

Corrigendum to

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The classification of static vacuum space-times containing an asymptotically flat spacelike hypersurface with compact interior

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There are two points which have not been handled properly in [3].

First, it has been pointed out to me by João Lopes Costa that neither the original proof, nor that given in [3], of the Vishveshwara-Carter Lemma, takes properly into account the possibility that the hypersurface \mathcal{N} of [3, Lemma 4.1] could fail to be embedded when it is degenerate. This problem arises whether or not the horizon is degenerate, since we do not know a priori whether or not \mathcal{N} has anything to do with the horizon. This issue is taken care of by [7] under the assumption of global hyperbolicity of the domain of outer communications. I am grateful to João for pointing out the problem, and for useful remarks on previous versions of this *corrigendum*.

A (wrong) solution to this problem has been proposed in the arXiv version 2 of [5] (that paper was intended as arXiv version 2 of [3], but has been posted as version 2 of [5] by an error of manipulation). The idea was to show that the family of hypersurfaces covering the set where the static Killing vector becomes null contains an outermost closed and embedded hypersurface. A family of curves in a plane that does not contain such a curve is drawn in Figure 0.1: the only reasonable candidate for an outermost curve there is the curve of infinite length, which is embedded, but does not form a closed subset of the plane. The example shows that the strategy proposed in the addendum to [5] has no chance of succeeding.

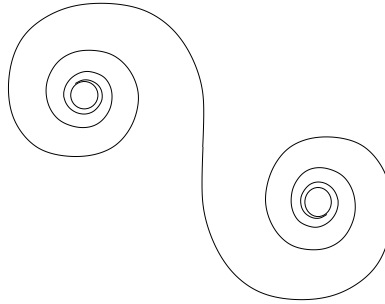


Figure 0.1: A family of three embedded curves in the plane consisting of two circles, together with a curve of infinite length that spirals towards the circles.

We note a corrected version of the Vishveshwara-Carter Lemma [2, 11]:

LEMMA 0.1 *Let (M, \mathbf{g}) be a smooth space-time with complete, static Killing vector X , set*

$$W := -\mathbf{g}_{\alpha\beta} X^\alpha X^\beta . \quad (0.1)$$

Then

1. $\{W = 0\} \cap \{X \neq 0\}$ is a union of integral leaves of the distribution X^\perp , which are totally geodesic within $M \setminus \{X = 0\}$.

2. Each connected component of

$$\{W = 0\} \cap \{dW \neq 0\} \cap \{X \neq 0\}$$

is a smooth, embedded, locally totally geodesic null hypersurface \mathcal{N} , with the Killing vector field X normal to \mathcal{N} .

REMARK 0.2 As the surface gravity is constant in electro-vacuum, point 2 covers adequately non-degenerate vacuum or electro-vacuum Killing horizons, but does not apply to degenerate ones. In [3, Lemma 4.1], the conclusions of point 2. were claimed for each connected component of the set

$$\{W = 0\} \cap \{X \neq 0\} \cap \partial\{W < 0\},$$

but the justification given there does not seem to be sufficient, and we do not know whether or not the result is correct without further hypotheses.

PROOF: The staticity condition $X^\flat \wedge dX^\flat = 0$ and the Frobenius theorem [9, Section 9.1] show that

$$\mathcal{O} := M \setminus \{X = 0\}$$

is foliated by immersed, but not necessarily embedded, hypersurfaces which are normal to X . Let

$$\hat{S}_p \subset \mathcal{O}$$

denote the maximally extended integral leaf of the distribution X^\perp passing through p .

Let $p \in \{W = 0\} \cap \{X \neq 0\}$ and let γ be an affinely parameterised geodesic starting at p with initial tangent $\dot{\gamma}(0)$ normal to X . A standard calculation along γ shows that

$$\frac{d(\mathbf{g}(\dot{\gamma}, X))}{ds} = \dot{\gamma}^\mu \nabla_\mu (\dot{\gamma}^\nu X_\nu) = \underbrace{\dot{\gamma}^\mu \nabla_\mu \dot{\gamma}^\nu}_{=0} X_\nu + \dot{\gamma}^\nu \dot{\gamma}^\mu \nabla_\mu X_\nu = \dot{\gamma}^\nu \dot{\gamma}^\mu \underbrace{\nabla_{(\mu} X_{\nu)}}_{=0} = 0,$$

hence $\dot{\gamma}$ remains normal to X , implying that \hat{S}_p is locally totally geodesic, as claimed. Clearly γ can exit \hat{S}_p only where that leaf ceases to be defined, namely at zeros of X . Hence the \hat{S}_p 's are totally geodesic within $M \setminus \{X = 0\}$.

The staticity condition can be rewritten as

$$2\nabla_{[\mu} X^\alpha X_{\nu]} = X^\alpha \nabla_{[\mu} X_{\nu]},$$

which, after a contraction with X_α , gives

$$\nabla_{[\mu} W X_{\nu]} = W \nabla_{[\mu} X_{\nu]}, \quad W \equiv X_\alpha X^\alpha. \quad (0.2)$$

Let ℓ_μ be any smooth covector field on \mathcal{O} such that $\ell_\mu X^\mu = 1$ and let γ be any differentiable curve contained in a leaf \hat{S}_p such that $W(\gamma(0)) = 0$; contraction of (0.2) with $\dot{\gamma}^\mu \ell^\nu$ gives

$$\frac{dW}{ds} = \dot{\gamma}^\mu \nabla_\mu W = 2W \dot{\gamma}^\mu \ell^\nu \nabla_{[\mu} X_{\nu]}. \quad (0.3)$$

Uniqueness of solutions of ODEs implies that $W \circ \gamma = 0$, and we conclude that $\hat{S}_p \subset \{W = 0\}$. This shows that if $\hat{S}_q \cap \{W = 0\} \neq \emptyset$ then $W \equiv 0$ on \hat{S}_q . Hence $\{W = 0\} \setminus \{X = 0\}$ is a union of leaves of the \hat{S}_p foliation.

At those points at which dW does not vanish, the set $\{W = 0\}$ is smooth, embedded hypersurface, and the proof is complete. \square

Next, the degenerate case in Boyer’s theorem [1] has been quoted incorrectly in [3, Theorem 3.1]: In spite of what is said there, there exist Killing vectors which have zeros at the closure of a degenerate horizon. The Minkowskian Killing vector

$$X = t\partial_x + x\partial_t + x\partial_y - y\partial_x = (t - y)\partial_x + x(\partial_t + \partial_y) \quad (0.4)$$

illustrates well the problem at hand. X vanishes at $\{t = y, x = 0\}$, and is null on $\{t = y, x \neq 0\}$. Recall that a Killing horizon associated to X is a null hypersurface \mathcal{N} on which X is null, tangent to \mathcal{N} . So, in this case, \mathcal{N} has two connected components

$$\mathcal{N}^\pm := \{t = y, \pm x > 0\}.$$

A key for the proof of [3, Theorem 1.1] is Proposition 3.2 there, which is *wrong* without the supplementary hypothesis that the Killing vector X has no zeros on $\partial\bar{\Sigma}$. Indeed, let $\Sigma = \{t = 0, y > 0\}$ in Minkowski space-time (\mathbb{R}^4, g) , and let X be given by (0.4). To see that Σ equipped with the “orbit space metric”¹

$$\forall Z_1, Z_2 \in T\Sigma \quad h(Z_1, Z_2) = g(Z_1, Z_2) - \frac{g(X, Z_1)g(X, Z_2)}{g(X, X)}$$

contains finite-proper length spacelike curves which reach the boundary $\partial\bar{\Sigma}$, consider the curve

$$(0, \infty) \ni s \mapsto \gamma(s) = (t = x = z = 0, y = s) \in \Sigma.$$

Then $g(X, \dot{\gamma}) = 0$, so $h(\dot{\gamma}, \dot{\gamma}) = g(\dot{\gamma}, \dot{\gamma}) = 1$, and the boundary $y = 0$ lies at h -distance along γ equal to s from any point $\gamma(s)$ on γ . But then this boundary lies to finite h -distance from any point $p \in \Sigma$, as one can reach $\partial\bar{\Sigma}$ from p by first going to γ , and then following γ until $\partial\bar{\Sigma}$ is reached. Here $\partial\bar{\Sigma}$ is not compact, but this seems irrelevant for the issue at hand.

More significantly, in this example X is spacelike near and away from $\{t = z\}$, so that h is *Lorentzian* there, while the orbit space metric *is* Riemannian in the context of the analysis of [3]. This observation is the key to the proof below that such zeros do not exist on boundaries $\partial\{\mathbf{g}(X, X) = 0\}$ in static space-times.

Let us thus show nonexistence of the offending zeros² of X under the hypotheses of [3, Theorem 1.1]. Recall that it is assumed there that a vacuum space-time (M, \mathbf{g}) has a hypersurface-orthogonal Killing vector X which is timelike on a spacelike hypersurface Σ , and vanishes on its boundary $\partial\bar{\Sigma} = \bar{\Sigma} \setminus \Sigma$, which is assumed to be a compact two-dimensional topological manifold. Now, as shown in [6], the set, say \mathcal{E} , where $\mathbf{g}(X, X)$ vanishes, is foliated by locally totally geodesic null hypersurfaces, away from the points where X vanishes. Hence each leaf of \mathcal{E} is smooth on an open dense set, so $\partial\Sigma$ is smooth on the open dense subset of $\partial\Sigma$ consisting of points at which X does not vanish. Note that \mathcal{E} might fail to be embedded in general, but this is irrelevant for the proof here because $\partial\Sigma$ is a compact embedded topological manifold by hypothesis. In vacuum, on every smooth leaf of \mathcal{E} , and hence on every smooth component of $\partial\Sigma$, the surface gravity κ is constant (see, e.g., [10, Theorem 2.1]). It follows that the problem with the incorrect [3, Theorem 3.1] is avoided by the following result:

PROPOSITION 0.3 *Let X be a Killing vector field, and suppose that*

$$\Omega := \partial\{p \in M \mid \mathbf{g}(X, X) < 0\}. \quad (0.5)$$

is a topological hypersurface. Suppose that

¹As already emphasised in [3], the metric h should *not* be thought of as the “metric on the space of orbits”, as we are not assuming anything about the manifold character of this last space; similarly transversality of X to Σ is *not* assumed.

²Zeros of X occurring at non-degenerate components of $\partial\bar{\Sigma}$ are allowed in [3, Theorem 1.1].

1. either X is hypersurface-orthogonal and Ω has vanishing surface gravity wherever defined,
2. or Ω is differentiable.

Then X has no zeros on Ω .

PROOF: The proof here is an adaptation to space-dimension $n = 3$ of a similar result proved in all dimensions in [4]. Let X be a non-trivial Killing vector, and suppose that X vanishes at $p \in \Omega$. Consider the anti-symmetric tensor $\lambda_{\mu\nu} = \nabla_\mu X_\nu|_p$; from [8, Section 7.2] or from [1] we have the following alternative:

1. There exists at p an orthonormal frame e_c , $c = 0, \dots, 3$, with e_0 timelike, such that in this frame we have

$$\lambda_{cd} = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & a & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (0.6)$$

with $a \neq 0$ unless $X \equiv 0$. Let \mathcal{U} be a geodesically convex neighborhood of p covered by normal coordinates (t, x, y, z) centred at p , and associated to e_a . Because the flow of X maps null geodesics to null geodesics, we have

$$X = a(x\partial_t + (t - y)\partial_x + x\partial_y). \quad (0.7)$$

This, together with elementary properties of normal coordinates, implies

$$\mathbf{g}(X, X) = a^2(t - y)^2 + O((t^2 + x^2 + y^2 + z^2)^2). \quad (0.8)$$

It follows from (0.7) that X is tangent to the two hypersurfaces

$$\mathcal{N}^\pm = \{t = y, \pm x > 0\},$$

non-vanishing there.

Assume that X is hypersurface-orthogonal. Consider any point $q \in \Omega$ at which X does not vanish. By Lemma 0.1 the hypersurface Ω is smooth near q , and any geodesic γ initially normal to X_q stays on Ω , except perhaps when it reaches a point at which X vanishes.

So, suppose that γ is such a geodesic from $q \in \Omega$ to p , with p being the first point on γ at which X vanishes. If $t \neq y$ at p , (0.8) shows that X is spacelike along γ near and away from p , contradicting the fact that X is null on Ω . We conclude that $\dot{\gamma}$ is tangent at p to the hypersurface $\{t = y\}$, but then $\gamma \cap \mathcal{U}$ is included in $\{t = y\}$. Consequently

$$\Omega \cap \mathcal{U} \subset \{t = y\}. \quad (0.9)$$

Since Ω is a topological hypersurface by hypothesis, we obtain that

$$\Omega \cap \mathcal{U} = \{t = y\}. \quad (0.10)$$

(In particular Ω is smooth near p .)

In the case where X is not necessarily hypersurface orthogonal, but we assume a priori that Ω is differentiable, the argument is somewhat similar, with a weaker conclusion: Let $\gamma \subset \Omega$ be any differentiable curve, then we must have $\dot{t} = \dot{y}$ at p . Since Ω is a hypersurface, this implies that

$$T_p\Omega = T_p\{t = y\}. \quad (0.11)$$

So, while (0.10) does not necessarily hold, the tangent spaces coincide at p in both cases.

Consider, now any differentiable curve σ through p on which $\dot{t} \neq \dot{y} \neq 0$ at p . As already noted, Equation (0.8) shows that on σ the Killing vector X is spacelike near and away from p . By (0.11) such curves are transverse to Ω , which shows that there exist points arbitrarily close to Ω at which X is *spacelike* on both sides of Ω . This contradicts (0.5), and shows that this case cannot happen under our hypotheses.

Note that X is *null future directed* on \mathcal{N}^+ , *null past directed* on \mathcal{N}^- ,³ and vanishes on the set

$$\mathcal{Y} := \{x = 0, t = y\} = \overline{\mathcal{N}^+} \cap \overline{\mathcal{N}^-}.$$

2. There exists at p an orthonormal frame e_c , $c = 0, \dots, 3$, with e_0 timelike, such that in this frame we have

$$\lambda_{cd} = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}, \quad (0.12)$$

with $a^2 + b^2 \neq 0$ unless $X \equiv 0$. As before, in normal coordinates (t, x, y, z) centred at p , and associated to e_a , we then have

$$X = a(t\partial_x + x\partial_t) + b(y\partial_z - z\partial_y), \quad (0.13)$$

leading to

$$\mathbf{g}(X, X) = a^2(t^2 - x^2) + b^2(y^2 + z^2) + O((t^2 + x^2 + y^2 + z^2)^2). \quad (0.14)$$

Suppose, first, that $a = 0$. Then $\text{Ker}\lambda = \text{Span}\{\partial_t, \partial_x\}|_p$. Now, because the flow of a Killing vector maps geodesics to geodesics, X vanishes on every geodesic γ with $\gamma(0) = p$ such that $\dot{\gamma}(0) \in \text{Ker}\lambda$. So X vanishes throughout the timelike hypersurface $\{y = z = 0\}$. At every point q of this hypersurface, in adapted normal coordinates centred at q the tensor $\nabla_c X_d|_q$ takes the form (0.12) with $a = 0$. This implies that X is spacelike or vanishing throughout a neighborhood of p , so $a = 0$ cannot occur.

If Ω is differentiable at p , an argument very similar to the one above shows that

$$T_p\Omega \subset E_+ \cup E_-, \quad \text{where } E_{\pm} := \{\dot{t} = \pm \dot{x}\}.$$

So either $T_p\Omega = E_+$ or $T_p\Omega = E_-$. But, the curves with $\dot{t} = 2\dot{x}$ at p are transverse both to E_- and to E_+ , with X spacelike on those curves near and away from p on both sides of E_{\pm} , contradicting the definition of Ω . Assuming differentiability of Ω we are done.

We continue with an analysis of the static case, and claim that $ab \neq 0$ is not possible. Indeed, let X^b be the field of one-forms defined as $X^b = \mathbf{g}(X, \cdot)$. Then

$$X^b = a(tdx - xdt) + b(ydz - zdy) + O((t^2 + x^2 + y^2 + z^2)^{3/2}), \quad (0.15)$$

$$dX^b = 2a dt \wedge dx + 2b dy \wedge dz + O(t^2 + x^2 + y^2 + z^2), \quad (0.16)$$

³This fact can be used to given an alternative justification that X has no zeros on degenerate components of $\partial\langle\langle M_{\text{ext}} \rangle\rangle$ if $\langle\langle M_{\text{ext}} \rangle\rangle$ is chronological, using the fact that Killing orbits through $\langle\langle M_{\text{ext}} \rangle\rangle$ are then future-oriented in the sense of [6]. But the current argument does not need the chronology hypothesis.

and the staticity condition $X^b \wedge dX^b = 0$ gives $ab = 0$.

It remains to consider $b = 0$. Arguments similar to the ones already given show that

$$\Omega \cap \mathcal{U} \cap \{y = z = 0, t = \pm x\} \neq \emptyset.$$

In this case from (0.14) we have

$$d(\mathbf{g}(X, X)) = 2a^2(tdx - xdt) + O((t^2 + x^2 + y^2 + z^2)^{3/2}).$$

Comparing with (0.15) with $b = 0$ at points lying on the surface $\{y = z = 0, t = \pm x\}$, with $|x|$ sufficiently small, we conclude that this case cannot occur if Ω is degenerate, and the proof is complete. \square

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