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# Strong cosmic censorship in polarised Gowdy spacetimes*

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**Abstract.** The strong cosmic censorship conjecture says that 'most' spacetimes developed as solutions of Einstein's equations from prescribed initial data cannot be extended outside of their domains of dependence. Here, we discuss results which show that if we restrict attention to the polarised Gowdy spacetimes, strong cosmic censorship holds. More specifically, these results show that in the space of Cauchy data for polarised Gowdy spacetimes, there is an open dense subset for which the maximal globally hyperbolic development is inextendible. Among the Gowdy spacetimes which can be extended, we find a set of them which admit a countable infinity of inequivalent classes of extensions. We also find that a polarised Gowdy spacetime ($T^3$ or $S^2 \times S^1$) may be extended as a solution of Einstein's equations (with the Gowdy isometries) across a compact Cauchy horizon only if it is analytic.

## 1. Introduction

While the well known Hawking-Penrose theorems [1] tell us that cosmological solutions of Einstein's equations generally contain singularities, they tell us very little about the generic nature of these singularities. The work of Clarke [2] suggests that a singularity is accompanied either by unbounded curvature and tidal forces, as in the Friedman-Robertson-Walker ('big bang') models, or by a breakdown in causality, as in the Taub-NUT ('Cauchy horizon') spacetimes; but it tells us little about when to expect either possibility. Setting this issue in the context of the initial value problem of general relativity, one may ask: which spacetimes developed as solutions of Einstein's equations from prescribed initial data can be extended outside of their maximal domains of dependence (across a Cauchy horizon) to include acausal regions? The conjecture of strong cosmic censorship (SCC) [3] asserts that such extendible spacetimes constitute only a negligible set in the space of allowed initial data. Determining the validity of the SCC conjecture is one of the more interesting current issues in classical general relativity.

In this paper, we discuss results which show rigorously that, at least for a certain small but non-trivial class of solutions of Einstein's equations—the polarised Gowdy

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spacetimes [4]—SCC holds. That is, we find that in the space $\mathscr{P}(\Sigma^3)$ of initial data for polarised Gowdy spacetimes (which we describe below), there is an open and dense subset $\hat{\mathscr{P}}(\Sigma^3)$ such that a spacetime developed from data in this subset cannot be extended outside its domain of dependence. Curvature singularities for all timelike observers occur for every spacetime with data in $\hat{\mathscr{P}}(\Sigma^3)$.

In addition to this verification of strong cosmic censorship in the class of polarised Gowdy spacetimes, we have gained some understanding of those Gowdy spacetimes which do admit smooth extensions. Our main result is that extensions which satisfy the vacuum Einstein equations, maintain the Gowdy form, and contain a compact Cauchy horizon, occur only in spacetimes which develop from analytic initial data (lying in $\mathscr{P}(\Sigma^3)-\hat{\mathscr{P}}(\Sigma^3)$). We also find examples of spacetimes which admit an infinite number of inequivalent classes of extensions.

We shall discuss these results below, after first describing the set of polarised Gowdy spacetimes and then sketching the main steps of the proof of SCC for this class of spacetimes.

## 2. The polarised Gowdy spacetimes

'Polarised' metrics with two commuting spacelike Killing vector fields have been exhaustively studied because they provide a useful arena in which to study gravitational radiation [5] and because with suitable topological restrictions they are among the simplest examples of inhomogeneous cosmological spacetimes [4, 6]. By definition, a coordinate system $(\theta, x, y, t)$ exists such that these metrics take the form [7]

$$g=L^2\{e^{-2U}[e^{2A}(-dt^2+d\theta^2)+R^2\,dy^2]+e^{2U}\,dx^2\} \tag{1}$$

where $L$ is a positive constant while $A$, $R$ and $U$ are functions of $\theta$ and $t$. The vacuum Einstein equations for $g$ then read

$$\partial_t^2R-\partial_\theta^2R=0 \tag{2a}$$

$$\partial_t(R\partial_tU)-\partial_\theta(R\partial_\theta U)=0 \tag{2b}$$

$$\partial_t^2A-\partial_\theta^2A=(\partial_\theta U)^2-(\partial_tU)^2 \tag{2c}$$

and

$$\partial_\theta\partial_+R=(\partial_+A)(\partial_+R)-R(\partial_+U)^2 \tag{3a}$$

$$\partial_\theta\partial_-R=(\partial_-A)(\partial_-R)-R(\partial_-U)^2 \tag{3b}$$

where we have for convenience used the null derivatives $\partial_+:=\partial_\theta+\partial_t$ and $\partial_-:=\partial_\theta-\partial_t$ in equation (3). Equation (2) is a system of dynamical (wavelike) equations for the functions $A$, $R$ and $U$; equation (3) is a system of constraints on the choice of these functions and their time derivatives at an initial time $t_0$.

To define the Gowdy polarised spacetimes, one adds the assumption that the $t=$ constant hypersurfaces $\Sigma^3$ are compact without boundary and orientable. It follows [8] that $\Sigma^3$ must be either $T^3$, $S^2\times S^1$, $S^3$, or a Lens space $L(p,q)$. Since the Lens spaces are covered by $S^3$, we need only work with the three topologies $T^3$, $S^2\times S^1$ and $S^3$. As we shall see, SCC holds for all three topologies.

The analysis of the system of field equations (2)-(3) for the Gowdy spacetimes may be greatly simplified. As observed by Gowdy, for almost all Gowdy metrics (solving the system (2)-(3)), one may use a diffeomorphism to cast the function $R(\theta, t)$

into a standard form. For $\Sigma^3 = T^3$, as long as the spacetime is *not* the $T^3$-identified Minkowski spacetime (that is, $M^4 = T^3 \times \mathbb{R}$ and $g = -dt^2 + \gamma_{ab}\, dx^a\, dx^b$, where $\gamma_{ab}$ is a positive definite matrix, constant in space and time), we get $R(\theta, t) = t$. For an open dense subset [9] of the $\Sigma^3 = S^3$ or $\Sigma^3 = S^2 \times S^1$ solutions, we have $R(\theta, t) = \sin\theta \sin t$ (with $\theta \in [0, \pi]$). Henceforth in referring to the polarised Gowdy spacetimes, we shall assume that the appropriate diffeomorphisms can, and have, been made so that $R(\theta, t)$ takes these forms.

With this treatment of the function $R(\theta, t)$, the Cauchy formulation for the polarised Gowdy spacetimes may be described as follows. Let $\mathcal{P}(\Sigma^3)$ denote the collection of smooth functions $\{(U, \partial_t U, A, \partial_t A)\}$ which satisfy the constraints (3) at some fixed time. Standard theory [10] shows that each set of data $(U, \partial_t U, A, \partial_t A) \in \mathcal{P}(\Sigma^3)$ may be evolved via equation (2) into a unique, maximal, globally hyperbolic spacetime (called the 'maximal development' of $(U, \partial_t U, A, \partial_t A)$) which solves the vacuum Einstein equations. Now a globally hyperbolic spacetime is necessarily causal (no closed, or almost closed, timelike or null paths) and physics proceeds in it deterministically. Hence, violations of causality and determinism may only occur if one extends a spacetime outside its globally hyperbolic region (the boundary of the globally hyperbolic region is called the Cauchy horizon). The claim of SCC is that, for most spacetimes, such extensions cannot be made.

Note that in most cases one may use the constraints (3) to solve for $A$ and $\partial_t A$, given $U$ and $\partial_t U$ (and $R$) and a single constant $\alpha$ ($\alpha$ is an integration constant for $A$). Hence, if we let $K := \partial_t U$, then $(U, K; \alpha)$ serve as essentially [11] free coordinates for $\mathcal{P}(\Sigma^3)$.

## 3. Main result

Before stating our main (SCC-type) theorem, which concerns the *nature* of singularities in polarised Gowdy spacetimes, we wish to note a result concerning the *existence* of singularities in these spacetimes. We can show that all polarised globally hyperbolic Gowdy spacetimes except for the $T^3$-identified Minkowski spacetime (described above), are singular in the sense of being timelike and null geodesically incomplete [12]. Moreover, in the $S^3$ and $S^2 \times S^1$ cases, the spacetimes are necessarily geodesically incomplete both to the future and to the past [13].

Do the polarised Gowdy spacetimes usually allow extensions past their singularities? Our main result says no.

*Theorem (Strong cosmic censorship for polarised Gowdy spacetimes.)* Let $\Sigma^3 = T^3$, $S^3$, or $S^2 \times S^1$, and let $\mathcal{P}(\Sigma^3)$ be the space of initial data for the polarised Gowdy spacetimes (with $C^\infty$ topology). There exists an open dense subset $\hat{\mathcal{P}}(\Sigma) \subset \mathcal{P}(\Sigma)$ such that the maximal development of any set of data in $\hat{\mathcal{P}}(\Sigma)$ is inextendible.

The proof of this theorem is rather long and somewhat technical. Here, we shall just sketch the main steps.

*Step 1 (Asymptotic expansion)*

Using a collection of energy estimates [13], we can control the behaviour of $U$ and its derivatives in the neighbourhood of the $t = 0$ and $t = \pi$ singularities of any polarised Gowdy spacetime. This control results in a power law type expansion for $U$ (and consequently for $A$). For the singularity set to occur at $t = 0$ (consistent with the choice of $R=t$ for $\Sigma^3=T^3$ and $R=\sin\theta\sin t$ for $\Sigma^3=S^3$ or $S^2\times S^1$), these expansions take the following form [13], [14].

$T^3$:

$$U(\theta,t)=\sum_{l=1}^{N}\left\{\frac{\partial_\theta^{2l}\pi}{(l!)^2}(\tfrac{1}{2}t)^{2l}\left(\ln t-\sum_{j=1}^{l}\frac{1}{j}\right)+\frac{\partial_\theta^{2l}\omega}{(l!)^2}(\tfrac{1}{2}t)^{2l}\right\}+R_N(\theta,t)$$
$$=\pi\ln t+\omega+\tfrac{1}{4}\partial_\theta^2\pi\, t^2\ln t+\tfrac{1}{4}(\partial_\theta^2\omega-\partial_\theta^2\pi)t^2+\ldots \tag{4a}$$

with the remainder term $R_N$ satisfying

$$|\partial_t^m\partial_\theta^n R_N|\leqslant C_1t^{2N+2-m}(1+|\ln t|)$$

for all $m\leqslant 2N+1$ and all $n$;

$$A(\theta,t)=\pi^2\ln t+\alpha+a_0+\sum_{l=1}^{N}\{a_lt^{2l}+b_lt^{2l}\ln t+d_lt^{2l}\ln^2 t\}+P_N(\theta,t) \tag{4b}$$

where $a_0$, $a_l$, $b_l$, and $d_l$ are functions of $\theta$ which are uniquely determined by $\omega$ and $\pi$, and the remainder term $P_N(\theta,t)$ satisfies

$$|\partial_t^m\partial_\theta^n P_N|\leqslant C_2t^{2N+2-m}(1+\ln^2 t)$$

for all $m\leqslant 2N+1$ and all $n$.

$S^2\times S^1$:

$$U(\theta,t)=\rho\ln(\sin t)+\mu+\sum_{l=1}^{N}\{\rho_l(\sin t)^{2l}\ln(\sin t)+\mu_l(\sin t)^{2l}\}+R_N(\theta,t) \tag{5a}$$

where $\rho_l$ and $\mu_l$ are functions of $\theta$ determined uniquely by the first $l$ derivatives of $\rho$ and $\mu$, and the remainder term satisfies

$$|\partial_t^m\partial_\theta^n R_N|\leqslant C_3(\sin t)^{2N+2-m}|\ln\sin t|$$

for all $m\leqslant 2N+1$ and all $n$;

$$A(\theta,t)=\rho^2\ln(\sin t)+\hat{a}_0$$
$$+\sum_{l=1}^{N}\{\hat{a}_l\sin^{2l}t+\hat{b}_l\sin^{2l}t\ln\sin t+\hat{d}_l\sin^{2l}t\ln^2\sin t\}+P_N(\theta,t) \tag{5b}$$

where $\hat{a}_0$, $\hat{a}_l$, $\hat{b}_l$, and $\hat{d}_l$ are functions of $\theta$ uniquely determined by $\mu$ and $\rho$, and the remainder term $P_N$ satisfies

$$|\partial_t^m\partial_\theta^n P_N|\leqslant C_4(\sin t)^{2N+2-m}|\ln^2(\sin t)|$$

$S^3$:

$$X(\theta,t)=\lambda\ln(\sin t)+\psi+\sum_{l=1}^{N}\{\lambda_l(\sin t)^{2l}\ln(\sin t)+\psi_l(\sin t)^{2l}\}+R_N(\theta,t) \tag{6a}$$

for $X:=2U-\ln(\sin\theta\sin t)+\ln(\tan\theta/2)$, where $\lambda_l$ and $\psi_l$ are functions of $\theta$ determined uniquely by the first $l$ derivatives of $\lambda$ and $\psi$, and the remainder term $R_N$ satisfies

$$|\partial_t^m\partial_\theta^n R_N|\leqslant C_5(\sin t)^{2N+2-m}|\ln\sin t|$$

for all $m\leqslant 2N+1$ and all $n$;

$$B(\theta,t)=\tfrac{1}{4}(\lambda^2-1)\ln(\sin t)+\bar{a}_0$$
$$+\sum_{l=1}^{N}\{\bar{a}_l\sin^{2l}(t)+\bar{b}_l\sin^{2l}t\ln(\sin t)+\bar{d}_l\sin^{2l}t\ln^2(\sin t)\}+P_N(\theta,t) \tag{6b}$$

for $B := A - U$, where $\bar{a}_0$, $\bar{a}_l$, $\bar{b}_l$, and $\bar{d}_l$ are functions uniquely determined by $\lambda$ and $\psi$, and the remainder term $P_N$ satisfies

$$|\partial_t^m \partial_\theta^n P_N| \leqslant C_6(\sin t)^{2N+2-m}|\ln^2 \sin t|$$

for all $m \leqslant 2N+1$ and all $n$.

Note that the constants $C_1, C_2, \ldots, C_6$ each depend upon $n$, $m$, and $N$ as well as on the particular solution under consideration.

In each case, the expansions are determined by a pair of functions of $\theta$: $\omega$ and $\pi$ for the $T^3$ models, $\rho$ and $\mu$ for the $S^2 \times S^1$ models, and $\lambda$ and $\psi$ for $S^3$ spacetimes. These functions act as coordinates for an effective 'asymptotic data space' $\mathcal{D}(\Sigma^3)$ for the polarised Gowdy spacetimes of each topology. This claim is based on a crucial fact [13, 14] (which we state here just for the $T^3$ case).

For every choice of the pair of functions $(\omega, \pi)$ (satisfying $\int_{S^1} \pi\partial_\theta\omega = 0$) along with the choice of a positive constant $\alpha$, there is a unique (up to diffeomorphism) polarised Gowdy spacetime which has the expansion form (5) in the neighbourhood of the singularity at $t = 0$.

Hence $\{(\omega, \pi; \alpha)\} = \mathcal{D}(T^3)$ labels the $T^3$ polarised Gowdy spacetimes just as well as does $\mathcal{P}(T^3) = \{(U, K; \alpha)\}$. Indeed, we find that the dynamics induces a smooth diffeomorphism from $\mathcal{P}(\Sigma^3)$ to $\mathcal{D}(\Sigma^3)$, for all three topologies $\Sigma^3 = T^3$, $S^2 \times S^1$, or $S^3$.

These expansion formulae are very useful for proving our SCC theorem, as we discuss below. They lead to other important results, as well. First, they allow us to rigorously prove [13] that these spacetimes are asymptotically velocity term dominated. This means that, as one approaches the $t = 0$ singularity, the metric field of any one of these solutions of Einstein's equations approaches a solution of a truncated system of equations in which, roughly speaking, spatial curvature terms have been dropped. Hence the gravitational fields seen by neighbouring observers essentially decouple near the singularity. Properties of this sort have been discussed by Belinsky, Khalatnikov and Lifschitz [15].

Second, we may use these expansion formulae to show that the polarised Gowdy spacetimes all admit foliations by spacelike constant mean curvature hypersurfaces. We do this by verifying (from calculations based on the expansions) that the Gerhardt 'barriers' [16]—appropriately bounded values for the mean curvature on each $t =$ constant slice—exist for each spacetime. A constant mean curvature foliation provides a useful standard 'choice of time' which one may use to compare the physics and geometry of different solutions of Einstein's equations.

Note that the expansion formulae (4)-(6) all hold in the neighbourhood of the spacetime singularities at $t = 0$—the 'big bang'. Each of the $S^2 \times S^1$ and $S^3$ polarised Gowdy spacetimes also has a 'big crunch' at $t = \pi$. Not surprisingly, there is a very similar expansion which holds in each case in the neighbourhood of $t = \pi$; just replace $t$ by $\pi - t$, and replace the pair of functions $(\rho, \mu)$ or $(\lambda, \psi)$ by a different pair $(\tilde{\rho}, \tilde{\mu})$ or $(\tilde{\lambda}, \tilde{\psi})$. Note that the data $(\tilde{\rho}, \tilde{\mu})$ may be explicitly related to $(\rho, \mu)$ (or $(\tilde{\lambda}, \tilde{\psi})$ explicitly related to $(\lambda, \psi)$) in a given spacetime by studying the evolution of spherical harmonic components of these functions.

For the $T^3$ spacetime, there is no big crunch. Rather, these spacetimes evolve toward $t = \infty$, with the volume expanding in time and the curvature diminishing. A different asymptotic expansion formula holds for large $t$; one can write [14]

$$U(\theta, t) = \beta \ln t + \gamma + \nu(\theta, t)t^{-1/2} + \kappa(\theta, t) \qquad (7a)$$

where $\beta$ and $\gamma$ are constants, $\nu$ is a bounded solution of the standard wave equation, $\partial_t^2\nu = \partial_\theta^2\nu$, and the remainder function $\kappa$ satisfies the inequalities

$$|\kappa(\theta, t)| \leqslant C_7 t^{-3/2}$$

$$|\partial_t\kappa(\theta, t)| \leqslant C_8 t^{-3/2}$$

and

$$A(\theta, t) = \begin{cases} \alpha + \beta^2 \ln t & \text{if } \nu(\theta, t) = 0, \text{all } \theta \text{ and } t \\ \delta t + \varepsilon(\theta, t) & \text{if } \nu(\theta, t) \neq 0, \text{for some } \theta \text{ and } t \end{cases} \tag{7b}$$

with $\alpha$ and $\delta$ constants, and $\varepsilon(\theta, t)$ satisfying the inequalities

$$|\varepsilon(\theta, t)| \leqslant C_8 t^{1/2} \qquad \text{and} \qquad |\partial_t\varepsilon(\theta, t)| \leqslant C_9.$$

This expansion is useful for establishing future geodesic completeness in polarised Gowdy spacetimes on $T^3 \times (0, \infty)$.

*Step 2 (Curvature behaviour near the singularity)*

Using the expansion formulae (4)-(6), we can relate the behaviour of the spacetime curvature in the vicinity of the singularity in a given polarised Gowdy spacetime to the data $(\omega, \pi) \in \mathcal{Q}(T^3)$ (or $(\mu, \rho) \in \mathcal{Q}(S^2 \times S^1)$, or $(\psi, \lambda) \in \mathcal{Q}(S^3)$) which labels that spacetime in asymptotic data space. This can be done by explicitly calculating the curvature invariants for the Gowdy metric (1) in terms of the expansions formulae (4)-(6) and then organising the resulting formula into certain terms with upper and lower bounds for $t \to 0$. These terms are turned on and off by properties of $(\omega, \pi)$ (or $(\mu, \rho)$, or $(\psi, \lambda)$). It turns out [13] that one of the pair of functions in $\mathcal{Q}(\Sigma^3)$—$\pi$ for $T^3$, $\rho$ for $S^2 \times S^1$, $\lambda$ for $S^3$—plays the major role in determining the behaviour of the curvature. Our results are as follows.

$T^3$:

Let $(\omega, \pi; \alpha)$ be the asymptotic data for some polarised Gowdy spacetime on $T^3 \times \mathbb{R}^1$. Let $I$ consist of the points $\theta$ in the circle $S^1$ at which either

$$\pi(\theta) = 0 \qquad \partial_\theta\pi(\theta) = 0 \qquad \partial_\theta^2\pi(\theta) = 0 \tag{8a}$$

or

$$\pi(\theta) = 1 \qquad \partial_\theta\pi(\theta) = 0 \qquad \partial_\theta^2\pi(\theta) = 0. \tag{8b}$$

($a$) If $I$ is empty, then the curvature invariant $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ blows up uniformly along all timelike or null paths approaching the singularity at $t = 0$.
($b$) If $I$ contains an open set $\hat{I}$, then the curvature invariants are bounded along all timelike and null paths which approach $\theta \in \hat{I}$ as $t \to 0$.

$S^2 \times S^1$:

The same as for $T^3$ but with $I$ defined as the set of points $\theta \in (0, \pi)$ at which either

$$\rho(\theta) = 0 \qquad \partial_\theta\rho(\theta) = 0 \qquad \partial_\theta^2\rho(\theta) = 0$$

or

$$\rho(\theta) = 1 \qquad \partial_\theta\rho(\theta) = 0 \qquad \partial_\theta^2\rho(\theta) = 0.$$

$S^3$:
The same as for $T^3$ but with $I$ defined as the set of points $\theta \in (0, \pi)$ at which either

$$\lambda(\theta)=1 \qquad \partial_\theta\lambda(\theta)=0 \qquad \partial_\theta^2\lambda(\theta)=0$$

or

$$\lambda(\theta)=-1 \qquad \partial_\theta\lambda(\theta)=0 \qquad \partial_\theta^2\lambda(\theta)=0.$$

Note that in the $S^2\times S^1$ and $S^3$ cases here, we are not (for brevity's sake) discussing what happens at the rotation axes $\theta=0$ and $\theta=\pi$. See reference [14].

One can considerably refine this description of the behaviour of the curvature, as controlled by $\pi$, $\rho$ or $\lambda$. However, for our present purposes—the SCC result—the above statement is sufficient.

It should be evident that, roughly speaking, for 'most' asymptotic data, the set $I$ is empty. One may prove this using results from Morse theory [17]. For example, in the case $\Sigma^3=T^3$, we may essentially ignore the integral constraint $\int_{S^1}\pi\partial_\theta\omega=0$ and focus on analysing the set

$$\mathscr{F}(S^1):=\{\pi\in C^\infty(S^1)|I \text{ is empty}\}.$$

But this set contains, as a subset, the collection of smooth functions $\mathcal{N}(S^1)$ on the circle which have no degenerate critical points (in the Morse sense). Since Morse theory shows that any smooth function $f$ may be approximated arbitrarily closely (in $C^\infty(S^1)$) by a sequence of functions in $\mathcal{N}(S^1)$, it follows that $\mathcal{N}(S^1)$ and hence $\mathscr{F}(S^1)$ are dense in $C^\infty(S^1)$. Since any sufficiently small perturbation of a function in $\mathscr{F}(S^1)$ also lies in $\mathscr{F}(S^1)$, we find that $\mathscr{F}(S^1)$ is open in $C^\infty(S^{1|}$

It immediately follows from the above argument that if we define $\hat{\mathscr{Q}}(T^3)$ to be the set of asymptotic data with $I$ empty, then $\hat{\mathscr{Q}}(T^3)$ is open and dense in $\mathscr{Q}(T^3)$. Thus among all $T^3$ polarised Gowdy spacetimes, those which have curvature blowing up uniformly in the neighbourhood of the $t=0$ singularity comprise an open and dense subset. The same sort of result for the $S^3$ and $S^2\times S^1$ Gowdy spacetimes may be proven in a similar way.

*Step 3 (Geodesic completeness at $t\to\infty$)*
As noted above, all of the globally hyperbolic polarised Gowdy spacetimes except for the $T^3$-identified Minkowski spacetime are singular somewhere. We have found [13] that singularities may possibly occur only at $t=0$ and $t=\pi$ in the generic $S^3$ and $S^2\times S^1$ models, and only at $t=0$ and $t\to\infty$ in the $T^3$ models. Can they in fact occur at $t\to\infty$?

In general relativity, one sometimes identifies singularities with geodesic incompleteness [1]. So it is of interest to determine whether the polarised $T^3$ Gowdy spacetimes are timelike and null geodesically complete towards the future ($t\to\infty$). Using the asymptotic expansion form (7) to study the behaviour of geodesics for large $t$, we establish [14] this geodesic completeness for all cases.

*Step 4 (Generic inextendibility)*
A spacetime $(M^4, g)$ is smoothly extendible if it can be properly and smoothly embedded isometrically in a spacetime $(\tilde{M}^4, \tilde{g})$. If a spacetime is geodesically complete or if all its incomplete geodesics have unbounded curvature on their incomplete ends, then it will not admit smooth extensions. As we have seen, for each of the three

topologies, there is an open and dense subset of the set of all polarised Gowdy spacetimes which has unbounded curvature on all incomplete ends of geodesics. Hence, our main theorem follows as a consequence.

## 4. Extendible Gowdy spacetimes

While most of the polarised Gowdy spacetimes do not admit smooth extensions, some do and we wish to comment on them. It is important to distinguish between two different types of extensions: those which satisfy Einstein's equations everywhere in the extended spacetime $(\tilde{M}, \tilde{g})$, and those which involve no restrictions on the metric $\tilde{g}$ other than smoothness. We shall discuss results concerning both types of extensions.

Consider a fixed polarised Gowdy spacetime $(M^4, g)$ with asymptotic data $(\omega, \pi) \in \mathscr{Q}(T^3)$, or $(\mu, \rho) \in \mathscr{Q}(S^2 \times S^1)$ or $(\psi, \lambda) \in \mathscr{Q}(S^3)$. As discussed above, unless $I$ is non-empty, the spacetime will not admit smooth extensions. Is a non-empty $I$ sufficient as well as necessary for extensions? This turns out to be the case, so long as $I$ contains an open interval. Indeed, if $\hat{I}$ is an open interval in $I$, then there exist at least two inequivalent smooth classes of extensions $\{(\tilde{M}_1, \tilde{g}_1)\}$ and $\{(\tilde{M}_2, \tilde{g}_2)\}$ of $(M, g)$ such that the region $\{t=0\} \times \hat{I}$ is a Cauchy horizon in the extended spacetimes. Here, we say that $\{(\tilde{M}_1, \tilde{g}_1)\}$ and $\{(\tilde{M}_2, \tilde{g}_2)\}$ are 'inequivalent classes of spacetime extensions', if each class includes all further extensions of all its members, and if no member of the first class is isometric to a member of the second class.

This result holds for all three topologies. The proof in each case is constructive. That is, we can introduce two sets of coordinates in the neighbourhood of each connected interval of $\{t=0\} \times \hat{I}$ (quite reminiscent of the coordinates used for Taub-NUT [1]) and then use each set of coordinates to smoothly extend across $\{t=0\} \times \hat{I}$, which then becomes (part of) the Cauchy horizon. The extensions based on the alternate choices of coordinates can be shown to lie in inequivalent classes.

An interesting consequence which can be deduced from this construction is the existence of spacetimes which admit an infinite collection of inequivalent classes of extensions. We argue as follows. Choose a function $f: S^1 \to \mathbb{R}^1$ which has an infinite (countable) number of disjoint plateaus on which $f$, $\partial_\theta f$, and $\partial_\theta^2 f$ vanish. As noted in step 1, we can always find a polarised Gowdy spacetime $(M, g)$ which has $\pi$ equal to this function $f$. Hence $I$ will consist of an infinite number of disjoint plateaus: $\{I_k\}$. On each $I_k$, we can independently construct a pair of inequivalent extensions. It follows that the entire spacetime has an infinite number of inequivalent classes. Note that if we choose $f$ to have $N$ plateaus, then we get at least $2^N$ inequivalent extension classes.

The extensions we have been discussing generally do not satisfy Einstein's equations. While it is not yet clear, in general, what conditions (on, say, the asymptotic data) guarantee the existence of Ricci flat extensions, we have interesting (rather surprising) results in certain special cases.

The key condition seems to be analyticity. For example, if $\pi$ has some collection of plateau regions (as above) and if $\omega$ is analytic on one or any number of them, then extensions across the analytic regions of the plateaus may be chosen to solve Einstein's equations (and there exist two inequivalent extensions across each analytic plateau). Similarly, for the other topologies, one has extensions which solve Einstein's equations in Gowdy form if $\rho$ has plateau regions (value 0 or 1) on which $\mu$ is analytic (for $S^2 \times S^1$), or if $\lambda$ has plateau regions (value $-1$ or $+1$) on which $\psi$ is analytic (for $S^3$).

If we wish to consider extensions on which the Cauchy horizon is compact and equal to $\{t=0\}\times\Sigma^3$, then analyticity becomes a necessary as well as sufficient condition for the existence of extensions which satisfy Einstein's equations and retain the polarised Gowdy form. More specifically, the conditions on the asymptotic data for such an extension are as follows.

$T^3$: $\pi = 0$ or 1 everywhere on $S^1$

$\omega$ is analytic

$S^2\times S^1$: $\rho = 1$ everywhere on $[0, \pi]$

$\mu$ is analytic

$S^3$: No such extensions exist.

Note that for the $T^3$ and $S^2\times S^1$ cases, the stated conditions guarantee that the entire spacetime is analytic. So if a polarised Gowdy spacetime is not analytic, it cannot extend across a compact Cauchy horizon as a solution of the Einstein equations, retaining the properties of a Gowdy spacetime.

For an $S^3$ polarised Gowdy spacetime, we note that there are no extensions, Ricci flat or otherwise, which admit a compact Cauchy horizon.

## 5. Conclusion

We have rigorously proven that strong cosmic censorship holds for the class of polarised Gowdy spacetimes. This is a small but non-trivial class of solutions of Einstein's equations, so our result lends support to the conjecture that SCC holds more generally. We are currently working on extending our ideas to larger classes of spacetimes such as the general Gowdy spacetime, and the set of solutions to Einstein's equations with one Killing field.

## References


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[3] Penrose R 1979 *General Relativity—An Einstein Centenary Survey* ed S W Hawking and W Israel (Cambridge: Cambridge University Press)

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[5] Einstein A and Rosen N 1937 *J. Franklin Inst.* **223** 43
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[6] They have often been used in this context for studies of quantum gravity. See Berger B 1984 *Ann. Phys., NY* **156** 155 and Husain V 1987 *Class. Quantum Grav.* **4** 1587, and also Husain V and Smolin L 1989 Exactly solvable quantum cosmologies from two Killing field reduction of general relativity *Syracuse preprint*

[7] This parametrisation of the metric differs slightly from the usual one presented (e.g., in Gowdy, ref [4]). The Einstein equations (2)-(5) are somewhat simplified when parametrisation (1) is used

[8] Mostert P S 1957 *Ann. Math.* **65** 447; 1957 errata in **66** 589

[9] An error appears in Gowdy's work (ref [4]), concerning the exact definition of this subset. See Chrusciel P 1989 On spacetimes with $U(1)\times U(1)$ symmetric compact Cauchy surfaces *Yale preprint*

[10] See Choquet-Bruhat Y 1952 *Acta. Math.* **88** 141, for a proof of well-posedness and see Choquet-Bruhat Y and Geroch R 1969 *Commun. Math. Phys.* **14** 329, for results on maximal developments

[11] There are certain extra conditions which $(U, K)$ must satisfy, depending upon the topology of $\Sigma^3$. For $\Sigma^3 = T^3$, one requires $\int_S K\partial_\theta U = 0$. This arises as an integrability condition for the constraints (3). For $\Sigma^3 = S^3$ and $\Sigma^3 = S^2 \times S^1$, in addition to a similar integral constraint, there are a number of boundary conditions which $U$ and $K$ must satisfy on the poles $\theta = 0$ and $\theta = \pi$
[12] Note that the singularity existence result here is a very strong version of a 'splitting type' theorem, as discussed by Yau and Galloway. See Galloway G 1989 *Conference on Mathematical Relativity*, CMA vol 19, ed R Bartnik, ANU publication and references cited therein
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