta-\mathrm{reg}}$ the right-hand side of (6.9) is in $L^1(C^0_{t_0}(\psi))$. The dominated convergence theorem applied to the usual integral representation of solutions of the one-dimensional wave equation,

$$u(t,x) = \mathring{u}(t,x) + \frac{1}{2}\int_{s=t_0}^{t}\int_{\theta=x-t+s}^{x+t-s} \Box u(s,\theta)\, d\theta ds \;,$$

where $\mathring{u}$ is the solution of the free wave equation with the same initial data, shows that the limit $\lim_{C^0_{t_0}(\psi)}(tf_2)^2$ exists. This limit has to be zero, otherwise the integral condition on f_2 in (6.4) wouldn't hold. If δg_1^ψ is in $L^1([t_0,0))$, then the right-hand side of (6.7) is also in $L^1(C^0_{t_0}(\psi))$, and by a similar argument $\lim_{C^0_{t_0}(\psi)} tf_1$ exists. $\Box$

We close this section with the following remark:

LEMMA 6.5 *In the definition of $\Omega_{\theta-\mathrm{reg}}$, the condition*

$$\int_{C^0_{t_0}(\psi)} f_2^2 \,dt\, d\theta < \infty \quad \textit{can be replaced by} \quad \int_{C^0_{t_0}(\psi)} |\partial_\theta f_1|\, dt\, d\theta < \infty \;. \quad (6.10)$$

Similarly, the condition

$$\int_{t_0}^{0} |\delta g_1^\psi|\, dt < \infty \quad \textit{can be replaced by} \quad \int_{C^0_{t_0}(\psi)} |\partial_\theta g_1|\, dt\, d\theta < \infty \;. \quad (6.11)$$

PROOF: Let

$$F(t) = \int_{\psi+t}^{\psi-t} f_1(t,\theta) d\theta \;,$$

then

$$\begin{aligned}\frac{dF}{dt} &= -f_1(t,\psi-t) - f_1(t,\psi+t) + \int_{\psi+t}^{\psi-t}\left(\partial_\theta g_1 + \frac{f_1}{|t|} + f_2^2 - g_2^2\right)\\ &= -f_1(t,\psi-t) - f_1(t,\psi+t) + \delta g_1^\psi(t) + \int_{\psi+t}^{\psi-t}\left(\frac{f_1}{|t|} + f_2^2 - g_2^2\right) \;.\end{aligned}$$

Integration by parts gives the identity

$$\frac{1}{|t|}\int_{\psi+t}^{\psi-t} u(\theta) d\theta = u(\psi-t) + u(\psi+t) + \int_{\psi+t}^{\psi-t} \frac{\psi-\theta}{|t|} u'(\theta) d\theta \;, \quad (6.12)$$

so that

$$\frac{dF}{dt} = \delta g_1^\psi(t) + \int_{\psi+t}^{\psi-t} \left(\frac{\psi-\theta}{|t|}\partial_\theta f_1 + f_2^2 - g_2^2\right) .$$

As F is bounded, integrating in t gives (6.10). Equation (6.11) is obvious. □

7 Power law in Sobolev spaces

As discussed in the Introduction, and as will be proved below in detail in any case, a power law inequality (2.1) implies existence of a velocity function. It turns out that one strategy for establishing (2.1) is to derive a power-law for t-weighted Sobolev norms. This is done in this section.

It is useful to introduce the following quantities

$$\mu_1 = \sup_{\Omega(a,b,t_0)} |tf_1| \; , \quad \mu_2 = \sup_{\Omega(a,b,t_0)} |tf_2| \; , \quad \lambda_2 = \sup_{\Omega(a,b,t_0)} |tg_2| \; . \tag{7.1}$$

These are finite by (5.3). (Recall that $\Omega(a,b,t_0)$ has been defined in (1.5).)

Let $t_0 \le t < 0$, $a < b$. Define the $k-$th order energy $E_k(t)$ by

$$E_k(t) = \int_{a+t}^{b-t} |t|^{2k+2} \sum_{i=1,2} ((\partial_\theta^k f_i)^2 + (\partial_\theta^k g_i)^2) \; d\theta. \tag{7.2}$$

Proposition 7.1 *1. If*

$$\beta := \sup_{\Omega(a,b,t_0)} \left(|tf_1| + \frac{|tf_2|}{2} + \frac{2|tg_2|^2}{\sqrt{1+4|tg_2|^2}+1} \right) < 1 \; , \tag{7.3}$$

then

$$E_k(t) \le C(t_0,k,\alpha)|t|^{2\alpha} \; , \qquad k \ge 1 \; , \tag{7.4}$$

where

$$\alpha = 1 - \beta \; . \tag{7.5}$$

2. Similarly if

$$\hat\beta := \sup_{\Omega(a,b,t_0)} \max(1 - |t|f_1, |t|f_1) < 1 \tag{7.6}$$

then (7.4) *holds with* $\alpha = 1 - \hat\beta$ *if* $\mu_1 < 1/2$, *and with* α *being any number strictly smaller than* $1 - \mu_1$ *if* $\mu_1 \ge 1/2$.

REMARK 7.2 Recall that, at fixed a and b, the constants μ_i and λ_i depend upon t_0. In several situations of interest μ_2 and λ_2 will tend to zero as t_0 tends to zero, in which case the essential restriction in (7.3) is that μ_1 be smaller than one sufficiently close to the singular boundary $t = 0$.

REMARK 7.3 We note that (7.3) will hold under the slightly stronger but simpler condition

$$\sup_{\Omega(a,b,t_0)} \left(|tf_1| + \frac{|tf_2|}{2} + \frac{|tg_2|}{\sqrt{2}} \right) < 1 \,. \tag{7.7}$$

Proposition 1.4 or Remark 1.5 can be used to replace (7.7) by a condition on initial data using the Cauchy-Schwarz inequality

$$\begin{aligned} |tf_1| + \frac{|tf_2|}{2} + \frac{|tg_2|}{\sqrt{2}} &\leq \sqrt{1 + \frac{1}{4} + \frac{1}{2}} \sqrt{|tf_1|^2 + |tf_2|^2 + |tg_2|^2} \\ &\leq \frac{\sqrt{7}}{2} \sqrt{|tX_t|^2 + |tX_\theta|^2} \\ &\leq \sqrt{7} \sqrt{|tX_t|^2 + |tX_\theta|^2} \Big|_{t=t_0} < 1 \,. \end{aligned}$$

PROOF: Differentiating $E_k(t)$ in t, using the field equations and integrating by parts one has

$$
\begin{aligned}
\frac{dE_k(t)}{dt} =& - t^{2k+2} \sum_{i=1,2} \Big((\partial_\theta^k (f_i - g_i))^2 (t, b-t) + (\partial_\theta^k (f_i + g_i))^2 (t, a+t) \Big) \\
& - (2k+2) \int_{a+t}^{b-t} |t|^{2k+1} \sum_{i=1,2} ((\partial_\theta^k f_i)^2 + (\partial_\theta^k g_i)^2) \\
& + 2|t|^{2k+2} \int_{a+t}^{b-t} \Big(\partial_\theta^k f_1 \cdot \partial_\theta^k (f_2^2 - g_2^2 - \frac{f_1}{t}) + \partial_\theta^k g_2 \cdot \partial_\theta^k (f_1 g_2 - g_1 f_2) \\
& \qquad + \partial_\theta^k f_2 \cdot \partial_\theta^k (-f_1 f_2 + g_1 g_2 - \frac{f_2}{t}) \Big) \, d\theta \\
\leq & - 2k \int_{a+t}^{b-t} |t|^{2k+1} \sum_{i=1,2} (\partial_\theta^k f_i)^2 - (2k+2) \int_{a+t}^{b-t} |t|^{2k+1} \sum_{i=1,2} (\partial_\theta^k g_i)^2 \qquad (7.8) \\
& + 2|t|^{2k+2} \int_{a+t}^{b-t} \Big(f_2 \partial_\theta^k f_2 \partial_\theta^k f_1 - g_2 \partial_\theta^k f_1 \partial_\theta^k g_2 - f_2 \partial_\theta^k g_2 \partial_\theta^k g_1 \qquad (7.9) \\
& \qquad + g_2 \partial_\theta^k g_1 \partial_\theta^k f_2 - f_1 (\partial_\theta^k f_2)^2 + f_1 (\partial_\theta^k g_2)^2 \Big) \qquad (7.10) \\
& + |t|^{2k+2} \int_{a+t}^{b-t} \partial_\theta^k u \cdot \sum_{\substack{i+j=k \\ i,j>0}} C(i,j,k) \partial_\theta^i u \cdot \partial_\theta^j u \; . \qquad (7.11)
\end{aligned}
$$

In (7.9) and (7.10) we have collected all those terms which contain undifferentiated functions f_i or g_i. In (7.11) we denote (f_i, g_i) by u and the $C(i,j,k)$'s are the coefficients of $k-$th binomial expansions; we will ignore those coefficients and replace them by an overall constant from now on — this is sufficient for estimation purposes. Note that when $k = 1$, then (7.11) does not appear. Mixed terms of the form $\partial_\theta^k f_1 \partial_\theta^k f_2$ and $\partial_\theta^k g_1 \partial_\theta^k g_2$ are estimated in the obvious way using $2ab \leq a^2 + b^2$. To take advantage of the different factors in front of the integrals appearing in (7.8) the mixed terms $\partial_\theta^k f_i \partial_\theta^k g_j$ are estimated using $2ab \leq \frac{a^2}{\sigma} + \sigma b^2$. Absorbing all the terms from

(7.9) and (7.10) into those appearing in (7.8) we obtain

$$\begin{aligned}\frac{dE_k(t)}{dt} \le & -|t|^{2k+1}\int_{a+t}^{b-t}(2k-2|tf_1|-|tf_2|-\frac{2|tg_2|}{\sigma})\sum_{i=1,2}(\partial_\theta^k f_i)^2\,d\theta\\ & -|t|^{2k+1}\int_{a+t}^{b-t}(2k+2-2|tf_1|-|tf_2|-2\sigma|tg_2|)\sum_{i=1,2}(\partial_\theta^k g_i)^2\,d\theta\\ & +|t|^{2k+2}C(k)\int_{a+t}^{b-t}\partial_\theta^k u\cdot\sum_{\substack{i+j=k\\ i,j>0}}\partial_\theta^i u\cdot\partial_\theta^j u\;.\end{aligned}$$

For $k=1$ this reads

$$\begin{aligned}\frac{dE_1(t)}{dt} \le & -|t|^3\int_{a+t}^{b-t}(2-2|tf_1|-|tf_2|-\frac{2|tg_2|}{\sigma})\sum_{i=1,2}(\partial_\theta^k f_i)^2\,d\theta\\ & -|t|^3\int_{a+t}^{b-t}(4-2|tf_1|-|tf_2|-2\sigma|tg_2|)\sum_{i=1,2}(\partial_\theta^k g_i)^2\,d\theta\;.\end{aligned}$$

It should be clear from what follows that the choice of $\sigma=\sigma(t,\theta)$ which is optimal for our purposes is that of equal factors in front of the sums, namely $\sigma=(\sqrt{1+4|tg_2|^2}+1)/(2|tg_2|)$. Choosing this value of σ leads to

$$\frac{dE_1(t)}{dt}\le-\frac{2\alpha}{|t|}E_1\;,$$

with α as in (7.5). This shows that $d((-t)^{-2\alpha}E_1)/dt\le 0$, and by integration one obtains

$$E_1(t)\le\left|\frac{t}{t_0}\right|^{2\alpha}E_1(t_0).$$

(This inequality holds whatever the sign of α, but for $\alpha\le 0$ it does not carry any new information.)

To cover the case (7.6) we shall need a Lemma:

Lemma 7.4 *Let*

$$F(t)=\int_{a+t}^{b-t}(f_2^2+g_2^2)\;.$$

Under (7.6) *we have*

$$F(t)\le F(t_0)\left|\frac{t}{t_0}\right|^{-2\hat\beta}\;.$$

Proof: We have

$$\begin{aligned}\frac{dF(t)}{dt} &= -(f_2-g_2)^2(t,b-t)-(f_2+g_2)^2(t,a+t)\\ &\quad +\frac{2}{|t|}\int_{a+t}^{b-t}(1-|t|f_1)f_2^2+|t|f_1g_2^2\\ &\le \frac{2\hat\beta F(t)}{|t|}\ , \end{aligned}\tag{7.12}$$

and the result follows by integration as before. □

Returning to the estimation of dE_1/dt, assume that (7.6) holds and consider any of the terms of the form $f_2\,\partial_\theta u\,\partial_\theta u$ or $g_2\,\partial_\theta u\,\partial_\theta u$ in (7.9) and (7.10); they are estimated as

$$\begin{aligned}\int_{a+t}^{b-t}\Big||t|^4\,f_2\,\partial_\theta u\,\partial_\theta u\Big| &\le \int_{a+t}^{b-t}\frac{\epsilon}{4C}|t|^7|\partial_\theta u|^4+\frac{16C}{\epsilon}|t|f_2^2\\ &\le \int_{a+t}^{b-t}\frac{\epsilon}{4}|t|^3|\partial_\theta u|^2+\frac{16C}{\epsilon}|t|f_2^2\\ &\le \frac{\epsilon E_1(t)}{4|t|}+C'(\epsilon)|t|^{1-2\hat\beta}\ ,\end{aligned}$$

where in the last line we have used Lemma 7.4; similarly for g_2. It follows that

$$\frac{dE_1(t)}{dt}\le-\frac{2(1-\mu_1-\epsilon)}{|t|}E_1+C'(\epsilon)|t|^{1-2\hat\beta}\ ,$$

Multiplying by $|\frac{t_0}{t}|^{2(1-\mu_1-\epsilon)}$ and integrating in t, one obtains

$$E_1(t)\le C(\alpha)\left|\frac{t}{t_0}\right|^{2\alpha}\ ,$$

with $\alpha=\mu_1$ if $\mu_1<1/2$, and α any number strictly smaller than $1-\mu_1$ if $\mu_1\ge 1/2$.

The cases $k\ge 2$ are established by induction: Suppose, thus, that (7.4) holds for $k=m-1$; we have already shown that it holds for $k=1$. Then the terms from line (7.11) in $\frac{dE_m(t)}{dt}$ are estimated as

$$|t|^{2m+2}\int_{a+t}^{b-t}\partial_\theta^m u\cdot\sum_{\substack{i+j=m\\ i,j>0}}\partial_\theta^i u\cdot\partial_\theta^j u\le C(t_0)|t|^{m+1}\int_{a+t}^{b-t}\sum_{\substack{i+j=m\\ 0<i,j<m}}|\partial_\theta^i u\cdot\partial_\theta^j u|,$$

letting $C(t_0) = \sup_{(t,\theta)} |t|^{m+1}|\partial_\theta^m u|$, which is finite by (5.3). Using the induction hypothesis for $i, j < m$ on $\partial_\theta^i u \cdot \partial_\theta^j u$, we get

$$\begin{aligned}
|t|^{m+1} \int_{a+t}^{b-t} \sum_{\substack{i+j=m \\ 0<i,j<m}} |\partial_\theta^i u| \cdot |\partial_\theta^j u| &= |t|^{-1} \int_{a+t}^{b-t} \sum_{\substack{i+j=m \\ 0<i,j<m}} |t|^{i+1}|\partial_\theta^i u| \cdot |t|^{j+1}|\partial_\theta^j u| \\
&\leq |t|^{-1} \int_{a+t}^{b-t} \sum_{\substack{i+j=m \\ 0<i,j<m}} |t|^{2i+2}(\partial_\theta^i u)^2 + |t|^{2j+2}(\partial_\theta^j u)^2 \\
&\leq C(m)|t|^{2\alpha-1} .
\end{aligned}$$

It follows that

$$\frac{dE_m(t)}{dt} \leq -\frac{2(m+\alpha-1)}{|t|} E_m(t) + C(t_0, m)|t|^{2\alpha-1} . \tag{7.13}$$

Multiplying by $|\frac{t_0}{t}|^{2(m+\alpha-1)}$ on both sides and integrating over (t_0, t) in t, we obtain

$$\begin{aligned}
E_m(t) &\leq |\frac{t}{t_0}|^{2(m+\alpha-1)} E_k(t_0) + C(t_0, m)|t|^{2(m+\alpha-1)} \int_{t_0}^{t} |s|^{1-2m} \, ds \\
&\leq C(t_0, m)|t|^{2\alpha} ,
\end{aligned}$$

as claimed. □

Corollary 7.5 *Under the conditions of Proposition 7.1, there exists a constant C such that*

$$\int_{a+t}^{b-t} t^2 |X_\theta|^2 d\theta = \int_{a+t}^{b-t} t^2 \left((g_1)^2 + (g_2)^2\right) d\theta \leq C|t|^{2\alpha} . \tag{7.14}$$

If

$$\sigma_1 := \inf_{\Omega(a,b,t_0)} |t| f_1 > 0 , \tag{7.15}$$

then we also have

$$\int_{a+t}^{b-t} t^2 f_2^2 d\theta \leq C|t|^{2\alpha'} , \tag{7.16}$$

with $\alpha' = \alpha$ if $\alpha < \sigma_1$, or α' any number smaller than σ_1 otherwise.

Proof: We calculate

$$
\begin{aligned}
\frac{d}{dt}\int_{a+t}^{b-t} t^2 g_2^2 &= -t^2 g_2^2(t,b-t) - t^2 g_2^2(t,a+t) \\
&\quad +2t\int_{a+t}^{b-t} g_2^2 + 2\int_{a+t}^{b-t} t^2 g_2 \Big(\underbrace{\partial_\theta f_2}_{I} + \underbrace{f_1 g_2}_{II} - \underbrace{g_1 f_2}_{III} \Big) \\
&\le -\frac{2}{|t|}\int_{a+t}^{b-t} t^2 g_2^2 + \underbrace{\epsilon\int_{a+t}^{b-t} |t| g_2^2 + \frac{1}{\epsilon}\int_{a+t}^{b-t} |t|^3 (\partial_\theta f_2)^2}_{I} \\
&\quad + \underbrace{\frac{2\mu_1}{|t|}\int_{a+t}^{b-t} t^2 g_2^2}_{II} + \underbrace{\frac{\mu_2}{|t|}\int_{a+t}^{b-t} t^2 \left(\frac{g_1^2}{\epsilon'} + \epsilon' g_2^2 \right)}_{III} \\
&\le -\frac{2-2\mu_1-\epsilon'\mu_2-\epsilon}{|t|}\int_{a+t}^{b-t} t^2 g_2^2 + C(\epsilon,\epsilon')|t|^{2\alpha-1} \;.
\end{aligned}
\tag{7.17}
$$

Choosing ϵ, ϵ' appropriately and arguing as in the paragraph following (7.13) one obtains the bound for $\int_{a+t}^{b-t} g_2^2$. A similar, but simpler, calculation with g_2 replaced by g_1 proves (7.16). In order to establish (7.16) we note that

$$
\begin{aligned}
\frac{d}{dt}\int_{a+t}^{b-t} t^2 f_2^2 &= -t^2 f_2^2(t,b-t) - t^2 f_2^2(t,a+t) \\
&\quad +2\int_{a+t}^{b-t} t^2 f_2 \Big(\underbrace{\partial_\theta g_2}_{I} - \underbrace{f_1 f_2}_{II} + \underbrace{g_1 g_2}_{III} \Big) \\
&\le \underbrace{\epsilon\int_{a+t}^{b-t} |t| f_2^2 + \frac{1}{\epsilon}\int_{a+t}^{b-t} |t|^3 (\partial_\theta g_2)^2}_{I} \\
&\quad \underbrace{-\frac{2\sigma_1}{|t|}\int_{a+t}^{b-t} t^2 f_2^2}_{II} + \underbrace{\frac{\mu_2}{|t|}\int_{a+t}^{b-t} t^2 \left(g_1^2 + g_2^2 \right)}_{III} \\
&\le -\frac{2\sigma_1-\epsilon}{|t|}\int_{a+t}^{b-t} t^2 f_2^2 + C(\epsilon)|t|^{2\alpha-1} \;,
\end{aligned}
$$

and we conclude as before. □

8 Pointwise power law

We start with the following observation, where $a < b$ is assumed (however, $a = b$ is allowed in the remaining results of this section):

LEMMA 8.1 (i) *Suppose that there exists* $\alpha > 0$ *such that*

$$\int_{a+t}^{b-t} t^2 |X_\theta|^2 d\theta \le C|t|^{2\alpha} \ . \tag{8.1}$$

Then for all multi-indices ν *we have*

$$\int_{a+t}^{b-t} \Big(|t|^{|\nu|+1} |D^\nu X_\theta| \Big)^2 d\theta \le C(\nu) |t|^{2\alpha} \ , \tag{8.2}$$

(ii) *If* $|tX_\theta| \le C|t|^{\alpha_p}$ *then we also have*

$$\sup_{\Omega(a,b,t_0)} |t|^{|\nu|+1} |D^\nu X_\theta| \le C'(\nu)|t|^{\alpha_p} \ . \tag{8.3}$$

(iii) *If the constant* α *in* (8.1) *satisfies* $\alpha > 1/2$ *then* (8.3) *holds with* $\alpha_p = \alpha - 1/2$.

PROOF: Equation (8.2) follows by a straightforward adaptation of the proof of [4, Proposition 3.3.1]. Equation (8.3) is obtained from [4, Remark, p. 73]. To establish point (iii) it remains to show that

$$\sup_{\Omega(a,b,t_0)} |t||X_\theta| \le C|t|^{\alpha - 1/2} \ . \tag{8.4}$$

That last inequality is obtained by applying the interpolation inequality

$$\|\partial_\theta u\|_{L^\infty([a+t,b-t])} \le C \|\partial_\theta^2 u\|^{3/4}_{L^2([a+t,b-t])} \|u\|^{1/4}_{L^2([a+t,b-t])}$$

(see, *e.g.,* [1, p. 94]; the condition there that u vanishes on the boundary is not necessary) to the functions tg_i, $i = 1, 2$. □

Thus, power-law blow-up in Sobolev spaces implies a pointwise one if the decay rate is larger than 1/2. The unpleasant feature of the above argument is the loss of 1/2 decay rate in point (iii) of Lemma 8.1. This can be avoided by working directly with L^∞ norms, as follows: Consider two fields f, g satisfying the symmetric hyperbolic set of equations

$$\begin{aligned} \partial_t f - \partial_\theta g &= S_f \ , \\ \partial_t g - \partial_\theta f &= S_g \ , \end{aligned} \tag{8.5}$$

set

$$T_{tt}[f,g]=T_{\theta\theta}[f,g]:=\frac{1}{2}(f^2+g^2)\ ,\quad T_{t\theta}[f,g]=T_{\theta t}[f,g]:=fg\ , \tag{8.6}$$

and define

$$j_\mu[f,g]=\partial_\nu(T_\mu{}^\nu[f,g])\ . \tag{8.7}$$

Writing $T_{\mu\nu}$ for $T_{\mu\nu}[f,g]$, *etc*, we have the identity [4, Equation (3.2.5)]

$$\begin{aligned}T_{tt}(t_1,\theta_1) \;=\; & -\frac{1}{2}\int_{t_0}^{t_1}\Big((j_t+j_\theta)(t,\theta_1+t_1-t)+(j_t-j_\theta)(t,\theta_1-t_1+t)\Big)dt \\ & +\frac{1}{2}\Big((T_{tt}+T_{t\theta})(t,\theta_1+t_1-t_0)+(T_{tt}+T_{t\theta})(t,\theta_1-t_1+t_0)\Big)\ .\end{aligned} \tag{8.8}$$

For (8.5) the j–terms appearing in (8.8) read

$$-\frac{1}{2}(j_t+j_\theta) \;=\; \frac{1}{2}(f+g)(S_f+S_g)\ , \tag{8.9}$$

$$-\frac{1}{2}(j_t-j_\theta) \;=\; \frac{1}{2}(f-g)(S_f-S_g)\ . \tag{8.10}$$

If we let $(f,g)=(\partial_\theta^k f_1,\partial_\theta^k g_1)$ we obtain

$$-\frac{1}{2}(j_t+j_\theta) \;=\; \frac{1}{2}(\partial_\theta^k f_1+\partial_\theta^k g_1)\left(\frac{\partial_\theta^k f_1}{|t|}+\partial_\theta^k\Big[(f_2-g_2)(f_2+g_2)\Big]\right)\ , \tag{8.11}$$

$$-\frac{1}{2}(j_t-j_\theta) \;=\; \frac{1}{2}(\partial_\theta^k f_1-\partial_\theta^k g_1)\left(\frac{\partial_\theta^k f_1}{|t|}+\partial_\theta^k\Big[(f_2-g_2)(f_2+g_2)\Big]\right)\ . \tag{8.12}$$

Similarly for $(f,g)=(\partial_\theta^k f_2,\partial_\theta^k g_2)$ one has

$$-\frac{1}{2}(j_t+j_\theta) \;=\; \frac{1}{2}(\partial_\theta^k f_2+\partial_\theta^k g_2)\left(\frac{\partial_\theta^k f_2}{|t|}-\partial_\theta^k\Big[(f_1+g_1)(f_2-g_2)\Big]\right)\ , \tag{8.13}$$

$$-\frac{1}{2}(j_t-j_\theta) \;=\; \frac{1}{2}(\partial_\theta^k f_2-\partial_\theta^k g_2)\left(\frac{\partial_\theta^k f_2}{|t|}-\partial_\theta^k\Big[(f_1-g_1)(f_2+g_2)\Big]\right)\ . \tag{8.14}$$

We define

$$E_k(t,\psi)=\sup_{t_0\le s\le t\ ,\ \psi+s-t\le\theta\le\psi-s+t}(T_{tt}[\partial_\theta^k f_1,\partial_\theta^k g_1]+T_{tt}[\partial_\theta^k f_2,\partial_\theta^k g_2])(s,\theta)\ . \tag{8.15}$$

It is useful to note that

$$T_{tt}[f,g] = \frac{1}{4}\Big((f-g)^2 + (f+g)^2\Big) \; . \tag{8.16}$$

Define

$$\mu_\pm = \sup_{\Omega(a,b,t_0)} |t(f_1 \pm g_1)| \; . \tag{8.17}$$

We have the following pointwise equivalent of Proposition 7.1:

PROPOSITION 8.2 *Suppose that*

$$\sup_{\Omega(a,b,t_0)} \Big(\mu_- + |tf_1 + tg_1| + \sqrt{(\mu_- + |tf_1+tg_1|)^2 + 4|tf_2+tg_2|^2}\Big) < 4 \; , \tag{8.18}$$

$$\sup_{\Omega(a,b,t_0)} \Big(\mu_+ + |tf_1 - tg_1| + \sqrt{(\mu_+ + |tf_1-tg_1|)^2 + 4|tf_2-tg_2|^2}\Big) < 4 \; . \tag{8.19}$$

Then there exist constants $C, \alpha_p > 0$ so that we have the inequality

$$|tX_\theta| \le C|t|^{\alpha_p} \; . \tag{8.20}$$

PROOF: We start by deriving an integral inequality for E_1 using (8.8). Let S denote the sum of (8.11) and (8.13) with $k=1$:

$$\begin{aligned}
S \;&=\; \frac{1}{2}(\partial_\theta f_1 + \partial_\theta g_1)\Bigg(\frac{\partial_\theta f_1 + \partial_\theta g_1 + \partial_\theta f_1 - \partial_\theta g_1}{2|t|} \\
&\qquad + \partial_\theta\Big[(f_2-g_2)(f_2+g_2)\Big]\Bigg) \\
&\quad + \frac{1}{2}(\partial_\theta f_2 + \partial_\theta g_2)\Bigg(\frac{\partial_\theta f_2 + \partial_\theta g_2 + \partial_\theta f_2 - \partial_\theta g_2}{2|t|} \\
&\qquad - \partial_\theta\Big[(f_2-g_2)(f_1+g_1)\Big]\Bigg) \\
&=\; \frac{1}{4|t|}\Big[(\partial_\theta f_1 + \partial_\theta g_1)^2 + (\partial_\theta f_2 + \partial_\theta g_2)^2 + (\partial_\theta f_1 + \partial_\theta g_1)(\partial_\theta f_1 - \partial_\theta g_1) \\
&\qquad + (\partial_\theta f_2 + \partial_\theta g_2)(\partial_\theta f_2 - \partial_\theta g_2)\Big] + \frac{1}{2}(f_2+g_2)(\partial_\theta f_1 + \partial_\theta g_1)(\partial_\theta f_2 - \partial_\theta g_2) \\
&\qquad - \frac{1}{2}(f_1+g_1)(\partial_\theta f_2 + \partial_\theta g_2)(\partial_\theta f_2 - \partial_\theta g_2).
\end{aligned} \tag{8.21}$$

A formula for the sum of (8.12) and (8.14) can be obtained by changing g_a to $-g_a$ in (8.21). We apply Young's inequality $ab \le \frac{c}{2}a^2 + \frac{1}{2c}b^2$ to estimate the mixed terms $(\partial_\theta f+\partial_\theta g)(\partial_\theta f-\partial_\theta g)$, using a c which might possibly depend upon t and θ. With a little work one finds that the integrand in the first line of (8.8) can be estimated by the sup, over θ in the relevant range, of the quantity

$$\begin{aligned}\frac{1}{2|t|}\Bigg[&\left(1+\frac{c_1|tf_2+tg_2|}{2}\right)(\partial_\theta f_1+\partial_\theta g_1)^2\\
&+\left(1+\frac{\mu_-}{2}+\frac{1}{2}\left(|tf_1+tg_1|+\frac{|tf_2+tg_2|}{c_1}\right)\right)(\partial_\theta f_2-\partial_\theta g_2)^2\\
&+\left(1+\frac{c_2|tf_2-tg_2|}{2}\right)(\partial_\theta f_1-\partial_\theta g_1)^2\\
&+\left(1+\frac{\mu_+}{2}+\frac{1}{2}\left(|tf_1-tg_1|+\frac{|tf_2-tg_2|}{c_2}\right)\right)(\partial_\theta f_2+\partial_\theta g_2)^2\Bigg]\end{aligned} \tag{8.22}$$

If all the factors in front of the derivative squared terms are strictly smaller than 2, say smaller than or equal to $2-2\alpha_p$, then (8.22) is smaller than or equal to $(4-4\alpha_p)E_1/|t|$. From (8.8) applied to E_1 and from Gronwall's Lemma (*cf., e.g.,* [4, Lemma 3.2.3]) one obtains an integral inequality for E_1, which translates into the inequality

$$|t^2\partial_\theta f_a|+|t^2\partial_\theta g_a|\le C|t|^{\alpha_p}\ . \tag{8.23}$$

It should be clear that an optimal estimate will be obtained in (8.22) if $c_1=c_1(t,\theta)$ is chosen so that

$$c_1|f_2+g_2|=\mu_-+|f_1+g_1|+\frac{|f_2+g_2|}{c_1}\ ,$$

similarly for c_2. This gives

$$c_1|f_2+g_2|\le\frac{1}{2}\left(\mu_-+|f_1+g_1|+\sqrt{(\mu_-+|f_1+g_1|)^2+4|f_2+g_2|^2}\right)\ ,$$

$$c_2|f_2-g_2|\le\frac{1}{2}\left(\mu_++|f_1-g_1|+\sqrt{(\mu_++|f_1-g_1|)^2+4|f_2-g_2|^2}\right)\ ,$$

with equalities if $(f_2+g_2)(f_2-g_2)\neq 0$, and the condition described immediately after equation (8.22) will be satisfied if (8.18)-(8.19) holds.

Equation (5.1) gives

$$\partial_t g_1=O(|t|^{\alpha_p-2})\ , \tag{8.24}$$

and by integration along rays $\theta=\psi+\lambda t$, $\lambda\in[-1,1]$ one obtains on $C^0_{t_0}(\psi)$

$$|tg_1|\le C|t|^{\alpha_p}\ .$$

Returning to (5.1) one finds

$$\partial_t g_2 - f_1 g_2 = O(|t|^{\alpha_p - 2}) \ , \tag{8.25}$$

which can be integrated to give

$$0 > t_2 > t_1 \geq t_0 \qquad g_2(t_2,\psi) \;=\; e^{\int_{t_1}^{t_2} f_1(s,\psi)ds} g_2(t_1,\psi) + \int_{t_1}^{t_2} e^{\int_u^{t_2} f_1(s,\psi)ds} O(|u|^{\alpha_p - 2}) du \ . \tag{8.26}$$

Since $|f_1| \leq (1-\gamma)\,|\ln|t|| + C$ one easily concludes that

$$g_2(t,\psi) = O(|t|^{\alpha_p - 1}) \ .$$

Integrating in θ from ψ to θ, at fixed t, the desired estimate for g_2 on $C^0_{t_0}(\psi)$ is obtained using (8.23). Since ψ was arbitrary in $[a,b]$, and since all the constants were uniform in ψ, the result follows. □

The main result of this section is the following:

Theorem 8.3 *Suppose that either*

$$\sup_{\Omega(a,b,t_0)} \left(|tf_1| + \frac{|tf_2|}{2} + \frac{2|tg_2|^2}{\sqrt{1+4|tg_2|^2}+1} \right) < \frac{1}{2} \ , \tag{8.27}$$

or that (8.18)-(8.19) *hold. Then the velocity* v_{weak} *has a continuous representative* $v<1$ *on* $[a,b]$*, and the weak convergence in Proposition 6.1 can be replaced by convergence in sup norm to* v*. In other words,* $\lim_{t\to 0} |tX_t|(t,\theta)$ *exists and is a continuous function on* $[a,b]$*. Moreover the solution satisfies a power law blow-up, Equation (8.20).*

Remark 8.4 Further information concerning the properties of the solutions considered in Theorem 8.3 can be found in Theorem 11.1 below.

Proof: Under (8.27) Corollary 7.5 applies, so that the conclusion of Lemma 8.1 point (iii) holds. It now follows from Proposition 8.2 that both under (8.27), or under (8.18)-(8.19), point (ii) of Lemma 8.1 applies, and thus there exists $\epsilon < 1$ such that $|tD_\theta X_\theta| + |tD_\theta X_t| \leq C|t|^{-\epsilon}$. Theorem 8.3 is now a straightforward consequence of the following: □

LEMMA 8.5 *Suppose that there exist positive constants C and α_p such that on $\Omega(a,b,t_0)$ we have*

$$|t^2 D_\theta X_\theta| + |t^2 D_\theta X_t| \le C|t|^{\alpha_p} \ . \tag{8.28}$$

Then there exists a continuous function v such that on $\Omega(a,b,t_0)$ it holds

$$\Big| |tX_t|^2(t,\theta) - v^2(\theta) \Big| \le C'|t|^{\alpha_p} \ . \tag{8.29}$$

Further, for every ψ such that (8.28) holds on $C^0_{t_0}(\psi)$ we also have (8.29) on $C^0_{t_0}(\psi)$.

PROOF: We have

$$\partial_t |tX_t|^2 = 2h\left(tX_t, tD_\theta X_\theta\right) \ . \tag{8.30}$$

By integration we obtain

$$|tX_t|^2(t_1,\theta) - |tX_t|^2(t_2,\theta) = \int_{t_2}^{t_1} \partial_t |tX_t|^2(s,\theta)ds = \int_{t_2}^{t_1} O(s^{\alpha_p - 1})ds \ . \tag{8.31}$$

It easily follows from this equation that

$$v(\theta) := \lim_{t\to 0} |tX_t|(t,\theta)$$

exists. By passing to the limit $t_1 \to 0$ in (8.31) we obtain on $[t_0, 0) \times [a,b]$

$$|v^2(\theta) - |tX_t|^2(t,\theta)| \le \int_t^0 C s^{\alpha_p - 1} ds \le C'|t|^{\alpha_p} \ . \tag{8.32}$$

This shows that $|tX_t|(t,\cdot)$ converges uniformly to v, and establishes continuity thereof. The same argument applies on $C^0_{t_0}(\psi)$ by integrating along rays $\theta = \psi + \lambda t$, $\lambda \in [-1,1]$; one easily checks, using (8.28), that the number $v(\psi)$ is λ–independent. This, together with the result already established on $[t_0, 0) \times [a,b]$, establishes (8.29) on $\Omega(a,b,t_0)$. □

In terms of the (f,g) variables the Gowdy-to-Ernst transformation (3.20) takes a remarkably simple form:

$$\begin{aligned} \hat{f}_1 &= -f_1 - \frac{1}{t} \ , \qquad & \hat{g}_1 &= -g_1 \ , \\ \hat{f}_2 &= -g_2 \ , \qquad & \hat{g}_2 &= -f_2 \ . \end{aligned} \tag{8.33}$$

This transformation immediately leads to the following counterpart of Theorem 8.3:

THEOREM 8.6 *Suppose that either*

$$\sup_{\Omega(a,b,t_0)} \left(|tf_1+1| + \frac{|tg_2|}{2} + \frac{2|tf_2|^2}{\sqrt{1+4|tf_2|^2}+1} \right) < \frac{1}{2}\ , \tag{8.34}$$

or

$$\sup_{\Omega(a,b,t_0)} \left(\hat\mu_- + |tf_1+1+tg_1| + \sqrt{(\hat\mu_- + |tf_1+1+tg_1|)^2 + 4|tf_2+tg_2|^2} \right) < 4\ , \tag{8.35}$$

$$\sup_{\Omega(a,b,t_0)} \left(\hat\mu_+ + |tf_1+1-tg_1| + \sqrt{(\mu_+ + |tf_1+1-tg_1|)^2 + 4|tf_2-tg_2|^2} \right) < 4\ , \tag{8.36}$$

where

$$\hat\mu_\pm = \sup_{\Omega(a,b,t_0)} |tf_1 + 1 \pm tg_1|\ .$$

Then there exists a continuous function $\hat v$ such that

$$|tf_1|^2 + |tg_2|^2 \to_{t\to 0} \hat v^2\ , \tag{8.37}$$

uniformly in θ. Further there exist constants $C,\epsilon>0$ such that we have

$$|tf_2|^2 + |tg_1|^2 \le C|t|^{2\epsilon}\ . \tag{8.38}$$

In Section 12 below we will see how to iterate the Gowdy-to-Ernst transformation to obtain information on more general solutions.

For further purposes it is convenient to restate the conclusions of Lemma 8.1 as higher derivative estimates for the g_a's and f_a's:

LEMMA 8.7 (i) *There exists a constant C such that*

$$|t|^{k+1}|\partial_\theta^k f_a| + |t|^{k+1}|\partial_\theta^k g_a| + |t|^{k+1}|\partial_t^k f_a| + |t|^{k+1}|\partial_t^k g_a| \le C\ . \tag{8.39}$$

(ii) *Under the conditions of point (i) of Lemma 8.1 we have*

$$\int_{a+t}^{b-t} \left(|t|^{l+k+1}\partial_t^k\partial_\theta^l u \right)^2 d\theta \le C(l,k)|t|^{2\alpha}\ , \tag{8.40}$$

for every l, $k\ge 0$ when $u=g_1$, g_2 or $l\ge 1$, $k\ge 0$ when $u=f_1,f_2$.

(iii) *If $|tX_\theta| \le C|t|^{\alpha_p}$ then we also have*

$$|t|^{l+k+1}|\partial_t^k\partial_\theta^l u| \le C(l,k)|t|^{\alpha_p}\ , \tag{8.41}$$

with the ranges of the (l,k)'s as in (ii).

PROOF: For the purposes of the proof let u stand for any of g_a, f_2.

(i): From (5.17) we have

$$(\partial_\theta^k g_1)^2 + (\partial_\theta^k g_2)^2 \le |D_\theta^k X_\theta|^2 + F_k^2 + G_k^2$$

then $|t|^{k+1}|\partial_\theta^k g| \le C$ inductively using part (ii) of Proposition 1.4. For $|t|^{k+1}|\partial_t^k f_a|$ we compute by induction

$$D_t^k X_\theta = (\partial_t^k g_1 + \tilde{F}_k(\partial^{k-1}u, \ldots, u))e_1 + (\partial_t^k g_2 + \tilde{G}_k(\partial^{k-1}u, \ldots, u))e_2, \tag{8.42}$$

$$D_t^k X_t = (\partial_t^k f_1 + \hat{F}_k(\partial^{k-1}f, \ldots, f))e_1 + (\partial_t^k f_2 + \hat{G}_k(\partial^{k-1}f, \ldots, f))e_2 \ , \tag{8.43}$$

where $\tilde{F}_k$, $\tilde{G}_k$, $\hat{F}_k$, $\hat{G}_k$ have the same properties as described in the proof of Proposition 5.2 for F_k and G_k. The remaining inequalities follow as before.

(ii): The proof is identical to that of Proposition 5.2.

(iii): We consider the g_a's first. Letting e stand for the basis vectors e_a's of (5.2), it is sufficient to show

$$\sup_{\Omega(a,b,t_0)} |t|^{i+j}|D_t^i D_\theta^j e| \le C\,, \tag{8.44}$$

then the assertion follows from

$$\begin{aligned}\partial_t^k \partial_\theta^l g &= \partial_t^k \partial_\theta^l \langle X_\theta, e\rangle = \partial_t^k \sum_{j=0}^{l} C(l,j)\langle D_\theta^j X_\theta, D_\theta^{l-j} e\rangle \\ &= \sum_{i=0}^{k}\sum_{j=1}^{l} C(l,k,i,j)\langle D_t^i D_\theta^j X_\theta, D_t^{k-i} D_\theta^{l-j} e\rangle\end{aligned}$$

together with Lemma 8.1. Now

$$D_\theta e_1 = g_2 e_2\ , \quad D_\theta e_2 = -g_2 e_1\ , \quad D_t e_1 = f_2 e_2\ , \quad D_t e_2 = -f_2 e_1\ , \tag{8.45}$$

which implies that the $D_\theta^k e_a$, $D_t^k e_a$ are of the form

$$D_t^k e_1 = \bar{F}_{k-1} e_1 + \left(\partial_t^{k-1} f_2 + \bar{G}_{k-1}\right) e_2\ ,\ D_t^k e_2 = \left(-\partial_\theta^{k-1} f_2 + \bar{F}'_{k-1}\right) e_1 + \bar{G}'_{k-1} e_2\ ,$$

$$D_\theta^k e_1 = \mathring{F}_{k-1} e_1 + \left(\partial_\theta^{k-1} g_2 + \mathring{G}_{k-1}\right) e_2\ ,\ D_\theta^k e_2 = \left(-\partial_\theta^{k-1} g_2 + \mathring{F}'_{k-1}\right) e_1 + \mathring{G}'_{k-1} e_2\ ,$$

with the $\bar{F}_{k-1}$'s, $\bar{G}_{k-1}$'s, *etc.*, of a similar structure as in (5.17). By (8.39) and by the above expression we see that $|t|^k|D_\theta^k e|$, $|t|^k|D_t^k e|$ are bounded.

For mixed derivatives of g, we write

$$\begin{aligned}
D_t^k D_\theta^l X_\theta &= D_t^k \Big((\partial_\theta^l g_1 + F_l) e_1 + (\partial_\theta^l g_2 + G_l) e_2 \Big) \\
&= \sum_{i=0}^{k} \partial_t^i (\partial_\theta^l g_1 + F_l) D_t^{k-i} e_1 + \partial_t^i (\partial_\theta^l g_2 + G_l) D_t^{k-i} e_2 \\
&= (\partial_t^k \partial_\theta^l g_1 + \partial_t^k F_l) e_1 + (\partial_t^k \partial_\theta^l g_2 + \partial_t^k G_l) e_2 \\
&+ \sum_{i=0}^{k-1} (\partial_t^i \partial_\theta^l g_1 + \partial_t^i F_l) D_t^{k-i} e_1 + (\partial_t^i \partial_\theta^l g_2 + \partial_t^i G_l) D_t^{k-i} e_2 \ ;
\end{aligned}$$

above, and in what follows, we ignore constants arising from binomial expansions. From above expression we get $|t|^{k+l+1} |\partial_t^k \partial_\theta^l g| \leq C$ inductively using (8.39) and boundedness of $|t|^i |D_t^i e|$. Finally (8.44) follows from writing

$$\begin{aligned}
D_t^k D_\theta^l e &= D_t^k \Big((\sigma \partial_\theta^{l-1} g_2 + \mathring{F}_{l-1}) e_1 + (\hat{\sigma} \partial_\theta^{l-1} g + \mathring{G}_{l-1}) e_2) \Big) \\
&= \sum_{i=0}^{k} (\sigma \partial_t^i \partial_\theta^{l-1} g_2 + \partial_t^i \mathring{F}_{l-1}) D_t^{k-i} e_1 + (\hat{\sigma} \partial_t^i \partial_\theta^{l-1} g + \partial^i \mathring{G}_{l-1}) D_t^{k-i} e_2
\end{aligned} \tag{8.46}$$

and then using boundedness of $|t|^{i+j+1} |\partial_t^i \partial_\theta^j g|, \ |t|^i |D_t^i e|$.

Let us turn now to the f_2's. According to (5.18) we write

$$\partial_\theta^l f_2 = \sum_{j=0}^{l} \langle D_\theta^j X_t, D_\theta^{l-j} e_2 \rangle,$$

so that

$$\begin{aligned}
\partial_t^k \partial_\theta^l f_2 &= \sum_{i=0}^{k} \sum_{j=0}^{l} \langle D_t^i D_\theta^j X_t, D_t^{k-i} D_\theta^{l-j} e_2 \rangle \,, \\
&= \sum_{i=0}^{k} \langle D_t^i X_t, D_t^{k-i} D_\theta^l e_2 \rangle + \sum_{i=0}^{k} \sum_{j=1}^{l} \langle D_t^i D_\theta^j X_t, D_t^{k-i} D_\theta^{l-j} e_2 \rangle \quad \Box
\end{aligned}$$

when $l \geq 1, \ k \geq 0$. Multiplying by $|t|^{k+l+1}$ we have

$$|t|^{k+l+1} |\partial_t^k \partial_\theta^l f_2| \leq \sum_{i=0}^{k} |t|^{k+l-i} |D_t^{k-i} D_\theta^l e_2| + \sum_{i=0}^{k} \sum_{j=1}^{l} |t|^{i+j+1} |D_t^i D_\theta^j X_t| \tag{8.47}$$

from Proposition 1.4 together with (8.44). Using the expression (8.46) we get

$$\begin{aligned}|t|^{k+l-i}|D_t^{k-i}D_\theta^l e_2| &\le \sum_{m=0}^{k-i} |t|^{k+l-i}\left(|\partial_t^m\partial_\theta^{l-1}g| + |\partial_t^m \mathring{F}_{l-1}| + |\partial_t^m \mathring{G}_{l-1}|\right)|D_t^{k-i-m}e| \\ &\le C\sum_{m=0}^{k-i} |t|^{l+m}\left(|\partial_t^m\partial_\theta^{l-1}g| + |\partial_t^m \mathring{F}_{l-1}| + |\partial_t^m \mathring{G}_{l-1}|\right)\end{aligned}$$

from the boundedness of $|t|^k|D_t^k e|$. The claim on f_2 follows now from Lemma 8.1 together with the assertions on g. Finally, the result for f_1 follows from what has been proved so far together with the equation

$$\partial_t(tf_1) = t(\partial_\theta g_1 + f_2^2 - g_2^2)\ .$$

9 Stability of the $(\frac{2}{3},\frac{2}{3},-\frac{1}{3})$ and $(1,0,0)$ Kasner metrics

The following result extends the *singularity stability theorem* for the $(\frac{2}{3},\frac{2}{3},-\frac{1}{3})$ Kasner metrics established in [4], by raising[9] the stability threshold there by a factor $6^{3/2}/2$

THEOREM 9.1 *Suppose that*

$$\sup_{\theta\in[a-t_0,b+t_0]} t_0^2\left(|X_t|^2+|X_\theta|^2\right)(t_0,\theta) < \frac{1}{2}\ . \tag{9.1}$$

Then the solution is of power-law type. Further, the curvature scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ blows up on every causal curve with endpoint on $\{0\}\times[a,b]\times S^1\times S^1$. In particular the associated Gowdy space-time is inextendible across the boundary $\{0\}\times[a,b]\times S^1\times S^1$.

PROOF: We wish to apply Proposition 8.2. Let $\mu_\pm$ be defined by (8.17), using the Cauchy-Schwarz inequality and some rather obvious estimations we have

$$\begin{aligned}\mu_- + |tf_1+tg_1| &+ \sqrt{(\mu_- + |tf_1+tg_1|)^2 + 4|tf_2+tg_2|^2} \\ &\le \sqrt{2}\sqrt{2(\mu_- + |tf_1+tg_1|)^2 + 4|tf_2+tg_2|^2} \\ &\le 2\sqrt{2}\sqrt{\mu_-^2 + |tf_1+tg_1|^2 + |tf_2+tg_2|^2}\ .\end{aligned} \tag{9.2}$$

[9]A similar result has been recently established by Ringström [18].

This shows that both conditions (8.18)-(8.19) will hold if

$$\sup_{\Omega(a,b,t_0)} \left(|tf_1 - tg_1|^2 + |tf_2 - tg_2|^2\right) < 1 ,$$

and if

$$\sup_{\Omega(a,b,t_0)} \left(|tf_1 + tg_1|^2 + |tf_2 + tg_2|^2\right) < 1 .$$

It follows from (1.8) that the last inequalities will hold if

$$\sup_{t=t_0,\theta\in[a-|t_0|,b+|t_0|]} |t| \left(|f_1 - g_1|^2 + |f_1 + g_1|^2 + |f_2 - g_2|^2 + |f_2 + g_2|^2\right) < 1 ,$$

which is equivalent to (9.1). The result follows now from the arguments in the proofs of Theorem 3.5.1 and Proposition 3.5.2 in [4]. □

Remark 9.2 We note that the Sobolev decay estimates of Section 7 lead to a similar somewhat weaker statement, with $1/2$ replaced by $\sqrt{3/19}$ in (9.1). This can be seen as follows: Equation (8.27) will hold under the slightly stronger but simpler condition

$$\sup_{\Omega(a,b,t_0)} \left(|tf_1| + \frac{|tf_2|}{2} + \frac{|tg_2|}{\sqrt{3}}\right) < \frac{1}{2} . \tag{9.3}$$

The Cauchy-Schwarz inequality,

$$|tf_1| + \frac{|tf_2|}{2} + \frac{|tg_2|}{\sqrt{3}} \le \sqrt{1 + \frac{1}{4} + \frac{1}{3}} \sqrt{|tf_1|^2 + |tf_2|^2 + |tg_2|^2} ,$$

together with point (i) of Proposition 1.4 show that Theorem 8.3 applies, and one concludes as before.

The solution

$$f_1 = 1/|t| , \quad f_2 = g_1 = g_2 = 0 ,$$

of (5.1) corresponds to the flat Kasner metric. Theorem 8.6 similarly implies complete control of the behavior of the solution for all data in a neighborhood of those for the flat Kasner metric. In this case the geometric interpretation is more complicated, because of occurrence of horizons. A further discussion of the latter can be found in [6].

10 Behavior of power-law solutions at $v = 0$ and $v = 1$

We consider solutions on $\Omega(a, b, t_0)$ such that

$$|tX_\theta| \leq C|t|^{\alpha_p} \; , \tag{10.1}$$

for some $\alpha_p > 0$. It follows from [4, Remark 7.3] and Lemma 8.5 that the velocity function v exists. The aim of this section is to study the behavior of such solutions at points, or intervals, on which v vanishes:

THEOREM 10.1 *Suppose that* (10.1) *holds with some* $\alpha_p > 0$ *and consider any point* $\psi \in [a, b]$ *such that* $v(\psi) = 0$. *Then:*

(i) *The restriction* $\bar{x} := x|_{C^0_{t_0}(\psi)}$ *of* x *to* $C^0_{t_0}(\psi)$ *can be extended to an* $AVTD^{(P,Q)}_\infty$ *map from* $\mathbb{R}^2$ *to* $\mathscr{H}_2$.

(ii) *If* $\lim_{C^0_{t_0}(\psi)} t\partial^j_\theta P_t = 0$ *for all* $j \in \mathbb{N}$, *then for* $i, k \in \mathbb{N}$ *we have*

$$\lim_{C^0_{t_0}(\psi)} \partial^{2i+1}_t \partial^k_\theta P = \lim_{C^0_{t_0}(\psi)} \partial^{2i+1}_t \partial^k_\theta Q = 0 \; . \tag{10.2}$$

(iii) *Further, if* v *vanishes on an interval* $[\theta_l, \theta_r]$, *then the restriction* $\tilde{x} := x|_{\Omega(\theta_l, \theta_r, t_0)}$ *of* x *to* $\Omega(\theta_l, \theta_r, t_0)$ *can be extended by continuity to a smooth map from* $\mathbb{R}^2$ *to* $\mathscr{H}_2$, *with* (10.2) *holding for all* $\psi \in [\theta_l, \theta_r]$.

Theorem 10.1 says, in essence, that x behaves on $C^0_{t_0}(\psi)$ as if it arose from an $\mathrm{AVTD}^{(P,Q)}_\infty$ map defined on $\mathbb{R}^2$. We emphasize, however, that the extensions mentioned above might fail to coincide with the original map x away from $C^0_{t_0}(\psi)$ (for $\bar{x}$), or away from $\Omega(\theta_l, \theta_r, t_0)$ (for $\tilde{x}$). Such a situation could arise when x has an infinite number of smaller and smaller spikes accumulating at a point at which $v(\psi) = 0$.

Equation (10.2) says, roughly speaking, that P and Q can be thought of as smooth functions of θ and t^2 (rather than t). This result is relevant to the question of extendibility of the associated metric across Cauchy horizons. If $\lim_{C^0_{t_0}(\psi)} t\partial^j_\theta P_t = 0$ for a finite number of j's, there will be a certain number of i's for which (10.2) will hold.

Using the Gowdy-to-Ernst transformation we obtain immediately the following counterpart of Theorem 10.1 at $v = 1$:

THEOREM 10.2 *Suppose that there exist constants $C, \alpha_p > 0$ such that*

$$|tg_1| + |tf_2| \le C|t|^{\alpha_p} \tag{10.3}$$

and consider any point $\psi \in [a,b]$ such that $v_1(\psi) = 1$. Then the functions $(\bar P, \bar Q) := (P + \ln|t|, Q)|_{C^0_{t_0}(\psi)}$ can be extended to an $AVTD^{(P,Q)}_\infty$ map from $\mathbb{R}^2$ to $\mathscr{H}_2$. If $\lim_{C^0_{t_0}(\psi)} t\partial^j_\theta P_t = 0$ for all $j \in \mathbb{N}$, then for all $i,k \in \mathbb{N}$ we have

$$\lim_{C^0_{t_0}(\psi)} \partial_t^{2i+1}\partial_\theta^k (P + \ln|t|) = \lim_{C^0_{t_0}(\psi)} \partial_t^{2i+1}\partial_\theta^k Q = \lim_{C^0_{t_0}(\psi)} \partial_\theta^{k+1} Q = 0\ . \tag{10.4}$$

Further, if $v_1 = 1$ on an interval $[\theta_l, \theta_r]$, then the restriction $(\tilde P, \tilde Q) := (P + \ln|t|, Q)|_{\Omega(\theta_l,\theta_r,t_0)}$ can be extended to a smooth map from $\mathbb{R}^2$ to $\mathscr{H}_2$, with (10.4) *holding for all $\psi \in [\theta_l, \theta_r]$.*

REMARK 10.3 The vanishing of the last term in (10.4) for all $k \ge 0$ is somewhat surprising. We emphasise that the power-law condition (10.1) in Theorem 10.1 is justified for small initial data by Theorem 8.3, and that the condition (10.3) is justified for initial data near the flat Kasner by Theorem 8.6.

PROOF OF THEOREM 10.2: The Gowdy-to-Ernst transformed map $\hat x$ satisfies the hypotheses of Theorem 10.1, and therefore (10.2) holds for the associated functions $\hat P$ and $\hat Q$. The claim about P in (10.4) is straightforward. From (3.20) we have

$$\partial_t\partial_\theta^k Q = -|t|\partial_\theta^k\left(e^{2\hat P}\partial_\theta \hat Q\right)\ ,\quad \partial_\theta^{k+1} Q = -|t|\partial_\theta^k\left(e^{2\hat P}\partial_t \hat Q\right)\ . \tag{10.5}$$

Integrating in t and using (10.2) one obtains (10.4). □

An iteration of isometries and Gowdy-to-Ernst transformations, as in the proof of Theorem 1.1 below, allows one to control the behavior of x near points $(0,\psi)$ at which $v(\psi) \in \mathbb{Z}$, the details are left to the reader.

PROOF OF THEOREM 10.1: We start with a lemma:

LEMMA 10.4 *Suppose that there exists $0 < \alpha_p$ such that* (10.1) *holds. Then there exists a function $v_1(\theta)$, with $|v_1| = v$, and constants $C_a(\theta)$, $a = 1,2$, such that on $C^0_{t_0}(\theta)$ we have*

$$|tf_2| \le C_2(\theta)\begin{cases} |t|^{\alpha_p}, & v(\theta) \le 0;\\ |t|^{v(\theta)} + |t|^{\alpha_p}, & 0 < v_1(\theta) \ne \alpha_p;\\ \big(1 + \big|\ln|t|\big|\big)|t|^{v(\theta)}, & v_1(\theta) = \alpha_p. \end{cases} \tag{10.6}$$

$$\Big||t|f_1 - v_1(\theta)\Big| \le C_1(\theta) \begin{cases} |t|^{\alpha_p}, & v_1(\theta) \le 0; \\ |t|^{2v_1(\theta)} + |t|^{\alpha_p}, & 0 < v_1(\theta) \ne \alpha_p; \\ |t|^{\alpha_p}, & v_1(\theta) = \alpha_p. \end{cases} \qquad (10.7)$$

The constants $C_a(\theta)$ are uniformly bounded on compact intervals on which v_1 is strictly positive, uniformly bounded away from zero.

REMARK 10.5 The examples discussed in Section 3 show that v_1 is not continuous in general. Further, the constants C_a are not uniformly bounded near points at which v_1 has discontinuities involving a change of sign. Similarly we do not expect uniformity at points at which v_1 crosses 0.

PROOF: As explained in the proof of Lemma 8.1, Equation (10.1) implies

$$|t|^{|\nu|+1}|D^\nu X_\theta| \le C(\nu)|t|^{\alpha_p} \ . \qquad (10.8)$$

We can use Lemma 8.5 to conclude that there exists a continuous function v such that

$$|v^2(\theta) - |tX_t|^2(t,\theta)| \le C|t|^{\alpha_p} \ . \qquad (10.9)$$

Equation (10.8) with $\nu = \theta$ and $\nu = t$ shows that the θ-derivatives conditions of Lemma 6.5 are satisfied, and Theorem 6.3 implies that there exists a function v_1 with $|v_1| = v$ such that, for all $\psi \in [a,b]$,

$$\lim_{C^0_{t_0}(\psi)}|t|f_1 = v_1(\psi) \ , \qquad (10.10)$$

together with

$$\lim_{C^0_{t_0}(\psi)}|t|f_2 = 0 \ . \qquad (10.11)$$

Equations (5.1) and Lemma 8.7 lead to

$$\begin{aligned} \partial_t(|t|f_1) &= |t|f_2^2 + O(|t|^{\alpha_p-1}) \ , & (10.12) \\ \partial_t(|t|f_2) &= -|t|f_1f_2 + O(|t|^{\alpha_p-1}) \ . & (10.13) \end{aligned}$$

Suppose, first, that θ is such that $v(\theta) = 0$. It follows from (10.9) that the right-hand sides of (10.12)-(10.13) are $O(|t|^{\alpha_p-1})$. The same is true for $\partial_\theta(|t|f_a)$ by Lemma 8.7, and by integration in t along rays $\Gamma_\lambda(t) = (t, \theta+\lambda t)$, $\lambda \in [-1,1]$, we find on $C^0_{t_0}(\theta)$

$$v(\theta) = 0 \implies |tf_1| \le C|t|^{\alpha_p} \ , \quad |tf_2| \le C|t|^{\alpha_p} \ . \qquad (10.14)$$

In general, integrating (10.12) shows first that $tf_2^2 \in L^1([t_0,0)$, and then

$$|t|f_1(t,\theta) = v_1(\theta) - \int_t^0 |s|f_2^2(s,\theta)ds + O(|t|^{\alpha_p}) \ . \qquad (10.15)$$

Similarly, (10.13) can be integrated to give

$$
\begin{aligned}
0>t_2>t_1\geq t_0 \qquad |t_2|f_2(t_2,\theta) &= e^{-\int_{t_1}^{t_2} f_1(s,\theta)ds}|t_1|f_2(t_1,\theta) \\
&\quad + \int_{t_1}^{t_2} e^{-\int_{u}^{t_2} f_1(s,\theta)ds}O(|u|^{\alpha_p-1})du \; .
\end{aligned}
\tag{10.16}
$$

If $v_1(\theta)>0$ one finds first from (10.16) that $|tf_2|\leq C|t|^\epsilon$, with ϵ equal to, say, $\min(v_1(\theta)/2,\alpha_p/4)$. Plugging this in (10.12) one obtains that $f_1-v_1=O(|t|^\epsilon)$. Returning to (10.16) one is then led to (10.6) at (t,θ) with $v_1(\theta)>0$. Inserting (10.6) into (10.15) we arrive at (10.7) at (t,θ), again for $v_1(\theta)>0$.

Suppose, finally, that $v_1(\theta)<0$, then we rewrite (10.16) as

$$
\begin{aligned}
0>t_2>t_1\geq t_0 \qquad |t_1|f_2(t_1,\theta) &= e^{\int_{t_1}^{t_2} f_1(s,\theta)ds}|t_2|f_2(t_2,\theta) \\
&\quad + \int_{t_1}^{t_2} e^{\int_{t_1}^{u} f_1(s,\theta)ds}O(|u|^{\alpha_p-1})du \; .
\end{aligned}
\tag{10.17}
$$

One readily checks that the integrand is in $L^1([t_0,0))$, and passing to the limit $t_2\to 0$ one obtains

$$
0>t_1\geq t_0 \qquad |t_1|f_2(t_1,\theta) = \int_{t_1}^{0} e^{\int_{t_1}^{u} f_1(s,\theta)ds}O(|u|^{\alpha_p-1})du \; .
\tag{10.18}
$$

Arguing as before one obtains the result at (t,θ). Finally, the result for $(t,\theta_1)\in C^0_{t_0}(\theta)$ is obtained from the one for (t,θ) by integrating $\partial_\theta(tf_a)$ in θ from (t,θ_1) to (t,θ), using point (iii) of Lemma 8.7. $\square$

We return to the proof of Theorem 10.1. To proceed further we need better control of the θ derivatives. Let E_k be defined by (8.15) and consider a point ψ such that $v(\psi)=0$. From (10.1), (10.14), together with Equations (8.8) and (8.11)-(8.14) one obtains

$$
E_1(t_1,\psi)\leq E_1(t_0,\psi)+\int_{t_0}^{t_1}\frac{2+O(|t|^{\alpha_p})}{|t|}E_1(t,\psi)\,dt\;;
\tag{10.19}
$$

we have also made use of (8.16). Gronwall's Lemma (*cf., e.g.,* [4, Lemma 3.2.3]) gives, for any $\epsilon>0$, decreasing $|t_0|$ if necessary,

$$
E_1(t,\psi)\leq C\left|\frac{t_0}{t}\right|^{2+\epsilon}\;.
\tag{10.20}
$$

This allows us to rewrite (10.19) as

$$E_1(t_1,\psi) \le E_1(t_0,\psi) + \int_{t_0}^{t_1} \Big(\frac{2}{|t|}E_1(t,\psi) + O(|t|^{\alpha_p-3-\epsilon})\Big)\,dt\ , \qquad (10.21)$$

which, together with Gronwall's Lemma, implies (10.20) with $\epsilon = 0$. It follows that on $C^0_{t_0}(\psi)$ we have

$$|t\partial_\theta f_a| + |t\partial_\theta g_a| \le C\ , \qquad (10.22)$$

and (5.1) gives

$$|t\partial_t g_1| \le C\ , \quad |t\partial_t g_2| \le C(1+|t|^{2\alpha_p-1})\ . \qquad (10.23)$$

Integrating in t over rays $\theta = \psi + \lambda t$, $\lambda \in [-1,1]$, one obtains

$$g_1 = O(\ln|t|)\ , \quad |g_2| \le C(\ln|t| + |t|^{2\alpha_p-1})\ . \qquad (10.24)$$

We have thus recovered (10.1) with the exponent α_p there replaced by $2\alpha_p$ if $2\alpha_p < 1$, or by a number as close to one as desired otherwise. Iterating the argument leading from (10.23) to (10.24) a finite number of times if needed one thus has

$$|tf_a| + |tg_a| \le C|t|^{1-\epsilon}\ , \qquad (10.25)$$

with ϵ as small as desired.

We continue by induction: suppose that for all $\epsilon > 0$ and for all $0 \le i \le k-1$ there exists $C_i(\epsilon)$

$$|t\partial^i_\theta f_a| + |t\partial^i_\theta g_a| \le C_i(\epsilon)|t|^{-\epsilon}\ . \qquad (10.26)$$

From (8.8), (8.11)-(8.14) one obtains

$$\begin{aligned} E_k(t_1,\psi) &\le E_k(t_0,\psi) + \int_{t_0}^{t_1}\Big(\frac{2+\epsilon}{|t|}E_k(t,\psi) + C|t|^{-2-2\epsilon}E_k^{1/2}(t,\psi)\Big)\,dt \\ &\le E_k(t_0,\psi) + \int_{t_0}^{t_1}\Big(\frac{2+2\epsilon}{|t|}E_k(t,\psi) + \frac{C}{4\epsilon}|t|^{-2-2\epsilon}\Big)\,dt\ , \end{aligned} \qquad (10.27)$$

leading to (10.26) with $i = k$ (and with ϵ replaced by 2ϵ.).

The equation

$$\partial_t \begin{pmatrix} \partial^k_\theta f_1 \\ \partial^k_\theta f_2 \\ \partial^k_\theta g_1 \\ \partial^k_\theta g_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \partial_\theta \begin{pmatrix} \partial^k_\theta f_1 \\ \partial^k_\theta f_2 \\ \partial^k_\theta g_1 \\ \partial^k_\theta g_2 \end{pmatrix} + \begin{pmatrix} \partial^k_\theta\Big(f_2^2 - g_2^2 - \frac{f_1}{t}\Big) \\ \partial^k_\theta\Big(-f_1f_2 + g_1g_2 - \frac{f_2}{t}\Big) \\ 0 \\ \partial^k_\theta\Big(f_1g_2 - g_1f_2\Big) \end{pmatrix} \qquad (10.28)$$

gives

$$|\partial_t \left(t\partial_\theta^i f_a\right)| + |t\partial_t\partial_\theta^i g_a| \leq C_i(\epsilon)|t|^{-\epsilon} \ . \tag{10.29}$$

Integrating in t over rays $\theta = \psi + \lambda t$, $\lambda \in [-1,1]$, from (10.26) and (10.29) one obtains

$$|\partial_\theta^i g_a| \leq C_i(\epsilon)|t|^{-\epsilon} \ . \tag{10.30}$$

Further, the same integration argument shows that there exist constants $v_{a,i}(\psi)$ such that

$$||t|\partial_\theta^i f_a - v_{a,i}(\psi)| \leq C_i(\epsilon)|t|^{1-\epsilon} \ . \tag{10.31}$$

We note that $v_{a,0}(\psi) = 0$ by (10.25). From (10.28) one obtains now the equation

$$\begin{aligned}
\partial_t(\partial_\theta^i|t|f_1) &= |t|\partial_\theta^i(f_2^2) + O(|t|^{-\epsilon}) \\
&= \left(\sum_{j=0}^{i}\binom{j}{i} v_{2,j}(\psi)v_{2,i-j}(\psi)\right)\frac{1}{|t|} + O(|t|^{-\epsilon}) \ ,
\end{aligned} \tag{10.32}$$

which is compatible with boundedness of $\partial_\theta^i(|t|f_1)$ if and only if the coefficient of $1/|t|$ vanishes. Suppose that we know that $v_{2,j}$ vanishes for $j = 1,\ldots,m-1$, then the condition of vanishing of the offending term in (10.32) with $i = 2m$ gives $v_{2,m} = 0$, and induction gives the vanishing of $v_{2,j}$ for all j. It then follows from (10.31) that

$$|\partial_\theta^i f_2| \leq C_i(\epsilon)|t|^{-\epsilon} \ . \tag{10.33}$$

Integrating the equations for $\partial_t\partial_\theta^i g_a$ one arrives at

$$\partial_\theta^i g_1 = -v_{1,i+1}(\psi)\ln|t| + g_{1,i}(\psi) + O(|t|^{1-\epsilon}) \ , \tag{10.34}$$

$$g_2 = g_{2,0}(\psi) + O(|t|^{1-\epsilon}) \ , \tag{10.35}$$

$$\partial_\theta^i g_2 = -v_{1,i}(\psi)g_{2,0}(\psi)\ln|t| + g_{2,1}(\psi) + O(|t|^{1-\epsilon}) \ , \tag{10.36}$$

for some constants $g_{a,i}(\psi)$. Similarly, as in (10.18),

$$\begin{aligned}
0 > t_1 \geq t_0 \qquad |t_1|\partial_\theta^k f_2(t_1,\psi) &= \int_{t_1}^0 e^{\int_{t_1}^u f_1(s,\psi)ds} O(|u|^{1-\epsilon})du \\
&= O(|t_1|^{2-\epsilon}) \ .
\end{aligned} \tag{10.37}$$

Integrating in θ from ψ to θ and using (10.37) one obtains, on $C_{t_0}^0(\psi)$,

$$\partial_\theta^k f_2(t,\psi) = O(|t|^{1-\epsilon}) \ . \tag{10.38}$$

Analogously to (10.15) one has

$$|t|\partial_\theta^k f_1(t,\psi) - v_{1,i}(\psi) = \int_t^0 O\left(|s\ln|s||\right) ds = O\left(\left|t^2\ln|t|\right|\right) \ . \qquad (10.39)$$

An iterative repetition of the arguments given leads to a full asymptotic expansion of the solution on $C^0_{t_0}(\psi)$. It should be clear that the expansions for the derivatives obtained in this way behave as though they arose from an $\mathrm{AVTD}^{(P,Q)}_\infty$ map, and the usual extension arguments can be used to provide an $\mathrm{AVTD}^{(P,Q)}_\infty$ extension of the restriction $x|_{C^0_{t_0}(\psi)}$ of x.

Suppose finally that all the $v_{1,j}(\psi)$ vanish. We note that it follows from the argument above that this will be the case when $v = 0$ on an interval containing ψ. It is then straightforward to show that all the $\partial_\theta^k f_a$'s and the $\partial_\theta^k g_a$'s can be extended by continuity to the tip $(0,\psi)$ of $C^0_{t_0}(\psi)$. By an abuse of notation we will denote the value at the tip of those extensions by $\partial^\alpha f_a(0,\psi)$, *etc.* Equation (10.37) gives, for all k,

$$\partial_\theta^k f_2(0,\psi) = 0 \ . \qquad (10.40)$$

Equation (10.39) implies

$$\partial_\theta^k f_1(0,\psi) = 0 \ . \qquad (10.41)$$

Arguing similarly one obtains that any mixed t and θ derivatives can also be extended by continuity to the tip $(0,\psi)$ of $C^0_{t_0}(\psi)$. For example, when $v_{1,j}(\psi) = 0$ (10.39) can be rewritten as

$$|t|\partial_\theta^k f_1(t,\psi) = \frac{\partial_\theta^k\left(g_2^2(0,\psi)\right)t^2}{2} + O\left(|t|^3\right) \ , \qquad (10.42)$$

which shows that $\partial_\theta^k f_1/t$ extends continuously to the tip. It then follows from (10.28) that $\partial_t\partial_\theta^k f_1$ extends continuously to the tip. A similar analysis shows that the same is true for $\partial_t\partial_\theta^k f_2$. It then immediately follows from (10.28) that $\partial_t\partial_\theta^k g_a$ extends continuously to the tip. Differentiating (10.28) with respect to t one can repeat this analysis for all higher-order t-derivatives.

An identical extension property is obviously true for P, Q, and all their derivatives.

In order to prove (10.2), suppose that

$$\partial_\theta^k f_a(t,\psi) = \sum_{j=0}^{i-1}\alpha_{a,j,k}t^{2j+1} + O(|t|^{2i}) \ , \qquad (10.43)$$

$$\partial_\theta^k g_a(t,\psi) = \sum_{j=0}^{i}\beta_{a,j,k}t^{2j} + O(|t|^{2i+1}) \ ; \qquad (10.44)$$

this clearly holds with $i = 0$. Then (10.28) gives

$$\partial_t \left(t \partial_\theta^k f_a(t,\psi) \right) = \sum_{j=0}^{i-1} \hat{\alpha}_{a,j,k} t^{2j+1} + O(|t|^{2i+2}) \ , \qquad (10.45)$$

and by integration one recovers (10.43) with i replaced by $i+1$. Inserting this into (10.28) one then finds

$$\partial_t \left(\partial_\theta^k g_a(t,\psi) \right) = \sum_{j=0}^{i} \hat{\beta}_{a,j,k} t^{2j+1} + O(|t|^{2i+2}) \ , \qquad (10.46)$$

and integration in t completes the induction step. The usual extension results finish the proof on $C^0_{t_0}(\psi)$. Clearly the same results remain valid if v vanishes on an interval I, we simply note that all the estimates so far were uniform in θ over I. $\square$

11 Solutions with $v < 1$ and with uniform power-law

We are ready now to prove Theorem 1.1 under the more restrictive hypotheses (11.1) and (11.2); those will be removed in Section 12:

THEOREM 11.1 *Let $\mathring{x}$ be a solution of the Gowdy equations on $\Omega(a,b,t_0)$ such that*

$$\sup_{\Omega(a,b,t_0)} |t\mathring{X}_t| < 1 \ . \qquad (11.1)$$

Suppose moreover that there exist positive constants ϵ and C_0 such that on $\Omega(a,b,t_0)$ we have

$$|t\mathring{X}_\theta| \le C_0 |t|^\epsilon \ . \qquad (11.2)$$

Then:

(i) *The solution belongs to the class $\mathscr{U}_1$ defined in Section 4, with a velocity function $v = |v_1|$ and a position function φ_∞ which are smooth except perhaps on the set*

$$\partial\{\theta : v(\theta) = 0\} \ .$$

(ii) *There exists $\eta > 0$ such that for all initial data $(x(t_0,\cdot), X_t(t_0,\cdot))$ satisfying*[1]

$$\|(x(t_0,\cdot) - \mathring{x}(t_0,\cdot), X_t(t_0,\cdot) - \mathring{X}_t(t_0,\cdot))\|_{H^3\oplus H^2} < \eta$$

the associated solution x also satisfies (11.1)-(11.2) *(with perhaps different values of ϵ and C). Further the conclusions of Theorem 9.1 apply.*

REMARK 11.2 The behavior of the solution near all points $(0,\psi)$ with $v(\psi) = 0$, including $\{0\}\times(\partial\{\theta : v(\theta) = 0\})$, is described exhaustively in Theorem 10.1. It is not clear whether or not the function φ_∞ of (3.24) is continuous at $\partial\{\theta : v(\theta) = 0\}$.

REMARK 11.3 Every $\mathrm{AVTD}_1^{(P,Q)}$ solution with v strictly smaller than one, and for which the error terms in (3.5)-(3.6) and in their derivative counterparts (see (3.10)) are uniform in θ satisfies (11.2), as well as (11.1) for some t_0 close enough to 0. This follows immediately from (3.5)-(3.6). Further, all initial data satisfying the hypotheses of Theorem 8.6, or of [17, Theorems 1 and 2], satisfy the hypotheses of Theorem 11.1.

PROOF: The arguments at the beginning of the proof of Theorem 8.6 show that v exists on $[a,b]$ and is continuous there. We continue with a lemma:

LEMMA 11.4 *Suppose that $|tX_\theta| \le C|t|^\epsilon$ for some $\epsilon > 0$ and*

$$\inf_{\Omega(a,b,t_1)} |tX_t| \ge \gamma > 0\ , \qquad \sup_{\Omega(a,b,t_1)} |tX_t| < 1-\gamma\ , \tag{11.3}$$

for some t_1. Then $\Omega(a,b,t_1)$ can be covered by a finite number N of domains of dependence on which x is $AVTD_\infty^{(P,Q)}$ in appropriate coordinates on hyperbolic space. (This implies smoothness of v and of φ_∞, as those properties are invariant under isometries of the hyperbolic space.) The number N depends only upon ϵ, C_0 and γ.

We will actually prove a slightly stronger statement:

LEMMA 11.5 *Under* (11.3)*, suppose that there exists a sequence t_i such that the function*

$$F(t_i) := \sup_{\theta\in[a+t_i,b-t_i]} \sum_{k=0}^{2} |t^{k+1}D_\theta^k X_\theta|(t_i,\theta) + \sum_{k=1}^{2} |t^{k+1}D_\theta^k X_t|(t_i,\theta) \tag{11.4}$$

goes to zero as t_i goes to zero. Then the conclusion of Lemma 11.4 holds, with the number N there depending upon γ and the sequences $\{t_i\}$, $\{F(t_i)\}$.

PROOF: We start by noting that under (11.2) we have $F(t_i) \le C|t_i|^\epsilon$ by (10.8), so that Lemma 11.4 does indeed follow from Lemma 11.5 by setting $t_i = 2^{-i}t_0$.

We wish to apply a result of Ringström [16, Theorem 9.1]. Set

$$\gamma = \frac{1}{8}\min(\inf_{\Omega(a,b,t_1)}|tX_t|, 1 - \sup_{\Omega(a,b,t_1)}|tX_t|) \ . \tag{11.5}$$

Let the constants ϵ_k, $k = 0,1,2$ in [16, Theorem 9.1] be equal to 10, decreasing $|t_1|$ if necessary we can suppose that $t_1 > -e^{-\tau_0}$, with τ_0 given by [16, Equation (9.1)]. Choose some $\psi \in (a,b)$, and define $\hat{C}^0_{t_1}(\psi)$ to be a triangle with the top vertex at $(0,\psi)$ and slopes $\pm 1/2$:

$$\hat{C}^0_{t_1}(\psi) := \{t \ge t_1 \ , \ \psi - 2|t| \le \theta \le \psi + 2|t|\} \ . \tag{11.6}$$

If $\psi = a$ we set

$$\hat{C}^0_{t_1}(\psi) := \{t \ge t_1 \ , \ \psi - |t| \le \theta \le \psi + 2|t|\} \ , \tag{11.7}$$

while for $\psi = b$ we set

$$\hat{C}^0_{t_1}(\psi) := \{t \ge t_1 \ , \ \psi - 2|t| \le \theta \le \psi + |t|\} \ . \tag{11.8}$$

For $0 < \lambda \le 1$ let $x^{(\psi,\lambda)}$ be defined as

$$\hat{C}^0_{t_1}(0) \ni (t,\theta) \to x^{(\psi,\lambda)}(t,\theta) := x(\lambda t, \psi + \lambda\theta) \ . \tag{11.9}$$

We have

$$X^{(\psi,\lambda)}_t(t,\theta) := \partial_t x^{(\psi,\lambda)}(t,\theta) = \lambda X_t(\lambda t, \psi + \lambda\theta) \ , \tag{11.10}$$

similarly

$$X^{(\psi,\lambda)}_\theta(t,\theta) := \partial_\theta x^{(\psi,\lambda)}(t,\theta) = \lambda X_\theta(\lambda t, \psi + \lambda\theta) \ , \tag{11.11}$$

with analogous formulae for higher derivatives. This, together with our hypothesis on the function F, shows that for $\lambda = \lambda_i = t_i/t_1$ we have for $\psi \in (a,b)$

$$\sup_{\theta\in[2t_1,-2t_1]}\sum_{k=0}^{2}|t^k D^k_\theta X^{(\psi,\lambda)}_\theta|(t_1,\theta) + \sum_{k=1}^{2}|t^k D^k_\theta X^{(\psi,\lambda)}_t|(t_1,\theta) \le F(\lambda_i t_1 = t_i)/|t_1| \to_{\lambda_i\to 0} 0 \ , \tag{11.12}$$

with similar results if $\psi = a$ or $\psi = b$. In particular at $t = t_1$ we have

$$\sup_{\theta\in[-2|t_1|,2|t_1|]} \sum_{k=0}^{2} |D_\theta^k X_\theta^{(\psi,\lambda)}|(t_1,\theta) + \sum_{k=1}^{2} |D_\theta^k X_t^{(\psi,\lambda)}|(t_1,\theta) \to_{\lambda\to 0} 0\ . \quad (11.13)$$

Now, it is easy to show, using (5.6), that

$$P_{t\theta}^2 + e^{2P} Q_{t\theta}^2 \le |D_\theta X_t|^2 + C|X_t|^2 |X_\theta|^2, \quad (11.14)$$

$$P_{\theta\theta}^2 + e^{2P} Q_{\theta\theta}^2 \le |D_\theta X_\theta|^2 + C|X_\theta|^4. \quad (11.15)$$

Similarly,

$$\begin{aligned} P_{t\theta\theta}^2 + e^{2P} Q_{t\theta\theta}^2 \le{}& |D_\theta^2 X_t|^2 + C(|X_t|^2(|D_\theta X_\theta|^2 + |X_\theta|^4 + |X_\theta|^2) \\ &+ |X_\theta|^2 |D_\theta X_t|^2), \quad (11.16) \\ P_{\theta\theta\theta}^2 + e^{2P} Q_{\theta\theta\theta}^2 \le{}& |D_\theta^2 X_\theta|^2 + C(|X_\theta|^2 |D_\theta X_t|^2 + |X_\theta|^6 + |X_\theta|^4). \quad (11.17) \end{aligned}$$

Equations (11.12)-(11.17) show that there exists λ_j such that Equations (6.2) and (9.3) with $k = 1, 2$, of [16] will be satisfied by the initial data for $x^{(\psi,\lambda)}$ for all $\lambda = \lambda_n \le \lambda_j$ with a multiplicative factor $1/(2C_1)$ at the right-hand sides there, with a constant C_1 to be made precise shortly. Decreasing λ_j if necessary Ringström's energy ϵ_2 corresponding to the current solution here will be smaller than $1/(2C_1)$. Again decreasing λ_j if necessary [16, Equation (9.4)] will hold with 2γ there replaced by 4γ. It remains to satisfy Ringström's equation (9.3) with $k = 0$. Recall that the group of isometries of the hyperbolic space $(\mathscr{H}_2, h)$ acts transitively on the unit tangent bundle of $\mathscr{H}_2$. This implies that there exists an isometry ψ_λ of $(\mathscr{H}_2, h)$ such that the map $\psi_\lambda \circ x^{(\psi,\lambda)}$, when written in the local coordinates (P, Q), will satisfy

$$P(t_2, \psi) = Q(t_2, \psi) = \partial_t Q(t_2, \psi) = 0\ , \quad \partial_t P(t_2, \psi) \ge 0.$$

Now, Ringström's norms are not invariant under isometries. However, the objects appearing in (11.12) and (11.13) are, and those have been used to control Ringström's norms, so that the inequalities which have already been fulfilled still hold (in any case we could decrease λ further to obtain the desired inequalities). It then follows by integration in θ, using the fact that the θ derivatives of P and Q are already known to be small, that there exists $d > 0$ such that Ringström's equation (9.3) with $k = 0$ and with a multiplicative factor $1/(2C_1)$ at the right-hand side there will hold on $[\psi - d + t_j, \psi + d - t_j]$. Decreasing d and λ_j if necessary, we can extend the initial data from that last interval to smooth periodic initial data, without increasing all the relevant quantities by more than a factor C_1. Applying Ringström's Theorem 9.1 the result follows for the map obtained by the evolution of the extended initial data. Uniqueness in domains of dependence establishes the claim on each $\Omega(\psi - d, \psi + d, t_j)$. Reverting to the original t_1, the result is obtained by covering $\Omega(a, b, t_1)$ by a finite number of sets $\Omega(\psi - d, \psi + d, t_1)$. $\square$

We note that an obvious modification of the argument just given establishes the following version of Ringström's result [16, Theorem 9.1]:

Proposition 11.6 *Let $a \le b$, $t_0 < 0$, $0 < \gamma < 1$, there exists $\epsilon(\gamma) > 0$ such that if*

$$\gamma \le t_0 P(t_0, \cdot) \le 1 - \gamma \ ,$$

$$\sum_{k=0}^{2} |t_0^k D_\theta^k X_\theta|(t_0, \cdot) + \sum_{k=1}^{2} |t_0^k D_\theta^k X_t|(t_0, \cdot) < \epsilon \ ,$$

on $[a - |t_0|, b + |t_0|]$, then the solution is $AVTD_\infty^{(P,Q)}$ in $\Omega(a, b, t_0)$, with velocity strictly positive and strictly smaller than one. $\square$

We return to the proof of point (ii) of Theorem 11.1. The reader will note that the above proof has been worded to leave room for perturbing the data at $t = t_1$, with t_1 as redefined in the paragraph following (11.5), while still satisfying Ringström's hypotheses; this is needed for the remainder of the argument.

We set now

$$\gamma = \frac{1}{8} \min\left(\frac{1}{3}, 1 - \sup_{\Omega(a,b,t_0)} |t\mathring{X}_t|\right) .$$

Since v is continuous, with $t\mathring{X}_t$ converging uniformly to v, the interval $[a, b]$ can be covered by a finite number of intervals $[a_i, b_i]$ on which either

$$\gamma \le \frac{1}{8} \inf_{\Omega(a_i,b_i,t_1)} |t\mathring{X}_t| \ ,$$

or on which

$$\sup_{\Omega(a_i,b_i,t_1)} |t\mathring{X}_t| < \frac{1}{12} \ .$$

Let us call the latter intervals of type II, and the former of type I. Each of the $[a_i, b_i]$'s of type I can further be chosen to coincide with one of the intervals $[\psi - d, \psi + d]$ of the proof of Lemma 11.4 such that $\mathring{x}$ is $\mathrm{AVTD}_{\infty}^{(P,Q)}$ on $\Omega(\psi - d, \psi + d, t_2)$, where t_2 is the time given in the proof of Lemma 11.4. A sufficiently small pertubation of $\mathring{x}$ at $t = t_2$ leads again to an $\mathrm{AVTD}_{\infty}^{(P,Q)}$ solution, thus satisfying (11.1)-(11.2) (with $\mathring{X}$ there replaced by X, with possibly different constants C and ϵ). The usual continuous dependence of solutions upon initial data on compact intervals of t shows that the same will remain true for sufficiently small perturbations of the initial data at $t = t_0$.

On each interval of type II we have at $t = t_2$

$$|t\mathring{X}_{\theta}| \leq Ct^{\epsilon} < \frac{1}{6} ,$$

making t_2 smaller if necessary, then any sufficiently small perturbation of the Cauchy data at $\{t_2\} \times [a_i + t_2, b_i - t_2]$ leads to a solution x such that

$$\sup_{\Omega(a_i,b_i,t_2)} |tX_t|^2 + |tX_{\theta}|^2 < 2(\frac{1}{12} + \frac{1}{6}) = \frac{1}{2},$$

where the factor 2 comes from Proposition 1.4. We can thus use Theorem 8.3 to conclude x will satisfy (11.1)-(11.2).

□

12 Existence and smoothness of v on an open dense set

The proof of Theorem 1.3 will run in parallel with that of the following, more precise, statement:

THEOREM 12.1 *Consider a solution x defined on $\Omega(a, b, t_0)$ and set*

$$n := \lfloor \sup |tX_t| \rfloor .$$

There exists an open dense set $\Omega \subset [a, b]$ such that for every $\psi \in \Omega$ there exist a neighborhood $\mathscr{O}_{\psi} \subset \Omega$ of ψ and an element G_{ψ} of the Geroch group, with the order of G_{ψ} less than or equal to n, such that $G_{\psi}x$ has on $\mathscr{O}_{\psi}$ a smooth velocity function $0 \leq v = |v_1| < 1$ and smooth position function Q_{∞}.

PROOF: We start with a covering argument: We choose arbitrarily two points $a \leq \theta_l < \theta_r \leq b$, we set $I_0 = [\theta_l, \theta_r]$ and we decompose $\Omega(\theta_l, \theta_r, -|I_0|/2)$ into a union of strips Ω_i, $\Omega(\theta_l, \theta_r, -|I_0|/2) = \cup_{i=1}^{\infty} \Omega_i$, where

$$\Omega_i = \{(t,x) \in \Omega(\theta_l, \theta_r, -|I_0|/2) \mid -\frac{|I_0|}{2^i} \leq t < -\frac{|I_0|}{2^{i+1}}\} .$$

Note that $|I_0|/2$ is the height of $\Omega(\theta_l, \theta_r, -|I_0|/2)$, and that the Ω_i's are pairwise disjoint. Let $t_i = -\frac{|I_0|}{2^i}$ and let B_i be the base line of Ω_i,

$$B_i = \{(t,x) \in \Omega_i \mid t = t_i \ , \ \theta_l - |t_i| \leq x \leq \theta_r + |t_i|\}, \quad i \geq 1 .$$

We denote by A_{i0} the left end point of B_i, and set $\theta_{l_i} = \theta_l - |t_i|$, $\theta_{r_i} = \theta_r + |t_i|$. We consider a partition $\mathscr{P}_i$ of B_i determined by the following sequence of points (see Figure 3)

$$\mathscr{P}_i = \{A_{i0}, \ldots, A_{ij}, \ldots, A_{i,m_i}\},$$

where $m_i = 2^i + 2$, $A_{ij} = (t_i, \theta_{l_i} + \frac{|I_0|}{2^i} j)$. Note that the length of each sub-interval of $\mathscr{P}_i$ is $\frac{|I_0|}{2^i}$.

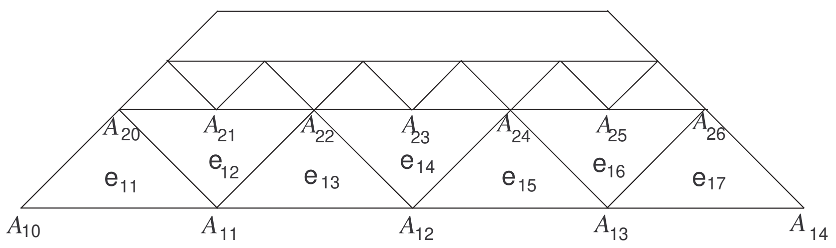

Figure 3: The points A_{ij} and the triangles e_{ij}.

Next, we decompose Ω_i into a union of triangles with 45 degrees slopes as follows: Let c_{ij} denote the triangle with vertices at the points $(A_{ij}, A_{i+1,2j}, A_{i,j+1})$ defined above, where $i \geq 1$, $j = 0, 1, \ldots, 2^i+1$. The set $\Omega_i - \bigcup_{j=0}^{2^i+1} c_{ij}$ is then the union of 'upside down' triangles with vertices $(A_{i+1,2j-2}, A_{ij}, A_{i+1,2j})$, where $i \geq 1$, $j = 1, \ldots, 2^i + 1$. We denote those last triangles by d_{ij}. It follows that

$$\Omega_i = (\cup_{j=0}^{2^i+1} c_{ij}) \bigcup (\cup_{j=1}^{2^i+1} d_{ij}) \ .$$

Finally we relabel the c_{ij}'s and the d_{ij}'s as e_{ij}:

$$\begin{aligned} e_{i,2j+1} &:= c_{ij},\ j = 0, \ldots, 2^i + 1 \\ e_{i,2j} &:= d_{ij},\ j = 1, \ldots, 2^i + 1 \end{aligned}$$

Lemma 12.2 *Let $\{\Omega_j\}_{j=0}^{\infty}$ be the decomposition of $\Omega(\theta_l, \theta_r, -|I_0|/2)$ described above. Let f be a nonnegative measurable function on $\Omega(\theta_l, \theta_r, -|I_0|/2)$ with*

$$\int_{\Omega} |t| f(t, \theta)\, d\theta dt < \infty \ .$$

Then for any $\epsilon > 0$ and $j_0 \in \mathbb{N}$ there exists $j \geq j_0$ such that Ω_j contains a set ω consisting of eight consecutive *triangles e_{ij} with*

$$\int_{\omega} f\, d\theta dt < \epsilon.$$

Proof: Let $\int_{\Omega} |t| f(t, \theta)\, d\theta dt = A$. Using the decomposition $\Omega(\theta_l, \theta_r, -|I_0|/2) = \cup_{i=1}^{\infty} \Omega_i$ we find

$$\begin{aligned} A &= \sum_{i=1}^{\infty} \int_{\Omega_i} |t| f(t, \theta)\, d\theta dt \\ &\geq |I_0| \sum_{i=0}^{\infty} \int_{\Omega_i} \frac{f(t, \theta)}{2^i}\, d\theta dt \ . \end{aligned}$$

Thus, since the last sum is finite, there exists $i(\epsilon)$ such that for all $i \geq i(\epsilon)$ we have

$$\frac{1}{2^{i+1}} \int_{\Omega_i} f(t, \theta)\, d\theta dt \leq \epsilon$$

for any given ϵ. Let us show that there exist j and at least $N = 8$ consecutive triangles starting at e_{ij} such that

$$\int_{\cup_{k=0}^{N} e_{i,j+k}} f(t, \theta)\, d\theta dt \leq \epsilon.$$

In fact we prove this assertion with any given $N \in \mathbb{N}$; we set

$$\begin{gathered} e_{i1} \cup e_{i2} \ldots \cup e_{iN} = c_1 \ , \\ \vdots \\ e_{i,j} \cup e_{i,j+1} \ldots \cup e_{i,j+N-1} = c_j \ , \\ \vdots \\ e_{i,m-N+1} \cup e_{i,m-N+2} \ldots \cup e_{im} = c_{m-N+1} \ , \end{gathered}$$

where $m = 2^{i+1} + 3$. Let

$$\int_{c_j} f = C_j \ , \quad \int_{e_{ij}} f = E_j \ ,$$

then

$$\begin{aligned} C_1 + \ldots + C_{m-N+1} &= E_1 + 2E_2 + \ldots + (N-1)E_{N-1} + N(E_N + \ldots \\ &\quad + E_{m-N+3}) + \ldots + 2E_{m-1} + E_m \\ &\leq N(E_1 + \ldots + E_m) = N \int_{\Omega_i} f(t,\theta)\, d\theta dt \leq N m \epsilon. \end{aligned}$$

Hence we obtain

$$\frac{C_1 + \ldots + C_{m-N+1}}{m-N+1} \leq \frac{N m \epsilon}{m-N+1} \leq N\epsilon,$$

which implies that there exists C_j such that $C_j \leq N\epsilon$. □

We apply Lemma 12.2 to the function

$$f := \sum_{k=0}^{\ell} |t^k D_\theta^k X_\theta|^2 + \sum_{k=1}^{\ell} |t^k D_\theta^k X_t|^2 \ ;$$

we are actually interested in $\ell = 4$, but the argument applies to any fixed $\ell \in \mathbb{N}$, $\ell \geq 2$. Point (iii) of Proposition 1.6 shows that f satisfies the hypotheses of Lemma 12.2, therefore there exists a sequence of domains $\omega_i \subset \Omega_i$ such that

$$\int_{\omega_i} f \to_{i\to\infty} 0 \ .$$

The base b_i of ω_i has length $4|t_i|$, so that it can be written as

$$b_i = \{t_i\} \times [\psi_i - 2|t_i|, \psi_i + 2|t_i|] \ ,$$

for some $\psi_i \in [\theta_l, \theta_r]$. We scale ω_i to a union of eight triangles with bottom edge lengths one, with the basis of the scaled set lying at $t=-1$, and (ψ_i, t_i) mapped to $(0,-1)$; we call $\tilde\omega$ the resulting set, and we note that the top of $\tilde\omega$ lies at $t=-1/2$. We have

$$\int_{\omega_i} f = \int_{\tilde\omega} \sum_{k=0}^{\ell} |t^k D_\theta^k X_\theta^{(\psi_i,t_i)}|^2 + \sum_{k=1}^{\ell} |t^k D_\theta^k X_t^{(\psi_i,t_i)}|^2 \;, \qquad (12.1)$$

with $x^{(\psi_i,t_i)}$, etc., defined in (11.9). There exists an isometry of the hyperbolic space into itself which maps $x^{(\psi_i,t_i)}(0,-1)$ to the origin in the (P,Q) coordinate system. We apply this isometry to $x^{(\psi_i,t_i)}$, and still use the same name for the resulting map. Since $|t| \in [1/2,1]$ on $\tilde\omega$, isometry-invariance of (12.1) gives

$$\|X_\theta^{(\psi_i,t_i)}\|_{H^\ell(\tilde\omega)} + \|D_\theta X_t^{(\psi_i,t_i)}\|_{H^{\ell-1}(\tilde\omega)} \to_{i\to\infty} 0 \;,$$

and the Sobolev inequality implies

$$\|X_\theta^{(\psi_i,t_i)}\|_{C^{\ell-2}(\tilde\omega)} + \|D_\theta X_t^{(\psi_i,t_i)}\|_{C^{\ell-3}(\tilde\omega)} \to_{i\to\infty} 0 \;.$$

Returning to the original ω_i one thus has

$$\sup_{\omega_i} \left(\sum_{k=0}^{\ell-2} |t^{k+1} D_\theta^k X_\theta| + \sum_{k=1}^{\ell-2} |t^{k+1} D_\theta^k X_t| \right) \to_{i\to\infty} 0 \;. \qquad (12.2)$$

Let $I_i = [\psi_i - 2|t_i|, \psi_i + 2|t_i|]$ and set

$$F_\ell(I_i) := \sup_{\theta \in [\psi_i - 2|t_i|, \psi_i + 2|t_i|]} \left(\sum_{k=0}^{\ell} |t^{k+1} D_\theta^k X_\theta| + \sum_{k=1}^{\ell} |t^{k+1} D_\theta^k X_t| \right) (t_i, \theta) \;.$$

Equation (12.2) shows that $F_2(I_i)$ approaches zero as t_i tends to zero. We choose i_0 large enough so that for all $i \ge i_0$ we have

$$F_2(I_i) < \frac{1}{100} \;.$$

Clearly the same bound will then also hold for $F_\ell(I_i)$ with $0 \le \ell \le 1$. Since the sequence $|tX_t|(t_i, \psi_i)$ is bounded, passing to a subsequence if necessary we can assume that there exists v_∞ such that $|tX_t|(t_i,\psi_i) \to v_\infty$. Suppose, first, that $v_\infty < 1/200$, then $|tX_t|(t_i,\psi_i) < 1/100$ for i large enough. By integration in θ we have for $\theta \in I_i$

$$\big| |tX_t|^2(t_i,\theta) - |tX_t|^2(t_i,\psi_i) \big| \le \pm 2 \int_{\psi_i}^{\theta} t^2 |h(X_t, D_\theta X_t)| d\theta \le 4 \sup |tX_t| F_1(I_i) \;, \qquad (12.3)$$

which goes to zero as i goes to infinity so that $|tX_t|(t_i,\theta) < 1/50$ for $\theta \in I_i$. Now $|tf_1| \le |tX_t| < 1/50$, similarly for tf_2, while $|tg_2| \le |tX_\theta| \le F_0(I_i) < 1/100$, which proves that Theorem 8.3 applies, and shows that for all $i \ge i_0$ (increasing i_0 if necessary) the solution satisfies a power law decay in each of the $\Omega(\psi_i - 2|t_i|, \psi_i + 2|t_i|, t_i)$. The conclusions of Theorem 11.1 apply to show that the solution is $\mathrm{AVTD}^{(P,Q)}_\infty$ on each of the $\Omega(\psi_i - 2|t_i|, \psi_i + 2|t_i|, t_i)$, except perhaps a) for points in $\partial\{\theta : v(\theta) = 0\}$ at which Q_∞ may have discontinuities and/or v_1 might fail to be smooth (even though it is continuous there), or b) for a set of discontinuities of v_1 introduced by applying back isometries to the isometry-transformed $\mathrm{AVTD}^{(P,Q)}_\infty$ solutions of Lemma 11.5. In any case Proposition 4.2 guarantees existence of an open dense subset of $[\theta_l, \theta_r]$ with $\mathrm{AVTD}^{(P,Q)}_\infty$ behavior there.

Suppose, next, that $1/200 \le v_\infty \le 1 - 1/200$. A scaling argument as in the proof of Lemma 11.5 shows that, after applying a suitable isometry, the solution is $\mathrm{AVTD}^{(P,Q)}_\infty$ in each of the $\Omega(\psi_i - 2|t_i|, \psi_i + 2|t_i|, t_i)$'s, for i large enough, and hence again in an open subset of $[\theta_l, \theta_r]$. Applying the isometry back to recover the original solution one obtains an open subset of $[\theta_l, \theta_r]$ with $\mathrm{AVTD}^{(P,Q)}_\infty$ behavior.

As the next possibility, consider the case in which $1 - 1/200 < v_\infty \le 1 + 1/200$. Applying an isometry ϕ_i we can assume that the map $\phi_i \circ x$, still denoted by x, satisfies $Q_t(t_i, \psi_i) = 0$ with $P_t(t_i, \psi_i)$ – positive, so that $1 - 1/100 < |t|P_t(t_i, \psi_i) < 1 + 1/100$. Note that F_ℓ is invariant under isometries, so that $F_2(I_i)$ remains unchanged. Now,

$$\partial_\theta f_a = \partial_\theta \left(h(X_t, e_a) \right) = h(D_\theta X_t, e_2) + h(X_t, D_\theta e_a) \;, \tag{12.4}$$

and by integration, making use of (8.45), one finds that

$$-1/100 < |t| f_2 = |t| e^P Q_t(t_i, \theta) < 1/100$$

on I_i for i large enough, similarly $1 - 1/50 < |t| f_1 = |t| P_t(t_i, \theta) < 1 + 1/50$. Applying a Gowdy-to-Ernst transformation (3.20) (compare (8.33)) we obtain on I_i

$$-1/50 < |t|\hat{P}_t = 1 - |t|P_t < 1/50 \;, \quad -1/100 < |t|\hat{g}_1 = -|t|g_1 < 1/100 \;,$$

as well as

$$-1/100 < |t|\hat{g}_2 = -|t|f_2 < 1/100 \;, \quad -1/100 < |t|\hat{f}_2 = -|t|g_2 < 1/100 \;.$$

This shows that the hypotheses of Theorem 8.3 are satisfied by the initial data for $\hat{x}$ on I_i, so that by Theorem 11.1 the map $\hat{x}$ will be $\mathrm{AVTD}^{(P,Q)}_\infty$ on an open dense subset of each of the $\Omega(\psi_i - 2|t_i|, \psi_i + 2|t_i|, t_i)$'s for i large enough. This gives an open subset of $[\theta_l, \theta_r]$ with AVTD$_\infty^{(P,Q)}$ behavior for $\hat{x}$. To analyse the behavior of x we need first to perform back the Gowdy-to-Ernst transformation. Since

$$P = -\hat{P} - \ln|t| ,$$

the existence and continuity properties of v are unchanged by the Gowdy-to-Ernst map. Next, we note that

$$|Q_t| = |te^{\hat{P}}\hat{g}_2| \le C|t|^{\epsilon-1} \tag{12.5}$$

for some $\epsilon > 0$, which by integration shows that the function Q_∞ of (3.6) exists and is a continuous function on each interval $[\psi_i - |t_i|, \psi_i + |t_i|]$ with i large enough. Hence the map (P, Q) belongs to the class $\mathscr{U}_1$ defined in Section 4. Now, the original map x is obtained from the one just described by composing with an isometry of the hyperbolic plane, and is thus again in the class $\mathscr{U}_1$ by Proposition 4.2. One then obtains an AVTD$_\infty^{(P,Q)}$ map in a neighborhood of the set obtained by removing from $[\psi_i - |t_i|, \psi_i + |t_i|]$ the set consisting of points where v has discontinuities, together with the boundary of the set where $v = 1$.

We continue by induction: suppose that we have already established the claim for $v_\infty \le k + 1/200$, and suppose that there exists $k \in \mathbb{N}$ such that $k + 1/200 \le v_\infty \le k + 1 - 1/200$. Applying an isometry ϕ_i to x, and still denoting by x the resulting map, we can assume that $f_2(t_i, \psi_i) = 0$, and that $k+1/50 < |t|P_t < k+1-1/50$ on I_i, while $|tX_\theta|+|tf_2| < 1/100$ there. Applying a Gowdy-to-Ernst transformation we obtain

$$-k + 1/50 < |t|\hat{P}_t < -(k-1) - 1/50 \ \text{ on } I_i, \text{ with } \ |t\hat{f}_2| < 1/100 \ . \tag{12.6}$$

It follows that $|t\hat{X}_t| < k$ on all I_i's for i large enough. We note that (12.4) with $a = 2$ shows, by integration, that on I_i we have

$$|t\hat{g}_2| = |tf_2| \le CF_1(I_i) \ , \quad \text{while } |t\hat{g}_1| = |tg_1| \le F_0(I_i) \ \text{ holds trivially.} \tag{12.7}$$

Next, by (5.7),

$$\begin{aligned} D_\theta \hat{X}_\theta &= (\partial_\theta \hat{g}_1 - \hat{g}_2^2)e_1 + (\partial_\theta \hat{g}_2 + \hat{g}_1\hat{g}_2)e_2 \\ &= (-\partial_\theta g_1 - f_2^2)e_1 + (-\partial_\theta f_2 + g_1 f_2)e_2 \ . \end{aligned} \tag{12.8}$$

Similarly,

$$\begin{aligned} D_\theta \hat{X}_t &= (\partial_\theta \hat{f}_1 - \hat{f}_2\hat{g}_2)e_1 + (\partial_\theta \hat{f}_2 + \hat{f}_1\hat{g}_2)e_2 \\ &= (-\partial_\theta f_1 - f_2 g_2)e_1 + (-\partial_\theta g_2 + f_1 f_2 - \frac{f_2}{|t|})e_2 \ . \end{aligned} \tag{12.9}$$

It is straightforward to check, using (12.7)-(12.9), that if one sets

$$\hat{F}_\ell(I_i) := \sup_{\theta\in[\psi_i-2|t_i|,\psi_i+2|t_i|]} \left(\sum_{k=0}^{\ell} |t^{k+1} D_\theta^k \hat{X}_\theta| + \sum_{k=1}^{\ell} |t^{k+1} D_\theta^k \hat{X}_t| \right)(t_i,\theta) ,$$

then $\hat{F}_1(I_i) \to 0$ as $i \to \infty$. A similar calculation shows that the same is true for $\hat{F}_2(I_i)$, and in fact for any higher order $\hat{F}_\ell$ if true for F_ℓ.

Returning to the proof of Theorem 12.1, it follows that the map $\hat{x}$ satisfies all the conditions needed to apply the result assumed to be true by the induction hypothesis. In particular the velocity function $\hat{v}$ and the position function $\hat{Q}_\infty$ are smooth on each interval $[\psi_i - |t_i|, \psi_i + |t_i|]$, with i large enough, except perhaps for a countable number of points. Performing back the Gowdy-to-Ernst transformation one obtains a map with a velocity function v which has the same properties. Equation (12.5) becomes

$$|Q_t| = |te^{\hat{P}}\hat{g}_2| \le Ce^{\hat{P}} \le C|t|^{k-1/50} , \tag{12.10}$$

which shows, as before, existence of a continuous function Q_∞ on each interval $[\psi_i - |t_i|, \psi_i + |t_i|]$, with i large enough. Proposition 4.2 implies again the $\mathrm{AVTD}_\infty^{(P,Q)}$ behavior in a neighborhood of an appropriate subset of $\{0\} \times [\psi_i - |t_i|, \psi_i + |t_i|]$.

The possibility $k+1-1/200 < v_\infty < k+1+1/200$ is handled similarly, and the induction step is completed.

Since we have a uniform bound on $|tX_t|$, the procedure stops in a finite number of steps.

We have thus shown that every subinterval $[\theta_l, \theta_r]$ of $[a, b]$ contains an open set so that the solution is $\mathrm{AVTD}_\infty^{(P,Q)}$ in a neighborhood thereof. The union of all such sets, as $[\theta_l, \theta_r]$ runs over all subintervals of $[a, b]$, provides the desired open dense set $\hat{\Omega}$ of Theorem 1.3. The set Ω of Theorem 12.1 can be taken to coincide with $\hat{\Omega}$. However, for the purpose of Proposition 12.3, one should keep in Ω those points at which the discontinuities in v arise from the action of the Geroch-group-inverse G^{-1} on Gx, when recovering x from Gx on $\Omega(\psi_i - 2|t_i|, \psi_i + 2|t_i|, t_i)$, by inverting the inductive method described above. □

The arguments given above together with Remark 1.2 can be used to obtain the following characterisation of points which are not in Ω:

PROPOSITION 12.3 *Let Ω be the largest open set for which the conclusions of Theorem 12.1 hold. Then $\psi \notin \Omega$ if and only if*

(i) *either there exists $\epsilon > 0$ such that for all t small enough*

$$\sup_{\theta\in[\psi-|t|,\psi+|t|]}\left(\sum_{k=0}^{1}|t^{k+1}D_\theta^k X_\theta| + |t^2 D_\theta X_t|\right)(t,\theta) > \epsilon\ ,$$

(ii) *or v exists and is continuous on an interval I containing ψ, with ψ belonging to the boundary of the set $\{v(\theta)=0\}$. Further*

$$\sum_{k=0}^{2}\sup_{\theta\in I}\left(|t^{k+1}D_\theta^k X_\theta| + |t^{k+1}D_\theta^k X_t|\right)(t,\theta)\to 0\ ,$$

but Q_∞ is discontinuous at ψ. □

The key open question is that of existence of solutions for which the set of points exhibiting the properties described in Proposition 12.3 is not empty. The behavior described in (i) above seems to be a much more serious problem than the one in (ii).

We are ready now to pass to the proof of the main result of our paper:

Proof of Theorem 1.1: Let $x=\mathring{x}$ and consider any point $\psi\in[a,b]$. The argument of the proof of Theorem 12.1 with $t_i=-2^{-i}$, $\psi_i=\psi$, shows the existence of a time $t_\psi<0$ and of an element G_ψ of the Geroch group such that $G_\psi x$ satisfies the hypotheses of Theorem 11.1 on $\Omega(\psi-|t_\psi|,\psi+|t_\psi|,t_\psi)$. (Note that the covering argument is not needed anymore in view of the hypothesis (1.6).) This proves (i) for $G_\psi x$, and what has been said concerning the action of the Geroch group for the solutions under consideration proves (i) and (ii) for x.

By compactness of $[a,b]$ a finite covering by intervals $(a_i,b_i):=(\psi_i-|t_{\psi_i}|,\psi+|t_{\psi_i}|)$ can be chosen, and (iii) readily follows. Point (iv) follows from the results in [4] (compare [6]). A small change of the initial data at $t=t_0$ will lead to a small change of initial data at $t=t_{\psi_i}$, hence to a small change of $G_{\psi_i}x(t_{\psi_i},\cdot)$ and its derivatives at t_{ψ_i}, and the claim about stability follows from Theorem 11.1. □

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