On fine differentiability properties of horizons and applications to Riemannian geometry

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Abstract

We study fine differentiability properties of horizons. We show that the set of end points of generators of a n-dimensional horizon $\mathcal{H}$ (which is included in a $(n+1)$-dimensional space-time M) has vanishing n-dimensional Hausdorff measure. This is proved by showing that the set of end points of generators at which the horizon is differentiable has the same property. For $1 \le k \le n+1$ we show (using deep results of Alberti) that the set of points where the convex hull of the set of generators leaving the horizon has dimension k is "almost a C^2 manifold of dimension $n+1-k$": it can be covered, up to a set of vanishing $(n+1-k)$-dimensional Hausdorff measure, by a countable number of C^2 manifolds. We use our Lorentzian

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geometry results to derive information about the fine differentiability properties of the distance function and the structure of cut loci in Riemannian geometry.

1 Introduction.

Horizons are amongst the most important objects that one encounters in causality theory: Cauchy horizons constitute boundaries beyond which predictability breaks down; event horizons are boundaries beyond which no return is possible. The key structural property of horizons is the existence of *generators*: recall that an embedded hypersurface $\mathcal{H} \subset M$ is said to be *future null geodesically ruled* if every point $p \in \mathcal{H}$ belongs to a future inextensible null geodesic $\Gamma \subset \mathcal{H}$; those geodesics are called generators of $\mathcal{H}$. One can then extract the essential properties of Cauchy horizons, or black hole event horizons, in the following definition: $\mathcal{H}$ is *a future horizon* if $\mathcal{H}$ is an *achronal, closed, future null geodesically ruled topological hypersurface.* It follows from the above definition (or from the properties of past Cauchy horizons, or from the properties of future event horizons) that the generators can have past endpoints on $\mathcal{H}$, but no future endpoints.

The set $\mathcal{H}_{\rm end}$ of end points of generators of $\mathcal{H}$ provides an important tool in the study of the structure of horizons; for simplicity we will refer to those points as *end points.* In particular one wants to know how "large" this set can be. One defines the multiplicity $N(p)$ of a point $p \in \mathcal{H}$ as the number of generators which pass through or exit $\mathcal{H}$ at p; it is well known that if $N(p) > 1$, then p is necessarily the endpoint of all relevant generators. The set of points with multiplicity $N(p) > 1$ determines the differentiability properties of horizons: as has been shown by Beem and Królak [2], horizons are non-differentiable precisely at this set. It is also well known that the set of points at which a horizon is non-differentiable has vanishing n-dimensional Hausdorff measure, and this gives one control over the size of the set of endpoints with multiplicity $N(p) > 1$. Thus, in order to control the dimension of $\mathcal{H}_{\rm end}$ it remains to estimate that of the set of endpoints with multiplicity $N(p) = 1$. Let us denote by $\mathcal{H}_{\rm diff}$ the set of points of $\mathcal{H}$ at which $\mathcal{H}$ is differentiable; what has been said shows that the set of endpoints with multiplicity $N(p) = 1$ coincides with the set $\mathcal{H}_{\rm end} \cap \mathcal{H}_{\rm diff}$. Królak and Beem [2] have displayed an example of a horizon with an end point with multiplicity one (at which $\mathcal{H}$ is, of course, differentiable), thus there exist horizons for which the set $\mathcal{H}_{\rm end} \cap \mathcal{H}_{\rm diff}$ is not empty. Our first main result is the following:

Theorem 1 *Let $\mathcal{H}$ be a future horizon in an $(n+1)$-dimensional spacetime (M, g). Then the set $\mathcal{H}_{\rm end} \cap \mathcal{H}_{\rm diff}$ has vanishing n-dimensional Hausdorff measure. Moreover, for any C^2 spacelike hypersurface $\mathscr{S}$ the set $\mathcal{H}_{\rm end} \cap \mathcal{H}_{\rm diff} \cap \mathscr{S}$ has vanishing $(n-1)$-dimensional Hausdorff measure.*

Remark 2 In fact we expect a stronger conclusion to hold, namely that the set $\mathcal{H}_{\rm end} \cap \mathcal{H}_{\rm diff}$ has vanishing $(n-1)$-dimensional Hausdorff measure in $\mathcal{H}$. The rationale for this is as follows: We first note that it is possible to weaken the regularity assumption on $\mathscr{S}$. For Theorem 1 to hold it is sufficient to require that $\mathscr{S}$ be a Lipschitz hypersurface that meets the horizon transversely at almost all points of the intersection. This point, taken in conjunction with the fact that the set $\mathcal{H}_{\rm end} \cap \mathcal{H}_{\rm diff}$ is necessarily acausal, suggests the possibility of constructing such a Lipschitz hypersurface (or perhaps a countable collection of such Lipschitz hypersurfaces) that contains $\mathcal{H}_{\rm end} \cap \mathcal{H}_{\rm diff}$, at least up to a set of vanishing $(n-1)$-dimensional Hausdorff measure. Theorem 1, under this weaker regularity assumption on $\mathscr{S}$ would then give the strengthened conclusion. Initial efforts to construct such a hypersurface have not been successful.

The set of points where the multiplicity $N(p)$ is large has a more precise structure. To describe this, equip M with an auxiliary complete Riemannian metric σ, and for each $p \in \mathcal{H}$ let $\mathcal{N}_p^+$ be the set of future pointing σ-unit vectors that are tangent to a generator of $\mathcal{H}$ at p. We call such vectors *semi-tangents* to $\mathcal{H}$. Then the number of vectors in $\mathcal{N}_p^+$ is just the multiplicity $N(p)$ of p. Define

$$\mathcal{C}_p := \text{convex cone generated by } \mathcal{N}_p^+ \ .$$

We can measure the size of the set of generators through p by $\dim(\mathcal{C}_p)$ (which is the dimension of the linear span of $\mathcal{C}_p$ in T_pM). This is a different measure than is $N(p)$; in particular this gives finer information when $N(p) = \infty$. We also set

$$\mathcal{H}[k] := \{p : \dim(\mathcal{C}_p) \geq k\}. \tag{1.1}$$

For $k = 1$, $\mathcal{H}[k] = \mathcal{H}$ as every point is on at least one generator. For $k = 2$, $\mathcal{H}[k]$ is the set of points of $\mathcal{H}$ that are on more than one generator. As $\dim(\mathcal{C}_p)$ is the dimension of the span of $\mathcal{N}_p^+$ and $\mathcal{N}_p^+$ contains $N(p)$ vectors, $\dim(\mathcal{C}_p) \leq N(p)$. This implies

$$\mathcal{H}[k] \subseteq \{p \in \mathcal{H} : N(p) \geq k\} \ .$$

Also, for $1 \leq k \leq 3$, any k distinct elements of $\mathcal{N}_p^+$ are linearly independent.[1] Therefore if $1 \leq k \leq 3$ and $N(p) \geq k$ then choosing k distinct, and thus linearly independent, elements of $\mathcal{N}_p^+$, shows that $\dim(\mathcal{C}_p) \geq k$. Whence

$$\mathcal{H}[k] = \{p \in \mathcal{H} : N(p) \geq k\} \qquad \text{for } 1 \leq k \leq 3 \ .$$

Our next main result, based on the deep results[2] in [1], is that the sets $\mathcal{H}[k]$ are "almost C^2 submanifolds of dimension $n+1-k$, up to singular sets of lower dimension".[3] To make this statement precise, let $\mathfrak{H}^m$ be the m-dimensional Hausdorff measure on M (defined with respect to some Riemannian metric σ on M). Recall, [1, Def. 1.1 p.19] a Borel set $\Sigma \subset M$ is a $(\mathfrak{H}^m, m)$ *rectifiable set of class* C^2 iff Σ can be covered, up to a set of vanishing $\mathfrak{H}^m$ measure, by

[1] The linear span of two future pointing σ-unit null vectors is a two dimensional timelike subspace and there are only two future pointing null rays in this subspace. So a third future pointing σ-unit vector can not be in the span of the first two.

[2] The reader is warned that the dimension index k, in $\mathcal{H}[k]$, is shifted by one, as compared to that used in Theorem 3 of [1]; compare the remarks following Definitions 1.5 and 1.7 in [1].

[3] We note a related result of [12], where it is shown that the set $\{N(p) = 2\}$ is, up to a lower dimensional set, a smooth submanifold of co-dimension two for the horizons considered there.

a countable collection of m-dimensional C^2 submanifolds of M. This definition is independent of the choice of the Riemannian metric σ. Following [1] we will shorten "($\mathfrak{H}^m, m$) rectifiable set of class C^2" to "C^2 rectifiable of dimension m".

Theorem 3 *For $1 \leq k \leq n+1$ the set $\mathcal{H}[k]$ is a C^2 rectifiable set of dimension $n+1-k$. Therefore $\mathcal{H}[k]$ has Hausdorff dimension $\leq n+1-k$.*

Using $k=1$, and that $\mathcal{H}[1] = \mathcal{H}$, this implies that horizons are C^2 rectifiable of dimension n. As they are also locally Lipschitz graphs they have the further property that $\mathfrak{H}^n(\mathcal{H} \cap K) < \infty$ for all compact sets $K \subseteq M$. When $k = n+1$ this implies that $\mathcal{H}[n+1]$ is a countable set (*cf.* Remark 7 below).

2 Proof of Theorem 1

We shall prove the second part of the theorem; the first part follows immediately from the second and the co-area formula. Let, then, $\mathscr{S}$ be as in Theorem 1; since the result is purely local, without loss of generality we may assume that $\mathscr{S}$ is the level set $\{t=1\}$ of a time function t, with range $\mathbb{R}$, the level sets of which are Cauchy surfaces. We use the constructions and notations of [4], with $\Sigma_1 = \mathscr{S}$ and $\Sigma_2 = \{t=2\}$. Let

$$\hat{S}_1 = \mathscr{S} \cap \mathcal{H}_{\mathrm{end}} \cap \mathcal{H}_{\mathrm{diff}},$$

and let A, ϕ be defined as at the beginning of the proof of Theorem 6.1 in [4]. Hence, A is the subset of $S_2 = \Sigma_2 \cap \mathcal{H}$ consisting of those points in S_2 that are met by the generators of $\mathcal{H}$ that meet $S_1 = \Sigma_1 \cap \mathcal{H}$, and $\phi : A \to S_1$ is the map that moves the points of A back along these generators to S_1. We can choose the auxiliary Riemannian metric σ on M so that dt has unit length with respect to this metric. Then A is an A_δ set as defined by Equation (6.6) of [4], with $\delta = 1$. We set

$$\hat{A} = \phi^{-1}(\hat{S}_1) \subset A \subset S_2;$$

thus, the points in $\hat{A}$ are precisely those points on $\Sigma_2 \cap \mathcal{H}$ the generators through which exit $\mathcal{H}$, when followed to the past, at the differentiable end points on $\mathscr{S}$.

For $i = 1, 2$, let $\mathfrak{H}^{n-1}_{h_i}$ denote the $(n-1)$-dimensional Hausdorff measure on Σ_i with respect to the distance function determined by the induced metric h_i on Σ_i. By a straightforward extension of the proof of [4, Proposition 6.14] one has for any $\mathfrak{H}^{n-1}_{h_2}$-measurable subset Ω of A,

$$\int_{S_1} N(p, S_2) \mathbf{1}_{\phi(\Omega)} d\mathfrak{H}^{n-1}_{h_1}(p) = \int_\Omega J(\phi)(q)\, d\mathfrak{H}^{n-1}_{h_2}(q), \tag{2.1}$$

where $\mathbf{1}_U$ denotes the characteristic function of the set U, and $J(\phi)$ is, in a suitably defined sense (*cf.* [4], Prop. 6.14), the Jacobian of the locally Lipschitz function ϕ.

In Proposition 4 below we show that there exists a $\mathfrak{H}^{n-1}_{h_2}$-negligible set $\hat{A}' \subset \hat{A}$ such that $J(\phi)(q) = 0$ for all $q \in \hat{A} \setminus \hat{A}'$. It then follows that,

$$\hat{A} \subset \Omega \equiv \{q \in A : J(\phi) = 0\} \cup \hat{A}'. \tag{2.2}$$

Ω, as defined above, is the union of a $\mathfrak{H}^{n-1}_{h_2}$-measurable set and a $\mathfrak{H}^{n-1}_{h_2}$-negligible set, and hence is itself $\mathfrak{H}^{n-1}_{h_2}$-measurable. Equation (2.1) then shows that $\phi(\Omega)$ is $\mathfrak{H}^{n-1}_{h_1}$-negligible. Now, since $\hat{S}_1 \subset \phi(\Omega)$, the result follows. $\square$

It thus remains to establish the following.

Proposition 4 *$J(\phi)(q) = 0$ for $\mathfrak{H}^{n-1}_{h_2}$-almost all $q \in \hat{A}$.*

PROOF: We use the definitions, constructions and notations of the proof of [4, Prop. 6.16]. Thus, let $\mathcal{U} \subset \Sigma_2$ be a coordinate neighborhood of the form $\mathcal{V} \times (a,b)$, with $\mathcal{V} \subset \mathbb{R}^{n-1}$ and $a, b, \in \mathbb{R}$, in which $\mathcal{U} \cap N$ is the graph of a $C^{1,1}$ function $g : \mathcal{V} \to \mathbb{R}$, and in which $\mathcal{H} \cap \mathcal{U}$ is the graph of a semi–convex function $f : \mathcal{V} \to \mathbb{R}$. Here $N = N_\delta$ is a locally $C^{1,1}$ hypersurface in Σ_2 into which A has been embedded. Let $\mathrm{pr}A$ denote the projection onto $\mathcal{V}$ of $A \cap \mathcal{U}$, thus $A \cap \mathcal{U}$ is the graph of g over $\mathrm{pr}A$.

Now, let $x_0 \in \hat{B} \cap \mathrm{pr}\hat{A}$, where $\hat{B}$ is the set of full measure in $\mathrm{pr}A$ constructed in the proof of [4, Prop. 6.16], and $\mathrm{pr}\hat{A}$ is the projection onto $\mathcal{V}$ of $\hat{A} \cap \mathcal{U}$. Since g is Lipschitz, the graph of g over $\hat{B} \cap \mathrm{pr}\hat{A}$ has full measure in $\hat{A}$. Let $q_0 = (x_0, f(x_0)) \in \hat{A}$ be the corresponding point on $\mathcal{H} \cap \Sigma_2$, thus the generator Γ of $\mathcal{H}$ passing through q_0 exits the horizon at a point $p \in \mathscr{S} \cap \mathcal{H}_{\rm diff}$. Let $\hat{\Gamma}$ be any null geodesic which extends Γ to the past, and let p_n be any sequence of points on $\hat{\Gamma}$ which are to the causal past of p and which approach p as n tends to infinity. Since $\hat{\Gamma}$ is a null geodesic which exits $\mathcal{H}$ at p, the p_n's lie to the timelike past $I^-(\mathcal{H})$ of $\mathcal{H}$. Thus the integral curve γ_n of $\frac{\partial}{\partial t}$ starting at p_n meets $\mathcal{H}$ at some point $r_n \in \gamma_n \cap \mathcal{H}$. One can then construct a causal curve from p_n to $\mathcal{H} \cap \Sigma_2$ by following γ_n from p_n to r_n, and any generator of $\mathcal{H}$ passing through r_n; as (M,g) is globally hyperbolic this generator will necessarily intersect Σ_2. It follows that there exists a timelike curve $\hat{\gamma}_n \subset J^-(\mathcal{H})$ from p_n to $\mathcal{H} \cap \Sigma_2$. By the compactness of the space of causal curves, passing to a subsequence if necessary, the $\hat{\gamma}_n$'s converge (in a well known sense) to a causal curve γ from p to a point $q \in \mathcal{H} \cap \Sigma_2$. The achronality of $\mathcal{H}$ shows that γ is a generator of $\mathcal{H}$ passing through p, hence $\gamma = \Gamma$ and $q = q_0$.

Now suppose that $J(\phi)(q_0) \neq 0$. By the construction of the set $\hat{B}$, there exists $g_j \in C^2(\mathcal{V})$, approximating g, such that S_j, the graph of g_j, is a C^2 hypersurface in Σ_2 which, in a well defined sense, makes second order contact with $\mathcal{H} \cap \Sigma_2$ at q_0. (More precisely, g_j and f, as well as their first derivatives, agree at q_0, and the second derivative of g_j agrees with the so-called second Alexandrov derivative of f at q_0.) Since S_j is tangent to $\mathcal{H} \cap \Sigma_2$ at q_0, the null geodesic $\hat{\Gamma}$ is normal to S_j at q_0. Let $\phi_j : S_j \to \mathscr{S}$ be the C^1 map which moves the points of S_j along the family of null geodesics normal to S_j which includes $\hat{\Gamma}$. Then we have $J(\phi_j(q_0)) = J(\phi)(q_0) \neq 0$, *cf.* [4, Equation (6.40)]. Define $g_{j,\varepsilon} \in C^2(\mathcal{V})$ by

$$g_{j,\varepsilon}(x) = g_j(x) + \varepsilon |x - x_0|^2 \,, \tag{2.3}$$

and let $S_{j,\varepsilon}$ be the graph of $g_{j,\varepsilon}$; for $\varepsilon > 0$, $S_{j,\varepsilon} \subset J^+(\mathcal{H})$ and $S_{j,\varepsilon} \backslash \{q_0\} \subset I^+(\mathcal{H})$. Note that as $S_{j,\varepsilon}$ is tangent to S_j at q_0, $\hat{\Gamma}$ is normal to $S_{j,\varepsilon}$ at q_0.

The fact that the Jacobian of ϕ_j is nonzero at q_0 implies that p is not a focal point to S_j along $\hat{\Gamma}$. Moreover, there can be no focal points to S_j along the segment of $\hat{\Gamma}$ to the future of p, *cf.* [4, Lemma 4.15]. It follows that by taking ε small enough and $\hat{\Gamma}$ short enough, there will be no focal points to $S_{j,\varepsilon}$ along $\hat{\Gamma}$. This implies by normal exponentiation that there exists an embedded C^2 null hypersurface $\mathcal{N}_{j,\varepsilon}$ which contains $\hat{\Gamma}$ and, by shrinking it if necessary, $S_{j,\varepsilon}$, as well, *cf.* [4, Prop. A3]. Moreover, there exists a neighborhood $\mathcal{O}$ of $\hat{\Gamma}$ in which $\mathcal{N}_{j,\varepsilon}$ is achronal: Indeed, since spacetime is time orientable, $\mathcal{N}_{j,\varepsilon}$ is a two-sided connected embedded hypersurface in M. As such $\mathcal{N}_{j,\varepsilon}$ admits a connected neighborhood $\mathcal{O}$ which is separated by $\mathcal{N}_{j,\varepsilon}$ ($\mathcal{N}_{j,\varepsilon} \subset \mathcal{O}$, and $\mathcal{O} \setminus \mathcal{N}_{j,\varepsilon}$ consists of two components). Then a future directed timelike curve joining points of $\mathcal{N}_{j,\varepsilon}$ would be a timelike curve from the future side of $\mathcal{N}_{j,\varepsilon}$ to the past side of $\mathcal{N}_{j,\varepsilon}$, which is impossible if the curve remains in $\mathcal{O}$. We conclude that $\mathcal{N}_{j,\varepsilon}$ is achronal in $\mathcal{O}$.

Consider now the timelike curves $\hat{\gamma}_n \subset J^-(\mathcal{H})$ constructed earlier in the proof; since the $\hat{\gamma}_n$'s converge to Γ there exists n_0 such that all the $\hat{\gamma}_n$'s are entirely contained in $\mathcal{O}$ for $n \geq n_0$. Moreover, by taking n_0 larger if necessary, it is clear that each such $\hat{\gamma}_n$ will meet the hypersurface P in M obtained by pushing $S_{j,\varepsilon}$ to the past along the integral curves of $-\frac{\partial}{\partial t}$. One can then construct a timelike curve from p_n to $S_{j,\varepsilon}$ contained in $\mathcal{O}$ by following $\hat{\gamma}_n$ from p_n to P, and then an integral curve of $\frac{\partial}{\partial t}$ to $S_{j,\varepsilon}$. This contradicts the achronality of $\mathcal{N}_{j,\varepsilon}$ in $\mathcal{O}$, and establishes Proposition 4. $\square$

3 Proof of Theorem 3

We start by showing that $\mathcal{H}$ has no worse regularity than being the boundary of a convex set:

Proposition 5 *For any point $p \in \mathcal{H}$ there is a coordinate system $x^1, \ldots, x^{n+1}$ defined on an open set $U \subseteq M$ so that $\mathcal{H} \cap U$ is given by the graph $x^{n+1} = h(x^1, \ldots, x^n)$ of a convex function h.*

PROOF: It is shown in [4, Theorem 2.2] that $\mathcal{H}$ is locally the graph of a semi-convex function. That is, there is a coordinate system $y^1, \ldots, y^{n+1}$ so that $U \cap \mathcal{H}$ is given by a graph $y^{n+1} = u(y^1, \cdots, y^n) + h(x^1, \cdots, x^n)$ where u is C^∞ and h is convex. Define new coordinates by $x^i = y^i$ for $i = 1, \ldots, n$ and $x^{n+1} = y^{n+1} - u(x^1, \ldots, x^n)$. In these coordinates $\mathcal{H}$ is given by $x^{n+1} = h(x^1, \ldots, x^n)$. $\square$

Recall that a convex body in a finite dimensional vector space has a well defined normal cone at each of its boundary points p. One definition of this tangent cone is the set of linear functionals on the vector space such that their restrictions to the body attain a maximum at the point p. The following is an adaptation of this definition to manifolds which allows us to define the tangent cones $\mathrm{Nor}_p(J^+(\mathcal{H}))$ to $J^+(\mathcal{H})$ at points $p \in \mathcal{H} = \partial J^+(\mathcal{H})$ in an invariant manner.

$$\begin{aligned}\mathrm{Nor}_p(J^+(\mathcal{H})) \quad := \quad & \{df(p) : f \in C^\infty(M, \mathbb{R}) \text{ and} \\ & f|_{J^+(\mathcal{H})} \text{ has a local max. at } p\} \ .\end{aligned}$$

The following is a special case of a main result[2] in the paper of Alberti [1, Thm 3 p. 18] adapted to our notation.

Proposition 6 (Alberti) *Let B be convex body in an $n+1$-dimensional vector space. For each $k = 1, 2, \dots, n+1$ let $\partial B[k]$ be the set of points $p \in \partial B$ so that $\dim \mathrm{Nor}_p(B) \geq k$. Then $\partial B[k]$ is a C^2 rectifiable set of dimension $n+1-k$. Therefore the Hausdorff dimension of $\partial B[k]$ is less then or equal to $n+1-k$.*$\square$

Remark 7 The top and bottom dimensional cases of this are worth remarking on. When $k = n+1$ this implies that $\partial B[n+1]$ is a C^2 rectifiable set of dimension 0. But then, [1, Theorem 3 and Definition 1.1], $\partial B[n+1]$ is a countable union of sets of finite $\mathfrak{H}^0$ measure. However the zero dimensional measure $\mathfrak{H}^0$ is just the counting measure, [9, p. 171], so that $\mathfrak{H}^0(A)$ is just the number of points in A. Therefore $\partial B[n+1]$ is countable as it is a countable union of finite sets. Consider, next, $k = 1$; as $\dim \mathrm{Nor}_p(J^+(\mathcal{H})) \geq 1$ for all $p \in \partial B$ we have $\partial B = \partial B[1]$ and therefore ∂B is a C^2 rectifiable set of dimension n. Because ∂B is also locally the graph of a Lipschitz function it has the further property that $\mathfrak{H}^n(K \cap \partial B) < \infty$ for all compact sets K. The corresponding fact is not true for $\partial B[k]$ when $2 \leq k \leq n-1$ (cf. [1, Thm. 2 p. 18]).

Theorem 3 follows immediately from this, Proposition 5, and

Lemma 8 *For each $p \in \mathcal{H}$ the normal cone $\mathrm{Nor}_p(J^+(\mathcal{H}))$ is given by*

$$\mathrm{Nor}_p(J^+(\mathcal{H})) = \{\langle v, \cdot \rangle : v \in \mathcal{C}_p\} \ .$$

PROOF: We choose a coordinate system $x^1, \dots, x^{n+1}$ on a open set U containing p as in Proposition 5 so that $\mathcal{H} \cap U$ is given by $x^{n+1} = h(x^1, \dots, x^n)$ where h is convex. We may assume that the point p has coordinates $(0, \dots, 0)$. We also assume that U is of the form $V \times (a, b)$ for V an open convex set in $= \mathbb{R}^n$ and that h takes values in the interval (a, b). Then h is locally Lipschitz and thus the Clarke differential $\partial h(0)$ exists and is a compact convex set of linear functionals on $\mathbb{R}^n$ ([7, pp. 27–28]). As h is convex $\partial h(0)$ is just the set of sub-differentials to h at 0 in the sense of convex analysis ([7, Prop. 2.2.7 p. 36]). It follows that for $q \in \mathcal{H} \cap U$ if we write $q = (x, h(x))$ with $x \in V$ that

$$\mathrm{Nor}_q(J^+(\mathcal{H})) = \{\lambda(\alpha - dx^{n+1}) : \lambda \geq 0, \alpha \in \partial h(x)\} \ .$$

There is another useful description of $\partial h(0)$. Let Ω_h be the set of points x in V where the classical derivative $dh(x)$ exists. As h is locally Lipschitz Ω_h has full measure in V. Let $\mathcal{L}_0$ be the set

$$\mathcal{L}_0 := \left\{ \lim_{\ell \to \infty} dh(x_\ell) : x_\ell \in \Omega_h,\ x_\ell \to 0, \text{ and } \lim_{\ell \to \infty} dh(x_\ell) \text{ exists.} \right\} \tag{3.1}$$

of limit points of sequences $\{dh(x_\ell)\}$ of sequences $\{x_\ell\} \subset \Omega_h$ with $x_\ell \to 0$. Then, [7, Thm 2.5.1 p. 63],

$$\partial h(0) = \text{convex hull of } \mathcal{L}_0 \ . \tag{3.2}$$

Letting, as in the introduction, $\mathcal{H}_{\text{diff}}$ be the set of points where $\mathcal{H}$ is differentiable, if $x \in V$ and $q = (x, h(x)) \in \mathcal{H}$, then $x \in \Omega_h$ if and only if $q \in \mathcal{H}_{\text{diff}}$. By the theorem of Beem and Królak [2] this is the case if and only if q is on exactly one generator of $\mathcal{H}$. If $q = (x, h(x)) \in \mathcal{H}_{\text{diff}}$ then let $v_q \in \mathcal{N}_q^+$ be the unique semi-tangent to $\mathcal{H}$ at q. Then at q the tangent plane to $\mathcal{H}$ can be defined either in terms of dh or in terms of v_q to be the set of vectors $X \in T_q(M)$ so that $(dh - dx^{n+1})(q)(X) = 0$ or $\langle v_q, X \rangle = 0$. Thus there is a positive scalar λ so that $(dh - dx^{n+1})(q) = \lambda \langle v_q, \cdot \rangle$. Therefore the normal cone at q is one dimensional and

$$\operatorname{Nor}_q(J^+(\mathcal{H})) = \{\lambda\langle v_q, \cdot\rangle : \lambda \geq 0\} = \{\lambda(dh - dx^{n+1}) : \lambda \geq 0\} .$$

It follows from this that if $\{x_\ell\} \subset \Omega_h$ and $q_\ell = (x_\ell, h(x_\ell))$ then $x_\ell \to 0$ if and only if $q_\ell \to p$ and $dh(x_\ell) \to \alpha$ if and only if $v_\ell \to v$ where $\langle v, \cdot \rangle = \alpha$. Unraveling all this and using (3.1) and (3.2) gives that in order to complete the proof it is enough to show

$$\mathcal{N}_p^+ = \left\{ \lim_{\ell\to\infty} v_{q_\ell} : q_\ell \in \mathcal{H}_{\text{diff}},\ q_\ell \to p, \text{ and } \lim_{\ell\to\infty} v_{q_\ell} \text{ exists.} \right\} . \tag{3.3}$$

Denote the right side of this equation by $\mathcal{T}_p$. Then, [4, Lemma 6.4], the set of semi-tangents $\mathcal{N}^+$ is a closed subset of $T(M)$ and therefore $\mathcal{T}_p \subseteq \mathcal{N}_p^+$. If $v \in \mathcal{N}_p^+$ then there is a generator $c\colon [0,\infty) \to M$ with $c(0) = p$, $c'(0) = v$ and parameterized so that it is unit speed with respect to the auxiliary Riemannian metric σ. For each positive integer ℓ, $c(1/\ell)$ is an interior point of the generator c and thus $c(1/\ell) \in \mathcal{H}_{\text{diff}}$. Then $v_{c(1/\ell)} = c'(1/\ell)$ and $\lim_{\ell\to\infty} v_{c(1/\ell)} = \lim_{\ell\to\infty} c'(1/\ell) = c'(0) = v$. Thus $v \in \mathcal{T}_p$ which yields $\mathcal{N}_p^+ \subseteq \mathcal{T}_p$. This shows (3.3) holds and completes the proof of the lemma and therefore of Theorem 3. □

4 Application to distance functions and cut loci in Riemannian Manifolds.

Let (S, h) be a connected Riemannian manifold which we do not assume to be complete. Let $C \subset S$ be a closed set. Then define the *distance function* $\rho_C\colon S \to [0,\infty)$ by

$$\rho_C(p) := \text{infimum of lengths of smooth cuves in } S \text{ connecting } p \text{ to } C .$$

This will be Lipschitz with Lipschitz constant one: $|\rho_C(p) - \rho_C(q)| \leq d(p,q)$ where $d(p,q)$ is the Riemannian distance between p and p.

We will see that regularity properties of ρ_C and the cut locus of C in S are closely related to the regularity properties of horizons, by looking at the graph of ρ_C (*cf.* Proposition 10). In this setting it is natural to consider the problem even when (S, h) is not complete. For example when S is the interior of a manifold P with boundary then ρ_C agrees with the distance from C defined by the infimum of the length of curves from p to C in P so that the results apply to that case as well. Also in the setting of Lorentzian geometry one can use the graphs of functions ρ_C to construct examples of horizons regardless of completeness of (S,h).

Let $I \subset \mathbb{R}$ be an interval (which may be open, closed, or half open). Then a *C-minimizing segment* on I is a unit speed geodesic $\gamma: I \to S$ so that

$$\rho_C(\gamma(s)) = s \quad \text{for all} \quad s \in I \ .$$

We emphasize that we do not assume that I contains 0. The Riemannian equivalent of the fact that horizons are null-geodesically-ruled is contained in the following proposition:

Proposition 9 *Every $p \in S \setminus C$ is on at least one C-minimizing segment.*

PROOF: Let U be a convex normal neighborhood of p disjoint from C. For $r > 0$ sufficiently small the distance sphere $S_r(p) = \{x \in M : d(p,x) = r\}$ is contained in U, is compact, and agrees with the geodesic sphere of radius r centered at p. Then, ρ_C restricted to $S_r(p)$ achieves a minimum at some point q, say. Let γ be the unique minimizing geodesic from q to p. From the choice of q on $S_r(p)$, and simple distance function considerations, one has for each x on γ,

$$d(C,q) + d(q,x) = d(C,x) \ , \tag{4.1}$$

where $d(C,x) = \rho_C(x)$. Since γ is minimizing on each segment, Equation (4.1) implies that γ, when suitably parameterized, is a C-minimizing segment. □

Each C-minimizing segment $\gamma: I \to S$ is contained in a maximal (with respect to the size of the interval of definition) C-minimizing segment and from now on we assume that all C-minimizing segments are defined on their maximal domain. We say that a C-minimizing segment $\gamma: I \to S$ has a *cut point* iff its interval of definition is of the form $[a,b]$ or $(a,b]$ with $b < \infty$, in which case $p = \gamma(b)$ is defined to be the cut point. A C-minimizing segment can fail to have a cut point either because its domain is unbounded, *i.e.* of the form $[a,\infty)$ or (a,∞), or because the domain is bounded, say (a,b) but the limit $\lim_{t\uparrow b}\gamma(t)$ does not exist in S. The later condition can not arise when S is complete. When S is complete the domains of C-minimizing segments are all of the form $[0,b]$ or $[0,\infty)$.

The collection of all cut points is the *cut locus* of C in S and denoted by Cut_C. The cut locus Cut_C is a subset of $S \setminus C$, and the definition here of Cut_C agrees with the usual definition when S is complete. For any $p \in S \setminus C$ let $N_C(p)$ be the number of C-minimizing segments on which p lies. Then for all $p \in S \setminus C$ we have $N_C(p) \geq 1$ and it is known that if $N_C(p) \geq 2$ then $p \in \mathrm{Cut}_C$. For $p \in S \setminus C$ let

$$\mathcal{M}_p := \left\{ -\gamma'(\rho_C(p)) : \gamma \text{ is a } C\text{-minimizing segment and } \gamma(\rho_C(p)) = p \right\}$$

be the set of the unit vectors at p which are tangent to C-minimizing segments and which point toward C. Then the number of vectors in $\mathcal{M}_p$ is just $N_C(p)$.

Let $M = S \times (-\infty, 0)$ and give M the Lorentzian metric $g = h - dt^2$. Let $\mathcal{H}$ be the graph of $-\rho_C$ in M, that is

$$\mathcal{H} := \{(x, -\rho_C(x)) : x \in S \setminus C\} \ .$$

We leave the proof of the following to the reader.

Proposition 10 *The set $\mathcal{H}$ is a future horizon in M. The null generators of $\mathcal{H}$ are (up to reparameterization) the curves $s \mapsto (\gamma(s), -\rho_C(\gamma(s)))$, where γ is C-minimizing segment of S and $N(p, -\rho_C(p)) = N_C(p)$. The set $\mathcal{H}_{\mathrm{end}}$ coincides with the set $\{(p, -\rho_C(p)) : p \in \mathrm{Cut}_C\}$.* □

The theorem of Beem and Królak [2] that a point p of a horizon is differentiable if and only if $N(p) = 1$ implies the Riemannian result:

Proposition 11 *With notation as above, the point $p \in S \setminus C$ is a differentiable point of the distance function ρ_C if and only if $N_C(p) = 1$.* □

Remark 12 While to best of our knowledge this result has not appeared in the literature on the regularity of Riemannian distance functions, it would be surprising if it were not known, at least in the case of of complete manifolds, to experts. It can also be deduced from general facts about the Clarke differential in non-smooth analysis. Explicitly, it follows from [6, Thm 2.1 (4) p. 251], a result about the generalized gradients of functions that are pointwise minimums of families of smooth functions with appropriate Hessian bounds, that the Clarke differential of ρ_C at $x \in N \setminus C$ is

$$\partial \rho_C(x) = \text{convex hull of } \{-\langle u, \cdot \rangle : u \in \mathcal{M}_x\}$$

(When S is Euclidean space this is [10, Lemma 4.2 pp. 1037–1038]. The extension to complete Riemannian manifolds is not hard.) The function ρ_C is semi-convex and a semi-convex function is differentiable at x if and only its Clarke differential at x is a singleton[4]. Therefore ρ_C is differentiable at x if and only if $N_C(x) = 1$. It is also possible to carry out a proof of the Beem and Królak result along these lines.

In [5] horizons in Lorentzian manifolds are constructed that are non-differentiable on a dense set. Another family of such examples, possessing genericity properties, is given in [3]. When translated into Riemannian terms the examples of [5] imply:

Proposition 13 *There exists a closed Lipschitz curve $C \subset \mathbb{R}^2$ such that its distance function ρ_C is non-differentiable on a dense subset of the set $\rho_C \leq 1$ of the unbounded component of $\mathbb{R}^2 \setminus C$.* □

[4]The general semi-convex case reduces to the case of convex functions. For a convex function the Clarke differential is the sub-differential in the sense of convex analysis [7, Prop 2.2.7 p. 36] and a convex function is differentiable at point if and only if its sub-differential is a singleton.

In both [3, 5] it is shown how to obtain higher dimensional examples.

Theorem 1 implies

Proposition 14 *If C is a closed set in an n-dimensional Riemannian manifold (S,h), then $\mathfrak{H}^n_H(\mathrm{Cut}_C)=0$.* $\square$

When C is a smooth submanifold of S (for example when C is a point) and the manifold (S,h) is complete then a recent result of Itoh and Tanaka [14, Thm. B p. 22] implies that Cut_C has Hausdorff dimension at most $n-1$. (When S is two dimensional and C is a point this had been done earlier by Hebda [11] and Itoh [13].) However for arbitrary closed sets the question of the Hausdorff dimension of Cut_C is open. In particular it is not known if there is a closed subset C of the Euclidean plane $\mathbb{R}^2$, with its usual metric, so that the Hausdorff dimension of Cut_C is 2. (Proposition 14 implies $\mathfrak{H}^2(\mathrm{Cut}_C)=0$, but this does not rule out the possibility that the Hausdorff dimension is 2.)

If $M=S\times(-\infty,0)$ is given as its auxiliary Riemannian metric $\sigma=h+dt^2$, then the set of semi-tangents to $\mathcal{H}$ at $p=(x,-\rho_C(x))$ is $\mathcal{N}^+_p=\{(2^{-1/2}u,2^{-1/2}\partial/\partial t):u\in\mathcal{M}_x\}$. Set

$$\mathrm{Cut}_C[k]:=\{x\in S\setminus C:\dim\mathrm{Span}\{(u,\partial/\partial t):u\in\mathcal{M}_x\}\geq k\}\ .$$

As in the introduction $\{x\in S\setminus C:N_C(p)\geq k\}\subseteq\mathrm{Cut}_C[k]$ and equality holds if $1\leq k\leq 3$. Note for $k\geq 2$ that $\mathrm{Cut}_C[k]\subset\mathrm{Cut}_C$, while for $k=1$ we have $\mathrm{Cut}_C[1]=S\setminus C$. The set $\{x\in N\setminus C:N_C(x)\geq 2\}=\mathrm{Cut}_C[2]$ is the *strict cut locus* and for some special choices of C, for example a point or a submanifold, its structure has been studied by several authors (*cf.* [11, 14] and the references therein). Using that the epi-graph of a distance function has locally positive reach it follows from results of Federer [8, Remark 4.15 (3) p. 447] that each set $\mathrm{Cut}_C[k]$ is countable rectifiable of dimension $n+1-k$. Theorem 3 allows us to refine this to C^2 rectifiablity.

Proposition 15 *For $1\leq k\leq n+1$ the set $\mathrm{Cut}_C[k]$ is a C^2 rectifiable set of dimension $n+1-k$. Therefore $\mathrm{Cut}_C[k]$ has Hausdorff dimension $\leq n+1-k$.* $\square$

Remark 16 By a theorem of Rademacher the set of non-differentiable points of a Lipschitz function on an n-dimensional manifold has $\mathfrak{H}^n$ measure zero. In the case that the function is a distance function, ρ_C, this can be improved. By Proposition 11 the set $\mathrm{Cut}_C[2]$ is exactly the set of points $p\in S\setminus C$ where the distance function ρ_C is not differentiable. Thus, by Proposition 15, the set of non-differentiable points of ρ_C is a C^2 rectifiable set of dimension $n-1$ and therefore has Hausdorff dimension at most $n-1$.

For each $r>0$ let $\tau_r(C):=\{p\in S:\rho_C(p)=r\}$ be the tube of radius r about C. In general these tubes can have singularities and need not be topological hypersurfaces in S. We now look at what the Riemannian versions of [4, Sec. 5] have to say about the regularity of $\tau_r(C)$ and ρ_C. By Proposition 5 the function ρ_C is semi-convex on $S\setminus C$. Therefore by Alexandrov's theorem for almost all

$p \in S \setminus C$ the function ρ_C has second Alexandrov derivatives. This means that in local coordinates $x^1, \dots, x^n$ centered at p the function ρ_C has a second order Taylor expansion

$$\rho_C = \rho_C(p) + d\rho_C(p)x + \frac{1}{2}D^2\rho_C(p)(x,x) + o(|x|^2)$$

where $x = (x^1, \dots, x^n)$, $d\rho_C(p)$ is a covector at p and $D^2\rho_C(p) \colon T_p(S) \times T_p(S) \to \mathbb{R}$ is a symmetric bilinear form. Denote by $S_{\mathcal{A}l}$ the set of Alexandrov points of ρ_C. At points $p \in S_{\mathcal{A}l}$ the function ρ_C is differentiable and therefore the discussion above yields that $N_C(p) = 1$ and that $d\rho_C(p) = -\langle u, \cdot \rangle$ where $u \in \mathcal{M}_p$. If $p \in S_{\mathcal{A}l}$ and $r = \rho_C(p)$ then the level set $\tau_r(C)$ has a well defined tangent space $T_p(\tau_r(C)) := \{X \in T_pS : d\rho_C(p)X = 0\}$ and a well defined second fundamental form $I\!I$ at p given by

$$I\!I_p(X,Y) = -D^2\rho_C(p)(X,Y) \quad \text{for} \quad X, Y \in T_p(\tau_r(C)) \ .$$

Let $A_p \colon T_p(\tau_r(C)) \to T_p(\tau_r(C))$ be the corresponding Weingarten map defined by

$$\langle A_p X, Y \rangle := I\!I_p(X,Y) \quad \text{for} \quad X, Y \in T_p(\tau_r(C)) \ .$$

That is A_p is the tensor of type $(1,1)$ corresponding to the tensor $I\!I$ of type $(0,2)$ and A_p is a self-adjoint linear map. With the choice of signs here, when $S = \mathbb{R}^n$ and C is the origin (so that $\tau_r(C)$ is the sphere of radius r) for $p \in \tau_r(C)$ we have $A_p = -\frac{1}{r}I$. Let the Riemannian connection on S be denoted by ∇ and let R be the curvature tensor of S with signs chosen so that $R(X,Y)Z = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z$. For $U \in T_pM$ define a linear map $R_U \colon T_pS \to T_pS$ be $R_U X = R(X,U)U$. The following summarizes the Riemannian versions of Proposition 3.1, Proposition 3.5, Theorem 5.1 and Theorem 5.6 of [4].

Theorem 17 *Let $\gamma \colon I \to S$ by a C-miminizing segment with $\gamma(0) \in C$. If $\gamma(t_0) \in S_{\mathcal{A}l}$ for some $t_0 \in I$, then $\gamma(t) \in S_{\mathcal{A}l}$ for all $t \in I_0 := \{t_0\} \cup (I \cap [t_0, \infty))^\circ$ (where J° is the interior of $J \subset \mathbb{R}$). The Weingarten maps $A_{\gamma(t)}$ of the tubes $\tau_{\rho_C(\gamma(t))}(C)$ vary smoothly on the interval I_0 and satisfy the usual Riccati equation for parallel hypersurfaces:*

$$\nabla_{\frac{d}{dt}} A_{\gamma(t)} = A^2_{\gamma(t)} + R_{\gamma'(t)} \ .$$

Moreover for all $r \in (0,\infty)$ the tube $\tau_r(C)$ has locally finite $\mathfrak{H}^{n-1}$-dimensional measure and for almost all $r \in (0,\infty)$ there is a set $P \subset \tau_r(C)$ with $\mathfrak{H}^{n-1}(P) = 0$ so that every C-miminizing segment $\gamma \colon I \to S$ which meets $\tau_r(C) \setminus P$ will have $\gamma(t) \in S_{\mathcal{A}l}$ for $t \in I^\circ$. □

In the terminology of [4] "Alexandrov points propagate to the past along generators" of horizons. In the Riemannian setting Alexandrov points of a distance function ρ_C propagate away from C along C-miminizing segments. The last sentence of Theorem 17 implies loosely that almost every C-miminizing segment is an *Alexandrov segment* in the sense that all of its points other than its endpoints are in $S_{\mathcal{A}l}$.

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