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On completeness of orbits of Killing vector fields

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Abstract. A theorem is proved which reduces the problem of completeness of orbits of Killing vector fields in maximal globally hyperbolic, say, vacuum spacetimes to some properties of the orbits near the Cauchy surface. In particular it is shown that all Killing orbits are complete in maximal developements of asymptotically flat Cauchy data, or of Cauchy data prescribed on a compact manifold.

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1. Introduction

In any physical theory a privileged role is played by solutions of the field equation which exhibit special symmetries. In general relativity there exist several ways for a solution to be symmetric: there might exist

(i) a Killing vector field X on the spacetime (M, g), or there might exist
(ii) an action of a group G on M by isometries, and finally there might perhaps exist
(iii) a Cauchy surface $\Sigma \subset M$ and a group G which acts on Σ while preserving the Cauchy data.

It is natural to enquire what are the relationships between those notions. Clearly (ii) implies (i), but (i) does not need to imply (ii) (remove e.g. points from a spacetime on which an action of G exists). With a little work one can show [12, 8, 6] that (iii) implies (i), and actually it is true [6] that (iii) implies (ii), when M is suitably chosen. The purpose of this paper is to address the question, *do there exist natural conditions on* (M, g) *under which (i) implies (ii)?* Such an implication has been recently established in [11] assuming timelikeness of X and timelike geodesic completeness of (M, g). Similarly in [13] [Proposition 30, p 254] it has been shown that spacelike and timelike geodesic completeness of (M, g) implies completeness of orbits of X without any hypotheses on the character of X. The hypotheses of [11] and of [13] do not seem satisfactory from a general relativistic point of view, as the hypothesis of geodesic completeness of any kind is incompatible with many situations of interest. Moreover the hypotheses of [11, 13] are difficult to control. Recall now that given a Cauchy data set (Σ, γ, K), where Σ is a three-dimensional manifold, γ is a Riemannian metric on Σ, and K is a symmetric 2-tensor on Σ, there exists a *unique up to*

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isometry vacuum spacetime (M, g), which is called the *maximal globally hyperbolic vacuum development of* (Σ, γ, K), with an embedding $i : \Sigma \to M$ such that $i^*g = \gamma$, and such that K corresponds to the extrinsic curvature of $i(\Sigma)$ in M [3]. (M, g) is *inextendible* in the class of globally hyperbolic spacetimes with a vacuum metric. This class of spacetimes is highly satisfactory to work with, as they can be characterized by their Cauchy data induced on some Cauchy surface. Let us also recall that in globally hyperbolic vacuum spacetimes (M, g), the question of existence of a Killing vector field X on M can be reduced to that of existence of appropriate Cauchy data for X on a Cauchy surface Σ (cf e.g. [6]). In this paper we show the following:

Theorem 1.1. Let (M, g) be a smooth, vacuum, maximal globally hyperbolic spacetime with Killing vector field X and Cauchy surface Σ. The following conditions are equivalent:

(i) There exists $\epsilon > 0$ such that for all $p \in \Sigma$ the orbits $\phi_s(p)$ of X through p are defined for all $s \in [-\epsilon, \epsilon]$.
(ii) The orbits of X are complete in M.

It should be said that though this result seems to be new, it is a relatively straightforward consequence of the results in [3].

The following example† shows that some conditions on the behaviour of the orbits on the Cauchy surface are necessary in general. Let Σ be a connected component of the unit spacelike hyperboloid in Minkowski spacetime $\mathbb{R}^{1+3}$, let (M, g) be the domain of dependence of Σ in $\mathbb{R}^{1+3}$ with the obvious flat metric and let X be the Killing vector $\partial/\partial t$. M is maximal globally hyperbolic (cf e.g. proposition 2.4 below), Σ is a complete Riemannian manifold, the Lorentzian length of X is uniformly bounded on Σ; nevertheless, no orbits of X are complete in M.

As a corollary of theorem 1.1 one obtains, nevertheless, the following (cf section 3 for precise definitions; here $\mathcal{D}(\Sigma)$ denotes the domain of dependence of an achronal set Σ):

Corollary 1.2. Let (M, g) be a smooth, vacuum, maximal globally hyperbolic spacetime with an achronal spacelike hypersurface Σ and with a Killing vector X (defined perhaps only on $\mathcal{D}(\Sigma)$). Suppose that either

(i) Σ is compact, or
(ii) (Σ, γ, K) is asymptotically flat, or
(iii) (Σ, γ, K) are Cauchy data for an asymptotically flat exterior region in a (non-degenerate) black-hole spacetime.

Then the orbits of X are complete in $\mathcal{D}(\Sigma)$.

The difference between the cases (ii) and (iii) above is, roughly speaking, as follows: in point (ii) above Σ is a complete Riemannian submanifold of M *without boundary*. On the other hand, in point (iii) above Σ is a complete Riemannian submanifold of M *with a compact boundary* $\partial\Sigma$, and the Killing vector is assumed to be tangent to $\partial\Sigma$; cf the beginning of section 3 for a longer discussion of the relevant notions. (It should also be pointed out that we necessarily have $M = \mathcal{D}(\Sigma)$ in point (i) above by [2]. In point (ii) we may, but do not need to, assume that $M = \mathcal{D}(\Sigma)$. In point (iii) above, however, $M = \mathcal{D}(\Sigma)$ *cannot* hold (cf definition 3.2, section 3).)

We have stated theorem 1.1 and corollary 1.2 for the vacuum, but they clearly hold for any kind of well posed hyperbolic system of equations for the metric g coupled with some

† I am grateful to R Wald and B Schmidt for discussions concerning this point.

matter fields. All that is needed is a local existence theorem for the coupled system, together with uniqueness of solutions in domains of dependence. In particular, theorem 1.1 will still be true e.g. for metrics satisfying the Einstein–Yang–Mills–Higgs equations. Corollary 1.2 will still hold for the Einstein–Yang–Mills–Higgs equations, provided both the gravitational field and the matter fields satisfy appropriate fall-off conditions in the asymptotically flat case. We plan to discuss this elsewhere.

The use and applicability of theorem 1.1 and corollary 1.2 are rather wide: all non-purely-local results about spacetimes with Killing vectors assume completeness of their orbits. Let us in particular mention the theory of uniqueness of black holes. Clearly it is essential to classify also those black holes in which the orbits of the Killing field are not complete and which are thus not covered by the existing theory. Consider then a stationary black-hole spacetime (M, g) in which an asymptotically flat Cauchy surface exists but in which the Killing orbits are *not* complete: corollary 1.2 shows that (M, g) can be enlarged to obtain a spacetime with complete Killing orbits. As another application, let us also mention the recent work of Wald and this author [7], where the question of existence of maximal hypersurface in asymptotically stationary spacetimes is considered. Corollary 1.2 shows that the hypothesis of completeness of the orbits of the Killing field made in [7] can be removed if the spacetime under consideration is vacuum (or satisfies some well behaved field equations) and e.g. maximal.

2. Proof of theorem 1.1

In this section we shall prove theorem 1.1. Let us start with a somewhat weaker result.

Theorem 2.1. Let (M, g) be a smooth, vacuum, maximal globally hyperbolic spacetime with Cauchy surface Σ and with a Killing vector field X, with $g, X \in C^\infty$. Then the orbits of X in M are complete if and only if

(i) there exists $\epsilon > 0$ such that for all $p \in \Sigma$ the orbits $\phi_s(p)$ of X are defined for all $s \in [-\epsilon, \epsilon]$, and
(ii) for $s \in [-\epsilon, \epsilon]$ the sets $\phi_s(\Sigma)$ are achronal.

Proof. Let us start by showing necessity. Point (i) is obvious, consider point (ii). As the orbits of X are complete, the flow of X (defined as the solution of the equations $\mathrm{d}\phi_s(p)/\mathrm{d}s = X \circ \phi_s(p)$, with the initial value $\phi_0(p) = p$) is defined for all $p \in M$ and all $s \in \mathbb{R}$. Suppose there exists $s_1 \in [-\epsilon, \epsilon]$ and a timelike path $\Gamma : [0, 1] \to M$ with $\Gamma(0) \in \phi_{s_1}(\Sigma)$ and $\Gamma(1) \in \phi_{s_1}(\Sigma)$. Then $\phi_{-s_1}(\Gamma)$ would be a timelike path with $\phi_{-s_1}(\Gamma(0)) \in \Sigma, \phi_{-s_1}(\Gamma(1)) \in \Sigma$, which is not possible as Σ is achronal. Hence (i) and (ii) are necessary.

To show sufficiency, we shall need the following proposition.

Proposition 2.2. Let (M_a, g_a), $a = 1, 2$, be vacuum globally hyperbolic spacetimes with Cauchy surfaces Σ_a, and suppose that (M_2, g_2) is maximal. Let $\mathcal{O} \subset M_1$ be a (connected) neighbourhood of Σ_1 and suppose there exists a one-to-one isometry $\Psi_\mathcal{O} : \mathcal{O} \to M_2$, such that $\Psi_\mathcal{O}(\Sigma_1)$ is achronal. Then there exists a one-to-one isometry

$$\Psi : M_1 \to M_2 \tag{2.1}$$

such that $\Psi|_\mathcal{O} = \Psi_\mathcal{O}$.

Remarks.

(i) When $\Psi_{\mathcal{O}}(\Sigma_1) = \Sigma_2$, this result can be essentially found in [3]. The proof below is a rather straightforward generalization of the arguments of [3], cf also [4, 9]. Although we assume smoothness of the metric throughout this paper for the sake of simplicity, we have taken some care to write the proof below in a way which generalizes with no essential difficulties to the case where low Sobolev-type differentiability of the metric is assumed.

(ii) The condition that $\psi_{\mathcal{O}}(\Sigma_1)$ is achronal is necessary, which can be seen as follows. Let $M_1 = \mathbb{R}^2$ with the standard flat metric, set $\Sigma_1 = \{t = 0\}$. Let $\sim_a$ be the equivalence relation defined as $(t, x) \sim_a (t + a, x + 1)$, where a is a number satisfying $|a| < 1$, $a \neq 0$. Define $M_2 = M_1/\sim_a$ with the naturally induced metric, $\mathcal{O} = (-a/3, a/3) \times \mathbb{R}$, $\psi_{\mathcal{O}} = i_{M_1}|_{\mathcal{O}}$, where i_{M_1} is the natural projection: $i_{M_1}(p) = [p]_{\sim_a}$. M_2 is causal geodesically complete; the function $t - ax : M_1 \to \mathbb{R}$ defines, by passing to the quotient, a time function on M_2, whose level sets are Cauchy surfaces. It follows that M_2 is maximal globally hyperbolic. Clearly $\psi_{\mathcal{O}}(\Sigma_1)$ is not achronal, and there is no one-to-one isometry from M_1 to M_2.

Proof. Consider the collection $\mathcal{X}$ of all pairs $(\mathcal{U}, \Psi_{\mathcal{U}})$, where $\mathcal{U} \subset M_1$ is a globally hyperbolic neighbourhood of Σ_1 (with Σ_1-Cauchy surface for $(\mathcal{U}, g_1|_{\mathcal{U}})$, and $\Psi_{\mathcal{U}} : \mathcal{U} \to M_2$ is an isometric diffeomorphism between $\mathcal{U}$ and $\Psi_{\mathcal{U}}(\mathcal{U}) \subset M_2$ satisfying $\Psi_{\mathcal{U}}|_{\Sigma_1} = \Psi_{\mathcal{O}}|_{\Sigma_1}$. $\mathcal{X}$ can be ordered by inclusion: $(\mathcal{U}, \Psi_{\mathcal{U}}) \leq (\mathcal{V}, \Psi_{\mathcal{V}})$ if $\mathcal{U} \subset \mathcal{V}$ and if $\Psi_{\mathcal{V}}|_{\mathcal{U}} = \Psi_{\mathcal{U}}$. Let $(\mathcal{U}_\alpha, \Psi_\alpha)_{\alpha\in\Omega}$ be a chain in $\mathcal{X}$, set $\mathcal{W} = \cup_{\alpha\in\Omega}\mathcal{U}_\alpha$, define $\Psi_{\mathcal{W}} : \mathcal{W} \to M_2$ by $\Psi_{\mathcal{W}}|_{\mathcal{U}_\alpha} = \Psi_\alpha$; clearly $(\mathcal{W}, \Psi_{\mathcal{W}})$ is a majorant for $(\mathcal{U}_\alpha, \Psi_\alpha)_{\alpha\in\Omega}$. From the set theory axioms (cf e.g. [10] [Appendix]) it is easily seen that $\mathcal{X}$ forms a set, we can thus apply Zorn's lemma [10] to conclude that there exist maximal elements $(\tilde{M}, \Psi)$ in $\mathcal{X}$. Let then $(\tilde{M}, \Psi)$ be any maximal element, by definition $(\tilde{M}, g_1|_{\tilde{M}})$ is thus globally hyperbolic with Cauchy surface Σ_1, and Ψ is a one-to-one isometry from $\tilde{M}$ into M_2 such that $\Psi|_{\Sigma_1} = \Psi_{\mathcal{O}}|_{\Sigma_1}$. By e.g. lemma 2.1.1 of [6] we have

$$\Psi|_{\tilde{M}\cap\mathcal{O}} = \Psi_{\mathcal{O}}|_{\tilde{M}\cap\mathcal{O}}. \tag{2.2}$$

We have the following lemma.

Lemma 2.3. Under the hypotheses of proposition 2.2, suppose that $(\mathcal{O}, \Psi_{\mathcal{O}})$ is maximal. Then the manifold

$$M' = (M_1 \sqcup M_2)/\Psi_{\mathcal{O}}$$

is Hausdorff.

Remark. Recall that $\sqcup$ denotes the disjoint union, while $(M_1 \sqcup M_2)/\Psi$ is the quotient manifold $(M_1 \sqcup M_2)/\sim$, where $p_1 \in M_1$ is equivalent to $p_2 \in M_2$ if $p_2 = \Psi(p_1)$.

Proof. Let $p, q \in M'$ be such that there exist no open neighbourhoods separating p and q; clearly this is possible only if (interchanging p with q if necessary) we have $p \in \partial\mathcal{O}$ and $q \in \partial\Psi_{\mathcal{O}}(\mathcal{O})$. Consider the set $\mathcal{H}$ of 'non-Hausdorff points' p' in M' such that $p' = i_{M_1}(p)$ for some $p \in M_1$, where i_{M_1} is the embedding of M_1 into M'; $\mathcal{H}$ is closed and we have $\mathcal{H} \subset \partial\mathcal{O}$.

Suppose that $\mathcal{H} \neq \emptyset$; changing time orientation if necessary, we may assume that $\mathcal{H} \cap I^+(\Sigma_1) \neq \emptyset$; let $p' \in \mathcal{H} \cap I^+(\Sigma_1)$. We wish to show that there necessarily exists $p \in \mathcal{H}$ such that

$$J^-(p) \cap \mathcal{H} \cap I^+(\Sigma_1) = \{p\}. \tag{2.3}$$

If (2.3) holds with $p = p'$ we are done, otherwise consider the (non-empty) set $\mathcal{Y}$ of causal paths $\Gamma : [0,1] \to I^+(\Sigma)$ such that $\Gamma(0) \in \mathcal{H}, \Gamma(1) = p'$. $\mathcal{Y}$ is directed by inclusion: $\Gamma_1 < \Gamma_2$ if $\Gamma_1([0,1]) \subset \Gamma_2([0,1])$. Let $\{\Gamma_\alpha\}_{\alpha\in\Omega}$ be a chain in $\mathcal{Y}$, set $\Gamma = \cup_{\alpha\in\Omega}\Gamma_\alpha([0,1])$, consider the sequence $p_\alpha = \Gamma_\alpha(0)$. Clearly $\Gamma \subset J^+(\Sigma_1) = I^+(\Sigma_1) \cup \Sigma_1$, and global hyperbolicity implies that Γ must be extendible, thus $\Gamma_\alpha(0)$ accumulates at some $p_* \in I^+(\Sigma_1) \cup \Sigma_1$. As $\mathcal{O}$ is an open neighbourhood of Σ_1 the case $p_* \in \Sigma_1$ is not possible, hence $p_* \in I^+(\Sigma_1)$ and consequently $\Gamma \in \mathcal{Y}$. It follows that every chain in $\mathcal{Y}$ has a majorant, and by Zorn's lemma $\mathcal{Y}$ has maximal elements. Let then Γ be any maximal element of $\mathcal{Y}$; setting $p = \Gamma(0)$, the equality (2.3) must hold.

We now claim that (2.3) also implies

$$J^-(p) \cap \partial\mathcal{O} \cap I^+(\Sigma_1) = \{p\}. \tag{2.4}$$

Suppose, on the contrary, that there exists $q \in (J^-(p) \cap \partial\mathcal{O} \cap I^+(\Sigma_1)) \setminus \{p\}$; let $q_i \in \mathcal{O}$ be a sequence such that $q_i \to q$. We can choose q_i so that $q_{i+1} \in I^+(q_i)$. Global hyperbolicity of $\mathcal{O}$ implies that for $i > i_o$, for some i_o, there exist timelike paths $\Gamma_i : [0,1] \to \bar{\mathcal{O}}$, $\Gamma_i([0,1)) \subset \mathcal{O}$, $\Gamma_i(0) = q_i$, $\Gamma_i(1) = p$. Let $\tilde{p} \in M_2$ be a non-Hausdorff partner of p such that the curves $\Psi_\mathcal{O}(\Gamma_i([0,1)))$ have $\tilde{p}$ as an accumulation point. We have $\Psi_\mathcal{O}(\Gamma_i) \subset J^+(\Psi_\mathcal{O}(q_{i_0})) \cap J^-(\tilde{p})$ which is compact by global hyperbolicity of M_2; hence there exists a subsequence $\Psi_\mathcal{O}(q_i)$ converging to some $\tilde{q} \in M_2$. This implies that q and $\tilde{q}$ constitute a 'non-Hausdorff pair' in M', contradicting (2.3), and thus (2.4) must be true.

Let $p_1 \in M_1$, $p_2 \in M_2$, $i_{M_1}(p_1) = i_{M_2}(p_2)$, be any non-Hausdorff pair in M' such that (2.3) holds with $p = p_1$. Let x^μ be harmonic coordinates defined in a neighbourhood $\mathcal{O}_2$ of p_2. [Such coordinates can be e.g. constructed as follows: let t_0 be any time function defined in some neighbourhood $\mathcal{O}_2$ of p_2, such that $t_0(p_2) = 0$, and set $\mathcal{I}_\tau = \{p \in \mathcal{O}_2 : t_0(p) = \tau\}$. Passing to a subset of $\mathcal{O}_2$, if necessary, there exists a global coordinate system x_0^i defined on $\mathcal{I}_0$; again passing to a subset of $\mathcal{O}_2$, if necessary, we may assume that $\mathcal{O}_2$ is globally hyperbolic with Cauchy surface $\mathcal{I}_0$. Let $x^\mu \in C^\infty(\mathcal{O}_2)$ be the (unique) solutions of the problem

$$\Box_{g_2} x^\mu = 0$$

$$x^0\Big|_{\mathcal{I}_0} = 0 \qquad \frac{\partial x^0}{\partial t_0}\Big|_{\mathcal{I}_0} = 1 \qquad x^i\Big|_{\mathcal{I}_0} = x_0^i \qquad \frac{\partial x^i}{\partial t_0}\Big|_{\mathcal{I}_0} = 0 \tag{2.5}$$

where $\Box_\gamma$ is the d'Alembert operator of a metric γ. Passing once more to a globally hyperbolic subset of $\mathcal{O}_2$ if necessary, the functions x^μ form a coordinate system on $\mathcal{O}_2$.] We can choose $\epsilon > 0$ such that

(i) $\mathrm{int}\mathcal{D}^+(\mathcal{I}_{-\epsilon}) \subset \mathcal{O}_2$,
(ii) $p \in \mathrm{int}\mathcal{D}^+(\mathcal{I}_{-\epsilon})$,
(iii) $\overline{\mathcal{I}_{-\epsilon}} \subset \Psi_\mathcal{O}(\mathcal{O})$.

Define

$$\hat{\mathcal{I}} = \Psi_\mathcal{O}^{-1}(\mathcal{I}_{-\epsilon}).$$

Let $y^\mu \in C^\infty\left(\mathcal{D}(\hat{\mathcal{I}})\right)$ be the (unique) solutions of the problem

$$\Box_{g_1} y^\mu = 0,$$

$$y^\mu\Big|_{\hat{\mathcal{I}}} = x^\mu \circ \Psi_\mathcal{O}\Big|_{\hat{\mathcal{I}}} \qquad \frac{\partial y^\mu}{\partial \hat{n}} = \frac{\partial\,(x^\mu \circ \Psi_\mathcal{O})}{\partial \hat{n}}\Big|_{\hat{\mathcal{I}}}$$

where $\partial/\partial\hat{n}$ is the derivative in the direction normal to $\hat{\mathcal{I}}$. By isometry invariance of the wave equation we have

$$y^\mu|_{\mathcal{D}(\hat{\mathcal{I}})\cap\mathcal{O}} = x^\mu \circ \Psi_\mathcal{O}|_{\mathcal{D}(\hat{\mathcal{I}})\cap\mathcal{O}}. \tag{2.6}$$

Set

$$\mathcal{U} = \mathcal{O} \cup \mathrm{int}\mathcal{D}^+(\hat{\mathcal{I}})$$

and for $p \in \mathcal{U}$ define

$$\Psi_\mathcal{U}(p) = \begin{cases} \Psi_\mathcal{O}(p) & p \in \mathcal{O} \\ q : \text{where } q \text{ is such that } x^\mu(q) = y^\mu(p) & p \in \mathrm{int}\mathcal{D}^+(\hat{\mathcal{I}}). \end{cases} \tag{2.7}$$

From (2.6) it follows that $\Psi_\mathcal{U}$ is a smooth map from $\mathcal{U}$ to M_2. Clearly $\mathcal{U}$ is a globally hyperbolic neighbourhood of Σ_1, and Σ_1 is a Cauchy surface for $\mathcal{U}$. Note that $\mathcal{O}$ is a proper subset of $\mathcal{U}$, as $p_1 \in \mathrm{int}D^+(\hat{\mathcal{I}})$ but $p_1 \notin \mathcal{O}$. It follows from uniqueness of solutions of the Einstein equations in harmonic coordinates that $\Psi_\mathcal{U}$ is an isometry. To prove that $\Psi_\mathcal{U}$ is one-to-one, consider $p, q \in \mathcal{U}$ such that $\Psi_\mathcal{U}(p) = \Psi_\mathcal{U}(q)$. Changing time orientation if necessary we may assume that $p \in I^+(\Sigma_1)$. By hypothesis we have $I^+\left(\Psi_\mathcal{O}(\Sigma_1)\right) \cap I^-\left(\Psi_\mathcal{O}(\Sigma_1)\right) = \emptyset$, hence $q \in I^+(\Sigma_1)$. Let $[0, 1] \ni s \to \Gamma(s)$ be a timelike path from Σ_1 to q, let $\Gamma_1(s)$ be a connected component of $\Psi_\mathcal{U}^{-1}(\Psi_\mathcal{U}(\Gamma))$ which contains $\{p\}$. Consider the set $\Omega = \{s \in [0, 1] : \Gamma(s) = \Gamma_1(s)\}$. Since $\Psi_\mathcal{U}|_\mathcal{O} = \Psi_\mathcal{O}$ which is one-to-one, Ω is non-empty. By continuity of Γ_1 and Γ, Ω is closed. Since $\Psi_\mathcal{U}$ is locally one-to-one (being a local diffeomorphism), Ω is open. It follows that $\Omega = [0, 1]$, hence $p = q$, and $\Psi_\mathcal{U}$ is one-to-one as claimed.

We have thus shown, that $(\mathcal{O}, \Psi_\mathcal{O}) \le (\mathcal{U}, \Psi_\mathcal{U})$ and $(\mathcal{O}, \Psi_\mathcal{O}) \neq (\mathcal{U}, \Psi_\mathcal{U})$ which contradicts maximality of $(\mathcal{O}, \Psi_\mathcal{O})$. It follows that M' is Hausdorff, as we desired to show. □

Returning to the proof of proposition 2.2, let $(\tilde{M}, \Psi)$ be maximal. If $\tilde{M} = M_1$ we are done, suppose then that $\tilde{M} \neq M_1$. Consider the manifold

$$M' = (M_1 \sqcup M_2)/\Psi.$$

By lemma 2.3, M' is Hausdorff. We claim that M' is globally hyperbolic with Cauchy surface $\Sigma' = i_{M_2}(\Sigma_2) \approx \Sigma_2$, where i_{M_a} denotes the canonical embedding of M_a in M'. Indeed, let $\Gamma' \subset M'$ be an inextendible causal curve in M', set $\Gamma_1 = i_{M_1}^{-1}(\Gamma' \cap i_{M_1}(M_1))$, $\Gamma_2 = i_{M_2}^{-1}(\Gamma' \cap i_{M_2}(M_2))$. Clearly $\Gamma_1 \cup \Gamma_2 \neq \emptyset$, so that either $\Gamma_1 \neq \emptyset$, or $\Gamma_2 \neq \emptyset$, or both. Let the index a be such that $\Gamma_a \neq \emptyset$. If $\hat{\Gamma}_a$ were an extension of Γ_a in M_a, then $i_{M_a}(\hat{\Gamma}_a)$ would be an extension of Γ' in M', which contradicts maximality of Γ', thus Γ_a is inextendible. Suppose that $\Gamma_1 \neq \emptyset$; as Γ_1 is inextendible in M_1 we must have $\Gamma_1 \cap \Sigma_1 = \{p_1\}$ for some $p_1 \in \Sigma_1$. We thus have $\Psi(p_1) \in \Gamma_2$, so that it always holds that $\Gamma_2 \neq \emptyset$. By global hyperbolicity of M_2 and inextendibility of Γ_2 it follows that $\Gamma_2 \cap \Sigma_2 = \{p_2\}$ for some $p_2 \in \Sigma_2$, hence $\Gamma' \cap i_{M_2}(\Sigma_2) = \{i_{M_2}(p_2)\}$. This shows that $i_{M_2}(\Sigma_2)$ is a Cauchy surface for M', thus M' is globally hyperbolic. As $\tilde{M} \neq M_1$ we have $M' \neq M_2$ which contradicts maximality of M_2. It follows that we must have $\tilde{M} = M_2$, and proposition 2.2 follows. □

Returning to the proof of theorem 2.1, choose $s \in [-\epsilon/2, \epsilon/2]$. There exists a globally hyperbolic neighborhood $\mathcal{O}_s$ of Σ such that the map $\phi_s(p)$ is defined for all $p \in \mathcal{O}_s$:

$$\mathcal{O}_s \ni p \to \phi_s(p) \in M.$$

$\phi_s(\Sigma)$ is achronal by hypothesis, and proposition 2.2 shows that there exists a map $\hat{\phi}_s : M \to M$ such that $\hat{\phi}_s|_{\mathcal{O}_s} = \phi_s$. For $s \in \mathbb{R}$ let k be the integer part of $2s/\epsilon$, define $\hat{\phi}_s : M \to M$ by

$$\hat{\phi}_s = \hat{\phi}_{s-k\epsilon/2} \circ \underbrace{\hat{\phi}_{\epsilon/2} \circ \ldots \circ \hat{\phi}_{\epsilon/2}}_{k \text{ times}} \quad \text{when} \quad s \in \mathbb{R}\backslash[-\epsilon, \epsilon].$$

It is elementary to show that $\hat{\phi}_s$ satisfies

$$\frac{\mathrm{d}\hat{\phi}_s}{\mathrm{d}s} = X \circ \hat{\phi}_s,$$

and theorem 2.1 follows. □

Let us point out the following useful result:

Proposition 2.4. Let (M, g) be a maximal globally hyperbolic vacuum spacetime. Suppose that $\tilde{\Sigma} \subset M$ is an achronal spacelike submanifold, and let $(\tilde{\gamma}, \tilde{K})$ be the Cauchy data induced by g on $\tilde{\Sigma}$. Then $(\mathcal{D}(\tilde{\Sigma}), g|_{\mathcal{D}(\tilde{\Sigma})})$ is isometrically diffeomorphic to the maximal globally hyperbolic vacuum development $(\tilde{M}, \tilde{\gamma})$ of $(\tilde{\Sigma}, \tilde{\gamma}, \tilde{K})$.

Proof. By maximality of $(\tilde{M}, \tilde{\gamma})$, there exists a map $\Psi : \mathcal{D}(\tilde{\Sigma}) \to \tilde{M}$ which is a smooth isometric diffeomorphism between $\mathcal{D}(\tilde{\Sigma})$ and $\Psi(\mathcal{D}(\tilde{\Sigma}))$. By standard local uniqueness results for vacuum Einstein equations there exists a globally hyperbolic neighborhood $\tilde{\mathcal{O}}$ of $\tilde{\imath}(\tilde{\Sigma})$ in $\tilde{M}$, where $\tilde{\imath}$ is the embedding of $\tilde{\Sigma}$ in $\tilde{M}$, and a map $\Phi_{\tilde{\mathcal{O}}} : \tilde{\mathcal{O}} \to \mathcal{D}(\tilde{\Sigma}) \subset M$ which is an isometric diffeomorphism between $\tilde{\mathcal{O}}$ and $\Phi_{\tilde{\mathcal{O}}}(\tilde{\mathcal{O}})$. By proposition 2.2, $\Phi_{\tilde{\mathcal{O}}}$ can be extended to a map $\Phi : \tilde{M} \to M$ which is an isometric diffeomorphism between $\tilde{M}$ and $\Phi(\tilde{M})$. Clearly we must have $\Phi(\tilde{M}) \subset \mathcal{D}(\tilde{\Sigma})$, so that one obtains $\Psi \circ \Phi = id_{\tilde{M}}$, $\Phi \circ \Psi = id_{\mathcal{D}(\tilde{\Sigma})}$, and the result follows. □

To prove theorem 1.1, i.e. to remove the hypothesis (ii) of theorem 2.1, more work is needed. Let $t_\pm(p) \in \mathbb{R} \cup \{\pm\infty\}$, $t_-(p) < 0 < t_+(p)$ be defined by the requirement that $(t_-(p), t_+(p))$ is the largest connected interval containing 0 such that the solution $\phi_s(p)$ of the equation $\mathrm{d}\phi_s(p)/\mathrm{d}s = X \circ \phi_s(p)$ with initial condition $\phi_0(p) = p$ is defined for all $s \in (t_-(p), t_+(p))$. From continuous dependence of solutions of ODEs upon parameters it follows that for every $\delta > 0$ there exists a neighborhood $\mathcal{O}_{p,\delta}$ of p such that for all $q \in \mathcal{O}_{p,\delta}$ we have $t_+(q) \geq t_+(p) - \delta$ and $t_-(q) \leq t_-(p) + \delta$. In other words, t_+ is a lower semicontinuous function and t_- is an upper semi-continuous function. We have the following lemma.

Lemma 2.5. Let $p \in I^+(\Sigma)$, $q \in J^-(p) \cap I^+(\Sigma)$, suppose that $t_+(p) \geq \tau_0$. If $t_+(q) < \tau_0$, then there exists $s \in [0, t_+(q))$ such that $\phi_s(q) \in \Sigma$.

Proof. Let $\gamma(s)$ be any *future-directed causal curve with* $\gamma(0) = q$, $\gamma(1) = p$. Suppose that $t_+(p) \geq \tau_0 > t_+(q)$ and let $(s_-, 1]$ be the largest interval such that $t_+(\gamma(s)) > t_+(q)$ for all $s \in (s_-, 1]$. By lower semicontinuity of t_+ we have $(s_-, 1] \neq \emptyset$. Consider the one-parameter family of causal paths

$$[0, t_+(q)] \times (s_-, 1] \ni (\tau, s) \to \tilde{\gamma}_\tau(s) = \phi_\tau(\gamma(s)).$$

Suppose that for all $s \in [0, t_+(q))$ we have $\phi_s(q) \notin \Sigma$. Global hyperbolicity of M implies that for all $s \in [0, t_+(p))$ we have $\phi_s(q) \in I^+(\Sigma)$, consequently for any $r \in I^+(\phi_s(q))$ it also holds that $r \in I^+(\Sigma)$; hence $\tilde{\gamma}_\tau(s) \in J^+(\Sigma)$ for all $\tau, s \in [0, t_+(q)] \times (s_-, 1]$. As Σ is a Cauchy surface, for each τ the curve $\tilde{\gamma}_\tau$ must be past-extendible. Let thus $\hat{\gamma}_\tau(s)$ be any past extension of $\tilde{\gamma}_\tau$, for $\tau \in [0, t_+(q)]$ define $\psi_\tau = \hat{\gamma}_\tau(s_-)$. It is elementary to show that $\psi_\tau = \phi_\tau(\gamma(s_-))$, so that $t_+(\gamma(s_-)) > t_+(q)$. This, however, contradicts the definition of s_-, and the result follows. □

Proof of theorem 1.1. Suppose there exists $s_0 \in [-\epsilon, \epsilon]$ such that $\phi_{s_0}(\Sigma)$ is *not* achronal. Let $\Gamma : [0, 1] \to M$ be a timelike curve such that $\Gamma(0), \Gamma(1) \in \phi_{s_0}(\Sigma)$. Changing X to $-X$ if necessary we may assume $s_0 < 0$; changing time orientation if necessary we may suppose that $\Gamma(1) \in J^+(\Sigma)$. We have $t_+|_\Sigma \geq \epsilon$, hence $t_+|_{\phi_{s_0}(\Sigma)} = (t_+ + |s_0|)|_\Sigma \geq \epsilon$.

Let $q \in I^-(\Gamma(1)) \cap J^+(\Sigma)$. By lemma 2.5 either $t_+(q) \geq \epsilon$, or there exists $s \in [0, t_+(q))$ such that $\phi_s(q) \in \Sigma$. In that last case we have $t_+(q) - s = t_+(\phi_s(q)) \geq \epsilon$, hence $t_+(q) \geq \epsilon$, and in either case we obtain $t_+(q) \geq \epsilon$. It follows that

$$t_+|_{\Gamma \cap J^+(\Sigma)} \geq \epsilon. \tag{2.8}$$

If $\Gamma(0) \in J^+(\Sigma)$ we thus obtain

$$t_+|_\Gamma \geq \epsilon. \tag{2.9}$$

Consider the case $\Gamma(0) \in J^-(\Sigma)$. We have $t_+(\Gamma(0)) \geq \epsilon$ and by an argument similar to that above (using the time-dual version of lemma 2.5) we obtain

$$t_+|_{\Gamma \cap J^-(\Sigma)} \geq \epsilon$$

and by global hyperbolicity we can again conclude that (2.9) holds. Equation (2.9) shows that $\phi_{-s_0}(\Gamma)$ is a timelike curve satisfying $\phi_{-s_0}(\Gamma(0)), \phi_{-s_0}(\Gamma(1)) \in \Sigma$. This, however, contradicts achronality of Σ. We therefore conclude that for all $s \in [-\epsilon, \epsilon]$ the hypersurfaces $\phi_s(\Sigma)$ are achronal. Theorem 1.1 follows now from theorem 2.1. □

3. Proof of corollary 1.2

Before passing to the proof of corollary 1.2, it seems appropriate to present some definitions.

Definition 3.1. We shall say that an initial data set (Σ, γ, K) for vacuum Einstein equations is asymptotically flat if (Σ, γ) is a complete Riemannian manifold (without boundary), with Σ of the form

$$\Sigma = \Sigma_{\text{int}} \bigcup_{i=1}^{I} \Sigma_i \tag{3.1}$$

for some $I < \infty$. Here we assume that Σ_{int} is compact, and each of the ends Σ_i is diffeomorphic to $\mathbb{R}^3 \setminus B(R_i)$ for some $R_i > 0$, with $B(R_i)$ being a coordinate ball of radius R_i. In each of the ends Σ_i the metric is assumed to satisfy the hypotheses† of the boost theorem, theorem 6.1 of [5].

The hypotheses of theorem 6.1 of [5] will hold if e.g. there exists $\alpha > 0$ such that in each of the ends Σ_i we have

$$0 \leq k \leq 4 \qquad |\partial_{i_1} \ldots \partial_{i_k}(\gamma_{ij} - \delta_{ij})| \leq C r^{-\alpha-k}$$

$$0 \leq k \leq 3 \quad |\partial_{i_1} \ldots \partial_{i_k} K_{ij}| \leq C r^{-\alpha-k-1}$$

for some constant C.

† The differentiability threshold of theorem 6.1 of [5] can actually be weakened to $s \geq 3$. Similarly, the differentiability threshold in theorem 6.2 of [5] can be weakened to $s \geq 4$, and probably also to $s \geq 3$.

To motivate the next definition, consider a spacetime with some number of asymptotically flat ends, and with a black hole region. In such a case there might be a Killing vector field defined in, say, the domain of outer communication of the asymptotically flat ends. It could, however, occur that there is no Killing vector field defined on the whole spacetime—a famous example of such a spacetime has been considered by Brill [1], yielding a spacetime in which no asymptotically flat maximal surfaces exist. Alternatively, there may be a Killing vector field defined everywhere; however, there may be some non-asymptotically flat ends in M. (As an example, consider a spacelike surface in the Schwarzschild–Kruskal–Szekeres spacetime in which one end is asymptotically flat, and the second is 'asymptotically hyperboloidal'.) In such cases one would still like to claim that the orbits of X are complete at least in the exterior region. We shall see that this is indeed the case, under some conditions which we spell out below:

Definition 3.2. Consider a stably causal Lorentzian manifold (M, g) with an achronal spacelike surface $\hat{\Sigma}$. Let $\Sigma \subset \hat{\Sigma}$ be a connected submanifold of $\hat{\Sigma}$ with smooth compact boundary $\partial\Sigma$, and let (γ, K) be the Cauchy data induced by g on Σ. Suppose finally that there exists a Killing vector field X defined on $\mathcal{D}(\Sigma)$. We shall say that (Σ, γ, K) are Cauchy data for an asymptotically flat exterior region in a (non-degenerate) black-hole spacetime if the following hold:

(i) The closure $\bar{\Sigma} \equiv \Sigma \cup \partial\Sigma$ of Σ is of the form (3.1), with $\Sigma_{\rm int}$ and Σ_i satisfying the requirements of definition 3.1.
(ii) (From equation (3.3) below it follows that X can be extended by continuity to $\overline{\mathcal{D}(\Sigma)}$.) We shall require that X be tangent to $\partial\Sigma$.

An example of the behaviour described in definition 3.2 can be observed in the Schwarzschild–Kruskal–Szekeres spacetime M, when $\hat{\Sigma}$ is taken as a standard $t = 0$ surface, Σ is the part of $\hat{\Sigma}$ which lies in one asymptotic end of M, and $\partial\Sigma$ is the set of points where the usual Killing vector X (which coincides with $\partial/\partial t$ in the asymptotic regions) vanishes. Such $\partial\Sigma$'s are usually called 'the bifurcation surface of a bifurcate Killing horizon'. An example in which X does not vanish on $\partial\Sigma$ is given by the Kerr spacetime, when X is taken to coincide with $\partial/\partial t$ in the asymptotic region, and $\partial\Sigma$ is the intersection of the black hole and of the white hole with respect to the asymptotic end under consideration.

The notion of *non-degeneracy* referred to in definition 3.2 above is related to the non-vanishing of the surface gravity of the horizon: indeed, it follows from [14] that in situations of interest the behaviour described in definition 3.2 can only occur if the surface gravity of the horizon is constant on the horizon, and does not vanish.

With the above definitions in mind, we can now prove corollary 1.2.

Proof of corollary 1.2. Suppose first that Σ is compact. We have

$$t_+|_\Sigma \geq \epsilon$$

for some $\epsilon > 0$ because a lower semi-continuous function attains its infimum on a compact set (cf e.g. [15]), and the result follows from theorem 1.1. (Here we could also use theorem 2.1: the hypersurfaces $\phi_s(\Sigma), s \in [-\epsilon, \epsilon]$, are compact and spacelike and hence achronal by [2].)

Consider next the case of (Σ, γ, K) being asymptotically flat. Let (M, g) be the maximal globally hyperbolic development of (Σ, γ, K). A straightforward extension of the boost theorem [5] using domain-of-dependence arguments shows that M contains a subset of the form

$$M_1 = ([-\delta, \delta] \times \Sigma_{\text{int}}) \bigcup_{i=1}^{I} \Omega_i \tag{3.2}$$

with some $\delta > 0$, where each of the Ω_is is a boost-type domain:

$$\Omega_i = \{(t, \vec{x}) \in \mathbb{R}^4 : |\vec{x}| \geq R_i, |t| \leq \delta + \theta(r - R_i)\}$$

with some $\theta > 0$. Let X be a Killing vector field on M. As is well known, X satisfies the equations

$$\nabla_\mu \nabla_\nu X_\alpha = R_{\lambda\mu\nu\alpha} X^\lambda. \tag{3.3}$$

Under the hypotheses of theorem 6.1 of [5], a simple analysis of (3.3) shows that in each Ω_i there exists $\alpha > 0$ and a constant (perhaps vanishing) matrix $\Lambda^\mu{}_\nu = \Lambda^\mu{}_\nu(i)$ such that

$$0 \leq j \leq 2 \quad \partial_{i_1} \ldots \partial_{i_j} [X^\mu - \Lambda^\mu{}_\nu x^\nu] = O(r^{1-\alpha-j}). \tag{3.4}$$

From equations (3.2) and (3.4) one easily shows that there exists $\epsilon > 0$ such that for all $p \in \Sigma$ the orbit $\phi_s(p)$ of X through p remains in M_1 for $|s| \leq \epsilon$. This shows that in the asymptotically flat case the hypotheses of theorem 1.1 are satisfied as well, and the second part of corollary 1.2 follows.

Consider finally point (iii) of corollary 1.2. Let $\hat{X}$ be any vector field (not necessarily Killing) defined in a neighbourhood $\mathcal{O}$ of $\partial\Sigma$ such that $\hat{X}|_{\mathcal{O}\cap\mathcal{D}(\Sigma)} = X$.

(Because $\mathcal{D}(\Sigma)$ is *not* a smooth manifold, a little work is needed to show that an extension $\hat{X}$ of X exists. A possible construction goes as follows: define $\psi^\mu = X^\mu|_\Sigma$, $\chi^\mu = n^\alpha \nabla_\alpha X^\mu|_\Sigma$, where n^α is the field of unit normals to Σ. Because $\partial\Sigma$ is smooth in $\hat{\Sigma}$, there exists smooth extensions $\hat{\chi}^\mu$ and $\hat{\psi}^\mu$ of χ^μ and of ψ^μ from Σ to $\hat{\Sigma}$. On $\mathcal{D}(\hat{\Sigma})$ let $\hat{X}$ be the unique solution of the problem

$$\left.\begin{array}{ll} \Box \hat{X}^\mu = -R^\mu{}_\alpha \hat{X}^\alpha & \\ \hat{X}^\mu \Big|_{\hat{\Sigma}} = \psi^\mu & \hat{n}^\alpha \nabla_\alpha \hat{X}^\mu \Big|_{\hat{\Sigma}} = \chi^\mu \end{array}\right\} \tag{3.5}$$

where $\hat{n}^\alpha$ is the field of unit normals to $\hat{\Sigma}$ and $R^\mu{}_\alpha$ is the Ricci tensor of g. We have $\hat{X}|_{\mathcal{D}(\Sigma)} = X$ by uniqueness of solutions of (3.5).)

Returning to the main argument, we may assume without loss of generality that the neighbourhood $\mathcal{O}$ of $\partial\Sigma$ is covered by normal geodesic coordinates based on $\partial\Sigma$:

$$\mathcal{O} = \{(q, t, x) : q \in \partial\Sigma, (t, x) \in B(\epsilon) \subset \mathbb{R}^2\}$$

for some $\epsilon > 0$, where $B(\epsilon)$ is a coordinate ball of radius ϵ. We have $\partial\Sigma \cap \mathcal{O} = \{(q, t, x) : t = x = 0\}$, and we can also assume that $\overline{\mathcal{O}}$ is a compact subset of M. For $p \in \mathcal{O}$ and $s \in (\hat{t}_-(p), \hat{t}_+(p))$ let $\hat{\phi}_s(p)$ be the orbit of $\hat{X}$ through p. There exists $\epsilon > 0$ such that $\hat{t}_+(p)|_{\mathcal{O}} \geq \epsilon$, $\hat{t}_-(p)|_{\mathcal{O}} \leq -\epsilon$. Consider† $p \in \mathcal{O} \cap \mathcal{D}(\Sigma)$, thus $p = (q, t, x)$, with $q \in \partial\Sigma$, $(t, x) \in B(\epsilon)$; changing x to $-x$ if necessary we also have $|t| < x$. By construction of the coordinates (q, t, x) the straight lines $q = q_0$, $t = \alpha s$, $x = \beta s$, $\alpha, \beta \in \mathbb{R}$, are affinely parametrized geodesics. Now for $|s| \leq \epsilon$ $\hat{\phi}_s : \mathcal{O} \cap \mathcal{D}(\Sigma) \to M$ are isometries, hence in $\mathcal{O} \cap \mathcal{D}(\Sigma)$ the maps $\hat{\phi}_s$ carry geodesics into geodesics and preserve affine parametrization. It follows that the $\hat{\phi}_s$s must be of the form

$$\mathcal{O} \cap \mathcal{D}(\Sigma) \ni (q, x^\mu) \to \phi_s(q, x^\mu) = \hat{\phi}_s(q, x^\mu) = (\psi_s(q), \Lambda(s, q)^\mu{}_\nu x^\nu)$$

† The argument that follows is essentially due to R Wald.

for some map $\psi_s : \partial\Sigma \to \partial\Sigma$, where we have set $x^\mu = (t, x)$ and where $\Lambda(s, q)$ is a Lorentz boost. Consequently, we can find $0 < \delta \le \epsilon$ and a conditionally compact neighbourhood $\mathcal{U}$ of $\partial\Sigma, \mathcal{U} \subset \mathcal{O}$, such that for all $p \in \mathcal{U} \cap \Sigma$ and for $s \in [-\delta, \delta]$ we have $\phi_s(p) \in \mathcal{D}(\Sigma)$. The result follows now from the arguments of the proof of parts 1 and 2 of this corollary. □

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