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LETTER TO THE EDITOR

A remark on the positive-energy theorem

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Abstract. The problem of the weakest possible boundary conditions for the positive-energy theorem is discussed. Examples of asymptotically flat (in some sense) metrics violating this theorem are given. A class of metrics which have well defined infinite ADM mass is presented.

One of the relatively recent major advances in general relativity was the proof of the positive-energy theorem [1, 2]. In order to make Witten's proof mathematically rigorous, Choquet-Bruhat [3] and Reula [4] (cf also [5]) have established the existence of solutions of Witten's equation under fairly weak conditions. The purpose of this letter is to discuss the weakest possible boundary conditions in the positive-energy theorem. This issue is clearly related to the following two problems.

(a) Under what conditions can the ADM mass, finite or infinite, be defined in a meaningful way?

(b) Can this mass, even though defined at spatial infinity, change in time?

As was shown in [6] (cf also [7]) the ADM mass of a fixed Cauchy data set is finite and well defined if the four-dimensional metric solution of Einstein equations satisfies the following conditions†:

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} = \mathcal{O}_1(r^{-\alpha}) \qquad 1 \geqslant \alpha > \tfrac{1}{2} \tag{1a}$$

$$T_{\mu\nu} = \mathrm{O}(r^{-3-\varepsilon}) \qquad \varepsilon > 0. \tag{1b}$$

It is not difficult to show that if (1) is satisfied between two boosted hypersurfaces Σ_1 and Σ_2 (with Σ_1 a $t=$ constant hypersurface), the ADM 4-momentum of Σ_2 is the Lorentz-transformed 4-momentum of Σ_1, and does not change if Σ_2 is time translated with respect to Σ_1 (these results both follow from the Stokes theorem and the Einstein–von Freud identity (cf [8] or, e.g., [6]) applied to a 'cylindrical hypersurface' spanned on $\partial\Sigma_1$ and $\partial\Sigma_2$). The coordinate invariance of the p_μ (cf [9] for probably the first fairly complete analysis of this problem in the ADM language) under coordinate transformations of the form

$$x^\mu \to x^\mu + \zeta^\mu(x) \qquad \zeta^\mu(x) = \mathcal{O}_2(r^{-\gamma}) \qquad \gamma > \tfrac{1}{2} \tag{2}$$

† The signature is +2, latin indices run from 0 to 3, greek ones from 1 to 3, $\eta_{\mu\nu}$ is the flat Minkowski metric, $\mathrm{d}V_\mu$, $\mathrm{d}S_{\mu\nu}$ and $\mathrm{d}S_i$ are the tensor-density-valued forms $\mathrm{d}V_\mu = \varepsilon_{\mu\alpha\beta\gamma}\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma/6$, $\mathrm{d}S_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta}\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta/2$, $\mathrm{d}S_i = \mathrm{d}S_{0i}$, $\varepsilon_{0123} = 1$. We use the notation $f = \mathrm{O}_n(r^\beta)$, $\beta \in \mathbb{R}$, if in the asymptotically flat 'end' under consideration f satisfies $|f| \leqslant C(1+r)^\beta$, $|\partial_i f| \leqslant C(1+r)^{\beta-1}, \ldots, |\partial_{i_1} \ldots \partial_{i_n} f| \leqslant C(1+r)^{\beta-n}$, for some positive constant C, $r^2 = \Sigma(x^i)^2$, x^i are any asymptotically rectangular coordinates, $\mathrm{O}(r^\beta) = \mathrm{O}_0(r^\beta)$. It is also convenient to write $f = \mathcal{O}_n(r^\beta)$ if $f = \mathrm{O}(r^\beta)$, $\partial_\mu f = \mathrm{O}(r^{\beta-1}), \ldots, \partial_{\mu_1} \ldots \partial_{\mu_n} f = \mathrm{O}(r^{\beta-n})$. r^0 is always understood to be $\ln r$.

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most easily follows from the following argument: if (1a) is satisfied p_μ can be written in the form

$$p_\mu X^\mu = \lim_{r\to\infty}(3/16\pi)\int_{\substack{r=\text{constant}\\ t=\text{constant}}} \delta^{\alpha\beta\gamma}_{\lambda\mu\nu} X^\nu \eta^{\lambda\rho}\eta_{\gamma\sigma} g^{\sigma\mu}{}_{,\rho}\, \mathrm{d}S_{\alpha\beta} \tag{3}$$

where X^μ is any asymptotically constant vector. It is easily found that under conditions (1) and (2), in the limit $r\to\infty$, the integrand of (3) changes by a complete divergence:

$$\Delta(p_\mu X^\mu) = \lim_{r\to\infty}(3/16\pi)\int_{r=\text{constant}} (X^\nu \delta^{\alpha\beta\gamma}_{\lambda\mu\nu}\eta^{\lambda\rho}\zeta^\mu{}_{,\rho})_{,\gamma}\, \mathrm{d}S_{\alpha\beta} = 0$$

(the Einstein–von Freud identity shows that there is no change of p_μ related to the change of surfaces of integration in the limit).

If one assumes that $h_{\mu\nu} = \mathcal{O}_2(r^{-\alpha})$, not only $\mathcal{O}_1(r^{-\alpha})$, an even simpler proof of coordinate invariance can be given: under (1) the ADM p_μ can be written in the Ashtekar–Hansen form [10]†:

$$\begin{aligned} p_\mu X^\mu = \lim_{r\to\infty}\Bigg(\bigg(\int_{r=\text{constant}} & (-\det g)^{1/2}\varepsilon_{\mu\nu\alpha\beta}X^\mu x^\nu R^{\alpha\beta}{}_{\rho\sigma}\,\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma \\ &+2\int_{r=\text{constant}} \mathrm{d}((-\det g)^{1/2}\varepsilon_{\mu\nu\alpha\beta}x^\nu X^\mu g^{\alpha\gamma}\Gamma^\beta_{\gamma\rho}\,\mathrm{d}x^\rho)\bigg)(32\pi)^{-1} \\ = \lim_{r\to\infty}\bigg(\int_{r=\text{constant}} & (-\det g)^{1/2}R_{\mu\nu\alpha\beta}X^\mu x^\nu\,\mathrm{d}S^{\alpha\beta}\bigg)(16\pi)^{-1} \end{aligned} \tag{4}$$

x^μ are the coordinate functions. Since $R_{\mu\nu\alpha\beta}$ is assumed to be $\mathrm{O}(r^{-2-\alpha})$, the change of the integrand induced by the transformation (2) identically vanishes in the limit $r\to\infty$. The expression (4) is particularly suitable for the proof of the equality of the ADM and Komar masses when X is a translational Killing vector‡:

$$\begin{aligned} p_\mu X^\mu &= \lim_{r\to\infty}\bigg(\int_{r=\text{constant}} X^{[\beta;\alpha]}{}_\gamma x^\gamma(-\det g)^{1/2}\,\mathrm{d}S_{\alpha\beta}\bigg)(16\pi)^{-1} \\ &= \lim_{r\to\infty}\bigg(2\int_{r=\text{constant}} X^{[\alpha;\beta]}(-\det g)^{1/2}\,\mathrm{d}S_{\alpha\beta} \\ &\qquad +3\int_{r=\text{constant}} (X^{[\alpha;\beta}x^{\gamma]}(-\det g)^{1/2})_{;\gamma}\,\mathrm{d}S_{\alpha\beta}\bigg)(16\pi)^{-1} \\ &= \lim_{r\to\infty}\bigg(\int_{r=\text{constant}} X^{[\alpha;\beta]}(-\det g)^{1/2}\,\mathrm{d}S_{\alpha\beta}\bigg)(8\pi)^{-1} \end{aligned}$$

where we have used $R_{\mu\nu}{}^{\alpha\beta}X^\mu = X^{\beta;\alpha}{}_\nu$; cf also [11] and [12].

If one is interested in a fixed Cauchy hypersurface, it is clear that some of the boundary conditions (1) are spurious because the ADM mass can be defined purely in

† Equation (4) is obtained by rewriting the Ashtekar–Hansen expression for p_μ in physical coordinates, with $\Omega = (x_\mu x^\mu)$.

‡ By a translational Killing vector we mean a Killing vector satisfying $X^\mu = \overset{0}{X}{}^\mu + Y^\mu$, $Y^\mu \doteq \mathcal{O}_2(r^{-\alpha})$, $\overset{0}{X}{}^\mu{}_{,\nu} = 0$, in the asymptotically flat 'end' under consideration, with X^μ being components of X in some asymptotically rectangular coordinates.

terms of Cauchy data. In this case to have a finite and unambiguously defined mass it is sufficient to require

$$\lim_{r\to\infty} r^{1/2}(g_{ij}-\delta_{ij}) = \lim_{r\to\infty} r^{3/2} g_{ij,k} = 0 \tag{5}$$

$$g_{ij,k} \in L^2(\Sigma) \qquad {}^3R(g_{ij}) \in L^1(\Sigma). \tag{6}$$

As has been shown by Denissov and Solovyev [13], condition (5) cannot be weakened (more precisely, any weakening of (5) must exclude metrics which differ from the flat one by terms of the form $k_{ij}(\theta, \varphi)/r^{1/2}$ unless some supplementary conditions forbidding coordinate transformations of the form $x^i \to x^i + f^i(\theta, \varphi) r^{1/2}$ are imposed on the metric and the coordinate system). Neither can the boundary conditions ($1a$) be weakened (in the sense of the above statement) while retaining a meaningful notion of energy-momentum, which we shall show by means of counterexamples. Let us first note the following†.

Proposition 1. If $T^\mu{}_\nu$ satisfies the dominant energy condition ($T_{\mu\nu} n^\mu X^\nu \geq 0$ for all timelike future directed n^μ and X^ν) and conditions (1) are satisfied, we have

(a) $\quad m^2 \geq (\Sigma(p^i)^2)^{1/2}$

(b) $\quad m = 0 \quad$ iff $T_{\mu\nu} = 0 \quad g_{\mu\nu} = \eta_{\mu\nu}$.

Outline of proof. Square-integrability of P_{ij} and of the first derivatives of g_{ij} together with the integrability of T^0_μ ensure existence of solutions of Witten's equation [4] (cf also [5]); the equality of the spinorial expression with its leading term (in the limit r going to infinity) holds if, moreover, $r\partial_k g_{ij} \in L^\infty(\Sigma)$, $rP^{ij} \in L^\infty(\Sigma)$, $rT^0{}_\mu \in L^2(\Sigma)$ and $T^0{}_\mu = \mathrm{O}(r^{-2})$ (this can be established by following the proof of lemma 1, § IV in [3], where the inequality $\|f/r\|_{L^2(\Sigma)} \leq C\|\nabla f\|_{L^2(\Sigma)}$ [14] (cf also [5]) should be made use of). The conditions ($1a$) finally ensure the equality of the leading term of the spinorial boundary integral and of the von Freud superpotential.

The boundary conditions ($1a$) require the existence of a four-dimensional coordinate system and it seems natural to ask whether one can reasonably speak of asymptotic flatness requiring only the existence of a foliation of M by hypersurfaces satisfying (5) and (6). The Kasner metrics [15], solutions of vacuum Einstein equations, show that such a requirement is too weak for the introduction of a meaningful notion of energy:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^{2p_1}\,\mathrm{d}x^2 + t^{2p_2}\,\mathrm{d}y^2 + t^{2p_3}\,\mathrm{d}z^2 \qquad p_1+p_2+p_3 = p_1^2+p_2^2+p_3^2 = 1. \tag{7}$$

It follows from (7) that each hypersurface $t =$ constant carries a flat metric for which an 'instantaneous coordinate system' satisfying (5) and (6) exists, but they cannot be patched together to a coordinate system in which ($1a$) is satisfied. The Cauchy data (g_{ij}, P^{ij}) for the Kasner metrics, on $t =$ constant hypersurfaces, consist of a flat metric δ_{ij} and a covariantly constant tensor P^{ij}—the ADM energy is zero although one would expect such Cauchy data to have infinite energy. Clearly point (b) of proposition 1 is not fulfilled by these metrics. A much finer example exhibiting similar features is given by the Schwarzschild metric in Lemaître coordinates [15]:

$$\begin{aligned} &\mathrm{d}s^2 = -\mathrm{d}t^2 + (\partial Y/\partial\rho\ \mathrm{d}\rho)^2 + Y^2\,\mathrm{d}\Omega^2 \qquad \mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\varphi^2 \\ &Y = (3(m/2)^{1/2}t + \rho^{3/2})^{2/3} \approx \rho + (2m/\rho)^{1/2}t + \mathrm{O}(\rho^{-2}). \end{aligned} \tag{8}$$

† This result has been independently established by P Bizoń and E Malec (private communication).

Again the hypersurfaces $t = \text{constant}$ are flat, so the ADM mass is zero. Boundary conditions (1) just fail to be satisfied—they would be if α were allowed to be $\frac{1}{2}$. It must be emphasised that (5) and (6) are satisfied simultaneously for all t, and not only in 'instantaneous spatial coordinates' as for the Kasner metrics.

An example of a metric (satisfying Einstein vacuum equations) with a negative ADM mass is given by the flat metric in which an unusual time coordinate $\tau = t - ar^{1/2}$ has been introduced†:

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 - a\,\mathrm{d}\tau\,\mathrm{d}r/r^{1/2} + (1 - a^2/4r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2. \tag{9}$$

The ADM mass of the metric (9) is clearly equal to $-a^2/8$. A common feature of the above three examples is that P_{ij} is not square integrable, and that is why Witten's argument fails (no solutions of Witten's equation with the required asymptotic behaviour exist). (It may be of some interest to note that, in terms of powers of fall-off in r, P_{ij} for the metrics (8), (9) and (19) is just at the limit of square integrability.) This shows that the condition of square integrability of P_{ij} in the positive-energy theorem cannot be weakened if the ADM mass is finite. We have the following proposition, which can be thought of as an extension of the positive-energy theorem.

Proposition 2. Suppose that the Cauchy data (g_{ij}, P^{ij}), satisfying the constraint equations with sources, satisfy also

$$g_{ij} - \delta_{ij} = \mathrm{O}_1(r^{-\alpha}) \qquad \alpha > \tfrac{1}{2} \tag{10a}$$

$$g_{ij}P^{ij} \in L^2(\Sigma) \tag{10b}$$

$$\int_\Sigma (16\pi\mu + P_{ij}P^{ij}) = \infty. \tag{10c}$$

The ADM mass, evaluated in any coordinate system in which (10) holds, is infinite.

Remark. To weaken the hypothesis on α one has to assume some supplementary conditions which ensure the infiniteness of m in any admissible coordinate system—a possible requirement is, for example,

$$\int_{B(R)} (16\pi\mu + P_{ij}P^{ij}) \geq C(R)R^{-2\alpha+1} \qquad \text{with } \lim_{R\to\infty} C(R) = \infty \tag{11}$$

$B(R)$ being a coordinate ball of radius R and R^0 is understood to be $\ln R$ ((11) will hold if, e.g., $P^{ij} \sim r^{-1-\alpha+\varepsilon}$). Along the same lines, the square integrability of $g_{ij}P^{ij}$ can be replaced by some conditions which imply a slower fall-off of $g_{ij}P^{ij}$, as compared to $(P_{ij}P^{ij})^{1/2}$. It must be noted that for $\alpha < \frac{1}{2}$ the Hamiltonian (cf [6]) does not coincide with the standard ADM expression, because it is not only the linearised terms in it that contribute. Condition (11) ensures also that both the ADM or the Dirac form of m give the same infinite answer; supplementary conditions on N and N^i can guarantee that the same is true for the von Freud superpotential. Only trivial modifications of the proof of proposition 2 are required if $\alpha < \frac{1}{2}$ and (11) are assumed. The hypotheses of proposition 3 below can be weakened in a similar way.

† The singularity of τ at $r = 0$ is irrelevant and can be removed by smoothing τ for small r—we are primarily interested in the large-r behaviour of the metric.

Proof.

$$m = -(1/16\pi) \lim_{r\to\infty} \int (\det g g^{ij}),_j (\det g)^{-1/2}\, \mathrm{d}S_i$$

$$= (1/16\pi)\left(\int_{\Sigma\setminus B(R)} (\det g)^{1/2} R + \text{`}\Gamma\cdot\Gamma \text{ terms'}\right.$$

$$\left. - \int_{\partial B(R)} (\det g g^{ij}),_j (\det g)^{-1/2}\, \mathrm{d}S_i\right)$$

$$= \int_{\Sigma\setminus B(R)} (\mu + P_{ij}P^{ij}/16\pi)(\det g)^{1/2}\, \mathrm{d}^3x + \text{`something finite'}. \tag{12}$$

The infinite value of the integral ($10c$) may result from an infinite matter energy or from an 'infinite gravitational kinetic energy'. As is shown by the metrics (8) and (9), the loss of square integrability of P_{ij} can be an artefact due to a poor choice of the t foliation. One can exhibit some supplementary conditions which will ensure that $m=\infty$ is an invariant property of the spacetime, at least under coordinate transformations of the form (2)†. It is simple to show, using $T^{\mu\nu}{}_{;\nu}=0$, that if ($1a$) holds (with any $\alpha>0$) and if

$$T^{\mu}{}_{\nu} = \mathrm{O}(r^{-\beta}) \qquad \beta > 3-\alpha \tag{13}$$

we have the implication

$$\int_{x^0=\text{constant}} T^{\mu}{}_{\nu} X^{\nu}\, \mathrm{d}V_\mu = \infty \Rightarrow \int_{y^0=\text{constant}} T^{\mu}{}_{\nu} X^{\nu}\, \mathrm{d}V_\mu = \infty \tag{14}$$

for all X^μ and y^μ satisfying $X^\mu - \delta^\mu_0 = \mathcal{O}_1(r^{-\alpha})$, $y^\mu = x^\mu + \zeta^\mu(x)$ and $\zeta^\mu = \mathcal{O}_2(r^{1-\alpha})$. This shows in particular that $m=\infty$ is indeed an invariant property if (13), ($10c$) and ($1a$) with $\alpha>\frac{1}{2}$ hold. An interesting situation arises when the gravitational field has by itself an infinite amount of energy.

Proposition 3. Suppose that the metric, solution of four-dimensional Einstein equations (possibly with sources) in Ω, $\Omega \supset \{(t,x)\colon t\leqslant a+br\}$, $a>0$, $b>0$, satisfies

$$g_{\mu\nu} - \eta_{\mu\nu} = \mathcal{O}_2(r^{-\beta}) \qquad \beta>\tfrac{1}{4} \tag{15a}$$

$$g_{ij} - \delta_{ij} = \mathrm{O}_1(r^{-\alpha}) \qquad \alpha>\tfrac{1}{2} \tag{15b}$$

$$g_{ij}P^{ij} = \mathrm{O}_1(r^{-\alpha}) \tag{15c}$$

$$T^{\mu}{}_{\nu} = \mathrm{O}(r^{-2-\chi}) \qquad \chi>1-\alpha \tag{15d}$$

(cf, e.g., [16] for the compatibility of ($15a$) with the Einstein equations). The property $m=\infty$ is invariant under coordinate transformations of the form

$$x^\mu \to y^\mu = x^\mu + \zeta^\mu(x) \qquad \zeta^\mu = \mathcal{O}_3(r^{1-\gamma}) \qquad \gamma>0 \tag{16}$$

if they preserve (15).

† Even under the usual boundary conditions ($1a$) with $\alpha=1$ there still does not exist a proof that p_μ is indeed well defined—the missing element being that all coordinate transformations preserving ($1a$) are of the form (2). It may be of some interest to note that such a result can be shown to be true if all the derivatives $\partial x^\alpha/\partial y^\beta$ (or $\partial y^\beta/\partial x^\alpha$) are assumed to be bounded.

Outline of proof. The proof consists of showing that if the transformation (16) preserves (15), ζ^μ must be $O_2(r^{1-\psi})$, $\psi=\min(\alpha, 2\beta)>\frac{1}{2}$. The square integrability of P_{ij} is preserved by such transformations—m is either finite or infinite for all foliations $x^0=$ constant related by these coordinate transformations. To establish this last estimate one proceeds as follows: first, it is elementary to show that preservation of (15a) requires $\gamma\geqslant\beta$. Equations (15b) and (15c) then lead to equations of the type:

$$\zeta^i{}_{,jk}=O_1(r^{-1-\psi}) \tag{17a}$$

$$\Delta\zeta^0=O_1(r^{-1-\psi}). \tag{17b}$$

From (17), the maximum principle for (17b) and the existence theorems for the Laplacian in weighted Hoelder spaces (cf [17]) one obtains the required result.

Remarks. (1) The C^2_β class of the coefficients of the metric and the C^1_α class of $g_{ij}P^{ij}$ can be weakened to Hoelder classes $C^{1,\varphi}_\beta$ and $C^{0,\varphi}_\alpha$ respectively, for some $\varphi>0$ (cf [17]).

(2) The author believes that the condition $\beta>\frac{1}{4}$ is not essential and can be relaxed. The method outlined above does not seem to work for $\beta<\frac{1}{4}$ due to the presence of terms $h_{ij,j}\,\partial\zeta^i/\partial t$ in the right-hand side of (17b)—a bootstrap of equations obtained from (17) by time differentiating seems to require some further restrictions on $h_{ij,j}$.

What has been said allows a reasonable discussion of boundary conditions which lead to a (finite) ADM mass varying in time. For the purpose of such a discussion we shall assume that, at least in the 'end' of M under consideration, there exists a coordinate system (t, x^i) such that (5) and (6) are satisfied for $t\in(-\varepsilon, \varepsilon)$, with some $\varepsilon>0$. Assuming that differentiation under the integral is allowed and that $\lim r^{1/2}\,\partial g_{ij}/\partial t=0=\lim r^{3/2}\,\partial^2 g_{ij}/\partial t\,\partial x^k$, one has

$$\begin{aligned}\partial m/\partial t&=-\left(\int_{S_\infty}\partial((\det g g^{ij})_{,j}(\det g)^{-1/2})/\partial t)\,\mathrm{d}S_j\right)(16\pi)^{-1}\\&=\left(\int_{S_\infty}({}^3R^i{}_jN^j+(NP^{il})_{,l})(\det g)^{1/2}\,\mathrm{d}S_i\right)(8\pi)^{-1}\\&=\left(\int_{S_\infty}({}^3R^i_jN^j+N_{,j}P^{ij}-8\pi NJ^i)(\det g)^{1/2}\,\mathrm{d}S_i\right)(8\pi)^{-1}.\end{aligned} \tag{18}$$

If μ is integrable and satisfies $\mu\geqslant|J|$, it is reasonable to assume $J^i=O(r^{-3-\varepsilon})$. Let us also suppose ${}^3R^i{}_j=O(r^{-2-\alpha})$, with $\alpha>\frac{1}{2}$, and $P^{ij}=O(r^{-1-\beta})$. To have $\partial m/\partial t\neq 0$ either $N^j\sim r^\alpha$ or $N_{,j}\sim r^{\beta-1}(\Rightarrow N\sim r^\beta)$. Such weak boundary conditions on N and N^i seem to be compatible with four-dimensional asymptotic flatness only if a very unnatural $x^0=$ constant foliation of the spacetime is given. An example is provided by the flat metric on $\mathbb{R}^4$, where a strange time $\tau=t/(1+ar^{1/2})$ has been introduced:

$$\mathrm{d}s^2=-(1+ar^{1/2})^2\,\mathrm{d}\tau^2-a\tau(a+r^{-1/2})\,\mathrm{d}\tau\,\mathrm{d}r+(1-\tau^2a^2/4r)\,\mathrm{d}r^2+r^2\,\mathrm{d}\Omega^2. \tag{19}$$

The metric (19) has negative ADM mass $-\tau^2a^2/8$. It must be noted that $P=P_{ij}g^{ij}$ is again not square integrable. If the condition $P\in L^2(\Sigma)$ is added one has to limit oneself to $\beta>\frac{1}{2}$, and these two requirements exclude examples of the type (19). It may well be possible that such restrictions lead to a family of metrics with unambiguously defined mass changing in time.

In a paper of fundamental importance Christodoulou and O'Murchadha [16] have provided physicists with a large amount of asymptotically flat spacetimes satisfying the Einstein vacuum equations. One of the features of the Christodoulou-O'Murchadha metrics is that they satisfy boundary conditions of the form (1a), with any $\alpha > 0$. In this letter we have pointed out some conditions which allow us to speak meaningfully of energy-momentum, whether finite or infinite, for large subset of this set of metrics. We have also shown that certain hypotheses allow us to weaken the positive-energy theorem to non-square-integrable P_{ij} and that these conditions are the best possible. We have finally pointed out that care must be taken when claiming that the ADM mass does not change in time, if one specifies the boundary conditions satisfied by P^{ij} and g_{ij} only. We wish to emphasise that, although all our examples (except the Kasner metrics) are coordinate artefacts, they show also that the ADM mass cannot be generically given an intrinsic meaning for metrics for which the weaker asymptotics is a real feature.

The boundary conditions ensuring $m = \infty$ we have presented are disappointing in a sense—in classical field theories in a Minkowski background (for which a positive Hamiltonian density usually exists) the concept of energy can play the useful ontological role of Ockham's razor: in Maxwell's electrodynamics monochromatic plane waves are solutions which cannot 'really exist' because they have infinite energy. The boundary conditions we have presented are certainly incompatible with such solutions as the pp gravitational waves, to which one would like to assign an infinite energy. Can such an assertion be given rigorous mathematical sense?

Note added. Once this letter was written the author was informed that part of the results presented here have independently been noted or established by N O'Murchadha (1986 *J. Math. Phys.* at press).

References

[1] Witten E 1981 *Commun. Math. Phys.* **80** 381
[2] Schoen R and Yau S T 1981 *Commun. Math. Phys.* **79** 231
[3] Choquet-Bruhat Y 1984 *Relativity, Groups and Topology II* ed B deWitt and R Stora (Amsterdam: North-Holland)
[4] Reula O 1982 *J. Math. Phys.* **23** 810
[5] Reula O and Tod K P 1984 *J. Math. Phys.* **25** 1004
[6] Chruściel P T 1986 *Boundary Conditions at Spatial Infinity from a Hamiltonian Point of View. Proc. 1985 Ettore Majorana School of Cosmology and Gravitation* ed P Bergmann (in press)
[7] Solovyev V O 1985 *Theor. Math. Phys.* **65** 400
[8] von Freud Ph 1939 *Ann. Math.* **40** 417
[9] York J W Jr 1980 *Essays in General Relativity* ed F J Tipler (New York: Academic)
[10] Ashtekar A and Hansen R O 1978 *J. Math. Phys.* **19** 1542
[11] Ashtekar A and Magnon-Ashtekar A 1979 *J. Math. Phys.* **20** 793
[12] Beig R 1978 *Phys. Lett.* **69A** 153
[13] Denissov V I and Solovyev V O 1983 *Theor. Math. Phys.* **56** 301
[14] Parker T H 1985 *Commun. Math. Phys.* **100** 471
[15] Kramer D, Stephani H, Herlt E, MacCallum M A H and Schmutzer E (ed) 1980 *Exact Solutions of Einstein Equations* (Cambridge: Cambridge University Press)
[16] Christodoulou D and O'Murchadha N 1981 *Commun. Math. Phys.* **80** 271
[17] Chaljub-Simon A and Choquet-Bruhat Y 1978 *Ann. Fac. Sci. Toulouse* **1** 9