

On solutions of the vacuum Einstein equation in the radiation regime

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Abstract

We review recent results by the author, in collaboration with Erwann Delay, Olivier Lengard, and Rafe Mazzeo, on existence and properties of space-times with controlled asymptotic behavior at null infinity.

The standard description of gravitational radiation proceeds as follows: one considers space-times (\mathcal{M}, g) which can be conformally completed by adding a boundary $\partial\mathcal{M} = \mathcal{I}$, so that an appropriate conformal rescaling of g leads to a metric which extends by continuity to a Lorentzian metric \tilde{g} on the new manifold with boundary $\tilde{\mathcal{M}} := \mathcal{M} \cup \mathcal{I}$. This raises several questions: do there exist non-trivial vacuum space-times in which this can be done? does this prescription cover all radiating space-times? or at least all interesting ones? is this the right way to proceed anyway? In this talk I will review some recent progress concerning those questions.

Recall [24] that a space-time is called *asymptotically simple* if the above conformal completion $(\tilde{\mathcal{M}}, \tilde{g})$ is smooth, and if every null geodesic of (\mathcal{M}, g) has precisely two end points on \mathcal{I} . It has been an open question whether there exist any vacuum asymptotically space-times other than the Minkowski one. A celebrated theorem of Christodoulou and Klainerman [4] proves existence of a large family of space-times which are close to being asymptotically simple: The Christodoulou-Klainerman metrics are geodesically complete, and admit conformal completions. However, the differentiability properties of the conformally rescaled metrics are poorly controlled. This last issue does play a role in the theory, as the properties of \mathcal{I} 's with low differentiability are rather different from those of the smooth ones. For instance, the peeling properties of the gravitational field, which are sometimes considered as a characteristic feature of gravitational radiation, are different for conformal completions which are, or which are not, of C^3 differentiability class (*cf.*, *e.g.* [1, 26]). Further, while it is rather likely that null geodesics will also have precisely two end points on \mathcal{I} for the Christodoulou-Klainerman space-times, no analysis of this question

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is known to the author. In conclusion, it is not *a priori* clear whether any of the spaces-times constructed by Christodoulou-Klainerman are asymptotically simple in the original sense of Penrose.¹

The main difficulty here can be traced back to the following question: how to control the asymptotic properties of the gravitational field near \mathcal{I} in terms of the asymptotic properties of the initial data in a neighborhood of i^0 . We note a recent important paper of Friedrich [21] which provides the first result in this direction for the linearised spin-2 equations. However, the transition from the linearised theory to the non-linear one does not appear to be straightforward, so that work remains to be done in order to go from [21] to a full answer to the question.

A pioneering construction of non-trivial, physically interesting, asymptotically simple space-times is due to Cutler and Wald [16], for gravitation interacting with an electromagnetic field. Cutler and Wald's idea is to construct a non-trivial family of initial data on \mathbb{R}^3 for the coupled Einstein-Maxwell equations which are exactly Schwarzschildian outside of a ball, and which are as close to Minkowski initial data as desired. The main interest of such data stems from the fact that the evolution of the initial data set provides a space-time which is exactly the Schwarzschild one near i^0 . For data small enough one then obtains an asymptotically simple space-time by invoking a stability theorem of Friedrich [20].

To repeat this argument in vacuum all one needs is appropriate initial data. In a recent work with Erwann Delay [6] we have been able to construct such data, using a variation of an important construction of Corvino and Schoen [13, 15]. In [6] we consider vacuum, time symmetric ($K_{ij} \equiv 0$) initial data on a ball $B(0, 1) \subset \mathbb{R}^3$ satisfying the parity condition

$$g_{ij}(\vec{x}) = g_{ij}(-\vec{x}) . \quad (0.1)$$

(Large families of such metrics can be constructed using the conformal method.) We show that all such metrics on $B(0, 1)$ which are close enough to Minkowski space-time can be extended to scalar flat metrics on \mathbb{R}^3 which are exactly Schwarzschildian on $\mathbb{R}^3 \setminus B(0, 2)$. As already mentioned, the technique is a gluing construction along the lines of that of [13]. The point of the parity condition is the following: one of the steps of the proof of the extension theorem requires an adjustment of the centre of mass \vec{c} of a tentative Schwarzschildian extension. This adjustment requires a translation of the Schwarzschild metric by a vector \vec{c}/m . When m gets small the vector \vec{c}/m could become very large, in which case the construction breaks down. The parity condition (0.1) guarantees that the centre of mass \vec{c} is zero regardless of m , and the problem with the division by the small parameter m goes away.

Summarising: one starts with any one-parameter family of time symmetric vacuum initial data on $B(0, 1)$ satisfying (0.1), such that the initial data approach the trivial ones as the parameter goes to zero. For values of the parameter small enough one can extend the initial data to Schwarzschildian ones

¹The space-times of [6] described below do actually belong to the Christodoulou-Klainerman class, so it is known in retrospect that some of the Christodoulou-Klainerman space-times will fit the Penrose scheme.

using the extension technique above. Making the parameter smaller if necessary one obtains an asymptotically simple² space-time using Friedrich’s conformal system of equations [19].

Exactly the same construction allows one to make an extension in which the Schwarzschild metric is replaced by any asymptotically flat metric with non-zero mass m , satisfying the parity condition (0.1), in the following sense: let $g(1)$ be any metric defined for $|\vec{x}|$ large enough, and consider the family of metrics $g(\lambda)$ obtained by scaling, $g(1)_{ij}(\vec{x}) \rightarrow g(\lambda)_{ij}(\vec{x}) := g(1)_{ij}(\lambda\vec{x})$. Then the mass $m(\lambda)$ of $g(\lambda)$ equals m/λ . If the initial vacuum metric on $B(0, 1)$ is close enough to the flat one, then there exists a scalar flat extension which coincides with $g(\lambda)$ on $\mathbb{R}^3 \setminus B(0, 2)$, for some λ large enough. This construction becomes most useful when $g = g(1)$ satisfies the static constraint equations; equivalently, when $g(1)$ arises from a static solution of the vacuum Einstein equations. Since stationary vacuum metrics have a smooth \mathcal{S} [17, 18], one thus obtains a reasonably large new family of asymptotically simple space-times.

In [6] yet another interesting application of the Corvino-Schoen technique has been pointed out: the extension theorem can be used to construct initial data sets for a “many-black-hole” space-time, as follows: one chooses any number of non-intersecting balls B_i , $i = 1, \dots, I$, which are symmetrically distributed around the origin. To each of those balls one assigns a small positive mass parameter m_i , again symmetrically with respect to the origin. If the parameters m_i are small enough, then [6] one can find a vacuum initial data set such that the initial Riemannian metric is exactly the (space) Schwarzschild metric, centred³ on the centre of B_i , with mass m_i , within each of the balls B_i , and also exactly the (space) Schwarzschild metric outside of a sufficiently large ball, with some mass m which is close to the sum of the masses m_i . The fact that the metric is exactly Schwarzschildian in each of the balls B_i guarantees that there will be I marginally trapped surfaces in the initial data set. Further, the evolution of the metric in the domain of dependence of the B_i ’s will produce exactly a Schwarzschild metric there, with an associated black hole region. The fact that the metric is exactly Schwarzschildian outside of a large ball guarantees that one has reasonably good control of the properties of the resulting \mathcal{S} . Those facts put together allow one to establish [12] (see also [5]) the following properties of the maximal globally hyperbolic development (\mathcal{M}, g) of so-constructed initial data:

1. If the mass parameters are small enough, then the only marginally trapped surfaces in the initial data set are the usual minimal surfaces occurring in each of the Schwarzschildian balls. Recall that a bounding marginally trapped surface within an initial data set always encloses a black hole region, and a non-connected *outermost* marginally trapped surface is usually

²In [6] we asserted that the construction leads to C^k conformal completions, where k can be chosen at will, but finite. However, one can show [7, 13] that if the starting metric on $B(0, 1)$ is smooth up-to-boundary, then things can be arranged so that the resulting conformal completion will also be smooth.

³Here we have in mind the conformally flat representation of the (space) Schwarzschild metric, in which the “other infinity” of the Einstein-Rosen bridge corresponds to the centre of the ball.

interpreted as reflecting existence of a non-connected black hole.

2. Making the mass parameters smaller if necessary, any configuration with two B_i 's will lead to a space-time in which the intersection of the event horizon with the initial data hypersurface will *not* be connected. Thus the initial data surface does indeed contain two distinct black hole regions.

We note that the paper [12] is the first one which proves rigorously that some families of vacuum initial data contain non-connected black hole regions. The proof again uses Friedrich's stability results. Other existing results [25] or methods [23] do not guarantee non-connectedness of the black hole.

The reader is referred to [7, 14] for further variations on the Corvino-Schoen technique.

The space-times discussed so far have smooth conformal completions, but they are also extremely special because of (0.1). In fact, there have been various indications that generic \mathcal{S} 's will not even be C^3 . First, this seems to occur in the Christodoulou-Klainerman space-times (compare [3]). Next, it has been observed in [1] that generic solutions of the constraint equations on hyperboloids, constructed by the conformal method, will not be C^3 at the surface where the hyperboloid intersects \mathcal{S} , even if the seed fields which enter the construction are smooth there. It has been similarly observed in [11] (compare [22, 27]) that generic initial data on outgoing null cones will not lead to smooth \mathcal{S} 's, even though the data induced on the initial light cone are smooth at \mathcal{S} . Now, those last two results leave open the possibility that the resulting space-times will not have a smooth \mathcal{S} for the simple reason that they will have no \mathcal{S} at all. In recent work with O. Lengard we have shown that this is not the case, for large classes of hyperboloidal initial data. In [9] one considers initial data in weighted Sobolev spaces, with regularity at the boundary compatible with that which is obtained in the conformal construction of [1, 2]. Roughly speaking, the weighting corresponds to the following behavior of the derivatives of the metric in terms of pseudo-Minkowskian coordinates:

$$|\partial_u^i \partial_r^j \partial_\theta^k \partial_\varphi^\ell g_{\mu\nu}(u, r, \theta, \varphi)| \leq C(i, j, k, \ell) r^{-3/2} \quad (0.2)$$

for $j \geq 1$, and for $u = t - r$ near u_0 . This should be contrasted with smooth conformal completions, which would lead to a power r^{-1-j} in (0.2). In those function spaces the derivatives of the fields are too singular for the usual Friedrich evolution theorems [19] to apply. It is shown in [9] that the maximal globally hyperbolic development of large classes of such initial data will possess a conformal completion in a neighborhood of the initial data hypersurface. The weighted Sobolev regularity (hence, also a weighted C^k regularity with a differentiability index k lower by two) of the metric is preserved during evolution.

Similar, and actually sharper, results can be proved for large classes of semi-linear wave equations, and for the wave-map equation, on Minkowski space-time [8, 10]. In particular for such equations one can show that polyhomogeneity of the initial data is preserved by evolution. We expect this to be also true for Einstein equations, and we are hoping to settle this issue in the near future.

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