The classification of static vacuum space–times containing an asymptotically flat spacelike hypersurface with compact interior

Piotr T. Chruściel*
Département de Mathématiques
Faculté des Sciences
Parc de Grandmont
F37200 Tours, France

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Abstract

We prove non–existence of static, vacuum, appropriately regular, asymptotically flat black hole space–times with degenerate (not necessarily connected) components of the event horizon. This finishes the classification of static, vacuum, asymptotically flat domains of outer communication in an appropriate class of space–times, showing that the domains of outer communication of the Schwarzschild black holes exhaust the space of appropriately regular black hole exteriors.

1 Introduction

•1.1 A classical question in general relativity, first raised and partially answered by Israel [38], is that of classification of black hole solutions of the vacuum Einstein equations satisfying some regularity conditions. The most complete result existing in the literature is that of Bunting and Masood–ul–Alam [11] who show, roughly speaking, that all appropriately regular such black holes which *do not contain degenerate horizons* belong to the Schwarzschild family. In this paper we remove the condition of non–degeneracy of the event horizon and show the following:

•*1.1:* **ptc:** *The main text here should be essentially identical with the version published in [19], and indeed with the original gr-qc v1 version, except for an extended bibliography, and a few margin notes pointing out the existing problems, and an addendum at the end of the paper showing how those can be solved in the globally hyperbolic case. Furthermore, the numbering of references is different because of the new references from the Addendum.*

*Alexander von Humboldt fellow. Supported in part by a grant from the Polish Committee for Scientific Research (KBN) No 2 PO3B 073 15 and by the Humboldt Foundation. *Email*: Chrusciel@Univ-Tours.Fr

Theorem 1.1 $\bullet^{1.2}$ *Let (M, g) be a static solution of the vacuum Einstein equations with defining Killing vector X. Suppose that M contains a connected spacelike hypersurface Σ the closure $\bar{\Sigma}$ of which is the union of a finite number of asymptotically flat ends and of a compact interior, such that:*

1. *We have $g_{\mu\nu}X^\mu X^\nu < 0$ on[1] Σ.*
2. *The topological boundary $\partial\Sigma \equiv \overline{\Sigma} \setminus \Sigma$ of Σ is a nonempty topological manifold, with $g_{\mu\nu}X^\mu X^\nu = 0$ on $\partial\Sigma$.*

Then Σ is diffeomorphic to $\mathbb{R}^3$ minus a ball, so that it is simply connected, it has only one asymptotically flat end, and its boundary $\partial\Sigma$ is connected. Further there exists a neighborhood of Σ in M which is isometrically diffeomorphic to an open subset of the Schwarzschild space–time.

$\bullet$*1.2:* **ptc:** *The proof as given in the main, original body of the paper does not correctly exclude the possibility that there could be components of $\partial\Sigma$ on which $\nabla(g(X, X))$ vanishes and on which X has zeros; Proposition A.10 in the Addendum takes care of the problem.*

The various notions used here are spelled out in detail in Section 2 below.

Theorem 1.1 gives a complete classification of asymptotically flat, static space–times with singularity–free space–like hypersurfaces and with boundaries defined by the condition that the Lorentzian norm squared of the Killing vector field vanishes there, where the notion of "singularity–free" is made precise in the statement above. In fact, together with the Lichnerowicz theorem[2] (reviewed in Section 4 below) it gives a complete classification of vacuum space–times which contain an asymptotically flat spacelike hypersurface Σ with compact interior (the meaning of that notion should be clear from the statement of Theorem 1.1 and from the discussion in Section 4 below) and a Killing vector field which is timelike on Σ and satisfies the staticity condition. We note that we do *not* assume any causal regularity conditions on (M, g) (in fact, even the hypothesis of time orientability imposed in Section 2 below is not needed here). The result is sharp: the extension of the Curzon space–time constructed in [48] contains space–like hypersurfaces Σ which satisfy all the hypotheses above except for the (implicit) condition of compactness of $\partial\Sigma$. It has been conjectured by M. Anderson [1, Conjecture 0.3] that this condition[3] is not necessary when Σ is taken to be normal to the Killing vector field X.

A loose way of stating the main point of the above result, as compared to the ones previously available, is that we are showing non–existence of static, vacuum, regular black hole space–times with degenerate (not necessarily connected) components of the event horizon. We further note that the Bunting and Masood–ul–Alam version of the above result requires (in addition to the non–degeneracy condition) Σ to be normal to the Killing vector field X. The hypothesis that Σ

[1]We use the signature $(-,+,+,+)$.

[2]We note that the sharpest version of the Lichnerowicz theorem currently available in the static case is that of [1, Theorem 1.1].

[3]I am grateful to M. Anderson for useful comments concerning those points.

is normal to the Killing vector field has been relaxed under various further hypotheses, including various global causality hypotheses on (M, g) [15, 25, 46, 50], however no statement with the generality above is available in the literature even in the case where no degenerate horizons are present.

It might be of some interest to mention that our conclusion will still hold for quite a larger class of manifolds Σ. A possible generalization is that with Σ being *e.g.* the union of a) a finite number of asymptotically flat ends with b) a neighborhood of the boundary $\partial\Sigma$ which has compact closure and c) a non–compact region on which we have $-1 + \epsilon < g_{\mu\nu}X^\mu X^\nu < -\epsilon$, provided that Σ with the induced metric is a *complete* Riemannian manifold. The proof carries through without any modifications to this case.

We note that the closure $\bar{\Sigma}$ in point 2 of Theorem 1.1 (and everywhere else in this paper) is taken in the space-time M, and that $\partial\Sigma$ defined in this way is sometimes called the edge of Σ in the physical literature. This should *not* be confused with the *metric boundary* of (Σ, γ), where γ is, *e.g.*, the metric induced by g on Σ, or some other metric on Σ. By considering spacelike hypersurfaces in Schwarzschild space–time it is easily seen that $\partial\Sigma$ will typically have corners at points at which the Killing vector field vanishes, and is therefore *not* a differentiable submanifold of M in such cases. Further, it is not *a priori* clear that Σ can always be chosen so that $\partial\Sigma$ is a smooth submanifold of M and/or of $\bar{\Sigma}$ even at points at which X does not vanish.

Our strategy is essentially the same as that of Bunting and Masood–ul–Alam [11], though our starting points differ: while Bunting and Masood–ul–Alam consider the metric induced on the hypersurface normal to the Killing vector field, we consider the *orbit space* metric h on Σ, as defined in Section 3 below. The key first step, which is new, is the analysis of the geometry of (Σ, h) near both the degenerate components of $\partial\Sigma$ (*cf.* Proposition 3.2) and the non–degenerate ones (*cf.* Proposition 3.3). Next, following [11], we consider a manifold which consists of two copies of (Σ, h) glued along all *non–degenerate* components of $\partial\Sigma$, equipped with an appropriate conformally deformed metric. The key next element of our proof is a new version of the positive energy theorem proved in[4] [4] (*cf.* Theorem 5.2 below). Using those results one shows that the metric on Σ is conformally flat. One can then use classical arguments to finish the proof[5] (*cf., e.g.*, [43, Section II], together with [42, Section 3] or [9, Lemma 4]); we present here a new argument, essentially due to Herzlich[6] (M. Herzlich, private communication), which gives a considerably simpler proof of this last step and avoids the problems related to uniqueness of analytic extensions.

[4]Once this paper was written we have realized that Theorem 5.2 can also be inferred from [29, Theorem 6].

[5]We note that it is usual in this last step of the proof to invoke analyticity to conclude. Because analytic extensions of manifolds are not unique this is not sufficient without a more thorough justification.

[6]We are grateful to M. Herzlich for allowing us to reproduce his unpublished proof here.

Under the hypotheses of Theorem 1.1 there is no chance of getting more information about the size of the set on which the metric is that of a Schwarzschild space–time (consider any hypersurface Σ in the Schwarzschild–Kruszkal–Szekeres space–time, and set M to be any neighborhood of Σ which does not coincide with the Schwarzschild space–time; alternatively, identify t with $t+1$ in the Schwarzschild space–time). Thus, to get more information about the size of this set some more hypotheses are needed. A simplest result of this kind is the following:

Corollary 1.2 *Under the hypotheses of Theorem 1.1, assume further that*

3. *The orbits of the Killing vector X through Σ are complete.*

Then the following properties are equivalent:

i. *$\Sigma_{\rm ext}$ is achronal*[7] *in $M_{\rm ext}$.*

ii. *$M_{\rm ext}$ is diffeomorphic to $\mathbb{R}\times\Sigma_{\rm ext}$ (which is equivalent to $\mathcal{J}$ having $\mathbb{R}\times S^2$ topology).*

iii. *There are no closed timelike curves through $\Sigma_{\rm ext}$ contained in $M_{\rm ext}$.*

Further, if one (and hence all) of the above conditions holds, then the Killing development[8] *$\mathcal{K}(\Sigma)$ of Σ defined as*

$$\mathcal{K}(\Sigma)\equiv\cup_{t\in\mathbb{R}}\phi_t(\Sigma)\ , \tag{1.1}$$

where ϕ_t is the action of the isometry group generated by X, equipped with the induced metric, is isometrically diffeomorphic to a domain of outer communications in the Schwarzschild–Kruszkal–Szekeres space–time.

The definition of the domain of outer communications used here is given in Section 2 below.

Strictly speaking, both Theorem 1.1 and Corollary 1.2 are not statements about black holes, because it is not *a priori* clear what is the relationship between their hypotheses and the existence of a black hole region. Moreover, in Corollary 1.2 it is not clear how much of the space–time is covered by the Killing development of Σ. Now, there are various goals one might wish to achieve: a) one might rest content with Corollary 1.2 (this is indeed suggested by the relatively weak hypotheses thereof); b) one might want to show that the d.o.c. of (M,g) is isometrically diffeomorphic to a d.o.c. of the Schwarzschild–Kruszkal–Szekeres

[7]By that we mean that there are no timelike curves from $\Sigma_{\rm ext}$ to itself which are entirely contained in $M_{\rm ext}$.

[8]The notion of Killing development used here differs slightly from the definition of [6] as we allow here a topology of $\mathcal{K}(\Sigma)$ which is not $\mathbb{R}\times\Sigma$.

space–time; c) one might wish to show that (M, g) is the Schwarzschild–Kruszkal–Szekeres space–time.

Concerning b) above, we are not aware of any construction which would show that more hypotheses than those of Corollary 1.2 are needed to obtain this conclusion. Let us however note that there is such an example in electro–vacuum space–times, which is obtained as follows: let $(\hat{M}, \hat{g})$ be the extension of the Reissner–Nordström space–time described by the Carter–Penrose diagram on p. 158 of [33]. The diffeomorphism ψ obtained by mapping a point on this diagram to a point "shifted by two blocks up" is an isometry of this space. Then $(M = \hat{M}/\psi, g)$, where g is the obvious metric on M, is an electro–vacuum space–time in which we can find a hypersurface Σ satisfying the conditions of Corollary 1.1. (We note that in this space–time there are closed time–like curves through every point, but there are *no closed timelike curves through* $\Sigma_{\rm ext}$ *contained in* $M_{\rm ext}$.) The d.o.c. associated with any asymptotically flat region $M_{\rm ext}$ in M is the whole space–time M, and is therefore *not* isometrically diffeomorphic to a d.o.c. in $(\hat{M}, \hat{g})$ (which consists of only one of the blocks of the Carter–Penrose diagram on p. 158 of [33]).

While this example does not satisfy the vacuum Einstein equations, it clearly shows that the standard global techniques of Lorentzian geometry, which assume at most some energy inequalities, cannot be sufficient to achieve the conclusion that the d.o.c. of (M, g) is isometrically diffeomorphic to a d.o.c. of the Schwarzschild–Kruszkal–Szekeres space–time. Thus some further hypotheses need to be imposed unless a careful study of the vacuum field equations is performed. We note that invoking analyticity will not help, unless one has a proof that the metric has to be analytic up to and beyond the event horizon. In any case non–uniqueness of analytic extensions of Lorentzian manifolds leads to problems even if one assumes that the whole space–time is analytic. For example, the vacuum examples constructed in [17] show that neither analyticity alone, nor analyticity together with the set of hypotheses in [33] will suffice to conclude that (M, g) must be the Schwarzschild–Kruszkal–Szekeres space–time. We further note that it is certainly of interest to completely classify the exterior regions of black hole space–times, while it is perhaps of limited interest to try to set up some heavy set of hypotheses which will allow us to say something about what is happening beyond the event horizons. In any case b) and c) are separate issues. Here we will ignore question c) and only address question b), using the usual global Lorentzian techniques, and prove the following result:

Theorem 1.3 *Let (M, g) be a solution of the vacuum Einstein equations containing a connected space-like hypersurface Σ, the closure $\bar{\Sigma}$ of which is the union of a finite number of asymptotically flat ends and of a compact interior. Let X be a Killing vector field on M which is timelike, future directed in all the asymptotically flat ends and satisfies the staticity condition (2.2). Let further $\mathcal{D}_{oc} \equiv \mathcal{D}_{oc}(M_{\rm ext})$ be a domain of outer communications in (M, g) associated to* *one of the asymptotically flat ends of Σ. Suppose that:*

1. *We have $\Sigma \subset \mathcal{D}_{oc}$.*
2. *The topological boundary $\partial\Sigma \equiv \overline{\Sigma}\setminus\Sigma$ of Σ is a nonempty topological manifold and satisfies $\partial\Sigma = \overline{\Sigma} \cap \partial\mathcal{D}_{oc}$.*
3. *X has complete orbits in $\mathcal{D}_{oc}$.*

In addition to the above, suppose that one of the following conditions holds:

4a) Either $(\mathcal{D}_{oc}, g|_{\mathcal{D}_{oc}})$ is globally hyperbolic, or

4b) (M, g) is globally hyperbolic, or
•1.3

4c) there are no closed timelike curves through $\Sigma_{\rm ext}$ contained in $\mathcal{D}_{oc}$, with $\mathcal{D}_{oc}$ being moreover simply connected, or

4d) $\mathcal{D}_{oc} \setminus \{X = 0\}$ is simply connected, or

4e) Σ is achronal[9] *in $\mathcal{D}_{oc}$ and the white hole region (*cf. *Equation (2.6)) is empty, or*

4f) Σ is achronal in $\mathcal{D}_{oc}$ and X has no zeros on $\partial\Sigma$, or

4g) Σ is achronal in $\mathcal{D}_{oc}$ and X is nowhere light-like on $\overline{\Sigma}$.

•1.3: **ptc:** *Points 4a-4f require the supplementary hypothesis of non-existence of degenerate non-embedded prehorizons within $\mathcal{D}_{oc}$. Under the hypotheses of points 4a or 4b this property is established in the Addendum under a supplementary hypothesis of analyticity or axisymmetry, see Theorem A.1 there, and holds trivially under the hypotheses of point 4g.*

Then the conclusions of Theorem 1.1 and Corollary 1.2 hold. Moreover $\mathcal{D}_{oc}$ is isometrically diffeomorphic to a domain of outer communications of the Schwarzschild–Kruszkal–Szekeres space–time.

The set of hypotheses 4*a)*–4*g)* will be referred to as *alternative* hypotheses, while the remaining conditions will be referred to as the *main* hypotheses.

To avoid ambiguities, we emphasize that in Theorem 1.3 it is *not* assumed that X is timelike throughout Σ.

It should be clear from the long list of alternative conditions we have given that we are not satisfied with any single one of them. We note that the alternative hypothesis 4*a)* is rather natural in many global problems in general relativity. It has the elegant feature that it makes no hypotheses on the global causal structure of M away from $\mathcal{D}_{oc}$. It is necessary since any d.o.c. of the Schwarzschild–Kruszkal–Szekeres space–time has this property. (It can be pointed out that each of the alternative conditions above eventually implies that $\mathcal{D}_{oc}$ is globally hyperbolic, hence that the alternative condition 4*a)* holds.) This is the hypothesis

[9]By that we mean that there are no timelike curves from Σ to itself which are entirely contained in $\mathcal{D}_{oc}$.

that we consider to be the most satisfactory amongst all alternative hypotheses above. Nevertheless, in view of the rather weak conditions of Corollary 1.2 one is tempted to look for *a priori* weaker conditions which would lead to a theorem of the kind of Theorem 1.3. We conjecture that the hypothesis that Σ is achronal in $\mathcal{D}_{oc}$ should be enough for the conclusion of Theorem 1.3 to hold, thus the remaining restrictions in $4e)$–$4g)$ are unnecessary.

We further note the following:

1. While the alternative hypothesis $4b)$ might also be natural for several purposes, we note that it seems to exclude degenerate horizons at the outset. For example, consider the extension $(\hat{M}, \hat{g})$ of the extreme Reissner–Nordström space–time as described by the Carter–Penrose diagram given on p. 160 of [33]. By inspection of this diagram one easily finds that the only globally hyperbolic subset (M, g) of $(\hat{M}, \hat{g})$ which contains asymptotically flat hypersurfaces and has complete Killing orbits has to be one of the d.o.c.'s of that space–time, hence will have no horizons, no black–hole regions, and no hypersurfaces as required in the main hypotheses of Theorem 1.3. (We note, however, that a space–time with the global structure of $(\hat{M}, \hat{g})$ could *a priori* satisfy the main hypotheses above together with any alternative hypotheses other than $4b)$ and $4g)$.)

2. The alternative hypothesis $4c)$ together with the accompanying proof is (up to minor improvements) due to Carter [13, Theorem 4.1]. We do not find the hypothesis $4d)$ especially natural, we have added it for completeness as it is closely related to the hypothesis $4c)$ while being shorter to formulate.

3. Theorem 1.3 with the alternative hypothesis $4e)$ gives a classification of appropriate space–times which have no white hole regions – that is, "pure black hole" solutions.

4. The requirement that Σ be normal to the Killing vector field, which occurs in the original form of the Israel–Bunting–Masood–ul–Alam theorem, implies that the alternative hypothesis $4g)$ holds; as already mentioned, it excludes degenerate horizons at the outset.

We note that while Theorems 1.1 and 1.3 are both new in the generality given here, the transition from Theorem 1.1 to Theorem 1.3 is in principle known. We shall give complete proofs because the arguments needed are scattered across the literature [13, 16, 23, 25, 33, 52], and because those arguments are often carried out under hypotheses which are different from the ones made here.

This paper is organized as follows: Section 2 contains definitions and some preliminary remarks. In Section 3 we analyze the boundary conditions satisfied by the orbit–space metric near Killing horizons. In that section neither staticity nor energy inequalities are assumed to hold. In Section 4 we recall an elementary and well known proof of Theorem 1.1, based on the Komar identity, under the hypothesis that all horizons are degenerate. Actually in that proof asymptotically flat stationary space–times are allowed provided that all the Killing horizons are *non–rotating*, and some further conditions are satisfied; this is discussed in detail there. Theorem 1.1 is proved in Section 5, while Theorem 1.3 is proved in Section 6. We close the paper with some concluding remarks in Section 7.

Acknowledgments: The author wishes to emphasize the key contribution of R. Bartnik to the results presented here through the joint proof of the version of the positive energy theorem in [4]. He acknowledges useful discussions with or comments from M. Anderson, R. Bartnik, R. Beig, H. Friedrich, G. Galloway, W. Simon, R. Wald and G. Weinstein. •1.4

•*1.4:* **ptc:** *Furthermore, I am grateful to J.L. Costa for pointing out the problem with Lemma 4.1, addressed in the Addendum by repeating the arguments in [22].*

2 Preliminaries

All the manifolds are assumed to be paracompact, Hausdorff and smooth. Space–times are equipped with smooth metrics and are always assumed to be time–orientable.

The Hawking–Ellis [33] notation for causal futures $J^\pm(\Omega)$, chronal futures $I^\pm(\Omega)$, *etc.*, is used throughout. Further, whenever needed, we use the notation $I^+(A;\Omega)$ to denote the chronological future of a set A in a space–time Ω, *etc.*

A space–like hypersurface $\Sigma_{\rm ext}$ will be called an *asymptotically flat end* if it is diffeomorphic to $\mathbb{R}^3$ minus a ball and if the fields (g_{ij}, K_{ij}) induced on $\Sigma_{\rm ext}$ by the space–time metric satisfy the fall–off conditions

$$|g_{ij}-\delta_{ij}|+r|\partial_\ell g_{ij}|+\cdots+r^k|\partial_{\ell_1\cdots\ell_k}g_{ij}|+r|K_{ij}|+\cdots+r^k|\partial_{\ell_1\cdots\ell_{k-1}}K_{ij}|\le C_{k,\alpha}r^{-\alpha}\ , \tag{2.1}$$

for some constants $C_{k,\alpha}$, $\alpha>0$, $k\ge 1$. We shall always implicitly assume $\alpha>1/2$ when the ADM mass will be invoked, as this condition makes it well defined in vacuum. It follows in any case from [41] or from [15, Section 1.3] that in stationary vacuum space–times there is no loss of generality in assuming $\alpha=1$, k – arbitrary. A hypersurface will be said to be *asymptotically flat* if it contains an asymptotically flat end $\Sigma_{\rm ext}$.

A space–time (M,g) containing an asymptotically flat end $\Sigma_{\rm ext}$ will be called *static* if there exists on M a Killing vector field X which is *timelike* in $\Sigma_{\rm ext}$ and which satisfies

$$X_{[\alpha}\nabla_\beta X_{\gamma]}=0\ . \tag{2.2}$$

It can be shown under fairly weak hypotheses [6, 7] (which are satisfied under the hypotheses of Theorem 1.1 by [34]) that $\Sigma_{\rm ext}$ can be boosted so that X asymptotes the unit future directed normal to $\Sigma_{\rm ext}$, and we shall always assume that this is the case. A Killing vector field satisfying the above requirements will be called the *defining* Killing vector field of the static space–time. We emphasize that unless explicitly indicated otherwise we do *not* assume that the orbits of X are complete in M.

Let X be a Killing vector field which asymptotically approaches the unit normal to $\Sigma_{\rm ext}$ in an asymptotically flat end $\Sigma_{\rm ext}$. Passing to a subset of $\Sigma_{\rm ext}$ we can without loss of generality assume that X is time-like on $\Sigma_{\rm ext}$, and we shall always assume that this is the case. If the orbits of X through $\Sigma_{\rm ext}$ are complete, then an exterior four–dimensional asymptotically flat region can be obtained by moving $\Sigma_{\rm ext}$ around with the flow ϕ_t;

$$M_{\rm ext} = \cup_{t\in\mathbb{R}}\phi_t(\Sigma_{\rm ext}) \ . \tag{2.3}$$

Following [25], the *domain of outer communications* (d.o.c.) $\mathcal{D}_{oc}(M_{\rm ext})$ associated with $\Sigma_{\rm ext}$ or with $M_{\rm ext}$ is then defined as

$$\mathcal{D}_{oc}(M_{\rm ext}) = J^+(M_{\rm ext}) \cap J^-(M_{\rm ext}) = I^+(M_{\rm ext}) \cap I^-(M_{\rm ext}) \ . \tag{2.4}$$

(The equality $J^\pm(M_{\rm ext}) = I^\pm(M_{\rm ext})$ is easily verified; *cf., e.g.*, [52, Section 12.2] for a proof in a $\mathscr{I}$ context.) It is shown[10], under appropriate conditions, in [16, Section 1.3] that for stationary vacuum space–times the above definition of the domain of outer communications is equivalent to the standard one using $\mathscr{I}$ (*cf. e.g.* [33, 52]). The definition (2.4) turns out to be more convenient for many purposes.

The black hole region $\mathcal{B}$ associated with the asymptotic end $\Sigma_{\rm ext}$ or with $M_{\rm ext}$ is defined as

$$\mathcal{B} = M \setminus J^-(M_{\rm ext}) = M \setminus I^-(M_{\rm ext}) \ , \tag{2.5}$$

while the white hole region $\mathcal{W}$ associated with the asymptotic end $\Sigma_{\rm ext}$ or with $M_{\rm ext}$ is defined as

$$\mathcal{W} = M \setminus J^+(M_{\rm ext}) = M \setminus I^+(M_{\rm ext}) \ . \tag{2.6}$$

Thus the occurrence of boundaries of $\mathcal{D}_{oc}(M_{\rm ext})$ signals that of black hole or white hole regions.

To avoid ambiguities, we define the *Schwarzschild* space–time $(M^{\rm Schw}, g^{\rm Schw})$ to be the manifold $\{t\in\mathbb{R}, r\in(2m,\infty), q\in S^2\}$, with the metric

$$g^{\rm Schw} = -(1-\frac{2m}{r})dt^2 + (1-\frac{2m}{r})^{-1}dr^2 + r^2 d\Omega^2 \ , \tag{2.7}$$

where $d\Omega^2$ is the standard round metric on a unit two–dimensional sphere S^2. We will refer to those coordinates as the standard coordinates on the Schwarzschild space–time. We shall call a *Schwarzschild–Kruszkal–Szekeres space–time* the extension of $(M^{\rm Schw}, g^{\rm Schw})$ described *e.g.* by the Carter–Penrose diagram on page

[10]We note that Conjecture 1.8 of [16, Section 1.3], needed for this equivalence, has been settled in [6].

154 of [33]. We note that each of the two copies of $(M^{\rm Schw}, g^{\rm Schw})$ which can be seen on that diagram forms a d.o.c with respect to the appropriate asymptotic region. In Section 5 we shall need the so–called *isotropic* coordinates on the Schwarzschild space–time $(t, \bar r, q) \in \mathbb{R} \times (m/2, \infty) \times S^2$, with $\bar r$ defined via the equation $r = \bar r(1 + m/(2\bar r))^2$, in which the Schwarzschild metric takes the form

$$g^{\rm Schw} = -\Big(\frac{1 - m/2\bar r}{1 + m/2\bar r}\Big)^2 dt^2 + \Big(1 + \frac{m}{2\bar r}\Big)^4 \Big(d\bar r^2 + \bar r^2 d\Omega^2\Big) \ . \tag{2.8}$$

3 Boundary conditions at Killing horizons

In this section we shall consider Killing horizons in *stationary*[11] space–times; thus we shall *not* assume that the staticity condition (2.2) holds. Further *no field equations* or *energy inequalities* are assumed in this section. A *null hypersurface* $\mathcal{N}$ will be called a *Killing horizon* if

1. X is nowhere vanishing on $\mathcal{N}$, and
2. $g_{\alpha\beta}X^\alpha X^\beta \equiv 0$ on $\mathcal{N}$.

In particular X is necessarily tangent to the generators of $\mathcal{N}$. We shall sometimes write $\mathcal{N}_X$ for $\mathcal{N}$ to emphasize that the Killing horizon in question is associated with the given Killing vector field X. (The reader should be warned that what is usually called a "bifurcate horizon" (*cf., e.g.*, [40]) is *not* a Killing horizon in our terminology, rather it is the union of four Killing horizons and of the "bifurcation surface".) Recall that the surface gravity κ of a Killing horizon is defined by the formula

$$(X^\alpha X_\alpha)_{,\mu}\Big|_{\mathcal{N}_X} = -2\kappa X_\mu \ . \tag{3.1}$$

A Killing horizon $\mathcal{N}_X$ is said to be degenerate if κ vanishes throughout $\mathcal{N}_X$. From what is said in [10] one can infer the following:

Theorem 3.1 (Boyer [10]) *Let $\mathcal{N}$ be a C^1 Killing horizon, with X tangent to the generators of $\mathcal{N}$. Then:*

1. *X is nowhere vanishing on the closure $\overline{\mathcal{N}}$ of any degenerate connected component of $\mathcal{N}$.* •*3.1*

•*3.1:* **ptc:** *This is wrong, a counterexample is given in the Addendum. The error does not lie with [10], but with the author of the current work. Proposition A.10 of the Addendum provides a correct statement needed for the remaining arguments in this work*

[11] Part of the results presented in this section have been originally obtained under the hypothesis that X satisfies the staticity condition. G. Weinstein pointed out to us that this analysis carries over to the stationary case when h is interpreted as the orbit space metric, as defined below. This remark provided the breakthrough which led to the proof of the Lemma 3.5 below. We are grateful to him for this suggestion, and for several useful discussions.

2. *Let* $p \in \overline{\mathcal{N}}$ *satisfy* $X(p) = 0$*, set*

$$\mathcal{S} = \{p \,|\, X(p) = 0\} \ . \tag{3.2}$$

Then there exists a neighborhood $\mathcal{V}$ *of* p *such that the set* $\mathcal{S} \cap \mathcal{V}$ *is a smooth, embedded, space–like, two–dimensional submanifold of* M*. Moreover the null geodesics normal to* $\mathcal{S} \cap \mathcal{V}$ *are Killing orbits such that* $\mathcal{S} \cap \mathcal{V}$ *is the accumulation set of those orbits in* $\mathcal{V}$*.*

The set $\mathcal{S} \cap \mathcal{V}$ of point 2 above is usually called a *bifurcation surface* of the Killing horizon $\mathcal{N}$. A good model for this behavior is provided by the set of zeros of the usual Killing vector field $\partial/\partial t$ in the Schwarzschild–Kruszkal–Szekeres space–time.

In this section we wish to analyze the behavior near a Killing horizon $\mathcal{N}$ of the orbit space metric on a space–like hypersurface Σ such that $g_{\mu\nu}X^\mu X^\nu < 0$ on Σ, with $\partial\Sigma$ being a subset of the closure $\overline{\mathcal{N}}$ of $\mathcal{N}$. Let us start by defining the orbit space metric. Consider a point $p \in M$ such that $g_{\mu\nu}X^\mu X^\nu(p) \neq 0$, then we have the decomposition

$$T_pM = L(X) \oplus X^\perp,$$

where $L(X)$ is the vector space spanned by $X(p)$ and $X^\perp$ the space orthogonal to X. Let Y be a vector tangent to M at p, we can thus write

$$Y = Y_\parallel + Y_\perp \ ,$$

with self–explanatory notation. We define the *orbit space metric* h at p by the formula

$$h(Y, Z) = g(Y_\perp, Z_\perp) \ .$$

(We note that h coincides at p with the metric induced by g on any hypersurface N which has the property that $T_pN = X^\perp$; this fact will play some role later. However, we do *not* assume that Σ has this property. Similarly in this section we do *not* assume that $X^\perp$ forms an integrable distribution. Finally we stress that the metric h should *not* be identified with the (natural) *metric on the space of orbits* because we are *not* assuming any regularity properties of that space; in particular we do *not* assume that the space of orbits is a differentiable manifold, which is a minimum requirement for the introduction of a metric on it.) We have

$$Y_\perp = Y - \frac{g(X, Y)}{g(X, X)} X \ ,$$

so that

$$h(Y, Z) = g(Y, Z) - \frac{g(X, Y)g(X, Z)}{g(X, X)} \ . \tag{3.3}$$

Consider, first, the case in which the connected component S of $\partial\Sigma$ under consideration corresponds to a degenerate component of the event horizon:

$$\kappa\Big|_S = 0 \ . \tag{3.4}$$

We have the following:

Proposition 3.2 •3.2 *Let Σ be a C^3 space–like hypersurface in a space–time (M, g) with Killing vector X, suppose that Σ is C^3 and space–like up to*[12] *its boundary $\partial\Sigma$, with*

$$g_{\mu\nu}X^\mu X^\nu < 0 \text{ on } \Sigma, \quad g_{\mu\nu}X^\mu X^\nu = 0 \text{ on } \partial\Sigma \ .$$

Then every compact connected component S of $\partial\Sigma$ which intersects a C^2 degenerate Killing horizon $\mathcal{N}_X$ corresponds to a complete *asymptotic end of (Σ, h).*

•3.2: **ptc:** *the proof given here requires absence of zeros of X on $\partial\Sigma$, which has been "justified" by the incorrect point 2 of Theorem 3.1. An example of a degenerate Killing horizon with zeros of X is given in the Addendum, but in that example X is spacelike on both sides of $\partial\Sigma$, so this does not provide a counterexample to the statement, which we find likely to be true. Proposition A.10 in the Addendum proves non-existence of zeros of X on $\partial\Sigma$ under the supplementary condition of hypersurface-orthogonality of X (which is sufficient for our purposes in this work), or of smoothness of $\partial\Sigma$.*

Remark: We note a discrepancy between the degree of differentiability of $\mathcal{N}_X$ assumed here and that asserted in Proposition 4.2 below. It is conceivable that with some effort one could weaken the hypotheses of Proposition 3.2 — we have not attempted to do that, as this is irrelevant for the purpose of this paper. In view of potential applications of Proposition 3.2 to the classification of stationary black holes it might be of interest to fill this gap.

Proof: Let $\mathcal{N}_X^0$ be the connected component of $\mathcal{N}_X$ which intersects S. Connectedness of S and of $\mathcal{N}_X^0$ shows that we must have $\partial\Sigma \subset \overline{\mathcal{N}_X^0}$. Boyer's Theorem 3.1 implies that the Killing vector field X is nowhere vanishing on S, which leads to $\partial\Sigma \subset \mathcal{N}_X$.

Consider any one–sided neighborhood $\mathcal{O} \subset \overline{\Sigma}$ of S covered by coordinates $(y^i) = (x, v^A)$ in which S is given by the equation $x = 0$; passing to a subset of $\mathcal{O}$ if necessary we can without loss of generality assume that $\mathcal{O}$ has compact closure. Let ϕ_s denote the (perhaps locally defined) flow of the Killing vector field X and set

$$\mathcal{U} = \cup_{s\in(-\epsilon,\epsilon)}\phi_s(\mathcal{O}) \ .$$

Now the Killing vector field X is non–spacelike in $\mathcal{U}$, hence transverse to $\mathcal{O}$, so that the flow parameter s along the orbits of X can be used as a coordinate on $\mathcal{U}$, at least for ϵ small enough. In the coordinate system (s, y^i) the metric takes the form

$$g_{\mu\nu}dx^\mu dx^\nu = g_{ss}ds^2 + 2g_{is}dy^i ds + g_{ij}dy^i dy^j \ ,$$

and the set $\mathcal{O}$ is given by the equation $\{s = 0\}$. In particular $g_{ij}dy^i dy^j$ is the metric induced by $g_{\mu\nu}dx^\mu dx^\nu$ on $\mathcal{O}$; by construction $\mathcal{O}$ is spacelike up–to–boundary so that $g_{ij}dy^i dy^j$ is uniformly non–degenerate and C^2 up to the boundary $\partial\Sigma \cap \mathcal{U} = \{s = x = 0\}$. We also have $X = X^\mu\partial_\mu = \partial/\partial s$. X is normal to $\mathcal{N}$

[12]Throughout this work "up to" means "up to and including".

by hypothesis, which implies that for all vectors Z^ν tangent to $\mathcal{N}$ we have

$$g_{\mu\nu}X^\mu Z^\nu = 0 \quad \Longrightarrow \quad g_{ss}\Big|_{x=0} = g_{sA}\Big|_{x=0} = 0 \ . \tag{3.5}$$

Since the space–time metric is non–degenerate up–to–boundary and since the closure of the set $\{s = x = 0\}$ is compact, there exists a constant C such that

$$C^{-1} \leq |g_{sx}|\Big|_{x=0} \leq C \ . \tag{3.6}$$

Let $\breve{\mathcal{U}}$ denote the set of points in $\mathcal{U}$ with $x > 0$. In local coordinates y^i the definition (3.3) gives

$$h = h_{ij}dy^i dy^j = \Big(g_{ij} + \frac{g_{is}g_{js}}{|g_{ss}|}\Big)dy^i dy^j \ . \tag{3.7}$$

We set

$$V^2\Big|_{\breve{\mathcal{U}}} = -g_{ss} > 0 \ . \tag{3.8}$$

Clearly $V^2 = -g_{\mu\nu}X^\mu X^\nu$, so that V can be extended by continuity to a function defined on $\mathcal{U}$, and hence on $\overline{\Sigma}$, still denoted by V, by setting $V\Big|_{x=0} = 0$. In the (x, s, v^A) coordinate system it holds that

$$g_{ss,x}\Big|_{x=0} = -2\kappa g_{sx} \ . \tag{3.9}$$

In the degenerate case, since the metric is C^2 we then have

$$|g_{ss}| \leq \hat{C}x^2 \ , \tag{3.10}$$

for some constant $\hat{C}$. We wish to show that all curves $\gamma \subset \Sigma$ that approach the boundary $\{x = 0\}$ have infinite length in the metric h. Let us note that h can be written in the form

$$h_{ij}dy^i dy^j = \chi dx^2 + h_{AB}(dv^A + f^A dx)(dv^B + f^B dx) \ , \tag{3.11}$$

where f^A is a solution of the equation

$$g_{AB}f^B = g_{Ax} + \frac{g_{As}}{|g_{ss}|}(g_{xs} - g_{Bs}f^B) \ , \tag{3.12}$$

and χ is given by

$$\begin{aligned} \chi &= g_{xx} + \frac{g_{xs}^2}{|g_{ss}|} - h_{AB}f^A f^B \\ &= \frac{g_{xs}}{|g_{ss}|}(g_{xs} - g_{Cs}f^C) - g_{Cx}f^C + g_{xx} \ . \end{aligned} \tag{3.13}$$

We wish to extract the most singular part of χ from the equations above. In order to do that, first note that (3.5) implies that we have

$$|g_{As}| \le Cx$$

for some constant C, so that one expects the terms $g_{sx}g_{sx}/|g_{ss}|$ to dominate in (3.7). This is, roughly speaking, the case, as can be seen as follows: Equation (3.12) leads to

$$\Big(|g_{ss}| + \hat g^{AB} g_{As} g_{Bs}\Big) g_{Cs} f^C = \hat g^{AB} g_{Bs} \Big(g_{Ax} |g_{ss}| + g_{xs} g_{As}\Big) ,$$

where $\hat g^{AB}$ is the matrix inverse to g_{AB}. It follows that

$$\begin{aligned}
g_{AB} f^B &= \frac{g_{As}(g_{sx} - \hat g^{CD} g_{Cx} g_{Ds})}{|g_{ss}| + \hat g^{CD} g_{Cs} g_{Ds}} + g_{Ax} \\
&= \frac{g_{As} g_{sx}}{|g_{ss}| + \hat g^{CD} g_{Cs} g_{Ds}} + \ O\Big(\frac{x^2}{|g_{ss}| + \hat g^{CD} g_{Cs} g_{Ds}}\Big) + O(1) , \\
\chi &= \frac{(g_{sx} - \hat g^{AB} g_{Ax} g_{Bs})^2}{|g_{ss}| + \hat g^{AB} g_{As} g_{Bs}} + g_{xx} - \hat g^{AB} g_{Ax} g_{Bx} \\
&= \frac{g_{sx}^2}{|g_{ss}| + \hat g^{AB} g_{As} g_{Bs}} + \ O\Big(\frac{x}{|g_{ss}| + \hat g^{AB} g_{As} g_{Bs}}\Big) + O(1) \ ,
\end{aligned}$$

so that there exists a constant $\epsilon > 0$ such that for x small enough we have

$$\chi \ge \frac{\epsilon}{x^2} \ .$$

Consider a curve $[0,1) \ni s \to \gamma(s) \subset \{t = 0\}$ such that $\lim_{s_i \to 1} x(\gamma(s_i)) = 0$ for some sequence s_i. Let $\ell(\sigma)$ denote the h–length of $\gamma([0,\sigma])$; by (3.11) we have

$$\begin{aligned}
\ell(\sigma_i) = \int_0^{\sigma_i} \sqrt{h_{ij}\dot\gamma^i\dot\gamma^j}\, ds \ &\ge \ \int_0^{\sigma_i} \sqrt{\chi}\Big|\frac{dx(\gamma(s))}{ds}\Big|\, ds \\
&\ge \ \int_0^{\sigma_i} \frac{\sqrt{\epsilon}}{x(s)}\Big|\frac{dx(\gamma(s))}{ds}\Big|\, ds \\
&\ge \ \sqrt{\epsilon}\Big(|\ln(x(\gamma(\sigma_i))| - |\ln(x(\gamma(0))|\Big) \ \longrightarrow_{\sigma_i \to 1} \ \infty
\end{aligned}$$

(this last inequality requires a not very difficult justification), which is what had to be established. ■

An example of the behavior described above is given by the extreme Reissner–Nordström space–time. In this case in appropriate coordinates the metric approaches the standard product metric on the cylinder $\mathbb{R} \times S^2$ in the asymptotic end constructed as above.

Let us turn our attention to the case where κ has no zeros:

Proposition 3.3 *Let Σ be a smooth space–like hypersurface in a space–time (M,g) with Killing vector X, suppose that Σ is smooth and space–like up to*[12] *its topological boundary $\partial\Sigma \equiv \bar{\Sigma} \setminus \Sigma$, except perhaps at those points of $\partial\Sigma$ at which X vanishes, with*

$$g_{\mu\nu}X^\mu X^\nu < 0 \text{ on } \Sigma, \quad g_{\mu\nu}X^\mu X^\nu = 0 \text{ on } \partial\Sigma \ .$$

Then every connected component S of $\partial\Sigma$ which intersects a smooth Killing horizon $\mathcal{N}_X$ on which

$$\kappa > 0$$

corresponds to a totally geodesic boundary of (Σ, h), with h being smooth up–to–boundary. Moreover

1. *a doubling*[13] *of (Σ, h) across S leads to a smooth metric on the doubled manifold,*
2. *with $\sqrt{-g_{\mu\nu}X^\mu X^\nu}$ extending smoothly to $-\sqrt{-g_{\mu\nu}X^\mu X^\nu}$ across S.*

Remarks: 1. We note that our proof does *not* require κ to be constant on $\partial\Sigma$, as long as it has no zeros. It is nevertheless worth mentioning that in *static* space–times κ is always constant on connected components of $\mathcal{N}$, independently of any field equations [46, Corollary 2.2] (*cf.* also [13, Theorem 8] in the vacuum case). Further, independently of staticity, κ is constant by Theorem 3.1 and by [40, p. 59] on any connected component $\mathcal{N}_X^0$ of $\mathcal{N}_X$ such that X has zeros on the closure of $\mathcal{N}_X^0$.

2. When Σ is orthogonal to the Killing vector and the space–time is vacuum or electro–vacuum the result is well known [38, 39]; in this case our approach seems to be simpler than elsewhere. The general result presented here seems to be new.

3. We emphasize that while the topological boundary $\partial\Sigma$ of Σ will typically have corners at those points at which X vanishes, Proposition 3.3 shows that one can introduce a differentiable structure on $\bar{\Sigma}$ such that the h–metric boundary of Σ will be a smooth submanifold of $\bar{\Sigma}$. .

Proof: We shall treat the case in which X has no zeros on $\partial\Sigma$ separately:

Lemma 3.4 *Under the hypotheses of Proposition 3.3, assume further that X has no zeros on some open connected set $\mathcal{O} \subset \partial\Sigma$. Then the conclusion of Proposition 3.3 holds near $\mathcal{O}$.*

Proof: As X has no zeros on $\mathcal{O}$ we can construct a coordinate system (s, x, v^A) in a neighborhood of $\mathcal{O}$ as in the proof of Proposition 3.2. From Eq. (3.9) we have

$$V^2 = 2\tilde{\kappa}x + O(x^2) \ , \qquad \tilde{\kappa} = \kappa g_{sx}\Big|_{x=0} \ .$$

[13]See the proof of Theorem 1.1, Section 5 below, for an explicit construction of the doubling.

Let $x = w^2$, it holds that

$$\begin{aligned} h_{ij}dy^i dy^j &= 4\Big(g_{sx}^2\Big(\frac{w}{V}\Big)^2 + w^2 g_{xx}\Big)\,dw^2 + 4w\Big(g_{xA} + g_{sx}\frac{g_{sA}}{w^2}\Big(\frac{w}{V}\Big)^2\Big)\,dv^A dw \\ &\quad + \Big(g_{AB} + g_{sA}\frac{g_{sB}}{w^2}\Big(\frac{w}{V}\Big)^2\Big)\,dv^A dv^B \;. \end{aligned} \tag{3.14}$$

Note that all the functions appearing in (3.14) are bounded, and that g_{AB}, g_{xx} and g_{sx} are smooth functions of (w^2, v^A) up to the boundary $\{w = 0\}$. We also have $g_{sA} = xg_A$ for some functions g_A which are smooth in (x, v^A), so that $\frac{g_{sA}}{w^2}$ is again a smooth function of (w^2, v^A) up to the boundary $\{w = 0\}$. Similarly $g_{ss} = -xg$ for some function g which is smooth up–to–boundary in (x, v^A), with $g|_{x=0} = 2\tilde\kappa \neq 0$. It follows that $w/V = 1/\sqrt{g}$ is a a smooth function of (w^2, v^A), up to $\{w = 0\}$. Equations (3.5)–(3.6) further show that the metric is uniformly non–degenerate up–to–boundary. The fact that the boundary is totally geodesic follows immediately from the definition of the extrinsic curvature and from the equality

$$\frac{\partial\phi(w^2, v^A)}{\partial w}\Big|_{w=0} = 2w\frac{\partial\phi(x, v^A)}{\partial x}\Big|_{x=0} = 0 \;,$$

for any differentiable function $\phi(x, v^A)$.

To justify the claims about the doubled manifold[13], recall that the double is constructed by allowing w to take negative values and by extending the relevant functions via $f(w) = f(-w)$. Thus w^2 is a smooth function on the doubled manifold in a neighborhood of S, and we have

$$4w\Big(g_{xA} + g_{sx}\frac{g_{sA}}{w^2}\Big(\frac{w}{V}\Big)^2\Big)\,dv^A dw = 2\Big(g_{xA} + g_{sx}\frac{g_{sA}}{w^2}\Big(\frac{w}{V}\Big)^2\Big)\,dv^A d(w^2) \;,$$

which is a smooth tensor field on the doubled manifold from what has been said above. The remaining coordinate components of h are obviously smoothly extendible to the double. Similarly V/w is a smooth scalar field when extended by an even function of w on the double, and our claims follow. ■

Proposition 3.3 follows immediately from Lemma 3.4 and from the following:

Lemma 3.5 *Under the hypotheses of Proposition 3.3, assume instead that X vanishes at a point $p \in \partial\Sigma$. Then there exists a neighborhood $\mathcal{O} \subset \partial\Sigma$ of p such that the conclusion of Proposition 3.3 holds near $\mathcal{O}$.*

Remark: We emphasize that we do *do not* assume that X vanishes throughout $\partial\Sigma$.

PROOF: By Theorem 3.1 there exists a neighborhood $\mathcal{V}$ of p such that the set $\{p \,|\, X(p) = 0\} \cap \mathcal{V}$ is a smooth, embedded, space–like, two–dimensional submanifold of M. Passing to a subset of $\mathcal{V}$ if necessary, $\mathcal{V}$ can be covered by a Rácz–Wald–Walker coordinate system[14] [45, 53] (u, v, x^a), with x^a being coordi-

[14] The coordinates (u, v, x^a) here correspond to the coordinates $(U, -V, x^\alpha)$ of [45]; we also use a different normalization of the Killing vector X.

nates on $\{p\,|\,X(p)=0\}\cap\mathcal{V}$, and with u and v covering the set $|uv|<\epsilon$, $|u|<\epsilon$, $|v|<\epsilon$, for some $\epsilon>0$. In this coordinate system the space–time metric g takes the form

$$g = 2G du\, dv + 2vH_a dx^a du + g_{ab} dx^a dx^b \ , \tag{3.15}$$

with G, h_a, g_{ab} being smooth functions of (uv, x^a). Here g_{ab} is a two by two strictly positive matrix, and G satisfies

$$C^{-1} \le G \le C \ ,$$

for some constant C. Orientations have been chosen so that $\Sigma\cap\mathcal{V}\subset\{u>0, v>0\}$, which implies $\partial\Sigma\cap\mathcal{V}\subset\{u=0\}\cup\{v>0\}$. In this coordinate system the Killing vector field X takes the following simple form:

$$X = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \ . \tag{3.16}$$

Since the vector field $\partial/\partial v$ is light-like throughout $\mathcal{V}$ it is transverse to Σ, hence in a neighborhood of any of its points $\Sigma\cap\mathcal{V}$ can be described as

$$\Sigma\cap\mathcal{V} = \{v = \phi(u, x^a)\} \ ,$$

for some smooth function ϕ. In this coordinate system the metric γ induced by g on Σ takes the form

$$\gamma = 2G du\, d\phi + 2\phi H_a dx^a du + g_{ab} dx^a dx^b \ . \tag{3.17}$$

The hypothesis that Σ is spacelike up to boundary implies that

$$0 < \frac{\partial\phi}{\partial u} \ . \tag{3.18}$$

We further have

$$g(X,\cdot) = G(u\, dv - v\, du) + uvH_a dx^a \ , \tag{3.19}$$

so that the orbit space metric h, as defined by Equation (3.3), reads

$$h = \frac{G}{2uv}(d(u\phi))^2 + 2H_a dx^a d(u\phi) + \Big(g_{ab} + \frac{u\phi}{2G}H_a H_b\Big) dx^a dx^b \ . \tag{3.20}$$

Define a function w on $\Sigma\cap\mathcal{V}$ by the formula

$$w^2 = \phi u \ .$$

We have $\phi u = (vu)|_\Sigma > 0$ so that w is well defined and takes real values. Moreover the derivative

$$\frac{\partial(w^2)}{\partial u} = u\frac{\partial\phi}{\partial u} + \phi$$

is strictly positive in virtue of Equation (3.18) and of the inequalities $\phi > 0$, $u|_\Sigma > 0$. This shows that (w, x^a) can be used as coordinates on $\Sigma \cap \mathcal{V}$. In this coordinate system we have

$$h = 2G(dw)^2 + 4wH_a dx^a dw + \Big(g_{ab} + \frac{w^2}{2G} H_a H_b\Big) dx^a dx^b \ . \tag{3.21}$$

We note that the functions G, H_a and g_{ab} appearing in this last equation are smooth functions of (w^2, x^a). The set $\mathcal{O}$ is defined now as $\partial\Sigma \cap \mathcal{V}$. The remaining claims follow now as in the proof of Lemma 3.4. ■

4 A proof based on the Komar identity

In the analysis below we shall need the Vishveshwara–Carter Lemma:

Lemma 4.1 (Vishveshwara–Carter Lemma [12, 51]) *Let (M, g) be a smooth space–time with Killing vector X satisfying the staticity condition (2.2), define $\hat{\mathcal{N}}$ to be the boundary of the set $\{g_{\alpha\beta}X^\alpha X^\beta < 0\}$, set*

$$\mathcal{N} \equiv \hat{\mathcal{N}} \cap \{X^\alpha \neq 0\}.$$

Then $\mathcal{N}$ is a smooth null hypersurface, with the Killing vector field X normal to $\mathcal{N}$. •*4.1*

•*4.1:* **ptc:** *there is a problem here, $\mathcal{N}$ could fail to be embedded; this is addressed in Theorem A.1 of the Addendum, under supplementary conditions.*

Remark: We emphasize that it is not assumed here that $\mathcal{N}$ satisfies some non–degeneracy conditions.

PROOF: Let $p \in \mathcal{N}$; from the staticity condition (2.2) and from the Frobenius theorem [36, Section 9.1] there exists a neighborhood $\mathcal{O}$ of p, with X nowhere vanishing on $\mathcal{O}$, foliated by a family of smooth hypersurfaces Σ_τ which are normal to X. By passing to a subset of $\mathcal{O}$ if necessary we may without loss of generality suppose that $\mathcal{O}$ has compact closure. The staticity condition (2.2) can be rewritten as

$$2\nabla_{[\mu}X^\alpha X_{\nu]} = X^\alpha \nabla_{[\mu}X_{\nu]} \ ,$$

which after a contraction with X_α gives

$$\nabla_{[\mu}W X_{\nu]} = W\nabla_{[\mu}X_{\nu]} \ , \qquad W \equiv X_\alpha X^\alpha \ . \tag{4.1}$$

On $\mathcal{O} \cap \{W < 0\}$ eq. (4.1) takes the form

$$\nabla_{[\mu}\Big(\ln(-W)\Big)X_{\nu]} = \nabla_{[\mu}X_{\nu]} \ . \tag{4.2}$$

Let ℓ_μ be any smooth covector field on $\mathcal{O}$ such that $\ell_\mu X^\mu = 1$ and let Z^μ be any smooth vector field tangent to a leaf Σ_τ such that $\Sigma_\tau \cap \{W < 0\} \neq \emptyset$; at points at which (4.2) holds contraction of this equation with $Z^\mu \ell^\nu$ gives

$$Z^\mu \nabla_\mu \ln(-W) = 2 Z^\mu \ell^\nu \nabla_{[\mu} X_{\nu]} \ . \tag{4.3}$$

The right–hand–side of the above equation is uniformly bounded on Σ_τ, so that $\ln(-W)$ has uniformly bounded gradient on $\Sigma_\tau \cap \{W < 0\}$. In particular $\ln(-W)$ is uniformly bounded on $\Sigma_\tau \cap \{W < 0\}$, which is only possible if $\Sigma_\tau \cap \{W = 0\} = \emptyset$. This shows that if $\Sigma_{\tau_0} \cap \{W = 0\} \neq \emptyset$ then $W \equiv 0$ on Σ_{τ_0}. Hence $\{W = 0\} \cap \mathcal{O}$ is a union of leaves of the Σ_τ foliation. In particular each connected component of $\mathcal{N} \cap \mathcal{O}$ coincides with a leaf of the Σ_τ foliation, hence is a smooth hypersurface normal to X. ■

Non–existence of black holes with *all* connected components of the event horizon degenerate, or with empty $\partial\Sigma$, can be established as follows: Suppose, as in Theorem 1.1, that the space–time is static, and that $\bar{\Sigma}$ is the union of a finite number of asymptotically flat ends and of a compact interior. We have the Komar identity

$$\begin{aligned} m_K &= \frac{1}{8\pi} \int_{S_\infty} \nabla^\mu X^\nu dS_{\mu\nu} \\ &= \frac{1}{8\pi} \Big\{ \frac{1}{2} \int_\Sigma \nabla_\mu \nabla^\mu X^\nu dS_\nu + \int_{\partial\Sigma} \nabla^\mu X^\nu dS_{\mu\nu} \Big\} \\ &= \frac{1}{8\pi} \int_{\partial\Sigma} \nabla^\mu X^\nu dS_{\mu\nu} \ . \end{aligned} \tag{4.4}$$

Here m_K is the sum of the Komar masses of the asymptotically flat ends, as defined by the first line of Equation (4.4), X^μ is the Killing vector field which asymptotes to $\partial/\partial t$ in the asymptotically flat ends, S_∞ is the "union of spheres at infinity" in the asymptotically flat ends, and we have used the equation $\nabla_\mu \nabla^\mu X^\nu = 0$ that holds for a Killing vector field in vacuum. A theorem of Beig [5] (*cf.* also [2, 14]) shows that the Komar mass of each static asymptotically vacuum end $(\Sigma_{\rm ext}, g|_{\Sigma_{\rm ext}})$ coincides with its ADM mass, so that m_K coincides with the sum of the ADM masses of the asymptotically flat ends. Boyer's Theorem 3.1 shows that X is nowhere vanishing on $\partial\Sigma$. Further, the Vishveshwara–Carter Lemma 4.1 •4.2 and staticity imply that $\partial\Sigma$ is a subset of the Killing horizon, in particular (slightly deforming Σ if necessary) $\partial\Sigma$ is a smooth submanifold of $\overline{\Sigma}$. Those facts and an easy calculation show (*cf., e.g.*, [3]) that

*•4.2: **ptc:** here $\partial\Sigma$ is assumed to be a compact embedded topological manifold, so the embededness problem in Lemma 4.1 does not arise by hypothesis*

$$\frac{1}{8\pi} \int_{\partial\Sigma} \nabla^\mu X^\nu dS_{\mu\nu} = \frac{1}{4\pi} \sum \kappa_i A_i = 0 \ . \tag{4.5}$$

Here the κ_i's are the surface gravities of the connected components of the event horizon (which vanish by hypothesis), and the A_i's are the areas of the corresponding components of $\partial\Sigma$; in the last line of (4.5) the sum is over an empty index set if $\partial\Sigma$ is empty. If $\partial\Sigma = \emptyset$ it follows from the rigid positive energy theorem proved in [6] that Σ can be embedded into Minkowski space–time so that the metric on Σ is the pull–back of the Minkowski space–time metric, with the extrinsic curvature tensor of Σ taking the appropriate values corresponding to the embedding. (Further the image of Σ has to be a Cauchy surface in Minkowski space–time [6]; in particular the maximal globally hyperbolic vacuum development of the initial data induced by g on Σ is the Minkowski space–time.) If $\partial\Sigma \neq \emptyset$ the positive mass theorem with marginally trapped boundary [34] gives a contradiction. This proves that the case in which all components of the boundary of Σ meet degenerate Killing horizons cannot occur under the hypotheses of Theorem 1.1.

It is of interest to enquire how to modify this argument if stationary asymptotically flat space–times are considered which do not necessarily satisfy the staticity condition. Clearly the Komar identity (4.4) still holds. If $\partial\Sigma = \emptyset$, the fact that (Σ, g, K) can be appropriately embedded in Minkowski space–time follows as above, whenever Σ is the union of a finite number of asymptotically flat regions and of a compact set. This is the classical Lichnerowicz theorem[2]. Now, if $\partial\Sigma \neq \emptyset$, the Vishveshwara–Carter lemma does not apply any more, so the Killing vector does not need to be tangent to the generators of $\mathcal{N}$. Further a degenerate non–static horizon could be non–differentiable. We note that in this case one can proceed as follows: Let (M, g) be an asymptotically flat solution of the vacuum Einstein equations with Killing vector field X. Suppose, as in Theorem 1.1, that M contains a connected space-like hypersurface Σ the closure $\bar{\Sigma}$ of which is the union of a finite number of asymptotically flat ends and of a compact interior, such that the topological boundary $\partial\Sigma \equiv \overline{\Sigma} \setminus \Sigma$ of Σ is a nonempty topological manifold. Assume that the Killing vector X is timelike future directed in all the asymptotic regions. Suppose further that $\partial\Sigma$ is a subset of a degenerate Killing horizon $\mathcal{N}$. This is certainly a supplementary hypothesis, as compared to the static case, which can be interpreted as the hypothesis that *all Killing horizons are non–rotating*. If one knew that the horizon is differentiable one could obtain the equality (4.5), and conclude as before that no such space–times exist. Let us show that the required differentiability must hold:

Proposition 4.2 *A Killing horizon is a* locally achronal *hypersurface of at least C^1 differentiability class. More precisely, the intersection of a connected component of a Killing horizon with any connected, globally hyperbolic open set $\mathcal{O}$ is a C^1 achronal hypersurface in $\mathcal{O}$.*

Remark: We note that a Killing horizon must be C^∞ (or as differentiable as the metric allows) in a neighborhood of any point thereof for which $\kappa \neq 0$ – this follows immediately from the fact that at such points $g_{\mu\nu}X^\mu X^\nu$ has non–zero gradient. Thus the only problematic points as far as differentiability is concerned are those at which κ vanishes.

PROOF: Let us start by showing that a Killing horizon $\mathcal{N}$ is necessarily *locally* achronal, that is, for every $p \in \mathcal{N}$ there exists a neighborhood $\mathcal{O}$ such that

$$\forall\, q \in \mathcal{N} \cap \mathcal{O} \qquad I(p;\mathcal{O}) \cap I(q;\mathcal{O}) = \emptyset \ .$$

Indeed, let $\mathcal{O}$ be any globally hyperbolic neighborhood $\mathcal{O}$ of p such that $\mathcal{N} \cap \mathcal{O}$ is connected; changing time orientation if necessary connectedness of $\mathcal{N} \cap \mathcal{O}$ shows that for all $q \in \mathcal{N} \cap \mathcal{O}$ and for all future directed timelike vectors T in T_qM we have

$$g(T,X) < 0 \ . \tag{4.6}$$

Let $q \in I(p;\mathcal{O})$, by global hyperbolicity there exists a timelike geodesic segment $\gamma \subset \mathcal{O}$ with q and p as endpoints. We have

$$\frac{dg(\dot\gamma, X)}{ds} = 0 \ ,$$

which together with (4.6) shows that γ can intersect $\mathcal{N}$ only at p.

We have thus shown that $\mathcal{N}$ is necessarily a *locally* achronal null hypersurface generated by the null geodesics tangent to X, such that every point on $\mathcal{N}$ is an interior point of a generator. Hence, by well known properties of such hypersurfaces, $\mathcal{N}$ has to be of C^1 differentiability class (*cf., e.g.* [18] for a simple proof). ■

5 Considerations global in space

We shall need the following, essentially obvious, result. For future reference, we state it in a context more general than the vacuum Einstein equations:

Lemma 5.1 *Suppose that (M,g) is static, and suppose that the couple $(\hat h, \hat V)$, where $\hat h$ is the metric induced on the hypersurfaces orthogonal to X and $-\hat V^2$ is the square of the Lorentzian norm of X on those hypersurfaces, satisfies some coordinate–independent system of equations. Then the orbit space–metric h together with the function V (such that $-V^2$ is the square of the Lorentzian norm of X on Σ) satisfies the same system of equations.*

PROOF: For any point $p \in \Sigma$ there exists a neighborhood $\breve{\mathcal{U}}$ on which we can construct coordinates as in the beginning of the proof of Proposition 3.2, with the difference that we are not assuming that we are near the boundary, so that the set $\mathcal{O} \subset \Sigma$ considered there is now some neighborhood of p in Σ. Passing to an appropriate smaller subset of $\mathcal{O}$ and then decreasing ϵ if necessary, where ϵ is as in the proof of Proposition 3.2, staticity and the Frobenius theorem imply that there exists a function $t \in C^\infty(\breve{\mathcal{U}})$ ($\breve{\mathcal{U}}$ again as in the proof of Proposition 3.2) such that the metric takes the form

$$ds^2 = -\hat{V}^2 dt^2 + \hat{h} \ , \tag{5.1}$$

with $X = \partial/\partial t$. The level sets of t are by definition normal to the Killing vector field which is time-like on $\breve{\mathcal{U}}$, so that $\hat{h}$ is Riemannian. This time function t is defined uniquely up to a constant. On $\breve{\mathcal{U}}$ we have $X = \partial/\partial s = \partial/\partial t$, which implies

$$t = s + f(y^i) \ , \quad dt = ds + \frac{g_{si}}{g_{ss}} dy^i \ , \tag{5.2}$$

$$\hat{V}^2\Big|_{\breve{\mathcal{U}}} = -g_{ss} = V^2 > 0 \ ,$$

$$\hat{h} = \hat{h}_{ij} dy^i dy^j = \Big(g_{ij} + \frac{g_{is}g_{js}}{|g_{ss}|}\Big) dy^i dy^j = h_{ij} dy^i dy^j \ , \tag{5.3}$$

and the result follows. ■

In the proof of Theorem 1.1 we shall need the following version of the positive energy theorem, proved[4] in [4]:

Theorem 5.2 *Let (Σ, h) be a smooth complete Riemannian manifold with an asymptotically flat end $\Sigma_{\rm ext}$ and with positive scalar curvature. If the Ricci scalar is integrable[15] on $\Sigma_{\rm ext}$ then the ADM mass of $\Sigma_{\rm ext}$ satisfies*

$$m \geq 0 \ .$$

Moreover, if the equality is attained, then there exists a diffeomorphism $\psi : \Sigma \to \mathbb{R}^3$ such that h is the pull–back by ψ of the standard Euclidean metric δ on $\mathbb{R}^3$.

We emphasize that in the result above Σ can have an arbitrary number (perhaps infinite) of asymptotic ends, and that *no hypotheses* are made on the asymptotic behavior of the metric in those ends except that the metric h is complete (and except, of course, that at least one of the ends is asymptotically flat so that its ADM mass is well defined). More general results, allowing for non–vanishing extrinsic curvature of the initial data hypersurface, poor differentiability of the metric, and boundaries, can be found in [4].

We are ready now to pass to the

Proof of Theorem 1.1: Consider the manifold Σ equipped with the metric h given in local coordinates by Equation (3.7). By Lemma 4.1 •5.1 the boundary $\partial\Sigma$ is a subset of the closure of the Killing horizon $\mathcal{N}$, which is a smooth

•5.1: **ptc:** *here $\partial\Sigma$ is assumed to be a compact embedded topological manifold, so the embededness problem arising in Lemma 4.1 does not arise by hypothesis*

[15]If the Ricci scalar is not integrable the conclusion still holds with $m = \infty$.

submanifold of M by that same lemma except at points at which X vanishes. Slightly deforming Σ in space–time if necessary we can without loss of generality assume that $\partial\Sigma$ is a smooth sub-manifold both of the space–time and of $\overline{\Sigma}$, except perhaps at those points at which X vanishes. Further deforming Σ if necessary we may assume that the metric on $\overline{\Sigma}$ is *spacelike up to boundary.* We can thus use Propositions 3.2 and 3.3 to conclude that the pair (Σ, h) is a complete Riemannian manifold with compact boundary and with at least one asymptotically flat end $\Sigma_{\rm ext}$. Let us denote by $\partial_{nd}\Sigma$ the collection of all those components of the boundary of Σ which correspond to non–degenerate components of the event horizon of the black hole; by Propositions 3.2 and 3.3 the metric boundary of (Σ, h) is $\partial_{nd}\Sigma$; it is compact and totally geodesic. (We note that the case where there are only degenerate horizons has already been excluded in Section 4.) By Lemma 5.1 the metric h and the function V satisfy the equations (*cf., e.g.*, [11])

$$\Delta_h V = 0 \ , \tag{5.4}$$

$$R_{ij} = V^{-1} D_i D_j V \ , \tag{5.5}$$

where Δ_h is the Laplace–Beltrami operator of the metric h and R_{ij} is its Ricci tensor. Following [11], set

$$\begin{aligned} \Sigma_+ &= \Sigma, \qquad h_+ = \left(\tfrac{1+V}{2}\right)^4 h \ , \\ \Sigma_- &= \Sigma \cup \{\Lambda_i\}, \qquad h_- = \left(\tfrac{1-V}{2}\right)^4 h \ , \\ \hat{\Sigma} &= \Sigma_+ \cup \Sigma_- \cup \partial_{nd}\Sigma \ , \qquad \hat{h}\Big|_{\Sigma_+} = h_+ \ , \qquad \hat{h}\Big|_{\Sigma_-} = h_- \ . \end{aligned} \tag{5.6}$$

Here $\Sigma \cup \{\Lambda_i\}$ denotes a one point compactification of all the asymptotically flat regions of Σ (with a point Λ_i for each asymptotically flat region). By the results of [8] the metric h_- can be extended across the "points at infinity" Λ_i in a smooth (even analytic) way to a Riemannian metric on Σ_- still denoted by h_-. The topological and differentiable structure of $\hat{\Sigma}$ are defined through the gluing of $\overline{\Sigma}_+ \equiv \Sigma_+ \cup \partial_{nd}\Sigma$ with $\overline{\Sigma}_- \equiv \Sigma_- \cup \partial_{nd}\Sigma$ by identifying $\partial_{nd}\Sigma$, considered as a subset of $\overline{\Sigma}_+$, with a second copy of $\partial_{nd}\Sigma$, considered as a subset of $\overline{\Sigma}_-$, using the identity map. Proposition 3.3 shows that the metric $\hat{h}$ defined in (5.6) can be extended by continuity to a smooth metric on $\hat{\Sigma}$. Note that near degenerate components of the event horizon, if any, we have $\frac{1\pm V}{2} \approx \frac{1}{2}$, which implies that the conclusion of Proposition 3.2 still holds for both h_+ and h_-, thus the ends of $\hat{\Sigma}$ corresponding to degenerate components of the event horizon are complete both for h_+ and h_-. It follows that $(\hat{\Sigma}, \hat{h})$ is a complete Riemannian manifold without boundary. On $\Sigma_{\rm ext}$ we have $\frac{1+V}{2} \approx 1$, so that $(\Sigma_{\rm ext}, \hat{h}|_{\Sigma_{\rm ext}})$ is an asymptotically flat end. A theorem of Beig [5] (*cf.* also [2, 14]) shows that the Komar mass of a static asymptotically vacuum end $(\Sigma_{\rm ext}, h|_{\Sigma_{\rm ext}})$ coincides with its ADM mass m, which gives

$$V = 1 - \frac{m}{r} + O(r^{-2}) \ ,$$

and which implies that the ADM mass of $(\Sigma_{\rm ext}, \hat{h}|_{\Sigma_{\rm ext}})$ vanishes. Equation (5.4) and the behavior of the Ricci scalar under conformal rescalings shows that the Ricci scalar $\hat{R}$ of $\hat{h}$ is non–negative (*cf., e.g.*, [11, Eq. 18]). The rigidity part of Theorem 5.2 shows that there exists a diffeomorphism $\psi : \hat{\Sigma} \to \mathbb{R}^3$ such that $\hat{h}$ is the pull–back by ψ of the standard Euclidean metric δ on $\mathbb{R}^3$. This shows that $\hat{h}$, and hence also h, are conformally flat.

Let S be any connected component of $\partial_{nd}\Sigma$ considered as a subset of $\hat{\Sigma}$, then $\psi(S)$ is an embedded sub-manifold of $\mathbb{R}^3$. S is totally geodesic with respect to the metric h, so that from the transformation formulae for the extrinsic curvature under conformal rescalings together with the constancy of the surface gravity on S it follows that $\psi(S)$ has constant mean curvature with respect to the flat metric on $\mathbb{R}^3$ (*cf., e.g.*, [11, Lemma 4]). By Alexandrov's theorem [27, Theorem 2.6] the manifold $\psi(S)$ is a coordinate sphere.

Suppose that $\partial_{nd}\Sigma$ had more than one component. Then $\psi(\partial_{nd}\Sigma)$ would consist of a finite union of disjoint coordinate spheres, and $\hat{\Sigma}\setminus\overline{\Sigma}_+ \approx \mathbb{R}^3\setminus\overline{\psi(\Sigma_+)} \approx \Sigma_-$ would be a union of disjoint balls, in particular Σ_- would not be connected, which contradicts connectedness of $\Sigma_- \approx \Sigma$. It follows that $\partial_{nd}\Sigma$ has precisely one connected component, with Σ diffeomorphic to $\mathbb{R}^3 \setminus B(0,R)$, so that (Σ, h) has only one asymptotic end and a connected compact boundary lying at finite h–distance. This establishes non–existence of degenerate event horizons in static vacuum space–times.

To finish the proof, that h has to be the metric induced on the standard $t = \text{const}$ slices of the Schwarzschild–Kruszkal–Szekeres space–time, we follow an idea of Herzlich (private communication). On $\mathbb{R}^3 \setminus B(0,R)$ consider the space–Schwarzschild metric of mass $2R$ (*cf.* Equation (2.8)):

$$\tilde{h} = \Omega_0^4\delta, \qquad \Omega_0 \equiv 1 + \frac{R}{r} \ .$$

The sphere $S(0,R)$ is a totally geodesic surface with respect to $\tilde{h}$. Let us still denote by h the pull–back by ψ^{-1} of h to $\mathbb{R}^3 \setminus B(0,R)$, and use V for $V \circ \psi^{-1}$; we have

$$h = \left(\frac{1+V}{2}\right)^{-4}\delta = \left(\frac{1+V}{2}\right)^{-4}\Omega_0^{-4}\tilde{h} = (1+\phi)^4\tilde{h} \ ,$$

with ϕ defined by the last equality above, $\phi = 2/((1+V)\Omega_0) - 1$. Both h and $\tilde{h}$ have vanishing scalar curvature, so that the transformation law for the Ricci scalar under conformal transformations shows that ϕ is $\tilde{h}$–harmonic:

$$\Delta_{\tilde{h}}\phi = 0 \ .$$

Here $\Delta_{\tilde{h}}$ is the Laplace–Beltrami operator of the metric $\tilde{h}$. Now $\psi(S) = S(0,R)$ is minimal for both h and $\tilde{h}$, and the transformation law for the mean extrinsic curvature shows that ϕ has vanishing Neumann data:

$$\left.\frac{\partial\phi}{\partial n}\right|_{S(0,R)} = 0 \ . \qquad (5.7)$$

Here it does not matter whether the partial derivative in the normal direction is taken with respect to the metric h or with respect to the metric $\tilde{h}$. It follows that

$$\begin{aligned}\int_{\Sigma} |d\phi|^2_{\tilde{h}} d^3\mu_{\tilde{h}} &= \int_{S_\infty} \phi \frac{\partial \phi}{\partial n} d^2\mu_{\tilde{h}} - \int_{S} \phi \frac{\partial \phi}{\partial n} d^2\mu_{\tilde{h}} \\ &= 0 \ .\end{aligned}$$

Here the integral over the sphere at infinity S_∞ vanishes by the well known asymptotic behavior of harmonic functions on asymptotically flat ends,

$$\phi = O(r^{-1}), \qquad \frac{\partial \phi}{\partial x^i} = O(r^{-2}) \ ,$$

while the integral over S vanishes because of the vanishing Neumann data (5.7). We thus have

$$\phi \equiv 0 \ ,$$

hence $h = \tilde{h}$, which immediately shows that the metric $-V^2dt^2 + h$ on $\mathbb{R} \times \Sigma$ is the Schwarzschild metric.

Consider any neighborhood $\mathcal{U}$ of Σ diffeomorphic to an open interval times Σ; the set $\mathcal{U}$ is simply connected because Σ has been shown to be simply connected. Let α be the one–form

$$\alpha = \frac{\hat{X}_\mu dx^\mu}{\hat{X}_\nu \hat{X}^\nu} \ ;$$

Equation (4.1) shows that α is closed, and simple–connectedness of $\mathcal{U}$ implies existence of a function $t \in C^\infty(\mathcal{U})$ such that $\alpha = dt$. As in the proof of Lemma 5.1 (*cf.* Eq. (5.2)) there exists a function $f : \Sigma \to \mathbb{R}$ such that

$$t = s + f \ , \tag{5.8}$$

except that now the function f is defined globally on Σ, while in Lemma 5.1 f was only locally defined on appropriately small open sets. Here s denotes the coordinate along the (perhaps only locally defined) orbits of the Killing vector field. Passing to a subset of $\mathcal{U}$ if necessary we may assume that every orbit of X in $\mathcal{U}$ intersects Σ precisely once. We can then extend f to a function on $\mathcal{U}$ by requiring that $X(f) = 0$. As the metric $-V^2dt^2 + h$ has already been shown to be the Schwarzschild metric, Equation 5.8 provides now the required embedding of $\mathcal{U}$ into an open subset of the Schwarzschild space–time. ∎

6 Considerations global in space–time

While the considerations of the previous section had an essentially Riemannian character, here we return to an analysis of the geometry of the space–times under consideration. Let us start with the

PROOF OF COROLLARY 1.2: Suppose, first, that each orbit of the flow ϕ_t of X through $\mathcal{K}(\Sigma)$ intersects Σ precisely once. Because X is transverse to Σ, it follows that the Killing development $\mathcal{K}(\Sigma)$ of Σ is diffeomorphic to $\mathbb{R}\times\Sigma$ in the standard manner. In this case the set $\mathcal{U}$ considered at the end of the proof of Theorem 1.1 can be taken to be equal to $\mathcal{K}(\Sigma)$. Consider the map

$$\mathcal{K}(\Sigma)\approx\mathbb{R}\times\Sigma\ni(s,q)\to\Psi(t,q)=(t=s+f(q),q)\in M^{\rm Schw}\ ,$$

where f is as at the end of the proof of Theorem 1.1. Ψ is injective because $\Psi(0,q)$ is injective from Σ to $M^{\rm Schw}$, and because all Killing orbits intersect both Σ and its image precisely once. Ψ is a local diffeomorphism by construction, hence an embedding. It follows from the proof of Theorem 1.1 and from the formula above that Ψ is surjective, thus Ψ is a diffeomorphism. This establishes the second part of the Corollary when all the orbits of X intersect Σ at most once. In particular our claims follow under the assumption that Σ is achronal in $\mathcal{D}_{oc}$.

Suppose, next, that some orbits of X intersect Σ more than once. In that case we can replace $\mathcal{K}(\Sigma)$ by its Hawking covering $\hat{\mathcal{K}(\Sigma)}$ as defined in [31] in the argument we just made, equipped with the metric π^*g, where $\pi:\hat{\mathcal{K}(\Sigma)}\to\mathcal{K}(\Sigma)$ is the covering map. In this covering one of the connected components of the pre-image of Σ under π will be homeomorphic to Σ [31]. (We note that this construction is equivalent to replacing $\mathcal{K}(\Sigma)$ by $\mathbb{R}\times\Sigma$ equipped with the obviously defined metric). We can then conclude, as before, that $(\hat{\mathcal{K}(\Sigma)},\pi^*g)$ is isometrically diffeomorphic to the Schwarzschild space–time. Hence $\mathcal{K}(\Sigma)$ is a quotient of $(M^{\rm Schw},g^{\rm Schw})$ by a discrete subgroup G of the isometry group of $(M^{\rm Schw},g^{\rm Schw})$.

Now, one easily finds that in standard coordinates on the Schwarzschild space–time every element g of its isometry group acts as follows:

$$\mathbb{R}\times(2m,\infty)\times S^2\ni(t,r,q)\longrightarrow g(t,r,q)=(\epsilon t+a,r,\omega(q))\ ,\qquad \epsilon=\pm1\ ,\quad(6.1)$$

where ω belongs to the isometry group of S^2 equipped with the standard metric. Our hypothesis that $\Sigma_{\rm ext}$ is diffeomorphic to $\mathbb{R}^3$ minus a ball implies that $a\neq0$ unless $\omega=\mathrm{id}$. If there exists $g\in G$ for which $a\neq0$, then $M_{\rm ext}$ will not have $\mathbb{R}\times\Sigma_{\rm ext}$ topology. This establishes the second part of the Corollary under the hypothesis that $M_{\rm ext}$ is diffeomorphic to $\mathbb{R}\times\Sigma_{\rm ext}$.

Let us finally show that if G is not the trivial group, then there exist closed timelike curves in $M_{\rm ext}$. The hypothesis that (M,g) is time orientable implies that $\epsilon=1$ in (6.1). Let $M^{\rm Schw}\ni p=(t,r,q)$ and consider the sequence $g^n(p)=(t+na,r,\omega^n(q))$. It is easily seen that for n large enough we will have $g^n(p)\in I^+(p)$, thus there exists n and a future directed timelike curve γ which starts at p and ends at $g^n(p)$. This holds for any $p\in M^{\rm Schw}$, in particular it will hold for p's which are in $\Sigma_{\rm ext}$. In that case γ can be chosen to lie in $\pi^{-1}(M_{\rm ext})$. Then $\pi(\gamma)$ will be a closed timelike curve in $M_{\rm ext}$, and the proof of the second part of the Corollary is complete. The first part of the Corollary follows immediately from what has been said above. ■

To prove Theorem 1.3 we shall need the following minor generalization of a result of Carter, which does not require staticity:

Lemma 6.1 (Carter [13, Corollary, p. 136]) *Consider a space–time (M, g) with a Killing vector field X which is timelike in an asymptotically flat end $\Sigma_{\rm ext}$, and suppose that the orbits of X are complete in $\mathcal{D}_{oc}$. If there are no closed timelike curves through $\Sigma_{\rm ext}$ contained in $\mathcal{D}_{oc}$, then the Killing vector field X is nowhere vanishing on $\mathcal{D}_{oc}$.*

PROOF: Since $\mathcal{D}_{oc}$ is an open subset of M invariant under ϕ_t we can without loss of generality assume that $M = \mathcal{D}_{oc}$. Suppose that there exists $p \in \mathcal{D}_{oc}$ such that $X(p) = 0$, then p is invariant under the flow of X, and so is $\partial J^+(p)$ because $\phi_t(\partial J^+(p)) = \partial(\phi_t(J^+(p))) = \partial J^+(\phi_t(p))$. Now since $p \in \mathcal{D}_{oc}$ there exists a future directed timelike curve γ which starts at $p_- \in M_{\rm ext}$, passes through p and finishes at $p_+ \in M_{\rm ext}$.

From $J^+(p) \ni p_+ \in M_{\rm ext}$ we have $J^+(p) \cap M_{\rm ext} \neq \emptyset$. Now $\partial J^+(p) \cap M_{\rm ext}$ is invariant under ϕ_t because both $\partial J^+(p)$ and $M_{\rm ext}$ are. The set $\partial J^+(p) \cap M_{\rm ext}$, if non–empty, is thus a null hypersurface in $M_{\rm ext}$, and since there are no ϕ_t invariant null hypersurfaces in $M_{\rm ext}$ we obtain $\partial J^+(p) \cap M_{\rm ext} = \emptyset$. Hence

$$J^+(p) \cap M_{\rm ext} = M_{\rm ext} \ .$$

It follows that there exists a future directed timelike curve γ' from p to p^-. Then the future directed timelike curve γ'' obtained by following γ from p_- to p and then γ' from p to p_- is a closed timelike curve through p_-. As $p_- \in M_{\rm ext}$ there exists $t \in \mathbb{R}$ such that $p_- = \phi_t(q)$, for some $q \in \Sigma_{\rm ext}$. Then $\phi_{-t}(\gamma'')$ is a future directed timelike curve from $\Sigma_{\rm ext}$ to itself, which gives a contradiction, and the result follows. ■

We are ready now to pass to the

PROOF OF THEOREM 1.3: Let $\dot\Sigma$ be that connected component of the set $\{q \in \Sigma \mid (X^\mu X_\mu)(q) < 0\}$ which contains the chosen asymptotic end $\Sigma_{\rm ext}$. $\overline{\dot\Sigma}$ clearly has compact interior, therefore $\dot\Sigma$ satisfies the hypotheses of Theorem 1.1. Moreover under all alternative conditions, except perhaps for condition 4*d)*, the hypotheses of Corollary 1.2 are satisfied with Σ replaced by $\dot\Sigma$. We can thus conclude, except perhaps in the case 4*d)*, that the Killing development $\mathcal{K}(\dot\Sigma)$ of $\dot\Sigma$ is isometrically diffeomorphic to the d.o.c. of the Schwarzschild–Kruszkal–Szekeres space–time. It follows from the proof of Corollary 1.2 that in the case 4*d)* the Killing development $\mathcal{K}(\dot\Sigma)$ of $\dot\Sigma$ is isometrically diffeomorphic to a quotient of the d.o.c. of the Schwarzschild–Kruszkal–Szekeres space–time by an appropriate (perhaps trivial) subgroup of its isometry group.

Next, let us observe that in a globally hyperbolic space–time any d.o.c. is globally hyperbolic, which is easily seen using the definition of global hyperbolicity that involves compactness of the sets of $J^+(p) \cap J^-(q)$. This shows that the alternative condition 4*b)* implies the alternative condition 4*a)*.

Further, a globally hyperbolic vacuum d.o.c. is simply connected by [26]. Thus the alternative condition 4*a)* implies the alternative condition 4*c)*. That last alternative condition is taken care of by the following:

Proposition 6.2 (Carter [13, Section 4]) *Under the main hypotheses of Theorem 1.3, suppose further that the alternative condition 4*c) *holds. Then the conclusion of Theorem 1.3 holds.*

PROOF: Let $\dot{\mathcal{D}}_{oc}$ denote that connected component of the set $\{q \in \mathcal{D}_{oc} \mid (X^\mu X_\mu)(q) < 0\}$ which contains the chosen asymptotic end $\Sigma_{\rm ext}$. Clearly

$$\dot{\mathcal{D}}_{oc} = \mathcal{K}(\dot\Sigma)(= \cup_{t\in\mathbb{R}}\phi_t(\dot\Sigma)),$$

where $\dot\Sigma$ is that connected component of the set $\{q \in \Sigma \mid (X^\mu X_\mu)(q) < 0\}$ which contains the chosen asymptotic end $\Sigma_{\rm ext}$. Suppose that $\mathcal{K}(\dot\Sigma) \neq \mathcal{D}_{oc}$, then $\partial_{\mathcal{D}_{oc}}\mathcal{K}(\dot\Sigma) \neq \emptyset$, where $\partial_{\mathcal{D}_{oc}}\Omega$ denotes the boundary of a set Ω in $\mathcal{D}_{oc}$. By Lemma 6.1 X has no zeros on $\partial_{\mathcal{D}_{oc}}\mathcal{K}(\dot\Sigma)$, and Lemma 4.1 •6.1 shows that $\partial_{\mathcal{D}_{oc}}\mathcal{K}(\dot\Sigma)$ is a Killing horizon. In particular $\partial_{\mathcal{D}_{oc}}\mathcal{K}(\dot\Sigma)$ is a smooth null hypersurface with $\mathcal{K}(\dot\Sigma)$ lying, locally, at one side of it, so that there exists a smooth null vector field ℓ which is transverse to $\partial_{\mathcal{D}_{oc}}\mathcal{K}(\dot\Sigma)$ and which points outwards from $\mathcal{K}(\dot\Sigma)$. (Such vector fields can be defined locally by using local two–dimensional cross–sections of the horizon, and requiring ℓ to be normal to those cross–sections. Those locally defined vector fields can be patched together to a globally defined one on any connected component of $\partial_{\mathcal{D}_{oc}}\mathcal{K}(\dot\Sigma)$ by using a partition of unity.) Now with our conventions X is everywhere future pointing on $\partial_{\mathcal{D}_{oc}}\mathcal{K}(\dot\Sigma)$, but ℓ can be future pointing at some points, and past pointing at some others, so we set

•*6.1:* **ptc:** *the argument here requires $\partial_{\mathcal{D}_{oc}}\mathcal{K}(\dot\Sigma)$ to contain a closed embedded hypersurface if non-empty; and it is not clear how to justify this without further hypotheses; compare Addendum*

$$N_+ = \{p \in \partial_{\mathcal{D}_{oc}}\mathcal{K}(\dot\Sigma) \mid g(X,\ell) < 0\}\ , \quad N_- = \{p \in \partial_{\mathcal{D}_{oc}}\mathcal{K}(\dot\Sigma) \mid g(X,\ell) > 0\}\ .$$

While N_+ and N_- do not have to be closed in M, Lemma 4.1 shows that N_+ and N_- are closed in $\mathcal{D}_{oc}$. Proposition 4.2 shows that the set $\mathcal{N}$ defined in the proof of Lemma 4.1 separates an appropriately small neighborhood $\mathcal{V}$ of p into the local future of $\mathcal{N}$ and its local past, with $\mathcal{K}(\dot\Sigma) \cap \mathcal{V}$ lying to the local past of $\mathcal{N}$:

$$I^+(\partial\dot\Sigma; \mathcal{V}) \cap \mathcal{K}(\dot\Sigma) = \emptyset\ . \tag{6.2}$$

Here $I^+(\Omega; N)$ denotes the chronological future of a set Ω in a space–time N. In particular no future directed causal curve through $\partial\dot\Sigma$ can enter $\mathcal{K}(\dot\Sigma)$ through

N_+. Similarly no future directed causal curve through $\partial\dot{\Sigma}$ can leave $\mathcal{K}(\dot{\Sigma})$ through N_-.

Let $p \in N_+$; since $N_+ \subset \mathcal{D}_{oc}$ there exists a timelike future directed curve γ from p to $M_{\rm ext}$. Now γ leaves $\mathcal{K}(\dot{\Sigma})$ at p so clearly it has to reenter $\mathcal{K}(\dot{\Sigma})$ again through some point $q \in N_-$. Let q be the first such point, consider the path $\mathcal{P}$ obtained by following γ from p until a little beyond q to $\mathcal{K}(\dot{\Sigma})$, and then following any curve contained entirely in $\mathcal{K}(\dot{\Sigma})$ until p. Since N_+ is closed the intersection number of $\mathcal{P}$ and N_+ is a well defined homotopy invariant (*cf., e.g.*, [30, Chapter 3][16]) and equals one. It follows that $\mathcal{P}$ cannot be deformed to a trivial loop the image of which is a point, as such loops have vanishing intersection number with N_+. This contradicts the hypothesis that $\mathcal{D}_{oc}$ is simply connected, and the proposition follows. ■

The alternative condition 4*d)* is taken care of by the following:

Proposition 6.3 *Under the main hypotheses of Theorem 1.3, suppose instead that the alternative condition 4*d) *holds. Then the conclusion of Theorem 1.3 holds.*

Proof: The result follows by a repetition of the proof of Proposition 6.2 as applied to the space–time $M' \equiv M \setminus \{p|X(p) = 0\}$, with the metric obtained by restriction of g to M'. We note that Lemma 6.1 is not needed anymore as X has no zeros in M' by construction. The argument of the proof of Proposition 6.2 shows that $\mathcal{K}(\dot{\Sigma}) = \mathcal{D}_{oc}\setminus\{p|X(p) = 0\}$. If $\mathcal{K}(\dot{\Sigma})$ were a non–trivial quotient of the d.o.c. of the Schwarzschild–Kruszkal–Szekeres space–time it would not be simply connected. As $\mathcal{D}_{oc} \setminus \{p|X(p) = 0\}$ is simply connected by hypothesis, it follows that $\mathcal{K}(\dot{\Sigma})$ is the d.o.c. of the Schwarzschild–Kruszkal–Szekeres space–time. Now $\mathcal{D}_{oc}$ is open, $\mathcal{K}(\dot{\Sigma})$ is open, and for any non–trivial Killing vector field X the set $\{p|X(p) = 0\}$ has no interior. Those remarks and elementary topological considerations imply that $\mathcal{D}_{oc} \setminus \{p|X(p) = 0\} = \mathcal{D}_{oc}$,

It remains to consider the alternative conditions 4*e)*, 4*f)* and 4*g)*. We note the following:

Lemma 6.4 *Under the main hypotheses of Theorem 1.3, suppose instead that the alternative condition 4*e) *holds. Then X has no zeros on $\partial\Sigma$.*

Proof: Suppose that there exists $p \in \partial\Sigma$ such that $X(p) = 0$. The hypothesis that there is no white hole region implies that $M = I^+(M_{\rm ext})$, hence $J^-(p) \cap M_{\rm ext} \neq \emptyset$. As in the proof of Lemma 6.1 we conclude that $J^-(p) \supset M_{\rm ext}$. It follows that there exists a future directed timelike curve γ from a point $q \in I^+(\Sigma_{\rm ext})$ to $p \in \partial\Sigma$, which contradicts achronality of $\overline{\Sigma}$; a contradiction with achronality of

[16]We are grateful to G. Galloway for pointing out this reference and for a simplification of a previous version of this argument.

Σ easily follows. ■

Lemma 6.4 reduces the alternative condition 4*e)* to the alternative condition 4*f)*, which we consider now:

Proposition 6.5 *Under the main hypotheses of Theorem 1.3, suppose instead that the alternative condition 4*f) *holds. Then the conclusion of Theorem 1.3 holds.*

PROOF: Let, as before, $\dot\Sigma$, denote that connected component of the set $\{q \in \Sigma \mid (X^\mu X_\mu)(q) < 0\}$ which contains the chosen asymptotic end $\Sigma_{\rm ext}$. By Lemma 6.1 the Killing vector does not vanish on Σ, and it does not vanish on $\partial\Sigma$ by hypothesis, therefore X has no zeros on $\dot\Sigma \cup \partial\dot\Sigma \subset \overline{\Sigma}$. By Theorem 1.1 $\partial\dot\Sigma$ is connected and has a well defined outwards pointing unit normal vector n with respect to the induced metric. Let T denote the future pointing unit normal to Σ, the non–vanishing of X and the connectedness of $\partial\dot\Sigma$ imply that X must be proportional either to $T+n$ or $T-n$ on $\partial\dot\Sigma$. Changing the time orientation and X to $-X$ if necessary we may without loss of generality suppose that X is proportional to $T-n$ on $\partial\dot\Sigma$, with a strictly positive proportionality factor. Proposition 4.2 shows that there exists a sufficiently small neighborhood $\mathcal{V}$ of $\partial\dot\Sigma$ such that the set $\mathcal{N}$ defined in the proof of Lemma 4.1 •6.2 separates $\mathcal{V}$ into the local future of $\mathcal{N}$ and its local past, with $\mathcal{K}(\dot\Sigma)\cap\mathcal{V}$ lying to the local past of $\mathcal{N}$:

•6.2: **ptc:** *the argument here requires $\partial_{\mathcal{D}_{oc}}\mathcal{K}(\dot\Sigma)$ to contain a closed embedded hypersurface if non-empty; and it is not clear how to justify this without further hypotheses*

$$I^+(\partial\dot\Sigma;\mathcal{V})\cap\mathcal{K}(\dot\Sigma)=\emptyset\ . \tag{6.3}$$

In particular every future directed causal curve through $\partial\dot\Sigma$ leaves $\mathcal{K}(\dot\Sigma)$ when crossing $\partial\dot\Sigma$.

Suppose that $\partial\dot\Sigma\cap\mathcal{D}_{oc}\neq\emptyset$, thus there exists a point $p\in\partial\dot\Sigma$ and a timelike curve $\hat\gamma:[0,2]\to\mathcal{D}_{oc}$ such that $\hat\gamma(0)=p_-\in M_{\rm ext}$, $\hat\gamma(1)=p\in\partial\dot\Sigma$, and $\hat\gamma(2)=p_+\in M_{\rm ext}$. Define $\gamma:\mathbb{R}\to\mathcal{D}_{oc}$ by

$$\gamma(s)=\begin{cases}\phi_s(p_-), & s\in(-\infty,0],\\ \hat\gamma(s), & s\in[0,2],\\ \phi_{s-2}(p_+), & s\in[2,\infty).\end{cases}$$

Then γ is an inextendible timelike curve through $p\in\partial\dot\Sigma$, with $\gamma(s)\in M_{\rm ext}\subset\mathcal{K}(\dot\Sigma)$ for $s\le 0$ and for $s\ge 2$, and with $\gamma(1)=p\notin\dot\Sigma$.

Let I_+ be that connected component of $\gamma\cap\mathcal{K}(\dot\Sigma)$ which contains $[2,\infty]$. Now $I_+\neq\mathbb{R}$ since $\gamma(1)=p$ and $p\notin\mathcal{K}(\dot\Sigma)$; this last assertion follows from the fact that X is timelike throughout $\mathcal{K}(\dot\Sigma)$ while X is null at p. Since $\mathcal{K}(\dot\Sigma)$ is open we obtain $I_+=(s_+,\infty)$, for some $s_+\in(1,2)$. Set

$$\tilde\gamma=\gamma\Big|_{(s_+,\infty)}\ ;$$

thus $\tilde{\gamma}$ is an inextendible timelike curve in $\mathcal{K}(\dot{\Sigma})$. As $\mathcal{K}(\dot{\Sigma})$ is ϕ_t invariant, we have that for all t the curve $\phi_t(\gamma)$ is an inextendible timelike curve in $\mathcal{K}(\dot{\Sigma})$.

We claim that $\tilde{\gamma}$ has to intersect $\dot{\Sigma}$. To see that this must be the case, define

$$I = \{t \in \mathbb{R} : \phi_t(\tilde{\gamma}) \cap \dot{\Sigma} \neq \emptyset\} \ .$$

Consider, first, the point $p_+ \in M_{\rm ext}$ defined above. By definition of $M_{\rm ext}$ we can write $p_+ = \phi_{t_+}(q_+)$ for some $q_+ \in \Sigma_{\rm ext} \subset \dot{\Sigma}$. It follows that $\phi_{-t_+}(\gamma) \ni \phi_{-t_+}(p_+) = q_+ \in \dot{\Sigma}$, so that $-t_+ \in I$. In fact by the definition of γ we have that $\phi_{-s}(\gamma) \cap \Sigma_{\rm ext} = q_+ \in \dot{\Sigma}$ for all $s > t_+$, thus $I \supset (-\infty, -t_+]$, in particular $I \neq \emptyset$.

Note that $\phi_t(\gamma)$ is timelike for all t, hence transverse to Σ for all $t \in I$, which implies that I is open. Let $\hat{I}$ be the connected component of I containing $(-\infty, -t_+]$ and suppose that $\hat{I} \neq \mathbb{R}$, then there exists $T_- \in \mathbb{R}$ such that $\hat{I} = (-\infty, T_-)$. Let $t_i \in \hat{I}$ be any sequence converging to T_-, set $q_i = \phi_{t_i}(\gamma) \cap \dot{\Sigma}$. By interior compactness of $\overline{\dot{\Sigma}}$, passing to a subsequence if necessary, we may suppose that q_i converges to $q \in \overline{\dot{\Sigma}}$. Clearly $q \in \partial\dot{\Sigma}$ by the definition of T_-, hence $\phi_{T_-}(\gamma)$ is a timelike curve through $q \in \partial\dot{\Sigma}$ which immediately enters $\mathcal{K}(\dot{\Sigma})$. There are however no such curves by (6.3), hence $\hat{I} = I = \mathbb{R}$. In particular $0 \in I$ so that $\tilde{\gamma} \cap \dot{\Sigma} = \phi_0(\tilde{\gamma}) \cap \dot{\Sigma} \neq \emptyset$. Let r be any point in $\tilde{\gamma} \cap \dot{\Sigma}$, we have $p \neq r$ as $p \notin \dot{\Sigma}$. It follows that γ is a timelike curve which meets Σ at two distinct points p and r, which contradicts achronality of Σ, and the Lemma follows. ■

It remains to show that Theorem 1.3 holds under the alternative condition 4*g)*. Suppose, thus, that there exists a point $p \in \partial\dot{\Sigma} \cap \mathcal{D}_{oc}$ at which the Killing vector vanishes. By Boyer's Theorem 3.1 p belongs to a "bifurcation surface" of a "bifurcate Killing horizon". In particular there are four Killing vector orbits which accumulate at p and which coincide with two geodesic generators of $\partial J(p) = \partial(J^+(p) \cup J^-(p))$ with p removed. We note the following:

Lemma 6.6 *Under the main hypotheses of Theorem 1.3, suppose that there exists a point $p \in \partial\dot{\Sigma} \cap \mathcal{D}_{oc}$ at which the Killing vector vanishes. If Σ is achronal, then the Killing orbits in $\partial J(p) \cap \partial\mathcal{K}(\dot{\Sigma})$ do* not *belong to $\mathcal{D}_{oc}$.*

PROOF: The local structure of the orbits of the flow of the Killing vector near p shows that we can find a point $q \in \partial J^+(p) \cap \partial\mathcal{K}(\dot{\Sigma})$ with the following properties:

1. The orbit of X through q coincides with the future directed null geodesic generator of the Killing horizon accumulating at p in the past.

2. $q \in I^+(\Sigma)$.

We wish to show that for all $t \in \mathbb{R}$ we will have

$$\phi_t(q) \in I^+(\Sigma) \ . \tag{6.4}$$

To establish (6.4), let γ be a timelike future directed curve from Σ to q, for $t \geq 0$ consider the future directed causal curve from Σ to q obtained by following γ from Σ to q and then following the orbit $\phi_s(q)$ of X from $q = \phi_0(q)$ to $\phi_t(q)$; the curve so constructed is not a null geodesic and can therefore be deformed to a timelike curve. On the other hand, for $t \leq 0$ the curve $\phi_t(\gamma) \cap J^+(\Sigma) \neq \emptyset$ provides the desired timelike curve from Σ to $\phi_t(q)$.

Suppose that $q \in \mathcal{D}_{oc}$, then there exists a future directed timelike curve $\hat\gamma$ from q to a point $p_+ \in M_{\rm ext}$. By definition of $M_{\rm ext}$ we have $p_+ = \phi_{t_+}(q_+)$ for some $q_+ \in \Sigma_{\rm ext}$ and some $t_+ \in \mathbb{R}$. Then $\phi_{-t_+}(\hat\gamma)$ is a future directed timelike curve passing through $\phi_{-t_+}(q) \in I^+(\Sigma)$ and $q_+ \in \Sigma_{\rm ext}$, which contradicts achronality of Σ. This establishes the result for $\partial J^+(p) \cap \partial\mathcal{K}(\dot\Sigma)$. The result for $\partial J^-(p) \cap \partial\mathcal{K}(\dot\Sigma)$ follows from this one by changing the time orientation of (M, g). ■

Lemma 6.6 immediately implies Theorem 1.3 under the alternative condition 4*g)*: indeed, under this condition the Killing vector field X vanishes throughout $\partial\dot\Sigma$. It follows that the set of those zeros of the Killing vector field which lie on $\partial\mathcal{K}(\dot\Sigma)$ is a compact connected manifold. It is then easily seen that each Killing orbit on $\partial\mathcal{K}(\dot\Sigma)$ is of the form $\partial J^+(p) \cap \partial\mathcal{K}(\dot\Sigma)$ or $\partial J^-(p) \cap \partial\mathcal{K}(\dot\Sigma)$ for some $p \in \partial\dot\Sigma$, and the proof of Theorem 1.3 is completed. ■

7 Concluding remarks

In this paper we have essentially finished the classification of asymptotically flat, static, vacuum, appropriately regular black hole space–times. It is natural to try to generalize our results to electro–vacuum space–times. Recall that Simon [49] and Masood–ul–Alam [44] have shown[17], roughly speaking, that all appropriately regular such black holes which *do not contain degenerate horizons* belong to the Reissner–Nordström family. In the case of a *connected* black hole the requirement of non–degeneracy of the event horizon has been removed by Heusler [35] (*cf.* also [16, 23]); in [35] some partial results concerning the case where *all horizons are degenerate* have also been obtained. Nevertheless the general case of a static electro–vacuum black hole containing both degenerate and non–degenerate horizons remains open. It turns out that the arguments used here can be generalized to exclude some further classes of electro–vacuum black holes without, so far, leading to a definitive classification of electro–vacuum static black holes. We hope to be able to improve those results in the future.

Let us close this paper with some comments concerning the classification problem of stationary black holes. Recall that the usual approach to the classification of *analytic* electro–vacuum black holes is via the so–called *Hawking*

[17] The paper by Ruback [47] with similar claims contains essential gaps.

rigidity theorem[18] which shows, under appropriate hypotheses including analyticity, that event horizons in stationary space–times with defining Killing vector X are Killing horizons for an appropriately defined Killing vector field Y. If $X = Y$ and if all horizons are *non–degenerate* it follows from the results in [25, 46, 50] •7.1 that the space–time must be static, so that the Israel — Bunting — Masood–ul–Alam theorem (or its extension here) can be used to analyze this case. The case with $X = Y$ and *all* horizons degenerate has been excluded in Section 4. The case $X = Y$ and *some* horizons degenerate is still open, and it would be of interest to fill this gap.

•*7.1:* **ptc:** *a simple proof can be found in [22, Section 7.2].*

Concerning the case $X \neq Y$, a complete classification *modulo existence of "struts" on the symmetry axis* has been given by Weinstein [54], again assuming *non–degeneracy* of all event horizons. •7.2 We note that even the question, whether a degenerate, connected, vacuum, stationary–axi–symmetric, regular black hole is an extreme Kerr black–hole has not been resolved so far. Our analysis in Section 3 provides, we believe, a good starting point for an extension of the results of [54] which allows degenerate components of the event horizon.

•*7.2:* **ptc:** *see [22] for a detailed exposition, filling in the gaps of the previous arguments*

A Addendum, May 2008

There are two points which have not been handled properly in the current work, published as [19]. First, it has been pointed out to me by João Lopes Costa that neither the original proof, nor that given in [19], of the Vishveshwara-Carter Lemma, takes properly into account the possibility that the hypersurface $\mathscr{N}$ of [19, Lemma 4.1] could fail to be embedded when it is degenerate. This problem arises whether or not the horizon is degenerate, since we do not know a priori whether or not $\mathscr{N}$ has anything to do with the horizon. This issue is handled below under the assumption of global hyperbolicity of the domain of outer communications. I am grateful to João for pointing out the problem, and for useful remarks on previous versions of this *corrigendum*.

A (wrong) solution to this problem has been proposed in the arXiv version 2 of [19]. The idea was to show that the family of hypersurfaces covering the set where the static Killing vector becomes null contains an outermost closed and embedded hypersurface. A family of hypersurfaces in a plane that does not contain such a hypersurface is drawn in Figure 1: the only reasonable candidate for an outermost curve there is the curve of infinite length, which is embedded, but does not form a closed subset of the plane. The example shows that the strategy proposed in [19] has no chance of succeeding.

In this corrigendum we two alternative sets of hypotheses which solve the problem: The first set assumes global hyperbolicity and analyticity, the second supposes axisymmetry together with some structure conditions, see Theorem A.1 below.

[18] This theorem is wrong as stated in [33], *cf.* [17]. A corrected version can be found in [16].

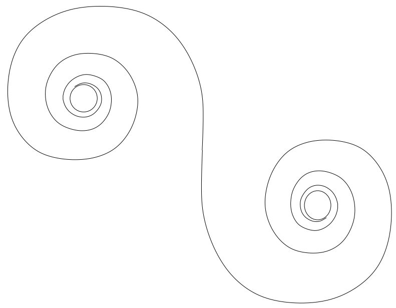

Figure 1: A family of three embedded curves in the plane consisting of two circles, together with a curve of infinite length that spirals around the circles.

Next, the degenerate case in Boyer's theorem [10] has been quoted incorrectly in [19, Theorem 3.1]: In spite of what is said there, there exist Killing vectors with a degenerate horizon which have zeros at its closure. The Minkowskian Killing vector

$$X = t\partial_x + x\partial_t + x\partial_y - y\partial_x = (t - y)\partial_x + x(\partial_t + \partial_y) \tag{A.1}$$

illustrates well the problem at hand. X vanishes at $\{t = y\ ,\ x = 0\}$, and is null on $\{t = y\ ,\ x \neq 0\}$. Recall that a Killing horizon associated to X is a null hypersurface $\mathscr{N}$ on which X is null, tangent to $\mathscr{N}$. So, in this case, $\mathscr{N}$ has two connected components

$$\mathscr{N}^{\pm} := \{t = y\ , \pm x > 0\}\ .$$

A key for the proof of [19, Theorem 1.1] is Proposition 3.2 there, which is *wrong* without the supplementary hypothesis that the Killing vector X has no zeros on $\partial\bar{\Sigma}$. Indeed, let $\Sigma = \{t = 0, y > 0\}$ in Minkowski space-time $(\mathbb{R}^4, g)$, and let X be given by (A.1). To see that Σ equipped with the "orbit space metric"[19]

$$\forall Z_1, Z_2 \in T\Sigma \qquad h(Z_1, Z_2) = g(Z_1, Z_2) - \frac{g(X, Z_1)g(X, Z_2)}{g(X, X)}$$

contains finite-proper length spacelike curves which reach the boundary $\partial\bar{\Sigma}$, consider the curve

$$(0, \infty) \ni s \mapsto \gamma(s) = (t = x = z = 0\ ,\ y = s) \in \Sigma\ .$$

Then $g(X, \dot\gamma) = 0$, so $h(\dot\gamma, \dot\gamma) = g(\dot\gamma, \dot\gamma) = 1$, and the boundary $y = 0$ lies at h–distance along γ equal to s from any point $\gamma(s)$ on γ. But then this boundary lies to finite h–distance from any point $p \in \Sigma$, as one can reach $\partial\bar{\Sigma}$ from p by first

[19]As already emphasised in [19], the metric h should *not* be thought of as the "metric on the space of orbits", as we are not assuming anything about the manifold character of this last space; similarly transversality of X to Σ is *not* assumed.

going to γ, and then following γ until $\partial\overline{\Sigma}$ is reached. Here $\partial\bar{\Sigma}$ is not compact, but this seems irrelevant for the issue at hand.

More significantly, in this example X is spacelike near and away from $\{t = z\}$, so that h is *Lorentzian* there, while the orbit space metric *is* Riemannian in the context of the analysis of [19]. This observation is the key to the proof below that such zeros do not exist on boundaries $\partial\{\mathbf{g}(X,X) = 0\}$ in static space-times.

In this corrigendum we assert, hopefully with any future errata:

Theorem A.1 *Let $(M,\mathbf{g})$ be a vacuum $(3+1)$–dimensional space-time containing a hypersurface Σ with boundary $\partial\Sigma := \overline{\Sigma}\setminus\Sigma$, the closure $\bar{\Sigma}$ of which is the union of a finite number of asymptotically flat ends and of a compact set. Let $\Sigma_{\rm ext}\subset\Sigma$ be a chosen asymptotically flat region of Σ, and suppose that X is a complete hypersurface-orthogonal Killing vector field on M which is timelike on $\Sigma_{\rm ext}$. Assume that*

i. either $(M,\mathbf{g})$ is analytic, with

$$\partial\Sigma\cap\mathcal{D}_{oc} = \emptyset\ ,$$

or

ii. that there exists another Killing vector on M generating a $\mathrm{U}(1)$ action. In addition, suppose Σ is acausal, with the boundary $\partial\bar{\Sigma} := \bar{\Sigma}\setminus\Sigma$ being a topological manifold satisfying

$$\partial\bar{\Sigma}\subset\mathscr{E}^+ := \partial\mathcal{D}_{oc}\cap I^+(M_{\rm ext})\ , \tag{A.2}$$

with $\partial\bar{\Sigma}$ meeting every generator of $\mathscr{E}^+$ precisely once. (See Figure 2.)

If $\mathcal{D}_{oc}$ is globally hyperbolic (which will necessarily be the case if $(M,\mathbf{g})$ is), then $\mathcal{D}_{oc}$ is isometrically diffeomorphic to the domain of outer communications of the Schwarzschild space–time.

Remarks A.2 1. Similarly, the gap in the proof of the remaining electro-vacuum uniqueness claims in [20], or their counterparts mentioned in [24], is removed when global hyperbolicity, together with analyticity or axisymmetry as above, are imposed.

2. The arguments below, to exclude null orbits of X in $\mathcal{D}_{oc}$, are essentially a repetition of the ones in [22], adapted to the somewhat simpler problem at hand.

3. A justification of the higher dimensional counterpart of Theorem A.1, requiring the validity of the positive energy theorem, can be found in [21].

We start with a revised version of the Vishveshwara–Carter Lemma [12, 51]:

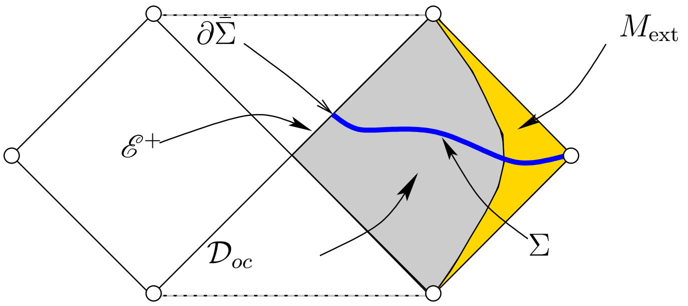

Figure 2: The hypersurface Σ from point 2 of Theorem A.1.

Lemma A.3 *Let $(M, \mathbf{g})$ be a smooth space–time with complete, static Killing vector X, set*

$$W := -\mathbf{g}_{\alpha\beta}X^\alpha X^\beta \ . \tag{A.3}$$

Then

i. $\{W = 0\} \cap \{X \neq 0\}$ *is a union of integral leaves of the distribution $X^\perp$, which are totally geodesic within $M \setminus \{X = 0\}$.*

ii. Each connected component of

$$\{W = 0\} \cap \{dW \neq 0\} \cap \{X \neq 0\}$$

is a smooth, embedded, locally totally geodesic null hypersurface $\mathscr{N}$, with the Killing vector field X normal to $\mathscr{N}$.

Remark A.4 As the surface gravity is constant in electro-vacuum, point 2 covers adequately non-degenerate vacuum or electro-vacuum Killing horizons, but does not apply to degenerate ones. In [19, Lemma 4.1], the conclusions of point ii. were claimed for each connected component of the set

$$\{W = 0\} \cap \{X \neq 0\} \cap \partial\{W < 0\} \ ,$$

but the justification given there does not seem to be sufficient, and we do not know whether or not the result is correct without further hypotheses.

Proof: The staticity condition $X^\flat \wedge dX^\flat = 0$ and the Frobenius theorem [36, Section 9.1] show that

$$\mathscr{O} := M \setminus \{X = 0\}$$

is foliated by immersed, but not necessarily embedded, hypersurfaces which are normal to X. Let

$$\hat{S}_p \subset \mathscr{O}$$

denote the maximally extended integral leaf of the distribution $X^\perp$ passing through p.

Let $p \in \{W = 0\} \cap \{X \neq 0\}$ and let γ be an affinely parameterised geodesic starting at p with initial tangent $\dot\gamma(0)$ normal to X. A standard calculation along γ shows that

$$\frac{d(\mathbf{g}(\dot\gamma, X))}{ds} = \dot\gamma^\mu \nabla_\mu (\dot\gamma^\nu X_\nu) = \underbrace{\dot\gamma^\mu \nabla_\mu \dot\gamma^\nu}_{=0} X_\nu + \dot\gamma^\nu \dot\gamma^\mu \nabla_\mu X_\nu = \dot\gamma^\nu \dot\gamma^\mu \underbrace{\nabla_{(\mu} X_{\nu)}}_{=0} = 0 ,$$

hence $\dot\gamma$ remains normal to X, implying that $\hat\Sigma_p$ is locally totally geodesic, as claimed. Clearly γ can exit $\hat\Sigma_p$ only where that leaf ceases to be defined, namely at zeros of X. Hence the $\hat\Sigma_p$'s are totally geodesic within $M \setminus \{X = 0\}$.

The staticity condition can be rewritten as

$$2\nabla_{[\mu} X^\alpha X_{\nu]} = X^\alpha \nabla_{[\mu} X_{\nu]} ,$$

which, after a contraction with X_α, gives

$$\nabla_{[\mu} W X_{\nu]} = W \nabla_{[\mu} X_{\nu]} , \qquad W \equiv X_\alpha X^\alpha . \tag{A.4}$$

Let ℓ_μ be any smooth covector field on $\mathscr{O}$ such that $\ell_\mu X^\mu = 1$ and let γ be any differentiable curve contained in a leaf $\hat\Sigma_p$ such that $W(\gamma(0)) = 0$; contraction of (A.4) with $\dot\gamma^\mu \ell^\nu$ gives

$$\frac{dW}{ds} = \dot\gamma^\mu \nabla_\mu W = 2W \dot\gamma^\mu \ell^\nu \nabla_{[\mu} X_{\nu]} . \tag{A.5}$$

Uniqueness of solutions of ODEs implies that $W \circ \gamma = 0$, and we conclude that $\hat\Sigma_p \subset \{W = 0\}$. This shows that if $\hat\Sigma_q \cap \{W = 0\} \neq \emptyset$ then $W \equiv 0$ on $\hat\Sigma_q$. Hence $\{W = 0\} \setminus \{X = 0\}$ is a union of leaves of the $\hat\Sigma_p$ foliation.

At those points at which dW does not vanish, the set $\{W = 0\}$ is smooth, embedded hypersurface, and the proof is complete. ∎

Next, we need to analyze what happens when a sequence of null leaves $\hat S_{p_n}$, all normal to the static Killing vector field X, has an accumulation point. We shall say that S is an *accumulation set* of a sequence of sets S_n if S is the collection of limits, as i tends to infinity, of sequences $q_{n_i} \in S_{n_i}$.

Lemma A.5 *Let $\hat S_{p_n} \subset \{W = 0\}$ be a sequence of integral leaves of $X^\perp$, and suppose that $p_n \to p$. If $X(p) \neq 0$, then p belongs to a leaf $\hat S_p$. Furthermore there exists a neighborhood $\mathscr{U}$ of p such that $\hat S_p \cap \mathscr{U}$ is the accumulation set of the sequence $\{\hat S_{p_n} \cap \mathscr{U}\}_{n\in\mathbb{N}}$.*

PROOF: Let $\mathscr{U}$ be a small, open, geodesically convex neighborhood of p which does not contain zeros of X. Let $\hat{S}_{p_n}$ be that leaf, within $\mathscr{U}$, of the distribution $X^\perp$ which contains p_n. The $\hat{S}_{p_n}$'s are totally geodesic submanifolds of $\mathscr{U}$ by Lemma A.3 and, therefore, are uniquely determined by prescribing $T_{p_n}\hat{S}_{p_n}$. Now, the subspaces $T_{p_n}\hat{S}_{p_n} = X_{p_n}^\perp$ obviously converge to $X_p^\perp$ in the sense of accumulation sets. By continuity $W = 0$ at p. Smooth dependence of geodesics upon initial values implies that $\exp_{\mathscr{U}}(X_{p_n}^\perp)$ converges in C^k, for any k, to $\exp_{\mathscr{U}}(X_p^\perp)$. Here we write $\exp_{\mathscr{U}}$ for the exponential map within the space-time $(\mathscr{U}, g|_{\mathscr{U}})$. Since $T_{q_n}\exp_{\mathscr{U}}(X_{p_n}^\perp) = X_{q_n}^\perp$ for any $q_n \in \exp_{\mathscr{U}}(X_{p_n}^\perp)$ we conclude that $T_r\exp_{\mathscr{U}}(X_p^\perp) = X_r^\perp$ for any $r \in \exp_{\mathscr{U}}(X_p^\perp)$. So $\exp_{\mathscr{U}}(X_p^\perp)$ is a leaf, within $\mathscr{U}$, through p of the distribution $X^\perp$ over the set

$$\{W = 0, g(X,X) = 0, X \neq 0\} ,$$

and $\exp_{\mathscr{U}}(X_p^\perp) = \hat{S}_p \cap \mathscr{U}$ is the accumulation set of the totally geodesic submanifolds $\hat{S}_{p_n} \cap \mathscr{U}$'s. ■

•A.1 We want to show that $\{W = 0\} \cap \mathcal{D}_{oc}$, if not empty, contains a closed embedded leaf of the distribution $X^\perp$, which will be denoted by S_0^+. This has been already established when dW does not vanish at some point $p \in \mathcal{D}_{oc} \cap \{W(p) = 0\}$, so the problem arises only whenever dW vanishes throughout the zero level set of W, if non-empty. •*A.1:* **ptc:** *new material starts here*

So, if there exists non-empty non-degenerate $\hat{\Sigma}_p$'s, we choose one and we set

$$S_0^+ = \hat{\Sigma}_p . \tag{A.6}$$

Otherwise, all non-empty $\hat{\Sigma}_p$'s are degenerate; to show that such prehorizons, if non-empty, are embedded, we will first invoke analyticity (which has not been used so far). Indeed, consider a degenerate component $\hat{\Sigma}_p$ and note, first, that $\hat{\Sigma}_p$ does not self-intersect, being a subset of the union of integral manifolds of a smooth distribution of hyperplanes. So suppose that $\hat{\Sigma}_p$ is not embedded. Then there exists a point $q \in \hat{\Sigma}_p$, a conditionally compact neighborhood $\mathscr{O}$ of q, and a sequence of points $p_n \in \hat{\Sigma}_p$ lying on pairwise disjoint components of $\mathscr{O} \cap \hat{\Sigma}_p$, with p_n converging to q. Now, Killing vectors are solutions of the overdetermined set of PDEs

$$\nabla_\mu \nabla_\nu X_\rho = R^\alpha{}_{\mu\nu\rho} X_\alpha ,$$

which imply that they are analytic if the metric is. So $g(X,X)$ is an analytic function that vanishes on an accumulating family of hypersurfaces. Consequently $g(X,X)$ vanishes everywhere, which is not possible. Hence the $\hat{\Sigma}_p$'s are embedded; it should be clear now that they are closed, coinciding with connected components of the set $\{g(X,X) = 0\}$. We define S_0^+ again using (A.6), choosing one non-empty $\hat{\Sigma}_p$.

We can finish the proof of Theorem A.1 in the analytic case. Suppose that W changes sign within $\mathcal{D}_{oc}$, then S_0^+ is a non-empty, closed, null hypersurface within $\mathcal{D}_{oc}$. Now, any connected embedded null hypersurface S_0^+ is locally two-sided, and we can assign an intersection number one to every intersection point of S_0^+ with a curve that crosses S_0^+ from its local past to its local future, and minus one for the remaining ones (this coincides with the oriented intersection number as in [30, Chapter 3]). Let $p \in S_0^+$, there exists a smooth timelike future directed curve γ_1 from some point $q \in M_{\rm ext}$ to p. By definition there exists a future directed null geodesic segment γ_2 from p to some point $r \in M_{\rm ext}$ intersecting S precisely at p. Since $M_{\rm ext}$ is connected there exists a curve $\gamma_3 \subset M_{\rm ext}$ (which, in fact, cannot be causal future directed, but this is irrelevant for our purposes) from r to q. Then the path γ obtained by following γ_1, then γ_2, and then γ_3 is closed. Since S_0^+ does not extend into $M_{\rm ext}$, γ intersects S_0^+ only along its timelike future directed part, where every intersection has intersection number one, and γ intersects S_0^+ at least once at p, hence the intersection number of γ with S_0^+ is strictly positive. Now, $\mathcal{D}_{oc}$ is simply connected by [26]. But, by standard intersection theory [30, Chapter 3], the intersection number of a closed curve with a closed orientable (embedded) hypersurface in a simply connected manifold vanishes, which gives a contradiction and proves that W cannot change sign on $\mathcal{D}_{oc}$.

It remains to show that W vanishes at the boundary of $\mathcal{D}_{oc}$. For this, note that $\partial\mathcal{D}_{oc} \subset \dot I^-(M_{\rm ext}) \cup \dot I^+(M_{\rm ext})$, and each of the boundaries $\dot I^-(M_{\rm ext})$ and $\dot I^+(M_{\rm ext})$ is invariant under the flow of X, and each is achronal, which is only possible if $W \le 0$ on $\partial\mathcal{D}_{oc}$.

Replacing Σ with $\Sigma\cap\mathcal{D}_{oc}$ we can without loss of generality assume $\partial\Sigma \subset \partial\mathcal{D}_{oc}$. Now, $\dot I^\pm(M_{\rm ext})$ are acausal boundaries, hence embedded hypersurfaces, and from the first part of Lemma A.3 we infer that $\dot I^\pm(M_{\rm ext})\cap\partial\mathcal{D}_{oc}$ are smooth, null, locally totally geodesic, embedded hypersurfaces, with X tangent to their generators, except perhaps at points at which X vanishes. Proposition 3.1 of [19] applies to those components of $\partial\Sigma$ which meet non-degenerate components of $\partial\mathcal{D}_{oc}$. We will prove shortly that X is nowhere vanishing near the degenerate components of $\partial\bar\Sigma$, so Proposition 3.2 of [19] applies as well.

The remaining arguments of proof of Theorem A.1 are as in [19], and the proof under the hypothesis of analyticity is complete.

Instead of analyticity, we assume now axisymmetry. Let

$$\pi : \mathcal{D}_{oc} \cup \mathscr{E}^+ \to \underbrace{\Big(\mathcal{D}_{oc} \cup \mathscr{E}^+\Big)/\Big(\mathbb{R}\times \mathrm{U}(1)\Big)}_{=:\mathcal{Q}}$$

denote the quotient map. As discussed in more detail in Sections 6.1 and 6.2 of [22] (keeping in mind that, by topological censorship, $\mathcal{D}_{oc}$ has only one asymptotically flat end), the orbit space $\mathcal{Q}$ is diffeomorphic to the half-plane $\{(x,y) \mid x \ge 0\}$ from which a finite number $\mathring{n} \ge 0$ of open half-discs, centred at the axis $\{x = 0\}$, have been removed. As explained at the beginning of Section 7 of [22], the case $\mathring{n} = 0$ leads to Minkowski space-time, in which case the result is clear, so from now on we assume $\mathring{n} \geq 1$.

Suppose that $\{W = 0\} \cap \mathcal{D}_{oc}$ is non-empty. We denote by $C_{\pi(p)}$ the image in $\mathcal{Q}$, under the projection map π, of $\hat{S}_p \cap (\mathcal{D}_{oc} \cup \mathscr{E}^+)$. Define

$$\mathring{\mathcal{Q}} = \mathcal{D}_{oc}/\Big(\mathbb{R} \times \mathrm{U}(1)\Big) ,$$

$$\mathscr{W}_0^\flat := \Big(\{W = 0\} \cap (\mathcal{D}_{oc} \cup \mathscr{E}^+)\Big)/\Big(\mathbb{R} \times \mathrm{U}(1)\Big) ,$$

Then $\mathscr{W}_0^\flat$ is a closed subset of $\mathcal{Q}$, with the following property: through every point q of $\mathscr{W}_0^\flat$ there exists a smooth maximally extended curve C_q, which will be called *orbit*, entirely contained in $\mathscr{W}_0^\flat$. The C_q's are pairwise disjoint, or coincide. Their union forms a closed set, and locally they look like a subcollection of leaves of a foliation. (Such structures are called laminations; see, e.g., [28].)

An orbit will be called a *Jordan orbit* if C_q forms a Jordan curve.

We need to consider several possibilities; we start with the simplest one:

Case I: If an orbit C_q forms a Jordan curve entirely contained in $\mathring{\mathcal{Q}}$, then the corresponding $\hat{S}_p = \pi^{-1}(C_q)$ forms a closed embedded hypersurface in $\mathcal{D}_{oc}$, and a contradiction arises as in the analytic case.

Case II: Consider, next, an orbit C_q which meets the boundary of $\mathcal{Q}$ at two or more points which belong to $\pi(\mathscr{A})$, and only at such points. Let $I_q \subset C_q$ denote that part of C_q which connects any two subsequent such points, in the sense that I_q meets $\partial\mathcal{Q}$ at its end points only. Now, every $\hat{S}_p$ is a smooth hypersurface in M invariant under $\mathbb{R} \times \mathrm{U}(1)$, and therefore meets the rotation axis $\mathscr{A}$ orthogonally. This implies that $\pi^{-1}(I_q)$ is a closed, smooth, embedded hypersurface in $\mathcal{D}_{oc}$, providing again a contradiction.

To handle the remaining cases, some preliminary work is needed. It is convenient to double $\mathcal{Q}$ across $\{x = 0\}$ to obtain a manifold $\widehat{\mathcal{Q}}$ diffeomorphic to $\mathbb{R}^2$ from which a finite number of open discs, centered at the axis $\{x = 0\}$, have been removed, see Figure 3. Connected components of the event horizon $\mathscr{E}^+$ correspond to smooth circles forming the boundary of $\widehat{\mathcal{Q}}$, regardless of whether or not they are degenerate. From what has been said, every C_q which has an end point at $\pi(\mathscr{A})$ is smoothly extended in $\widehat{\mathcal{Q}}$ across $\{x = 0\}$ by its image under the map $(x, y) \mapsto (-x, y)$. We will continue to denote by C_q the orbits so extended in $\widehat{\mathcal{Q}}$.

The analysis of Cases I and II also shows:

Lemma A.6 *An orbit C_q which does not meet $\partial\widehat{\mathcal{Q}}$ can cross the axis $\{x = 0\}$ at most once.* ■

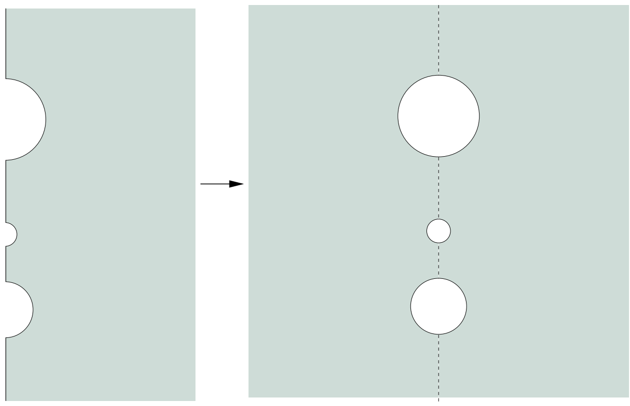

Figure 3: The quotient space $\mathcal{Q}$ and its double $\widehat{\mathcal{Q}}$.

An orbit C_q will be called an *accumulation orbit* of an orbit C_r if there exists a sequence $q_n \in C_r$ such that $q_n \to q$. Every orbit is its own accumulation orbit. It is a simple consequence of the accumulation Lemma A.5 that:

Lemma A.7 *Let C_q be an accumulation orbit of C_r. Then for every $p \in C_q$ there exists a sequence $p_n \in C_r$ such that $p_n \to p$.* ■

We will need the following:

Lemma A.8 *Let $r_n \in C_r$ be a sequence accumulating at $p \in \pi(\mathscr{A}) \setminus \partial\widehat{\mathcal{Q}}$. Then $p \in C_r$, and C_r continues smoothly across $\{x = 0\}$ at p.*

PROOF: By Lemma A.5 there exists an orbit C_p crossing the axis $\{x = 0\}$ transversally at p. Lemma A.7 shows that C_r crosses the axis. But, by Lemma A.6, C_r can cross the axis only once. It follows that $C_r = C_p$ and that $p \in C_r$. ■

Abusing notation, we continue to denote by W the function $W \circ \pi$. If W vanishes at a point lying at the boundary $\partial\widehat{\mathcal{Q}}$, then the corresponding circle forms a Jordan orbit. We have:

Lemma A.9 *The only orbits accumulating at $\partial\widehat{\mathcal{Q}}$ are the boundary circles.*

PROOF: Suppose that $r_n \in C_q$ accumulates at $p \in \partial\widehat{\mathcal{Q}}$. Then, by continuity, $W(p) = 0$, which implies that the boundary component through p is a Jordan orbit. But it follows from Lemma A.7 that any orbit accumulating at $\partial\widehat{\mathcal{Q}}$ has to cross the axis more than once, and the result follows from Lemma A.6. ■

The remaining possibilities will be excluded by a lamination version of the Poincaré-Bendixson theorem. We will make use of a smooth transverse orientation of all the $\hat{S}_p$'s; such a structure is not available for a general lamination, but exists in the problem at hand. More precisely, we will endow $\mathcal{D}_{oc} \cup \mathscr{E}^+$ with a smooth vector field Z transverse to all $\hat{S}_p$'s. The construction proceeds as follows: Choose any decomposition of $\mathcal{D}_{oc} \cup \mathscr{E}^+$ as $\mathbb{R} \times \overline{\mathring{\Sigma}}$, as in [22, Theorem 4.5]: thus each level set $\overline{\mathring{\Sigma}_t}$ of the time function t is transverse to the stationary Killing vector field K_0, with the periodic Killing vector K_1 tangent to $\overline{\mathring{\Sigma}_t}$. Let $q \in \hat{S}_p \cap \mathring{\Sigma}_0$; as the null leaf $\hat{S}_p$ is transversal to $\mathring{\Sigma}_0$, the intersection $\mathring{\Sigma}_0 \cap \hat{S}_p$ is a hypersurface in M of co-dimension two. There exist precisely two null directions at q which are normal to $\mathring{\Sigma}_0 \cap \hat{S}_p$, one of them is spanned by $l_0(q)$; we denote by $\tilde{Z}_q$ the unique future directed null vector spanning the other direction and satisfying $\mathring{Z}_q = T_q + \tilde{Z}_q$, where T_q is the unit timelike future directed normal to $\mathring{\Sigma}_0$ at q, and $\tilde{Z}_q$ is tangent to $\mathring{\Sigma}$.

The above definition of Z_q extends by continuity to $q \in \hat{S}_p \cap \overline{\mathring{\Sigma}_0}$.

Transversality and smoothness of l_0 imply that there exists a neighborhood $\mathscr{O}_q$ of q and an extension $\hat{Z}_q$ of $\tilde{Z}_q$ to $\mathscr{O}_q$ with the property that $\hat{Z}_q(r)$ is transverse to $\hat{S}_r$ for every $r \in \mathscr{O}_q$ satisfying $W(r) = 0$. The neighborhood $\mathscr{O}_q$ can, and will, be chosen to be invariant under $\mathbb{R} \times \mathrm{U}(1)$.

Consider the covering of $\overline{\mathring{\Sigma}_0} \cap \{W = 0\}$ by sets of the form $\mathscr{O}_q \cap \overline{\mathring{\Sigma}_0}$. Asymptotic flatness implies that $\overline{\mathring{\Sigma}_0} \cap \{W = 0\}$ is compact, which in turn implies that a finite subcovering $\mathscr{O}_i := \mathscr{O}_{q_i}$ can be chosen. Let φ_i be a partition of unity subordinated to the covering of $\mathcal{D}_{oc} \cup \mathscr{E}^+$ by the $\mathscr{O}_i$'s together with

$$\mathscr{O}_0 := \Big(\mathcal{D}_{oc} \cup \mathscr{E}^+\Big) \setminus \{W = 0\} \; .$$

The φ_i's can, and will, be chosen to be $\mathbb{R} \times \mathrm{U}(1)$–invariant. Set

$$Z := \sum_{i \geq 1} \varphi_i \hat{Z}_{q_i} \; .$$

Then Z is smooth, tangent to $\mathring{\Sigma}_0$, and transverse to all $\hat{S}_p$'s.

Choose an orientation of $\widehat{\mathcal{Q}}$. The vector field Z projects under π to a vector field $Z^\flat$ on $\widehat{\mathcal{Q}}$ transverse to each C_q. For each $r \in C_q$ we define a vector $V_q(r)$ by requiring $V_q(r)$ to be tangent to C_1 at r, with $\{V_q, Z^\flat\}$ positively oriented, and with V_q having length one with respect to some auxiliary Riemannian metric on $\widehat{\mathcal{Q}}$. Then V_q varies smoothly along C_q, and each C_q is in fact a complete integral curve of its own V_q. The vector field V_p along C_p defines an order, and diverging sequences, on C_p in the obvious way: we say that a point $r' \in C_p$ is subsequent to $r \in C_p$ if one flows from r to r' along V_p in the forward direction. A sequence $r_n \in V_p$ is diverging if $r_n = \phi(s_n)(p)$, where $\phi(s)$ is the flow of V_p along C_p, with $s_n \nearrow \infty$ or $s_n \searrow -\infty$.

By Lemma A.5, if a sequence $r_n \in C_{q_n}$ tends to $r \in C_q$, then the tangent spaces TC_{q_n} accumulate on TC_q. This implies that there exist numbers $\epsilon_n \in \{\pm 1\}$ such that $\epsilon_n V_{q_n}(r_n) \to V_q(r)$, and this is the best one can say in general. However, the existence of Z guarantees that $V_{q_n}(r_n) \to V_q(r)$.

We are ready now to pass to the analysis of

Case III: In view of Lemmata A.6 and A.9, it remains to exclude the existence of orbits C_q which are entirely contained within $\widehat{\mathcal{Q}} \setminus \partial\widehat{\mathcal{Q}}$, and which do not intersect $\pi(\mathscr{A})$, or which intersect $\pi(\mathscr{A})$ only once, and which do *not* form Jordan curves in $\mathring{\mathcal{Q}}$. Since $\{W = 0\} \cap \overline{\mathring{\Sigma}_0}$ is compact, there exists $p \in \widehat{\mathcal{Q}}$ and a diverging sequence $q_n \in C_q$ such that $q_n \to p$. Again by Lemmata A.6 and A.9, $p \notin \partial\widehat{\mathcal{Q}}$. The fact that C_p is a closed embedded curve follows now by the standard arguments of the proof of the Poincaré–Bendixson theorem, as e.g. in [37]. The orbit C_p does not meet $\partial\widehat{\mathcal{Q}}$ by Lemma A.9. If C_p met $\pi(\mathscr{A})$, it would have an intersection number with $\{x = 0\}$ equal to one by Lemma A.6, which is impossible for a Jordan curve in the plane. Thus C_p is entirely contained in $\mathring{\mathcal{Q}}$, which has already been shown to be impossible in Case I, and the result is established. ■

It remains to exclude the offending zeros[20] of X under the hypotheses of Theorem A.1, or under those of [19, Theorem 1.1], recall that it is assumed there that a vacuum space-time $(M, \mathbf{g})$ has a hypersurface-orthogonal Killing vector X which is timelike on a spacelike hypersurface Σ, and vanishes on its boundary $\partial\bar{\Sigma} = \bar{\Sigma} \setminus \Sigma$, which is assumed to be a compact two-dimensional topological manifold. As already pointed out above, Lemma A.3, shows that the set, say $\mathscr{E}$, where $\mathbf{g}(X, X)$ vanishes, is foliated by locally totally geodesic null hypersurfaces, away from the points where X vanishes. Hence each leaf of $\mathscr{E}$ is smooth on an open dense set, so $\partial\Sigma$ is smooth on the open dense subset of $\partial\Sigma$ consisting of points at which X does not vanish. Note that $\mathscr{E}$ might fail to be embedded in general, but this is irrelevant for the proof here because $\partial\Sigma$ is a compact embedded topological manifold by hypothesis. In vacuum, on every smooth leaf of $\mathscr{E}$, and hence on every smooth component of $\partial\Sigma$, the surface gravity κ is constant (see, e.g., [46, Theorem 2.1]). It follows that the problem with the incorrect [19, Theorem 3.1] is avoided by the following result:

Proposition A.10 *Let X be a Killing vector field, and suppose that*

$$\Omega := \partial\{p \in M \mid \mathbf{g}(X, X) < 0\} . \tag{A.7}$$

is a topological hypersurface. Suppose that

i. either X is hypersurface-orthogonal and Ω has vanishing surface gravity wherever defined,

[20]Zeros of X occurring at non-degenerate components of $\partial\bar{\Sigma}$ are allowed in [19, Theorem 1.1].

ii. or Ω is differentiable.

Then X has no zeros on Ω.

PROOF: The proof here is an adaptation to space-dimension $n = 3$ of a similar result proved in all dimensions in [21]. Let X be a non-trivial Killing vector, and suppose that X vanishes at $p \in \Omega$. Consider the anti-symmetric tensor $\lambda_{\mu\nu} = \nabla_\mu X_\nu|_p$; from [32, Section 7.2] or from [10] we have the following alternative:

i. There exists at p an orthonormal frame e_c, $c = 0, \ldots, 3$, with e_0 timelike, such that in this frame we have

$$\lambda_{cd} = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & a & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{A.8}$$

with $a \neq 0$ unless $X \equiv 0$. Let $\mathscr{U}$ be a geodesically convex neighborhood of p covered by normal coordinates (t, x, y, z) centred at p, and associated to e_a. Because the flow of X maps null geodesics to null geodesics, we have

$$X = a\Big(x\partial_t + (t - y)\partial_x + x\partial_y\Big) . \tag{A.9}$$

This, together with elementary properties of normal coordinates, implies

$$\mathbf{g}(X, X) = a^2(t - y)^2 + O((t^2 + x^2 + y^2 + z^2)^2) . \tag{A.10}$$

It follows from (A.9) that X is tangent to the two hypersurfaces

$$\mathscr{N}^{\pm} = \{t = y \ , \ \pm x > 0\} ,$$

non-vanishing there.

Assume that X is hypersurface-orthogonal. Consider any point $q \in \Omega$ at which X does not vanish. By Lemma A.3 the hypersurface Ω is smooth near q, and any geodesic γ initially normal to X_q stays on Ω, except perhaps when it reaches a point at which X vanishes.

So, suppose that γ is such a geodesic from $q \in \Omega$ to p, with p being the first point on γ at which X vanishes. If $\dot t \neq \dot y$ at p, (A.10) shows that X is spacelike along γ near and away from p, contradicting the fact that X is null on Ω. We conclude that $\dot\gamma$ is tangent at p to the hypersurface $\{t = y\}$, but then $\gamma \cap \mathscr{U}$ is included in $\{t = y\}$. Consequently

$$\Omega \cap \mathscr{U} \subset \{t = y\} . \tag{A.11}$$

Since Ω is a topological hypersurface by hypothesis, we obtain that

$$\Omega \cap \mathscr{U} = \{t = y\} . \tag{A.12}$$

(In particular Ω is smooth near p.)

In the case where X is not necessarily hypersurface orthogonal, but we assume a priori that Ω is differentiable, the argument is somewhat similar, with a weaker conclusion: Let $\gamma \subset \Omega$ be any differentiable curve, then we must have $\dot{t} = \dot{y}$ at p. Since Ω is a hypersurface, this implies that

$$T_p\Omega = T_p\{t = y\} \ . \tag{A.13}$$

So, while (A.12) does not necessarily hold, the tangent spaces coincide at p in both cases.

Consider, now any differentiable curve σ through p on which $\dot{t} \neq \dot{y} \neq 0$ at p. As already noted, Equation (A.10) shows that on σ the Killing vector X is spacelike near and away from p. By (A.13) such curves are transverse to Ω, which shows that there exist points arbitrarily close to Ω at which X is *spacelike* on both sides of Ω. This contradicts (A.7), and shows that this case cannot happen under our hypotheses.

Note that X is *null future directed* on $\mathscr{N}^+$, *null past directed* on $\mathscr{N}^-$,[21] and vanishes on the set

$$\mathscr{Y} := \{x = 0 \ , \ t = y\} = \overline{\mathscr{N}^+} \cap \overline{\mathscr{N}^-} \ .$$

ii. There exists at p an orthonormal frame e_c, $c = 0, \ldots, 3$, with e_0 timelike, such that in this frame we have

$$\lambda_{cd} = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix} \ , \tag{A.14}$$

with $a^2 + b^2 \neq 0$ unless $X \equiv 0$. As before, in normal coordinates (t, x, y, z) centred at p, and associated to e_a, we then have

$$X = a(t\partial_x + x\partial_t) + b(y\partial_z - z\partial_y) \ , \tag{A.15}$$

leading to

$$\mathbf{g}(X, X) = a^2(t^2 - x^2) + b^2(y^2 + z^2) + O((t^2 + x^2 + y^2 + z^2)^2) \ . \tag{A.16}$$

Suppose, first, that $a = 0$. Then $\mathrm{Ker}\lambda = \mathrm{Span}\{\partial_t, \partial_x\}|_p$. Now, because the flow of a Killing vector maps geodesics to geodesics, X vanishes on

[21] This fact can be used to given an alternative justification that X has no zeros on degenerate components of $\partial\mathcal{D}_{oc}$ if $\mathcal{D}_{oc}$ is chronological, using the fact that Killing orbits through $\mathcal{D}_{oc}$ are then future-oriented in the sense of [22]. But the current argument does not need the chronology hypothesis.

every geodesic γ with $\gamma(0) = p$ such that $\dot\gamma(0) \in \mathrm{Ker}\lambda$. So X vanishes throughout the timelike hypersurface $\{y = z = 0\}$. At every point q of this hypersurface, in adapted normal coordinates centred a q the tensor $\nabla_c X_d|_q$ takes the form (A.14) with $a = 0$. This implies that X is spacelike or vanishing throughout a neighborhood of p, so $a = 0$ cannot occur.

If Ω is differentiable at p, an argument very similar to the one above shows that

$$T_p\Omega \subset E_+ \cup E_- \ , \quad \text{where } E_\pm := \{\dot t = \pm \dot x\} \ .$$

So either $T_p\Omega = E_+$ or $T_p\Omega = E_-$. But, the curves with $\dot t = 2\dot x$ at p are transverse both to E_- and to E_+, with X spacelike on those curves near and away from p on both sides of $E_\pm$, contradicting the definition of Ω. Assuming differentiability of Ω we are done.

We continue with an analysis of the static case, and claim that $ab \neq 0$ is not possible. Indeed, let $X^\flat$ be the field of one-forms defined as $X^\flat = \mathbf{g}(X, \cdot)$. Then

$$X^\flat = a(tdx - xdt) + b(ydz - zdy) + O((t^2 + x^2 + y^2 + z^2)^{3/2}) \ , \tag{A.17}$$
$$dX^\flat = 2a\, dt \wedge dx + 2b\, dy \wedge dz + O(t^2 + x^2 + y^2 + z^2) \ , \tag{A.18}$$

and the staticity condition $X^\flat \wedge dX^\flat = 0$ gives $ab = 0$.

It remains to consider $b = 0$. Arguments similar to the ones already given show that

$$\Omega \cap \mathscr{U} \cap \{y = z = 0 \ , \ t = \pm x\} \neq \emptyset \ .$$

In this case from (A.16) we have

$$d\left(\mathbf{g}(X, X)\right) = 2a^2(tdx - xdt) + O((t^2 + x^2 + y^2 + z^2)^{3/2}) \ .$$

Comparing with (A.17) with $b = 0$ at points lying on the surface $\{y = z = 0 \ , \ t = \pm x\}$, with $|x|$ sufficiently small, we conclude that this case cannot occur if Ω is degenerate, and the proof is complete. ■

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