Remarks on rigidity of the de Sitter metric

Piotr T. Chruściel*

Abstract

We discuss various proofs of the theorem, that the de Sitter metric is the only non-singular static metric with positive cosmological constant $\Lambda = n(n-1)/2$, $n = \dim \mathscr{M} - 1$, such that the metric induced on the set where the Killing vector vanishes is the canonical metric on S^{n-1}.

AMS 83C57; PACS 04.70.Bw

1 Introduction

The understanding of any dynamical systems requires knowledge of its time-independent solutions. In general relativity the *non-singular static* metrics with cosmological constant $\Lambda = 0$ are completely understood [1, 13] (the pioneering results going back to [9, 21, 23]), and there has been progress recently in the understanding of such solutions with negative cosmological constant [2,3,16,26] (the pioneering result being [7]). For $\Lambda > 0$ the expectation seems to be the following:

CONJECTURE 1.1 *Let $(\mathscr{M}, g)$ be a four dimensional vacuum space-time with $\Lambda > 0$, suppose that there exists on $\mathscr{M}$ a Killing vector field X with complete orbits and vanishing twist, and assume that*

$$\mathscr{M}^- := \{g(X,X) < 0\} \neq \emptyset \; . \tag{1.1}$$

If $\mathscr{M}$ contains a compact spacelike hypersurface $\mathscr{S}$, then the metric g restricted to $\mathscr{M}^-$ is locally isometric to the Kottler metric.

(Recall that the twist form ω is defined as $*(X^\flat \wedge dX^\flat)$, where $X^\flat = g(X, \cdot)$.)

Candidates for a higher dimensional analogue of Conjecture 1.1 include the *generalised Kottler metrics*,

$$g = -V dt^2 + V^{-1} dr^2 + r^2 \tilde{g} \; , \tag{1.2}$$

where $\tilde{g}$ is an Einstein metric on a $(n-1)$-dimensional manifold N, with

$$R(\tilde{g})_{ij} = (n-1)(n-2)k \; , \quad k \in \{0, \pm 1\} \; , \tag{1.3}$$

*Département de Mathématiques, Faculté des Sciences, Parc de Grandmont, F37200 Tours, France. Partially supported by a Polish Research Committee grant 2 P03B 073 24; email `piotr@gargan.math.univ-tours.fr`, URL `www.phys.univ-tours.fr/~piotr`

while

$$V = -\frac{2\Lambda}{n(n-1)}r^2 + k - \frac{2m}{r^{n-2}} \ . \tag{1.4}$$

The metrics g solve the Einstein vacuum equations with cosmological constant Λ,

$$R(g)_{ij} = \frac{2\Lambda}{n-2} g_{ij} \ , \tag{1.5}$$

cf., e.g., [5, 10]. It appears, however, that for $n = \dim \mathscr{M} - 1 \geq 4$ there exist space-times as in Corollary 1.1, which are not in the generalised Kottler class [17].

One could likewise attempt to formulate conjectures where the isometry is global on $\mathscr{M}^-$, or global on $\mathscr{M}$, though the main point of interest seems to be the local statement anyway.

Some very partial results in the literature lending support to Conjecture 1.1 are the following: If one further assumes that there are no degenerate Killing horizons in $\mathscr{M}$, and that the orbit-space metric on $\mathscr{S}$ is conformally flat then the result, on every component of $\mathscr{M}^-$ that intersects $\mathscr{S}$, follows from the analysis of Lafontaine [22]. In [15] (compare [6]) a uniqueness result is proved under the assumption of existence of a smooth conformal completion at future timelike infinity; further, no hypotheses about the twist are made.

To put the results below in proper context, let us start by recalling the Vishveshwara-Carter lemma [11, 31]: under the hypotheses of Conjecture 1.1, the set

$$\partial \mathscr{M}^- \setminus \{X = 0\}$$

coincides with the *Killing horizon* $\mathcal{H}$, which is a union of smooth null hypersurfaces, with X being tangent to the null geodesics threading $\mathcal{H}$. A component of $\mathcal{H}$ is called *non-degenerate* if $g(X, X)$ has non-zero gradient at some point of $\mathcal{H}$, it is called *degenerate* otherwise.

Consider an $(n + 1)$-dimensional space-time containing a *non-degenerate* Killing horizon $\mathcal{H}$ associated with a Killing vector field X, since X changes type across $\mathcal{H}$ the region $\mathscr{M}^-$ of (1.1) is not empty. Let $\mathscr{S}$ be a compact spacelike hypersurface in $\mathscr{M}$, set

$$M = \mathscr{S} \cap \mathscr{M}^- \ ,$$

let $\varphi : M \to \mathbb{R}^+$ be defined as

$$\varphi = \sqrt{-g(X, X)} \ ,$$

and let h be the *orbit space metric*[1] on M,

$$TM \ni Y, Z \qquad h(Y, Z) = g(Y, Z) + \varphi^{-2} g(X, Y) g(X, Z) \ .$$

Consider a non-empty connected component $\mathscr{H}_0$ of the set

$$\mathscr{H} := \overline{\mathcal{H}} \cap \mathscr{S} \subset \{p \in \mathscr{S} : g(X, X)(p) = 0\} \ .$$

[1]The metric h coincides obviously with the metric induced on M by g when M is orthogonal to X, but we are not making this hypothesis.

If $\mathscr{H}_0$ is associated with a *non-degenerate* component of the closure $\overline{\mathcal{H}}$ of the horizon $\mathcal{H}$, then it is well known (*cf., e.g.*, [13]) that the metric h smoothly extends to a smooth Riemannian metric[2] $\gamma_{\mathscr{H}_0}$ on $\mathscr{H}_0$.

By a constant rescaling of the metric we can without loss of generality assume that

$$\Lambda = n(n-1)/2 \ , \quad \text{where } n := \dim \mathscr{M} - 1 \ . \tag{1.6}$$

In our arguments below we shall make use of the positive energy theorem for asymptotically flat space-times. This theorem has been established under the hypothesis that $3 \leq n \leq 7$ in [27], or under the hypothesis that the manifold is spin in [33]. A proof in all dimensions, without the spin restriction, has been announced [12], but the details have not been made available so far. We shall say that a manifold (M, g) is of *positive energy type* if the positive energy theorem, in a version which allows boundaries with non-negative outer mean curvature, can be applied to an asymptotically flat extension of (M, g). From what has been said it follows that every Riemannian manifold with boundary which is either spin and complete (compare [4]), or is compact and has dimension $3 \leq n \leq 7$, is of positive energy type. (If there are no asymptotically flat extensions of (M, g) then the condition is trivially satisfied.) It is expected that positivity holds for all complete manifolds admitting asymptotically flat extensions, in all dimensions.

Our main result here is the following result related to Conjecture 1.1, without the restriction $n = \dim M = 3$ made there:

THEOREM 1.2 *Let $(\mathscr{M}, g)$ be an $(n+1)$-dimensional vacuum space-time with a hypersurface-orthogonal Killing vector field X, $n \geq 3$, with cosmological constant Λ as in* (1.6)*, suppose that $\mathscr{M}$ contains a compact spacelike hypersurface $\mathscr{S}$. Assume that the Killing horizon $\mathcal{H}$ associated with X has only non-degenerate components, and that for some component $\mathcal{H}_0$ it holds that*

$$(\mathscr{H}_0, \gamma_{\mathscr{H}_0}) \ \text{ is isometric to a sphere } S^{n-1} \text{ with the canonical metric} \ . \tag{1.7}$$

If the connected component M_0 of M associated with $\mathscr{H}_0$ is of positive energy type, then M_0 is diffeomorphic to an open ball in Euclidean space, and there exists a neighborhood of M_0 in $\mathscr{M}$ which is isometrically diffeomorphic to an open subset of de Sitter space-time.

Further, each connected component of $\mathscr{M}^-$ intersecting M_0 which does not contain closed timelike curves is isometrically diffeomorphic to $\mathbb{R} \times B^n(1)$ with the de Sitter metric there.

As already pointed out, the metric $\gamma_{\mathscr{H}}$ coincides with the metric induced by g on (the relevant component of) the set $\{p \in \mathscr{M} : X(p) = 0\}$, whenever this set exists and is compact.

[2]The set $\mathscr{H}$ can be thought of as the "bifurcation surface of the Killing horizon $\mathcal{H}$"; in the familiar Schwarzschild example this would be the sphere at the intersection of the past and the future event horizons, which coincides with the set where the Killing vector vanishes, compare [8]. We emphasise, however, that we do not need the hypothesis of existence of such a surface in $\mathscr{M}$ for our analysis here. If such a surface exists, then $\gamma_{\mathscr{H}_0}$ coincides with the metric induced on $\mathscr{H}$ by g.

In Section 2 below we show that the hypothesis of non-degeneracy of $\mathcal{H}$ is not needed when $\mathcal{H}$ is connected. Thus, a corollary of Theorem 1.2 is that if the set $\{p \in \mathscr{M} : X(p) = 0\}$ is diffeomorphic to S^n, and if the metric induced by g there is the canonical metric on S^n, then g is, at least locally on $\mathscr{M}^-$, the $(n+1)$-dimensional de Sitter metric.

The question of *staticity* of solutions is addressed in Section 3 below. Using Theorem 3.1 there, the hypothesis of staticity in Theorem 1.2 can be replaced by the assumption that the boundary of $\mathscr{M}^-$ is a Killing horizon. However, it is not clear how to justify that $\partial\mathscr{M}^-$ should be a Killing horizon when staticity is not assumed. It is conceivable that Sudarsky-Wald type arguments [30] could be used here, but we have not attempted to analyse that.

A version of Theorem 1.2 has been previously established in dimension $n = 4$ in [7], under the restrictions that $\mathscr{S}$ is orthogonal to X, and assuming a single non-degenerate horizon. In [7] two proofs were given, one using techniques specific to that dimension. We review the other proof from [7] in Section 6 below, showing in particular that it generalises to higher dimensions.

We note that if $(\mathscr{M}, g)$ is a space-periodic identification of the Gibbons-Hawking extension [18] (compare [32]) of a Kottler metric with $m > 0$, with the normalisation (1.6), then $\mathscr{H}$ is diffeomorphic to several copies of S^n, with the metric $\gamma_{\mathscr{H}}$ being proportional, but not equal to, the canonical metric on each S^n.

The geometry of the Kottler metrics shows that a necessary condition for Conjecture 1.1 to be true is that $\gamma_{\mathscr{H}}$ has constant scalar curvature under the hypotheses made, and a proof of such a statement could be a useful step towards the proof of the whole conjecture. In Section 5 we relate this condition to the properties of the mean curvature of the level sets of φ near $\mathscr{H}$, providing a somewhat more general version of Theorem 1.2.

The proof of Theorem 1.2 which is presented in Section 4 is essentially a repetition of an argument of Qing [26] for $\Lambda < 0$, and which we view as a modification of an argument of Walter Simon [29]. That last, unpublished, proof proceeds as follows: Assuming space dimension three, Simon considered a conformal transformation which maps the set $\mathscr{H}$ to the conformal infinity of a Riemannian metric $\tilde{h}$. Using the maximum principle he showed that his transformation leads to a metric $\tilde{h}$ with $R(\tilde{h}) \geq -6$. Assuming that the metric on $\mathscr{H}$ is the round sphere, he obtained an asymptotically hyperbolic metric with zero hyperbolic mass, which implied that $\tilde{h}$ was the hyperbolic metric, hence conformally flat. The analysis of Lafontaine [22] finishes the proof. Qing's argument for anti-de Sitter space is essentially identical, except that hyperbolic space is replaced by a subset of the Euclidean one. The version presented in Section 4 is a straightforward variation of the above.

2 A non-existence result

In this section we exclude the possibility that *all* horizons are degenerate: suppose, for contradiction, that $\mathscr{M}^-$ contains a spacelike hypersurface $\mathscr{S}$ with compact closure $\overline{\mathscr{S}}$ such that $\partial\mathscr{S} \subset \mathcal{H}$, with all components of $\mathcal{H}$ – degenerate.

The Killing equations imply the identity

$$\nabla_\sigma \nabla^\sigma X^\beta = -R^\beta{}_\lambda X^\lambda = -\frac{2\Lambda}{n-2} X^\beta \ .$$

We thus have

$$\int_{\partial\mathscr{S}} \nabla^\sigma X^\beta dS_{\sigma\beta} = \frac{\Lambda}{n-2}\int_{\mathscr{S}} X^\beta dS_\beta = \frac{\Lambda}{n-2}\int_{\mathscr{S}} g(X,n) d^n\mu \ ,$$

where n is the unit normal to $\mathscr{S}$, and $d^n\mu$ the induced volume form. The last integral does not vanish when $\Lambda \neq 0$ since X is strictly timelike on $\mathscr{S}$. It is standard that

$$\int_{\partial\mathscr{S}} \nabla^\sigma X^\beta dS_{\sigma\beta} = \sum_i \kappa_i A_i \ ,$$

where the sum runs over the connected component of $\partial\mathscr{S}$, with A_i being the respective areas, which is clearly not possible with all κ_i's vanishing when $\Lambda \neq 0$.

3 Staticity

We continue by showing that stationary solutions satisfying the hypotheses of Theorem 1.2 are static. The argument is standard by now, and follows a calculation of Heusler [20], as generalised to higher dimensions by Qing [26]. The identity

$$d\left(\frac{X^\flat \wedge \omega}{\varphi^2}\right) = -\frac{|\omega|^2 i_X(\mathrm{Vol})}{\varphi^4} \qquad (3.1)$$

is integrated on the set

$$M_\epsilon := \{p \in M : \phi(p) \geq \epsilon\} \ .$$

Here i_X denotes contraction with the volume form Vol. Assuming non-degeneracy of all horizons, we will show that the boundary integral over ∂M_ϵ that arises from the left-hand-side of (3.1) will vanish in the limit as ϵ tends to zero. Since X is transverse to M, the vanishing of the volume integral over M will then imply the vanishing of ω.

To analyse the behavior of the left-hand-side of (3.1) as the horizon is approached, one needs to consider separately those points on $\partial(\mathscr{S} \cap \mathscr{M}^-) \subset \mathscr{M}$ which are in $\mathcal{H}$, and those at which X vanishes. In the former case we use the coordinate system of [13, Lemma 3.4]. Thus, $X = \partial_s$, $\mathscr{S}$ is given by $\{s = 0\}$, and the space-time metric g near a point $p \in \mathcal{H}$ takes the form

$$g_{\mu\nu}dx^\mu dx^\nu = g_{ss}ds^2 + 2g_{is}dy^i ds + g_{ij}dy^i dy^j \ ,$$

with $(y^i) = (x, x^A) \equiv (w^2, x^A)$, and with $\mathscr{H}$ given by $\{s = w = 0\}$. The coordinates (s, x, x^A) are regular coordinates in space-time, while the coordinates (w, x^A) are regular coordinates near the boundary $\mathscr{H}$ for the metric h on M. One has

$$\varphi^2 = -g_{ss} = 2\kappa g_{sx} x + O(x^2) \ , \qquad (3.2)$$

where κ (the surface gravity) is a constant, with $g_{sx} >0$ and $g_{sA} = O(x)$. This gives

$$\begin{aligned} X^\flat &:= g(X,\cdot) = g_{ss}ds + g_{sx}dx + g_{sA}dx^A \\ &= 2g_{sx}w\,dw + O(w^2)\ , \qquad (3.3) \\ dX^\flat &= 2wd(g_{sx})\wedge dw + O(w)\ , \qquad (3.4) \end{aligned}$$

where $O(\cdot)$ denotes the (s,w,x^a)-coordinate-behavior of the components of a tensor. After taking the wedge product one thus has

$$X^\flat \wedge dX^\flat = O(w^2)\ .$$

Now, in the (s,x,x^A) coordinate system we have $-\det g_{\mu\nu} > 0$ uniformly, but the transition to the (s,w,x^i) coordinate system introduces a factor w^2 in this determinant, hence $\sqrt{|\det g^{\mu\nu}|}$ behaves as c/w in the (s,w,x^A) coordinates. This gives

$$\omega^{\alpha_1\ldots\alpha_{n-3}} = \sqrt{|\det g^{\mu\nu}|}\,\epsilon^{\alpha_1\ldots\alpha_{n-3}\beta\gamma\delta}X_\beta\partial_\gamma X_\delta = O(w)\ ,$$

(here the alternating symbol $\epsilon^{\alpha_1\ldots\alpha_{n-3}\beta\gamma\delta}$ has been normalised so that $\epsilon^{\alpha_1\ldots\alpha_{n-3}\beta\gamma\delta}\in\{0,\pm1\}$), so finally

$$\omega_{\alpha_1\ldots\alpha_{n-3}} = O(w)\ .$$

By (3.3) the pull-back of $X^\flat$ to the surfaces $\{w = \text{const}\}$ is $O(w^2)$, which together with (3.2) gives

$$\frac{X^\flat\wedge\omega}{\varphi^2}\Big|_{\partial M_\epsilon} = O(\epsilon)\ ,$$

giving zero in the limit $\epsilon\to 0$, as desired.

It remains to consider neighborhoods of points $p\in\overline{\mathcal{H}}\cap\mathscr{S}$ at which X vanishes. This is done by a calculation very similar to the above, using the coordinate system of [13, Eq. (3.15)], the details are left to the reader.

Note that the argument applies irrespective of the topology of $\mathscr{H}$, in particular irrespective of whether $\mathscr{H}$ is connected or not. Similarly it is not necessary for $\mathscr{H}$ to bound M. However, it is not clear what happens with the boundary term when $\mathcal{H}$ has degenerate components, so that non-degeneracy seems to be essential in the argument so far.

We emphasise that the above calculations, showing the vanishing of the boundary integral on each non-degenerate Killing horizon, apply irrespective of the value of the cosmological constant.

Summarising, we have proved:

THEOREM 3.1 *Let $(\mathscr{M},g)$ be an $(n+1)$-dimensional vacuum space-time, $n\geq 3$, with a Killing horizon $\mathcal{H}$ associated with a Killing vector field X, with cosmological constant $\Lambda\neq 0$, suppose that $\mathscr{M}$ contains a compact spacelike hypersurface $\mathscr{S}$. Then:*

1. *At least one connected component of $\mathcal{H}$ is non-degenerate.*
2. *If the boundary of a connected component $\mathscr{M}_0^-$ of $\mathscr{M}^-$ is the union of non-degenerate components of $\mathcal{H}$, then X is hypersurface-orthogonal on $\mathscr{M}_0^-$.*

4 Rigidity from Euclidean extensions

The vacuum Einstein equations with cosmological constant $\Lambda = (n-1)\lambda/2$ imply that the triple (M, φ, h), as described above, satisfies the following set of equations, often referred to as the *static vacuum equations*,

$$\Delta_h \varphi = -\lambda\varphi \;, \tag{4.1a}$$

$$\varphi(R(h)_{ij} - \lambda h_{ij}) = D_i D_j \varphi \;. \tag{4.1b}$$

Here D is the Levi-Civita connection of h, $\Delta_h := D_k D^k$ its Laplace-Beltrami operator, $R(h)_{ij}$ the Ricci tensor of h and $R(h)$ the scalar curvature of h. We now forget the whole discussion of Section 1, and we take, as the starting point of our further considerations, any triple (M, φ, h) satisfying (4.1) with $\lambda \in \mathbb{R}$.

One has the standard formula

$$R(u^2 h) = u^{-2}\Big\{R(h) - 2(n-1)u^{-1}\Delta_h u - (n-1)(n-4)u^{-2}|Du|_h^2\Big\} \;. \tag{4.2}$$

Let $\beta \in \mathbb{R}$, setting

$$u = \frac{1}{\beta + \varphi} \tag{4.3}$$

one obtains

$$R(u^2 h) = (n-1)\lambda F \;, \tag{4.4}$$

where

$$F := \beta^2 - \varphi^2 - \frac{n}{\lambda}|D\varphi|_h^2 \;. \tag{4.5}$$

A calculation presented in detail in [26] gives the following useful version of Lindblom's [24] identity

$$\Delta_h F - \varphi^{-1} h(D\varphi, DF) = -\frac{2n}{\lambda}|\mathrm{Hess}\,\varphi + \frac{\lambda}{n}\varphi h|_h^2 \;. \tag{4.6}$$

Suppose from now on that

$$\lambda > 0 \;,$$

rescaling h by a constant we can without loss of generality assume that

$$\lambda = n \iff R(h) = n(n-1) \;.$$

The right-hand-side of (4.6) is then non-positive, and the strong maximum principle implies that F has no interior minimum.[3] Suppose that M is compact without boundary, (4.1a) shows that

$$\mathscr{H} := \{\varphi = 0\} \tag{4.7}$$

is non-empty. Equation (4.1b) implies that if $\varphi(p) = D\varphi(p) = 0$ at some point $p \in M$, then φ vanishes identically on M. From this one concludes that $D\varphi$ is nowhere vanishing on $\mathscr{H}$, so that $\mathscr{H}$ is a smooth embedded submanifold of M.

[3] It is striking that the signs in (4.4) and (4.6) conspire towards the desired conclusion regardless of the sign of Λ: for $\lambda < 0$ the function F has no interior maximum, but $R(u^2 h)$ has then the same sign as the negative of F.

It follows then from Equation (4.1b) that $\mathscr{H}$ is totally geodesic with respect to the metric h. Replacing M by $\{\varphi > 0\}$ we can without loss of generality assume that

$$\varphi > 0 \tag{4.8}$$

on M. Then M is a smooth manifold with non-empty boundary $\partial M = \mathscr{H}$, and with $\overline{M}$ – compact. As discussed in Section 1, we would also have arrived at this setting assuming, in addition to the hypotheses of Conjecture 1.1, that all the horizons are non-degenerate. On the other hand, existence of a degenerate horizon would lead to asymptotic regions in which φ approaches zero as one recedes to infinity. We expect that one would then also have $|D\varphi|_h \to 0$, but this is not completely clear. A proof of this fact would allow one to remove the hypothesis of non-degeneracy of horizons in Theorem 1.2.

It follows immediately from (4.1b) that $|D\varphi|_h$ is a non-zero constant on each connected component of $\mathscr{H}$. Let $\mathscr{H}_i$ be any such component, and let[4]

$$\kappa_i := |D\varphi|_h \Big|_{\mathscr{H}_i}$$

be the surface gravity of $\mathscr{H}_i$, with φ normalised so that

$$|D\varphi|_h \geq 1 \quad \text{on} \quad \mathscr{H}\ , \tag{4.9}$$

with equality holding on at least one connected component of $\mathscr{H}$. Whenever $\beta \geq \max \kappa_i$ we have $F \geq 0$ on $\mathscr{H}$ and therefore

$$F \geq 0 \implies R(u^2 h) \geq 0\ . \tag{4.10}$$

The same conclusion can be achieved if M has asymptotic regions with φ and $|D\varphi|_h$ asymptotic to zero there.

Let $\tilde{H}_i$ denote the mean curvature of $\mathscr{H}_i$ in the metric $u^2 h$ with respect to the outwards normal, since $\mathscr{H}$ is totally geodesic in the metric h, we have that the second fundamental form of $\mathscr{H}_i$ is proportional to $\gamma_{\mathscr{H}_i}$, with trace

$$\tilde{H}_i = -\frac{1}{\sqrt{\det(u^2 h_{ij})}}\partial_k \left(\sqrt{\det(u^2 h_{ij})}\ u^{-1}\frac{D^k\varphi}{|D\varphi|_h}\right)\Big|_{\mathscr{H}_i} = (n-1)\kappa_i\ . \tag{4.11}$$

Suppose that $\gamma_{\mathscr{H}_i}$ is proportional to the canonical metric on S^n, with proportionality constant r_i^2. Recall that the mean extrinsic curvature of the level sets of r in the (space) Schwarzschild metric, with "mass parameter α",

$$h_{\mathrm{Schw}} = (1 - \frac{\alpha}{r^{n-2}})^{-1} dr^2 + r^2 d\Omega^2;\ ,$$

where $d\Omega^2$ is the canonical metric on S^{n-1}, equals

$$H_{h_{\mathrm{Schw}}}(r) = \frac{n-1}{r}\left(1 - \frac{\alpha}{r^{n-2}}\right)^{\frac{1}{2}}\ .$$

[4]It is well known, and follows in any case immediately from (4.1b), that $|D\varphi|_h$ is constant on any connected component of $\mathscr{H}$.

This shows that at any fixed $r > \alpha^{1/(n-2)}$ the function $H_{h_{\rm Schw}}(r)$, considered as a function of α, covers $\mathbb{R}^+$. It follows that the metric u^2h can be $C^{1,1}$ extended to a Schwarzschild metric on $\mathbb{R}^n \setminus B(1)$ with some mass parameter $\alpha_i \in \mathbb{R}$. In fact, $\alpha_i = 0$ if $\kappa_i = 1/r_i$, $\alpha_i > 0$ if $\kappa_i < 1/r_i$ and $\alpha_i < 0$ if $\kappa_i > 1/r_i$.

Equation (4.11) shows that all the remaining components of $\mathscr{H}$, if any, are trapped as seen from the asymptotically flat region. The hypothesis that M is of positive energy type, and the positive energy theorem (in its version with trapped boundary) implies that $\alpha_i \geq 0$, leading to

$$(n-1)\kappa_i = H_{h_{\rm Schw}}(r_i) \leq \frac{(n-1)}{r_i} \;, \tag{4.12}$$

so that

$$r_i \leq \kappa_i r_i \leq 1 \;, \tag{4.13}$$

with equality if and only if M is diffeomorphic to a ball in $\mathbb{R}^n$, with a flat metric. We have thus shown

Proposition 4.1 *Under the hypotheses of Theorem 1.2, let $\mathscr{H}_i$ be a component of $\mathscr{H}$ with induced metric $\gamma_{\mathscr{H}_i} = r_i^2\gamma_{can}$, where γ_{can} is the canonical metric on S^{n-1}. Then $r_i \leq 1$.*

If $r_i = 1$ we have $F \equiv 0$ by flatness of u^2h and by (4.4), hence $\operatorname{Hess}\varphi = -\varphi h$ by (4.6). The results of Lafontaine [22], or [14, Lemma VII.3], show that (M, h, φ) correspond to the de Sitter metric. Alternatively, the calculations of [26, Lemma 3.3] show that h is a space-form near a maximum of φ, and analyticity of solutions of (4.1) proves that h is a space-form everywhere. In particular in dimension $n = 3$ one finds $\varphi = \sqrt{1-r^2}$, with $h = \varphi^{-2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$, as desired.

This proves the local part of Theorem 1.2. To obtain the global diffeomorphism one can use a theorem of Harris [19] which, under the hypothesis of non-existence of closed timelike curves, shows that the connected component of $\mathscr{M}^-$ containing M is diffeomorphic to $\mathbb{R} \times M$. This leads to the global representation of g as

$$g = -\varphi^2(dt + \theta)^2 + h \;, \quad X = \partial_t \;,$$

where θ is a one-form on $B^n(1)$. Staticity implies that θ is closed, and since $B^n(1)$ is simply connected there exists a function τ such that $d\tau = dt + \theta$, giving the result.

When the metric on $\mathscr{H}$ is not proportional to the canonical metric on the sphere, one can still obtain some information, as follows: Suppose that a connected component $\mathscr{H}_i$ of $\mathscr{H}$ can be isometrically embedded into $\mathbb{R}^n$ as a convex surface.[5] Proceeding as in Shi-Tam [28] one can construct an asymptotically flat extension of (M, h) across $\mathscr{H}_i$, with positive scalar curvature. Applying the positive energy theorem with a trapped boundary we conclude that

$$\int_{\mathscr{H}_i} (\tilde{H}_i^0 - \tilde{H}_i) d\mu_{u^2h} \geq 0 \;, \tag{4.14}$$

[5] Recall that in dimension $n = 3$ the embedding exists when $\gamma_{\mathscr{H}}$ has positive Gauss curvature [25].

where $\tilde H_i^0$ is the mean curvature of the isometric embedding of $(\mathscr{H}_i, u^2 h|_{\mathscr{H}_i})$ into Euclidean space, with equality if and only if $\mathscr{H}$ is connected and M is a domain in $\mathbb{R}^n$ with the flat metric. Using (4.11) this can be rewritten as

$$\frac{\kappa_i}{\max_j \kappa_j} \le \frac{1}{(n-1)A_h(\mathscr{H}_i)} \int_{\mathscr{H}_i} \tilde H_0 d\mu_h \ . \tag{4.15}$$

Note that the right-hand-side depends only upon the image of $\mathscr{H}_i$ in $\mathbb{R}^n$. In case of equality we recover the conclusions of Theorem 1.2.

5 The geometry near $\mathscr{H}$

The object of this section is to relate the hypotheses of Theorem 1.2 to other geometric quantities characterising $\mathscr{H}$. We use geodesic coordinates (x, v^A) near $\mathscr{H}$, so that $x \ge 0$ is the distance to $\mathscr{H}$, with the v^A's being constant along the integral curves of $D\varphi$. The metric takes then the following form

$$h = dx^2 + \gamma \ , \tag{5.1}$$

so that ∂_x is the *inwards pointing* normal to $\mathscr{H}$. Further, *e.g.* by [13, Prop. 3.3], φ is an odd function of x, while γ_x is even. Equation (5.1) leads to the following formulae

$$\Gamma^x_{xx} = \Gamma^x_{Ax} = \Gamma^B_{xx} = 0 \ , \tag{5.2a}$$
$$k_{AB} := -\Gamma^x_{AB} = \tfrac{1}{2}\partial_x \gamma_{AB} \ , \tag{5.2b}$$
$$\Gamma^A_{xB} = \tfrac{1}{2}\gamma^{AC}\partial_x\gamma_{CB} =: -k^A{}_B \ , \tag{5.2c}$$

with the Γ^A_{BC} coinciding with the Christoffel symbols of γ_{AB}. Thus, k_{AB} is the extrinsic curvature of the level sets of x with respect to the inwards-pointing normal. Further,

$$\Delta_h\varphi = \ddot\varphi + \Delta_\gamma\varphi + \mathrm{tr}_\gamma k\ \dot\varphi \ , \tag{5.3a}$$
$$D_x D_A\varphi = D_A D_x \varphi = \partial_A\dot\varphi + k_A{}^C\partial_C\varphi \ , \tag{5.3b}$$
$$D_x D_x\varphi = \ddot\varphi \ , \tag{5.3c}$$
$$D_A D_B\varphi = \mathscr{D}_A\mathscr{D}_B\varphi + k_{AB}\dot\varphi \ , \tag{5.3d}$$

where a dot denotes an x-derivative, while $\mathscr{D}$ is the Levi-Civita covariant derivative of γ. Equation (4.1) gives thus

$$\ddot\varphi = -\Delta_\gamma\varphi - \lambda\varphi - \mathrm{tr}_\gamma k\ \dot\varphi \ , \tag{5.4a}$$
$$\varphi R_{xA} = \partial_A\dot\varphi + k_A{}^C\partial_C\varphi \ , \tag{5.4b}$$
$$\varphi R_{xx} = \ddot\varphi + \lambda\varphi \ , \tag{5.4c}$$
$$\varphi R_{AB} = \mathscr{D}_A\mathscr{D}_B\varphi + k_{AB}\dot\varphi + \lambda\varphi\gamma_{AB} \ . \tag{5.4d}$$

Since $\mathscr{H}$ is totally geodesic we have $k_{AB} = 0$ on $\mathscr{H}$. Let

$$\kappa := \dot\phi|_{\mathscr{H}} \ . \tag{5.5}$$

As $\varphi = \kappa x + O(x^3)$, (5.4a) gives

$$\ddot{\varphi} = 0 \quad \text{on} \quad \mathscr{H} \ . \tag{5.6}$$

Differentiating (5.4) with respect to x one obtains on $\mathscr{H}$

$$\kappa^{-1}\dddot{\varphi} = -\lambda - \mathrm{tr}_{\gamma}\dot{k} \ , \tag{5.7a}$$
$$R_{xA} = 0 \ , \tag{5.7b}$$
$$R_{xx} = \kappa^{-1}\dddot{\varphi} + \lambda = -\mathrm{tr}_{\gamma}\dot{k} \ , \tag{5.7c}$$
$$R_{AB} = \dot{k}_{AB} + \lambda\gamma_{AB} \ . \tag{5.7d}$$

From the Codazzi-Mainardi equations we have, at $x = 0$,

$$R_{ABCD} = \mathscr{R}_{ABCD} \ , \tag{5.8a}$$
$$(n-1)\lambda = R = 2R^{xA}{}_{xA} + R^{AB}{}_{AB} = 2R_{xx} + \mathscr{R} = -2\mathrm{tr}_{\gamma}\dot{k} + \mathscr{R} \ , \tag{5.8b}$$

where $\mathscr{R}_{ABCD}$ is the curvature tensor of γ, and $\mathscr{R}$ the Ricci scalar thereof. Thus,

$$\mathscr{R} = (n-1)\lambda + 2\mathrm{tr}_{\gamma}\dot{k} \ . \tag{5.9}$$

Next, again on $\mathscr{H}$,

$$\begin{aligned} R^{x}{}_{AxB} &= \partial_x \Gamma^x_{AB} - \partial_B \Gamma^x_{Ax} + \Gamma^x_{ix}\Gamma^i_{AB} - \Gamma^x_{iB}\Gamma^i_{Ax} \\ &= -\dot{k}_{AB} \ , \end{aligned}$$

which gives

$$R_{AB} = R^{x}{}_{AxB} + R^{C}{}_{ACB} = -\dot{k}_{AB} + \mathscr{R}_{AB} \ . \tag{5.10}$$

Comparing with (5.7d) one finds

$$2\dot{k}_{AB} = \mathscr{R}_{AB} - \lambda\gamma_{AB} \ . \tag{5.11}$$

In dimension two all metrics are Einstein, so that when $\mathscr{H}$ is two-dimensional we conclude that $\dot{k}_{AB}$ is proportional to the metric:

$$\dot{k}_{AB} = \frac{1}{2}\mathrm{tr}_{\gamma}\dot{k}\ \gamma_{AB} \ . \tag{5.12}$$

If we suppose that $\lambda = n = 3$, Equations (5.9)-(5.12) show that γ will be a unit round metric at $x = 0$ if and only if

$$\mathrm{tr}_{\gamma}\dot{k} = -2 \ . \tag{5.13}$$

Equation (5.13) and the arguments of Section 4 prove thus the following version of Theorem 1.2:

THEOREM 5.1 *Let $(\bar{M}, h)$ be a complete three-dimensional manifold with non-empty compact boundary, let M be the interior of $\bar{M}$ and suppose that φ and h solve* (4.1) *with $\lambda = 3$, and with $\varphi = 0$ precisely on ∂M. If $\bar{M}$ is not compact, assume moreover that φ and $|D\varphi|_h$ approach zero in the asymptotic regions. Then the following are equivalent:*

1. *The metric induced by h a connected component $\mathscr{H}$ of ∂M is the unit round metric γ_{can} on S^2.*

2. *Let $\mathrm{tr}_{\gamma} k$ be the mean inwards extrinsic curvature of the level sets of φ, then the inwards normal derivative of $\mathrm{tr}_{\gamma} k$ on $\mathscr{H}$ equals minus two.*

3. *Let $\kappa = |D\varphi|_h$ on $\mathscr{H}$, then the third inwards normal derivative of φ at $\mathscr{H}$ equals minus κ.*

4. *$\bar{M}$ is diffeomorphic to $B^3(1)$ with $\varphi = \sqrt{1-r^2}$ and $h = \varphi^{-2}dr^2 + r^2\gamma_{can}$.*

□

More generally, in any dimension, Equation (5.11) with $\lambda = n$ implies that the boundary metric will be Einstein, $\mathscr{R}_{AB} = \frac{\mathscr{R}}{n-1}\gamma_{AB}$, if and only if

$$\dot{k}_{AB} = \frac{\mathscr{R} - n(n-1)}{2(n-1)}\gamma_{AB} \ . \tag{5.14}$$

This leads to the following:

THEOREM 5.2 *Let $(\bar{M}, h)$ be a complete n–dimensional manifold with non-empty compact boundary, let M be the interior of $\bar{M}$ and suppose that φ and h solve (4.1) with $\lambda = n$, and with $\varphi = 0$ precisely on ∂M. Assume that M is of positive energy type. If $\bar{M}$ is not compact, assume moreover that φ and $|D\varphi|_h$ approach zero in the asymptotic regions. Then the following are equivalent:*

1. *The metric induced by h on a connected component $\mathscr{H}$ of ∂M is the canonical metric γ_{can} on S^{n-1}.*

2. *$\bar{M}$ is diffeomorphic to B^n with $\varphi = \sqrt{1-r^2}$ and $h = \varphi^{-2}dr^2 + r^2\gamma_{can}$.*

A necessary condition for the above is

$$\dot{k}_{AB}|_{\mathscr{H}} = -\frac{1}{n-1}\gamma_{AB} \ .$$

6 The Boucher-Gibbons-Horowitz inequality

For completeness we review the Boucher-Gibbons-Horowitz proof of Theorem 1.2, further showing how that proof generalises to higher dimensions. Equation (4.6) is rewritten as

$$D_i(\varphi^{-1}D^iF) = -\frac{2n}{\lambda\varphi}|\mathrm{Hess}\,\varphi + \frac{\lambda}{n}\varphi h|_h^2 \ . \tag{6.1}$$

Integrating over the set $\{\varphi \geq \epsilon\}$ one therefore finds

$$\lim_{\epsilon\to 0}\int_{\{\varphi=\epsilon\}} \varphi^{-1}D^iF dS_i \leq 0 \ . \tag{6.2}$$

Let n be the outwards-pointing normal to the level sets of φ, we have $n = -D\varphi/|D\varphi|_h$, so that the integrand in (6.2) reads, by l'Hospital's rule, together with (5.7a) and (5.9),

$$\begin{aligned}
-\lim_{\varphi\to 0}\varphi^{-1}h(\frac{D\varphi}{|D\varphi|_h}, DF) &= -\frac{1}{\kappa}\partial_x\left(\partial_x F + O(x^2)\right)\Big|_{\mathscr{H}} \\
&= -\frac{1}{\kappa}\partial_x\left(-2\varphi\dot\varphi - \frac{2n}{\lambda}\dot\varphi\ddot\varphi + O(x^2)\right)\Big|_{\mathscr{H}} \\
&= 2\kappa\left(1 + \frac{n}{\lambda\kappa}\dddot\varphi\right)\Big|_{\mathscr{H}} \\
&= \kappa\left((n-1)(n-2) - \frac{n}{\lambda}\mathscr{R}\right)\Big|_{\mathscr{H}} ,
\end{aligned}$$

We have therefore obtained

$$\sum_i \kappa_i \int_{\mathscr{H}_i}\left((n-1)(n-2) - \frac{n}{\lambda}\mathscr{R}\right) \le 0 .$$

In dimension $n = 2$ this, together with the Gauss-Bonnet theorem, implies

$$4\pi\lambda^{-1}\sum_i \kappa_i(1-g_i) \ge \sum \kappa_i A(\mathscr{H}_i) .$$

(Since the right-hand-side is positive, one can conclude that at least one component of $\mathscr{H}$ has spherical topology.) Further, equality holds if and only if $F \equiv 0$, and as before one recovers Theorem 1.2. If M has connected boundary $\partial M_1 = \mathscr{H}_1$ we can normalise φ so that $\kappa_1 = 1$, then g_1 is necessarily zero and one obtains the Boucher-Gibbons-Horowitz inequality,

$$A(\mathscr{H}_1) \le \frac{4\pi}{\lambda} ,$$

with equality holding if and only if h is the space-part of the de Sitter metric.

Acknowledgements: The author acknowledges useful discussions with Walter Simon. He is grateful to Gary Horowitz and Eric Woolgar for bibliographical advice.

References

[1] M.T. Anderson, *On the structure of solutions to the static vacuum Einstein equations*, Annales H. Poincaré **1** (2000), 995–1042, gr-qc/0001018.

[2] M.T. Anderson, P.T. Chruściel, and E. Delay, *Non-trivial, static, geodesically complete vacuum space-times with a negative cosmological constant*, JHEP **10** (2002), 063, 22 pp., gr-qc/0211006.

[3] ———, *Non-trivial, static, geodesically complete vacuum space-times with a negative cosmological constant II: $n \ge 5$*, in preparation (2003).

[4] R. Bartnik and P.T. Chruściel, *Boundary value problems for Dirac-type equations*, (2003), `math.DG/0307278`.

[5] D. Birmingham, *Topological black holes in anti-de Sitter space*, Class. Quantum Grav. **16** (1999), 1197–1205, hep-th/9808032.

[6] W. Boucher, *Cosmic no hair theorems*, Classical general relativity (W.B. Bonnor and M.A.H. MacCallum, eds.), Cambridge University Press, Cambridge, 1984, pp. 43–52.

[7] W. Boucher, G.W. Gibbons, and G.T. Horowitz, *Uniqueness theorem for anti–de Sitter spacetime*, Phys. Rev. **D30** (1984), 2447–2451.

[8] R.H. Boyer, *Geodesic Killing orbits and bifurcate Killing horizons*, Proc. Roy. Soc. London A **311** (1969), 245–252.

[9] G. Bunting and A.K.M. Masood–ul–Alam, *Nonexistence of multiple black holes in asymptotically euclidean static vacuum space-time*, Gen. Rel. Grav. **19** (1987), 147–154.

[10] C. Cadeau and E. Woolgar, *New five dimensional black holes classified by horizon geometry, and a Bianchi VI braneworld*, Class. Quantum Grav. (2001), 527–542, gr-qc/0011029.

[11] B. Carter, *Killing horizons and orthogonally transitive groups in space–time*, Jour. Math. Phys. **10** (1969), 70–81.

[12] U. Christ and J. Lohkamp, in preparation (2003).

[13] P.T. Chruściel, *The classification of static vacuum space–times containing an asymptotically flat spacelike hypersurface with compact interior*, Class. Quantum Grav. **16** (1999), 661–687, gr-qc/9809088.

[14] P.T. Chruściel and W. Simon, *Towards the classification of static vacuum spacetimes with negative cosmological constant*, Jour. Math. Phys. **42** (2001), 1779–1817, gr-qc/0004032.

[15] H. Friedrich, *Existence and structure of past asymptotically simple solutions of Einstein's field equations with positive cosmological constant*, Jour. Geom. Phys. **3** (1986), 101–117.

[16] G.J. Galloway, S. Surya, and E. Woolgar, *A uniqueness theorem for the AdS soliton*, Phys. Rev. Lett. **88** (2002), 101102, hep-th/0108170.

[17] G.W. Gibbons, S. A. Hartnoll, and C.N. Pope, *Bohm and Einstein-Sasaki metrics, black holes and cosmological event horizons*, Phys. Rev. **D67** (2003), 084024, hep-th/0208031.

[18] G.W. Gibbons and S.W. Hawking, *Cosmological event horizons, thermodynamics, and particle creation*, Phys. Rev. **D15** (1977), 2738–2751.

[19] S. Harris, *Conformally stationary spacetimes*, Class. Quantum Grav. **9** (1992), 1823–1827.

[20] M. Heusler, *Black hole uniqueness theorems*, Cambridge University Press, Cambridge, 1996.

[21] W. Israel, *Event horizons in static vacuum space-times*, Phys. Rev. **164** (1967), 1776–1779.

[22] J. Lafontaine, *Sur la géométrie d'une généralisation de l'équation différentielle d'Obata*, Jour. de Math. Pures et Appli. **62** (1983), 63–72.

[23] A. Lichnerowicz, *Théories relativistes de la gravitation et de l'électromagnétisme. Relativité générale et théories unitaires*, Masson et Cie, Paris, 1955.

[24] L. Lindblom, *Static uniform-density stars must be spherical in general relativity*, Jour. Math. Phys. **29** (1988), 436–439.

[25] L. Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large*, Commun. Pure Appl. Math. **6** (1953), 337–394.

[26] J. Qing, *On the uniqueness of AdS spacetime in higher dimensions*, Annales Henri Poincaré (2003), in press, math.DG/0310281.

[27] R. Schoen and S.-T. Yau, *Proof of the positive mass theorem II*, Commun. Math. Phys. **79** (1981), 231–260.

[28] Y. Shi and L.-F. Tam, *Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature*, (2003), math.DG/0301047.

[29] W. Simon, private communication, 1998.

[30] D. Sudarsky and R.M. Wald, *Extrema of mass, stationarity and staticity, and solutions to the Einstein–Yang–Mills equations*, Phys. Rev. **D46** (1993), 1453–1474.

[31] C.V. Vishveshwara, *Generalization of the "Schwarzschild Surface" to arbitrary static and stationary metrics*, Jour. Math. Phys. **9** (1968), 1319–1322.

[32] M. Walker, *Bloc diagrams and the extension of timelike two–surfaces*, Jour. Math. Phys. **11** (1970), 2280–2286.

[33] E. Witten, *A simple proof of the positive energy theorem*, Commun. Math. Phys. **80** (1981), 381–402.