IC/84/4

INTERNATIONAL CENTRE FOR THEORETICAL PHYSIC

THE ISOMETRY GROUP AND KILLING SPINORS
FOR THE pp WAVE SPACE TIME IN D = 11 SUPERGRAVITY

P.T. Chruściel

and

J. Kowalski-Glikman

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1984 MIRAMARE-TRIES

IC/84/47

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

THE ISOMETRY GROUP AND KILLING SPINORS FOR THE pp WAVE SPACE TIME IN D = 11 SUPERGRAVITY *

P.T. Chruściel **
International Centre for Theoretical Physics, Trieste, Italy,
and
Institute for Theoretical Physics, Polish Academy of Sciences,
Warsaw, Poland,

and

J. Kowalski-Glikman ***
International Centre for Theoretical Physics, Trieste, Italy,
and
Institute of Theoretical Physics, University of Warsaw,
Warsaw, Poland.

ABSTRACT

We study the isometry group of the pp wave solution of D = 11 supergravity found in Ref.1. We also find explicitly the Killing spinors in the pp wave space time.

MIRAMARE - TRIESTE
April 1984

* To be submitted for publication.

** Present address: Département de Mécanique, Université Pierre et Marie Curie, 4 pl., Jussieu, Tours 65-66, 75005 Paris, France.

*** Work supported in part by Polish Research Project MR 1 7.
Permanent address: Institute of Theoretical Physics, University of Warsaw, ul. Hoza 69, 00-681 Warszawa, Poland.

I. INTRODUCTION

The first - and the most important - step in building a supersymmetr Kaluza-Klein model is to formulate conditions that the eleven-dimensional background space-time should satisfy. Between all possible background configurations a privileged role is played by these ones which can be considere vacuum states. These background configurations must certainly be pseudo-Riemannian eleven-dimensional manifolds with a vanishing Rarita-Schwinger field. In order to describe a vacuum state these backgrounds should have ze energy in a quantum sense, from which it follows that all supersymmetries are unbroken in a quantum sense (i.e. the quantum supersymmetry operator has zero eigenvalue on these states). We shall assume that this is equivalent to the existence of a maximal (namely 32) number of independent solutions of the Killing spinor equation in the classical background space time

$$\left[D_M(\omega) + i\alpha\left(\Gamma_M{}^{NPQR} - 8\delta_M^N \Gamma^{PQR}\right)F_{NPQR}\right]\epsilon = 0,$$
$$D_M(\omega) = \partial_M + \tfrac{1}{4}\omega_M{}^{AB}\Gamma_{AB},$$
$$\Gamma_{AB} = \tfrac{1}{2}[\Gamma_A, \Gamma_B], \quad \alpha = \frac{\sqrt{2}}{288}. \tag{1.1}$$

It was shown in [2] that the integrability conditions of (1.1) imply that the background satisfies the classical field equations. In [1] all the classical field configurations satisfying these integrability conditions were found. It was shown that apart from the trivial Riemann flat space-times and the well-known Freund-Rubin solutions, these conditions allow for only one more family of solutions, namely the pp wave space-times, the metric being of the form (there is an error in coefficients in the metric in Ref. 1):

$$ds^2 = -\sum_{i=2}^{4}(dx^i)^2 - \sum_{a=5}^{11}(dx^a)^2 + 2du\,dv + 2G\,du^2$$
$$G = \frac{f^2}{9}\left(\sum_i x^{i\,2} + \frac{1}{4}\sum_a x^{a\,2}\right) \tag{1.2}$$

and the four index electromagnetic field taking the form (in above coordinate

$$F_{uijk} = \xi \epsilon_{ijk} \, , \quad \xi = \text{const.} \tag{1.3}$$

Although the pp-wave solutions do not have a natural decomposition into a product of a compact internal space and a four-dimensional space-time, such a decomposition can be done by writing the metric (1.2) in spherical coordinates:

$$ds^2 = \frac{2\xi^2}{g}(r_1^2 + \tfrac{1}{4} r_2^2)du^2 + 2du\,dv + - r_1^2 d\Omega_2^2 - r_2^2 d\Omega_5^2 \tag{1.4}$$

where $d\Omega_n^2$ is the standard metric on S^n .

This space time can be considered as a product $S^2 \times S^5 \times R^4$, with the radii of spheres depending upon the physical space-time $R^4 = (u,v,r_1,r_2)$ coordinates. Such a compactification introduces a coordinate singularity into the internal space metric, but the space-time metric remains perfectly regular. The theory derived by dimensional reduction may exhibit interesting features due to an r-dependence of coupling constants of Yang-Mills fields.

In this paper we explicitly find the Killing spinors and study the geometric structure of the pp wave type solution.

II. KILLING SPINORS IN pp WAVE SPACE TIMES

The Killing spinor equation takes a particularly simple form in a null vielbein defined as follows:

$$\begin{aligned} \Theta^0 &= du \\ \Theta^1 &= dv + G du \\ \Theta^\alpha &= dx^\kappa \; ; \; \alpha = i, a \, . \end{aligned} \tag{2.1}$$

From (2.1) we have

$$ds^2 = \eta_{AB} \Theta^A \Theta^B , \tag{2.2}$$

with

$$\eta_{\alpha\beta} = -\delta_{\alpha\beta} , \; \eta_{01} = 1 , \; \eta_{00} = \eta_{11} = 0 . \tag{2.3}$$

(2.3) leads to

$$\Gamma^{0\,2} = \Gamma^{1\,2} = 0 \, , \quad \Gamma^0\Gamma^1 + \Gamma^1\Gamma^0 = 2 \tag{2.4}$$

Relations (2.4) will be eseential in solving the Killing spinor equation. From (2.1) and (2.2) one easily evaluates the connection coefficients

$$\begin{aligned} \omega^{AB} &= \omega^{AB}{}_{\mu}\, dx^\mu \\ \omega^{1\alpha} &= -\omega^{1}{}_{\alpha} = -\omega_{0\alpha} = -\omega^{\alpha}{}_{0} = -G_{,\alpha}\, du \end{aligned} \tag{2.5}$$

Inserting (1.2) and (2.5) into (1.1) leads to the following set of equations for ϵ :

$$\epsilon_{,a} = i\omega \Gamma_A{}^{234}\Gamma^0\epsilon \, , \quad \omega = \frac{\sqrt{2}}{12}\xi , \tag{2.6}$$

$$\epsilon_{,i} = -i\omega\, \epsilon_{ijk} x^i \Gamma^{jk}\Gamma^0 \epsilon \, , \tag{2.7}$$

$$\begin{aligned} \epsilon_{,u} &= 3i\omega\Gamma^{234}\epsilon + B(x,y)\Gamma^0\epsilon , \\ B &= \tfrac{1}{2} G_{,\alpha}\Gamma^\alpha + i\omega\,\Gamma^{1234} , \end{aligned} \tag{2.8}$$

$$\epsilon_{,v} = 0 . \tag{2.9}$$

From (2.6) and (2.7) one easily obtains, using $\Gamma^{0^2} = 0$

$$\epsilon_{,\alpha\beta} = 0 , \tag{2.10}$$

from which it follows that

$$\epsilon = \overset{\circ}{\epsilon}_\alpha(u)\, x^\alpha + \overset{\circ}{\epsilon}(u). \tag{2.11}$$

Eqs.(2.6) and (2.7) imply that $\Gamma^0 \epsilon_{,a} = 0$ which implies also

$$\Gamma^0 \overset{\circ}{\epsilon}_\alpha = 0. \tag{2.12}$$

Making use of this fact one finds from (2.6) and (2.7)

$$\epsilon = \overset{\circ}{\epsilon} + A\Gamma^0 \overset{\circ}{\epsilon}, \tag{2.13}$$

with

$$A = -i\omega\left(\epsilon_{ijk}\, x^i \Gamma^{jk} - x^a \Gamma_a{}^{234}\right). \tag{2.14}$$

Inserting the expression (2.13) into Eq.(2.8) multiplied from the left by Γ^0 leads to

$$\Gamma^0 \overset{\circ}{\epsilon}_{,u} = -i\omega\, \Gamma^{234}\Gamma^0 \overset{\circ}{\epsilon}. \tag{2.15}$$

Therefore (2.13), (2.15) and (2.8) imply

$$\overset{\circ}{\epsilon}_{,u} = 3i\omega\,\Gamma^{234}\overset{\circ}{\epsilon} + (i\omega A\Gamma^{234} + 3i\omega\Gamma^{234}A + B)\Gamma^0 \overset{\circ}{\epsilon} \tag{2.16}$$

It is easily seen that the x^α-dependent terms in (2.16) cancel (this was in fact guaranteed by the fact that the integrability conditions of the Killing spinor equation are satisfied identically in the pp solution) and (2.16) reduces to

$$\overset{\circ}{\epsilon}_{,u} = i\omega\,\Gamma^{234}(3 - \Gamma^1\Gamma^0)\overset{\circ}{\epsilon} \tag{2.17}$$

This equation may be integrated explicitly to give

$$\overset{\circ}{\epsilon}(u) = \{\cos 3\omega u + i \sin 3\omega u\, \Gamma^{234} + \sin\omega u \sin 2\omega u\, \Gamma^1\Gamma^0 - i \sin\omega u \cos 2\omega u\, \Gamma^{234}\Gamma^1\Gamma^0\}\overset{\circ}{\eta} \tag{2.18}$$

where $\overset{\circ}{\eta} = \overset{\circ}{\epsilon}(u = 0)$. From (2.18) and (2.13) one obtains all the independent solutions of the Killing spinor equation, labelled by $\overset{\circ}{\eta}$.

The time dependence of the Killing spinors takes a very simple form, if one decomposes the space of spinors with the help of the projection operators $\frac{1}{2}\Gamma^1\Gamma^0$ and $\frac{1}{2}\Gamma^0\Gamma^1$:

$$\epsilon = \epsilon_0 + \epsilon_1,$$
$$\epsilon_0 = \tfrac{1}{2}\Gamma^0\Gamma^1\epsilon \iff \Gamma^0\epsilon_0 = 0,$$
$$\epsilon_1 = \tfrac{1}{2}\Gamma^1\Gamma^0\epsilon \iff \Gamma^1\epsilon_1 = 0. \tag{2.19}$$

From (2.19), (2.18) and (2.13) one deduces

$$\epsilon_1(u,x) = \overset{\circ}{\epsilon}_1(u) = (\cos\omega u + i\sin\omega u\,\Gamma^{234})\overset{\circ}{\eta}_1,$$
$$\epsilon_0(u,x) = \overset{\circ}{\epsilon}_0(u) + A\Gamma^0\overset{\circ}{\epsilon}_1(u),$$
$$\overset{\circ}{\epsilon}_0(u) = (\cos 3\omega u + i\sin 3\omega u\,\Gamma^{234})\overset{\circ}{\eta}_0 \tag{2.20}$$

From (2.20) one concludes that all the spinors are u-dependent, and they are x^α-independent if and only if $\overset{\circ}{\eta}_1 = 0$.

III. THE ISOMETRY GROUP

The Killing equations can easily be integrated, and the Killing vector algebra can be spanned by the following set of vector fields:

$$\frac{\partial}{\partial u},\ \frac{\partial}{\partial v},\ \Sigma_\alpha{}^\beta x^\alpha \frac{\partial}{\partial x^\beta},$$
$$\xi_-^\alpha = \cos(\epsilon_\alpha\omega u)\frac{\partial}{\partial x^\alpha} - \epsilon_\alpha\omega\sin(\epsilon_\alpha\omega u)\, x^\alpha\frac{\partial}{\partial v},$$
$$\xi_+^\alpha = \sin(\epsilon_\alpha\omega u)\frac{\partial}{\partial x^\alpha} + \epsilon_\alpha\omega\cos(\epsilon_\alpha\omega u)\, x^\alpha\frac{\partial}{\partial v},$$
$$\epsilon_i = 4,\ \epsilon_a = 2 \tag{3.1}$$

$\Sigma_\alpha{}^\beta$ are $SO(3) \times SO(6)$ matrices that preserve the quadratic form $x^{i2} + \frac{1}{4}x^{a2}$ and there is no summation in formulas where ϵ_α appears. The non-trivial commutation relations are

$$\left[\frac{\partial}{\partial u}, \xi^\alpha_\pm\right] = \pm \varepsilon_\alpha \omega\, \xi^\alpha_\mp ,$$
$$\left[\xi^\alpha_-, \xi^\beta_+\right] = \varepsilon_\alpha \omega\, \delta^{\alpha\beta} \frac{\partial}{\partial v},$$
$$\left[\Sigma_\alpha{}^\beta x^\alpha \frac{\partial}{\partial x^\beta}, \xi^\gamma_\pm\right] = \Sigma_\beta{}^\gamma\, \xi^\beta_\pm ,$$
$$\left[\Sigma_\alpha{}^\beta x^\alpha \frac{\partial}{\partial x^\beta}, \Sigma'_\mu{}^\nu x^\mu \frac{\partial}{\partial x^\nu}\right] = \left([\Sigma, \Sigma']\right)_\alpha{}^\beta x^\alpha \frac{\partial}{\partial x^\beta}. \tag{3.2}$$

The radical of the algebra is a subalgebra

$$\mathcal{F} = \mathcal{F}_+ + \mathcal{F}_- + \left\{\frac{\partial}{\partial v}\right\} \qquad \text{(vector space sum)} \tag{3.3}$$

with $\mathcal{F}_+ = \text{span}\,\{\xi^\alpha_+\}$ (span denotes the real vector space spanned by the set of generators). This leads to the following Levi-Malcev decomposition of the isometry group G: *)

$$G = \left(SO(3) \otimes SO(6) \otimes SO(2)\right) \circledS F, \tag{3.4}$$

where $\otimes$ denotes a direct product of groups, $\circledS$ a semidirect product, and F is a group having $\mathcal{F}$ as Lie algebra. The semidirect product is obtained through:

a) the natural vector representation of SO(3) $\otimes$ SO(6) acting on the vector index α of ξ^α_+ and ξ^α_-,

b) the natural vector representation of SO(2) rotating the planes span $\{\xi^\alpha_+, \xi^\alpha_-\}$, independently for each α.

It is interesting that the group F has also a Levi-Malcev type decomposition of the following form:

$$F = (\mathbb{R}^9, +) \circledS (\mathbb{R}^{10}, +), \tag{3.5}$$

where for example

$$\mathbb{R}^9 = \text{span}\,\{\xi^\alpha_-\},$$
$$\mathbb{R}^{10} = \text{span}\,\left\{\xi^\alpha_+, \frac{\partial}{\partial v}\right\},$$

* The SO(2) factor will be replaced by $(\mathbb{R},+)$ if one does not identify the points u with $u + \frac{\pi}{\omega}$ (as we implicitly did in this formula).

with the Abelian composition law. The semi-direct product structure in (3.5) is obtained by the following representation of $\mathbb{R}^9$ acting on $\mathbb{R}^{10}$:

$$\mathbb{R}^9 \ni x^\alpha \to T_{x^\alpha}(y^\alpha, v) = \left(y^\alpha, v - \omega \sum_\alpha \varepsilon_\alpha x^\alpha y^\alpha \right) \tag{3.6}$$

The only topological identifications which do not reduce the isometry group are the following:

a) v may be identified with v+b for any $b \neq 0$,

b) u may be identified with $u + \frac{\pi n}{\omega}$, $n \in \mathbb{Z}$.

The stability group of $(x^i = 0,\ x^a = 0,\ u = 0,\ v = 0)$ is the group

$$SO(3) \otimes SO(6) \circledS (\mathbb{R}^9, +),$$

where

$$(\mathbb{R}^9, +) = (\text{span}\,\{\xi^\alpha_-\},\ \text{Abelian composition law}),$$

therefore the pp wave space time (or any space time obtained from it by identifications preserving the isometry group, as discussed above) has the structure of the coset space

$$M = G / SO(3) \otimes SO(6) \circledS (\mathbb{R}^9, +).$$

The one-parameter subgroups generated by ξ^α_+ and ξ^α_- can easily be found, and this allows to find coordinates in which the metric is explicitly x^α independent. If we choose as a coordinate the group parameter along e.g. ξ^2_-, the corresponding coordinate transformation may be chosen to be

$$x^2 = \sin(4\omega u)\, x^{2'}$$
$$v = \omega \sin(8\omega u) |x^{2'}|^2 + v' \tag{3.7}$$

and it leads to the following metric:

$$ds^2 = 2du\,dv' + 2\frac{\xi^2}{9}\left[(x^3)^2 + (x^4)^2 + \frac{1}{4}\Sigma (x^a)^2\right] du^2 +$$
$$- \sin^2(4\omega u)(dx^{2'})^2 - \sum_{\alpha=3}^{10} (dx^\alpha)^2. \tag{3.8}$$

Such a transformation induces an apparent singularity of the metric at $4\omega u = n\pi$ (which is, of course, a coordinate artefact). It seems moreover, that a compactification of $x^{2'}$ with a period $\frac{\pi}{2\omega}$ cannot lead to a non-singular metric. Proceeding as in (3.7) with the remaining x^{α} coordinates, one can transform the metric to the form:

$$ds^2 = \sin^2(4\omega u)\sum(dx^i)^2 - \sin^2(2\omega u)\sum{}'(dx^a)^2 + 2du\,dv \tag{3.9}$$

IV. CONCLUSIONS

In the usual dimensional reduction process in supergravity the unbroken supersymmetries, at the D = 4 level, are determined by the number of internal space Killing spinors. Because of the mixing of internal and space-time coordinates it seems very likely that after dimensional reduction the theory will not possess any residual supersymmetry. This would be a desirable property of the pp wave space-time, since we expect that supersymmetry is broken in four-dimensional space-time at low energy scale.

It may be of some interest to mention that both maximally symmetric, non-trivial families of backgrounds (Freund-Rubin+pp waves) show the property which could be called "spontaneous time compactification". In these solutions a characteristic time length in which the metric is periodic appears, and which allows a natural compactification of the time-line to a circle. If one wants to take this observation seriously, this time compactification phenomenon could be a new physical prediction of D = 11 supergravity.

ACKNOWLEDGMENTS

The authors benefitted from discussions with B. Milewski, P. Morawiec and P. Spindel. They would also like to thank Professor Abdus Salam, the International Atomic Emergy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, where this work was performed. J.K-G is grateful to the ICTP, and P.T.C. to the French Government for financial support.

REFERENCES

[1] J. Kowalski-Glikman, Phys. Lett. B134, 194 (1984).

[2] B. Brian, F. Englert, B. de Wit and H. Nicolai, Phys. Lett. B124, 45 (1983).