Corrigendum to

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The classification of static vacuum space–times containing an asymptotically flat spacelike hypersurface with compact interior

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There are two points which have not been handled properly in [3].

First, it has been pointed out to me by João Lopes Costa that neither the original proof, nor that given in [3], of the Vishveshwara-Carter Lemma, takes properly into account the possibility that the hypersurface $\mathscr{N}$ of [3, Lemma 4.1] could fail to be embedded when it is degenerate. This problem arises whether or not the horizon is degenerate, since we do not know a priori whether or not $\mathscr{N}$ has anything to do with the horizon. This issue is taken care of by [7] under the assumption of global hyperbolicity of the domain of outer communications. I am grateful to João for pointing out the problem, and for useful remarks on previous versions of this *corrigendum*.

A (wrong) solution to this problem has been proposed in the arXiv version 2 of [5] (that paper was intended as arXiv version 2 of [3], but has been posted as version 2 of [5] by an error of manipulation). The idea was to show that the family of hypersurfaces covering the set where the static Killing vector becomes null contains an outermost closed and embedded hypersurface. A family of curves in a plane that does not contain such a curve is drawn in Figure 0.1: the only reasonable candidate for an outermost curve there is the curve of infinite length, which is embedded, but does not form a closed subset of the plane. The example shows that the strategy proposed in the addendum to [5] has no chance of succeeding.

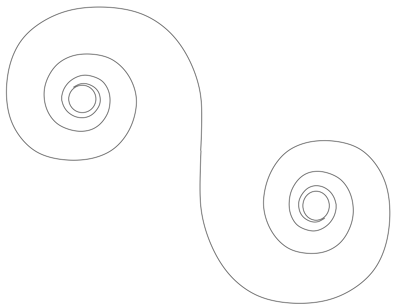

Figure 0.1: A family of three embedded curves in the plane consisting of two circles, together with a curve of infinite length that spirals towards the circles.

We note a corrected version of the Vishveshwara–Carter Lemma [2, 11]:

LEMMA 0.1 *Let $(M, \mathbf{g})$ be a smooth space–time with complete, static Killing vector X, set*

$$W := -\mathbf{g}_{\alpha\beta} X^\alpha X^\beta \,. \tag{0.1}$$

Then

1. *$\{W = 0\} \cap \{X \neq 0\}$ is a union of integral leaves of the distribution $X^\perp$, which are totally geodesic within $M \setminus \{X = 0\}$.*

2. *Each connected component of*

$$\{W=0\}\cap\{dW\neq 0\}\cap\{X\neq 0\}$$

is a smooth, embedded, locally totally geodesic null hypersurface $\mathscr{N}$, with the Killing vector field X normal to $\mathscr{N}$.

REMARK 0.2 As the surface gravity is constant in electro-vacuum, point 2 covers adequately non-degenerate vacuum or electro-vacuum Killing horizons, but does not apply to degenerate ones. In [3, Lemma 4.1], the conclusions of point 2. were claimed for each connected component of the set

$$\{W=0\}\cap\{X\neq 0\}\cap\partial\{W<0\}\ ,$$

but the justification given there does not seem to be sufficient, and we do not know whether or not the result is correct without further hypotheses.

PROOF: The staticity condition $X^\flat\wedge dX^\flat=0$ and the Frobenius theorem [9, Section 9.1] show that

$$\mathscr{O}:=M\setminus\{X=0\}$$

is foliated by immersed, but not necessarily embedded, hypersurfaces which are normal to X. Let

$$\hat S_p\subset\mathscr{O}$$

denote the maximally extended integral leaf of the distribution $X^\perp$ passing through p.

Let $p\in\{W=0\}\cap\{X\neq 0\}$ and let γ be an affinely parameterised geodesic starting at p with initial tangent $\dot\gamma(0)$ normal to X. A standard calculation along γ shows that

$$\frac{d(\mathbf{g}(\dot\gamma,X))}{ds}=\dot\gamma^\mu\nabla_\mu(\dot\gamma^\nu X_\nu)=\underbrace{\dot\gamma^\mu\nabla_\mu\dot\gamma^\nu}_{=0}X_\nu+\dot\gamma^\nu\dot\gamma^\mu\nabla_\mu X_\nu=\dot\gamma^\nu\dot\gamma^\mu\underbrace{\nabla_{(\mu}X_{\nu)}}_{=0}=0\ ,$$

hence $\dot\gamma$ remains normal to X, implying that $\hat S_p$ is locally totally geodesic, as claimed. Clearly γ can exit $\hat S_p$ only where that leaf ceases to be defined, namely at zeros of X. Hence the $\hat S_p$'s are totally geodesic within $M\setminus\{X=0\}$.

The staticity condition can be rewritten as

$$2\nabla_{[\mu}X^\alpha X_{\nu]}=X^\alpha\nabla_{[\mu}X_{\nu]}\ ,$$

which, after a contraction with X_α, gives

$$\nabla_{[\mu}WX_{\nu]}=W\nabla_{[\mu}X_{\nu]}\ ,\qquad W\equiv X_\alpha X^\alpha\ . \tag{0.2}$$

Let ℓ_μ be any smooth covector field on $\mathscr{O}$ such that $\ell_\mu X^\mu=1$ and let γ be any differentiable curve contained in a leaf $\hat S_p$ such that $W(\gamma(0))=0$; contraction of (0.2) with $\dot\gamma^\mu\ell^\nu$ gives

$$\frac{dW}{ds}=\dot\gamma^\mu\nabla_\mu W=2W\dot\gamma^\mu\ell^\nu\nabla_{[\mu}X_{\nu]}\ . \tag{0.3}$$

Uniqueness of solutions of ODEs implies that $W\circ\gamma=0$, and we conclude that $\hat S_p\subset\{W=0\}$. This shows that if $\hat S_q\cap\{W=0\}\neq\emptyset$ then $W\equiv 0$ on $\hat S_q$. Hence $\{W=0\}\setminus\{X=0\}$ is a union of leaves of the $\hat S_p$ foliation.

At those points at which dW does not vanish, the set $\{W=0\}$ is smooth, embedded hypersurface, and the proof is complete. □

Next, the degenerate case in Boyer's theorem [1] has been quoted incorrectly in [3, Theorem 3.1]: In spite of what is said there, there exist Killing vectors which have zeros at the closure of a degenerate horizon. The Minkowskian Killing vector

$$X = t\partial_x + x\partial_t + x\partial_y - y\partial_x = (t-y)\partial_x + x(\partial_t + \partial_y) \tag{0.4}$$

illustrates well the problem at hand. X vanishes at $\{t = y\ ,\ x = 0\}$, and is null on $\{t = y\ ,\ x \neq 0\}$. Recall that a Killing horizon associated to X is a null hypersurface $\mathscr{N}$ on which X is null, tangent to $\mathscr{N}$. So, in this case, $\mathscr{N}$ has two connected components

$$\mathscr{N}^{\pm} := \{t = y\ , \pm x > 0\}\ .$$

A key for the proof of [3, Theorem 1.1] is Proposition 3.2 there, which is *wrong* without the supplementary hypothesis that the Killing vector X has no zeros on $\partial\bar{\Sigma}$. Indeed, let $\Sigma = \{t = 0, y > 0\}$ in Minkowski space-time $(\mathbb{R}^4, g)$, and let X be given by (0.4). To see that Σ equipped with the "orbit space metric"[1]

$$\forall Z_1, Z_2 \in T\Sigma \qquad h(Z_1, Z_2) = g(Z_1, Z_2) - \frac{g(X, Z_1)g(X, Z_2)}{g(X, X)}$$

contains finite-proper length spacelike curves which reach the boundary $\partial\bar{\Sigma}$, consider the curve

$$(0, \infty) \ni s \mapsto \gamma(s) = (t = x = z = 0\ ,\ y = s) \in \Sigma\ .$$

Then $g(X, \dot\gamma) = 0$, so $h(\dot\gamma, \dot\gamma) = g(\dot\gamma, \dot\gamma) = 1$, and the boundary $y = 0$ lies at h–distance along γ equal to s from any point $\gamma(s)$ on γ. But then this boundary lies to finite h–distance from any point $p \in \Sigma$, as one can reach $\partial\bar{\Sigma}$ from p by first going to γ, and then following γ until $\partial\bar{\Sigma}$ is reached. Here $\partial\bar{\Sigma}$ is not compact, but this seems irrelevant for the issue at hand.

More significantly, in this example X is spacelike near and away from $\{t = z\}$, so that h is *Lorentzian* there, while the orbit space metric *is* Riemannian in the context of the analysis of [3]. This observation is the key to the proof below that such zeros do not exist on boundaries $\partial\{\mathbf{g}(X, X) = 0\}$ in static space-times.

Let us thus show nonexistence of the offending zeros[2] of X under the hypotheses of [3, Theorem 1.1]. Recall that it is assumed there that a vacuum space-time $(M, \mathbf{g})$ has a hypersurface-orthogonal Killing vector X which is timelike on a spacelike hypersurface Σ, and vanishes on its boundary $\partial\bar{\Sigma} = \bar{\Sigma} \setminus \Sigma$, which is assumed to be a compact two-dimensional topological manifold. Now, as shown in [6], the set, say $\mathscr{E}$, where $\mathbf{g}(X, X)$ vanishes, is foliated by locally totally geodesic null hypersurfaces, away from the points where X vanishes. Hence each leaf of $\mathscr{E}$ is smooth on an open dense set, so $\partial\Sigma$ is smooth on the open dense subset of $\partial\Sigma$ consisting of points at which X does not vanish. Note that $\mathscr{E}$ might fail to be embedded in general, but this is irrelevant for the proof here because $\partial\Sigma$ is a compact embedded topological manifold by hypothesis. In vacuum, on every smooth leaf of $\mathscr{E}$, and hence on every smooth component of $\partial\Sigma$, the surface gravity κ is constant (see, e.g., [10, Theorem 2.1]). It follows that the problem with the incorrect [3, Theorem 3.1] is avoided by the following result:

Proposition 0.3 *Let X be a Killing vector field, and suppose that*

$$\Omega := \partial\{p \in M \mid \mathbf{g}(X, X) < 0\}\ . \tag{0.5}$$

is a topological hypersurface. Suppose that

[1] As already emphasised in [3], the metric h should *not* be thought of as the "metric on the space of orbits", as we are not assuming anything about the manifold character of this last space; similarly transversality of X to Σ is *not* assumed.

[2] Zeros of X occurring at non-degenerate components of $\partial\bar{\Sigma}$ are allowed in [3, Theorem 1.1].

1. *either X is hypersurface-orthogonal and Ω has vanishing surface gravity wherever defined,*

2. *or Ω is differentiable.*

Then X has no zeros on Ω.

PROOF: The proof here is an adaptation to space-dimension $n = 3$ of a similar result proved in all dimensions in [4]. Let X be a non-trivial Killing vector, and suppose that X vanishes at $p \in \Omega$. Consider the anti-symmetric tensor $\lambda_{\mu\nu} = \nabla_\mu X_\nu|_p$; from [8, Section 7.2] or from [1] we have the following alternative:

1. There exists at p an orthonormal frame e_c, $c = 0, \ldots, 3$, with e_0 timelike, such that in this frame we have

$$\lambda_{cd} = \left(\begin{array}{cccc} 0 & a & 0 & 0 \\ -a & 0 & a & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) , \tag{0.6}$$

with $a \neq 0$ unless $X \equiv 0$. Let $\mathscr{U}$ be a geodesically convex neighborhood of p covered by normal coordinates (t, x, y, z) centred at p, and associated to e_a. Because the flow of X maps null geodesics to null geodesics, we have

$$X = a\Big(x\partial_t + (t - y)\partial_x + x\partial_y\Big) . \tag{0.7}$$

This, together with elementary properties of normal coordinates, implies

$$\mathbf{g}(X, X) = a^2(t - y)^2 + O((t^2 + x^2 + y^2 + z^2)^2) . \tag{0.8}$$

It follows from (0.7) that X is tangent to the two hypersurfaces

$$\mathscr{N}^{\pm} = \{t = y\ ,\ \pm x > 0\} ,$$

non-vanishing there.

Assume that X is hypersurface-orthogonal. Consider any point $q \in \Omega$ at which X does not vanish. By Lemma 0.1 the hypersurface Ω is smooth near q, and any geodesic γ initially normal to X_q stays on Ω, except perhaps when it reaches a point at which X vanishes.

So, suppose that γ is such a geodesic from $q \in \Omega$ to p, with p being the first point on γ at which X vanishes. If $\dot t \neq \dot y$ at p, (0.8) shows that X is spacelike along γ near and away from p, contradicting the fact that X is null on Ω. We conclude that $\dot\gamma$ is tangent at p to the hypersurface $\{t = y\}$, but then $\gamma \cap \mathscr{U}$ is included in $\{t = y\}$. Consequently

$$\Omega \cap \mathscr{U} \subset \{t = y\} . \tag{0.9}$$

Since Ω is a topological hypersurface by hypothesis, we obtain that

$$\Omega \cap \mathscr{U} = \{t = y\} . \tag{0.10}$$

(In particular Ω is smooth near p.)

In the case where X is not necessarily hypersurface orthogonal, but we assume a priori that Ω is differentiable, the argument is somewhat similar, with a weaker conclusion: Let $\gamma \subset \Omega$ be any differentiable curve, then we must have $\dot t = \dot y$ at p. Since Ω is a hypersurface, this implies that

$$T_p\Omega = T_p\{t = y\} . \tag{0.11}$$

So, while (0.10) does not necessarily hold, the tangent spaces coincide at p in both cases.

Consider, now any differentiable curve σ through p on which $\dot t \neq \dot y \neq 0$ at p. As already noted, Equation (0.8) shows that on σ the Killing vector X is spacelike near and away from p. By (0.11) such curves are transverse to Ω, which shows that there exist points arbitrarily close to Ω at which X is *spacelike* on both sides of Ω. This contradicts (0.5), and shows that this case cannot happen under our hypotheses.

Note that X is *null future directed* on $\mathscr{N}^+$, *null past directed* on $\mathscr{N}^-$,[3] and vanishes on the set

$$\mathscr{Y} := \{x = 0\ ,\ t = y\} = \overline{\mathscr{N}^+} \cap \overline{\mathscr{N}^-}\ .$$

2. There exists at p an orthonormal frame e_c, $c = 0, \dots, 3$, with e_0 timelike, such that in this frame we have

$$\lambda_{cd} = \left(\begin{array}{cccc} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{array} \right) , \tag{0.12}$$

with $a^2 + b^2 \neq 0$ unless $X \equiv 0$. As before, in normal coordinates (t, x, y, z) centred at p, and associated to e_a, we then have

$$X = a(t\partial_x + x\partial_t) + b(y\partial_z - z\partial_y)\ , \tag{0.13}$$

leading to

$$\mathbf{g}(X, X) = a^2(t^2 - x^2) + b^2(y^2 + z^2) + O((t^2 + x^2 + y^2 + z^2)^2)\ . \tag{0.14}$$

Suppose, first, that $a = 0$. Then $\mathrm{Ker}\lambda = \mathrm{Span}\{\partial_t, \partial_x\}|_p$. Now, because the flow of a Killing vector maps geodesics to geodesics, X vanishes on every geodesic γ with $\gamma(0) = p$ such that $\dot\gamma(0) \in \mathrm{Ker}\lambda$. So X vanishes throughout the timelike hypersurface $\{y = z = 0\}$. At every point q of this hypersurface, in adapted normal coordinates centred a q the tensor $\nabla_c X_d|_q$ takes the form (0.12) with $a = 0$. This implies that X is spacelike or vanishing throughout a neighborhood of p, so $a = 0$ cannot occur.

If Ω is differentiable at p, an argument very similar to the one above shows that

$$T_p\Omega \subset E_+ \cup E_-\ , \quad \text{where } E_\pm := \{\dot t = \pm \dot x\}\ .$$

So either $T_p\Omega = E_+$ or $T_p\Omega = E_-$. But, the curves with $\dot t = 2\dot x$ at p are transverse both to E_- and to E_+, with X spacelike on those curves near and away from p on both sides of $E_\pm$, contradicting the definition of Ω. Assuming differentiability of Ω we are done.

We continue with an analysis of the static case, and claim that $ab \neq 0$ is not possible. Indeed, let $X^\flat$ be the field of one-forms defined as $X^\flat = \mathbf{g}(X, \cdot)$. Then

$$\begin{aligned} X^\flat &= a(tdx - xdt) + b(ydz - zdy) + O((t^2 + x^2 + y^2 + z^2)^{3/2})\ , && (0.15)\\ dX^\flat &= 2a\, dt \wedge dx + 2b\, dy \wedge dz + O(t^2 + x^2 + y^2 + z^2)\ , && (0.16) \end{aligned}$$

[3]This fact can be used to given an alternative justification that X has no zeros on degenerate components of $\partial\langle\langle M_{\rm ext}\rangle\rangle$ if $\langle\langle M_{\rm ext}\rangle\rangle$ is chronological, using the fact that Killing orbits through $\langle\langle M_{\rm ext}\rangle\rangle$ are then future-oriented in the sense of [6]. But the current argument does not need the chronology hypothesis.

and the staticity condition $X^\flat \wedge dX^\flat = 0$ gives $ab = 0$.

It remains to consider $b = 0$. Arguments similar to the ones already given show that

$$\Omega \cap \mathscr{U} \cap \{y = z = 0\ ,\ t = \pm x\} \neq \emptyset\ .$$

In this case from (0.14) we have

$$d\left(\mathbf{g}(X,X)\right) = 2a^2(tdx - xdt) + O((t^2 + x^2 + y^2 + z^2)^{3/2})\ .$$

Comparing with (0.15) with $b = 0$ at points lying on the surface $\{y = z = 0\ ,\ t = \pm x\}$, with $|x|$ sufficiently small, we conclude that this case cannot occur if Ω is degenerate, and the proof is complete. □

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