Space-time Engineering

Piotr T. Chruściel[*] James Isenberg[†]
Université de Tours University of Oregon

Daniel Pollack[‡]
University of Washington

December 15, 2003

A fascinating possibility opened-up by general relativity is that of deforming space by creating wormholes connecting distant points, or bridges which connect distinct universes. If the points in question are distant in space, a wormhole provides a shorter path between them. If the points are distant both in space and time, a wormhole provides half of a potential time machine. However, Einstein's theory places strong restrictions (the Einstein "constraint equations") on the configuration of the gravitational field at any time. These equations tie the geometry of the space to energy and momentum of the matter fields, so that it is conceivable that attempts to create wormholes would always lead to negative energy densities, something which has never been observed on mezzoscopic or macroscopic scales. Here we show that this does not happen, and that in generic situations it is possible in principle to create wormholes connecting arbitrary points in space while maintaining the requirement of energy positivity. In fact, this can even be done using vacuum wormholes if there is no matter near the points that we wish to connect.

A second, closely related question is the following: can a physicist determine, from the knowledge of the geometry of the nearby region, if the universe she lives in contains wormholes away from this region? Because of the Einstein constraint equations the presence of wormholes or other complicated topology far away could perhaps be reflected in the local gravitational field. Here we also prove that this is not the case. More specifically, we show that for generic gravitational field configurations (consistent with the constraints), and for any chosen pair of points, there are other field configurations which are identical except for a wormhole connecting that pair of points. The same holds true for an arbitrary number of wormholes, placed in arbitrary locations anywhere in

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[*] Partially supported by a Polish Research Committee grant 2 P03B 073 24; email `piotr@gargan.math.univ-tours.fr`, URL www.phys.univ-tours.fr/˜piotr

[†] Partially supported by the NSF under Grant PHY-0099373; email `jim@newton.uoregon.edu`, URL www.physics.uoregon.edu/˜jim

[‡] Partially supported by the NSF under Grant DMS-0305048 and the UW Royalty Research Fund; email `pollack@math.washington.edu`, URL: www.math.washington.edu/˜pollack

space at a given time. The construction can also be used to construct bridges connecting two distinct surfaces of simultaneity, whether in the same universe, or in two distinct ones, without violating the requirement that the energy be positive, or (in the absence of matter) that the vacuum constraint equations be satisfied.

The fact that Einstein's theory of general relativity places strong restrictions on the gravitational field configuration at a given moment of time should not be too surprising. The same is true for electromagnetic fields, governed by Maxwell's theory. Assuming, for simplicity, that no charges are present, an electromagnetic field configuration may be specified, at a given moment in time, by a choice of the two vector fields $\vec{E}$ and $\vec{B}$. Two of Maxwell's equations, $\partial_t \vec{E} = \nabla \times \vec{B}$ and $\partial_t \vec{B} = -\nabla \times \vec{E}$ describe how the fields $\vec{E}(x,t)$ and $\vec{B}(x,t)$ evolve in time from the initial data specification $(\vec{E}(x,0), \vec{B}(x,0))$ of them. The others, however, restrict one's choice of the initial data: $\nabla \cdot \vec{E}(x,0) = 0$ and $\nabla \cdot \vec{B}(x,0) = 0$.

The appropriate information to describe the gravitational field at a given moment, according to Einstein's theory, consists of a three dimensional manifold $\mathscr{S}^3$, together with a pair of symmetric tensor fields $\gamma = \gamma_{ab}$ and $K = K_{cd}$ on $\mathscr{S}^3$. The manifold $\mathscr{S}^3$ describes the space of the universe and its topology. It might be the unbounded Euclidean space $\mathbb{R}^3$ of the flat Minkowski spacetime, it might be the closed and bounded S^3 of the standard positively curved "Friedman-Robertson-Walker" big bang cosmology, or it might be any other of an infinite set of possibilities. The geometry for this snapshot of the universe, including the information regarding distances, angles, and curvature on $\mathscr{S}^3$, is provided by the specification of the tensor field γ, the spatial metric. The last piece of the gravitational initial data, K, essentially prescribes the time derivative of the spatial metric γ.

As with Maxwell's theory, a portion of Einstein's equations determines how a set of initial data for the gravitational field $(\mathscr{S}^3, \gamma, K)$ evolves in time, while the rest puts restrictions on this data at any given time. Specifically, $(\mathscr{S}^3, \gamma, K)$ must satisfy the constraint equations

$$R - K^{cd}K_{cd} + (trK)^2 = 16\pi\rho \,, \tag{1}$$

and

$$\nabla_a K^a{}_b - \nabla_b(trK) = 8\pi J^a, \tag{2}$$

where ∇_a is the spatial covariant derivative related to γ, R is its scalar curvature, and ρ and J^a describe the energy density and momentum density of any matter fields that are present. We note a key difference between the Maxwell constraint equations and those of Einstein: the former are linear, while the latter are not. Thus, unlike electromagnetic fields, gravitational fields cannot be superposed.

While there are only four Einstein constraint equations restricting the values of the twelve functions contained in the initial data set $(\mathscr{S}^3, \gamma, K)$, these constraints are known to possess elliptic features (like the constraints from Maxwell's theory), and one might therefore naively expect them to tie the behavior of the gravitational fields at one point in the space $\mathscr{S}^3$ (at a given moment) to that of the fields elsewhere in $\mathscr{S}^3$ at the same moment. Striking recent results of Corvino [5] and Corvino-Schoen [6] have, by exploiting the underdetermined nature of the Einstein constraint equations, shown that these naive expectations are incorrect. This flexibility among solutions is one of the key ingredients in the results we present here. Note that for fairly weak gravitational fields, of the sort that can be adequately modeled using Newton's equations, the constraints reduce to the familiar Poisson equation, $\Delta\phi = \rho$. This single elliptic equation for the single gravitational potential ϕ enforces the familiar "action at a distance" property of Newtonian gravity.

We would like to use the Einsteinian $(\mathscr{S}^3, \gamma, K)$ description of the gravitational field configuration to restate in a mathematically precise manner the questions which we asked at the beginning of this letter. The topological part of the mathematical procedure associated with the creation of wormholes is called a "connected sum": one removes a ball around each of the points, with a radius as small as desired; this results in a manifold in which a spherical boundary S^2 replaces each solid ball. Then a "handle" $[0,1] \times S^2$ is glued in, with each of the boundaries $\{0\} \times S^2$ and $\{1\} \times S^2$ being attached to the spheres left after the removal of the balls. If the original points belonged to the same manifold, a wormhole is created. (See Figure 1.) If those points belonged to two different manifolds, a connecting bridge arises. (See Figure 2.)

If we now go beyond purely topological considerations to include handling the geometry and the physical fields, our first question takes the following form: Can this procedure be done while preserving the positive energy condition, ρ not smaller than the length of $\vec{J}$? or while preserving the vacuum equations, $\rho = \vec{J} = 0$? We show that the answers are yes. The second question raised at the beginning of this letter can be restated as follows: Say we have a set of data $(\mathscr{S}^3, \gamma, K)$ which satisfies the constraint equations (1)-(2), and say we know the exact values of γ and K in some open subset $\mathscr{W}$ of $\mathscr{S}^3$. Does the knowledge of γ and K in $\mathscr{W}$ allow us to control the number of wormholes that might connect points away from $\mathscr{W}$? The results we present here show that the answer is no.

These answers are obtained using a very strong, mathematical "gluing theorem". This new result involves analytically gluing a wormhole onto $(\mathscr{S}^3, \gamma, K)$ between small balls about points p_1 and p_2 in $\mathscr{S}^3$ in such a way to obtain a new solution of the constraint equations $(\hat{\gamma}, \hat{K})$ on the topologically distinct manifold $\hat{\mathscr{S}}^3$ obtained from $\mathscr{S}^3$ via the connected sum procedure described above. Here we are now augmenting the topological construction with a gluing procedure for both the geometric data given by γ and K together with the matter fields, if present, as represented by ρ and $\vec{J}$. Moreover we do this in such a way so that the new data $(\hat{\gamma}, \hat{K})$ is identical to the original data (γ, K) outside of these small balls (similarly for ρ and $\vec{J}$ in the non-vacuum setting). Our only requirement to carry this out is that the initial data $(\mathscr{S}^3, \gamma, K)$ satisfy the following *nondegeneracy* condition: when attempting to connect regions containing matter, we require that ρ be *strictly larger* than the length of $\vec{J}$ at the gluing points. When gluing vacuum regions, we require that there exist spatial neighborhoods $\mathscr{U}_a$ of the points p_a $(a = 1, 2)$ being glued so that there are no local space-time isometries in some space-time neighborhoods of those points.

For generic solutions one can find arbitrarily small sets $\mathscr{U}_1$ and $\mathscr{U}_2$ satisfying the no-isometries condition [2], and therefore the perturbation introduced in the original initial data set can generically be localised in sets as small as desired. Iterating the construction leads to solutions with an arbitrary number of wormholes present. This implies indeed that an exhaustive knowledge of $(\mathscr{S}^3, \gamma, K)$ away from the vicinity of p_1 and p_2 does not allow us to detect the presence or absence of handles connecting regions around p_1 and p_2.

While the details of the proof are very technical [4] the overall argument is simple enough to be given here. We choose balls $B(p_1, r_1) \subset \mathscr{U}_1$ and $B(p_2, r_2) \subset \mathscr{U}_2$ within which to do the gluing. In [2] it is shown that, under the nondegeneracy assumption, we can ϵ-perturb the data on $\mathscr{U}_1$ and $\mathscr{U}_2$, without changing them away from those regions, so that the constraint equations still hold, and so that there are no space-time isometries in any connected open set intersecting $B(p_a, r)$, for sufficiently small r. The next step is to use a theorem of Bartnik [1] to deform the balls $B(p_a, r)$ in space-time so that the trace of K is constant on $B(p_a, r)$, reducing r if necessary. The non-existence of space-time isometries is preserved under this deformation. This deformation is done so that we are in the setting in which a generalisation of the gluing theorem of [8] to compact manifolds with boundary, and to include matter fields, may be applied. This constitutes the third step in the construction and can be done by repeating the arguments of [7, 8] in this new setting. We thus obtain a one parameter family of initial data which satisfies the constraint equations, and which contains a neck connecting the spheres $S(p_a, r)$, parameterised by a parameter ϵ, which have the property that the initial data approach the original ones in a neighborhood of the $S(p_a, r)$'s when ϵ tends to zero. However, the transverse derivatives of those data do not match those of the original ones at the spheres. This problem is cured, for ϵ small enough, by a theorem in [3], which holds precisely under the "no local space-time isometries" condition. This provides the desired gluing, localised within the sets $\mathscr{U}_a$.

In addition to the conceptual implications of our results, another one is the existence of spatially compact space-times which contain no closed constant mean curvature surfaces whatsoever [4]. One would have preferred this to be wrong, because constant mean curvature foliations provide a preferred time parameter, in a space-time, which is very useful for studying the dynamical properties of the metric.

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