Uniqueness of the Trautman-Bondi mass

Piotr T. Chruściel*
Département de Mathématiques, Faculté des Sciences, Université de Tours, Parc de Grandmont, F-37200 Tours, France

Jacek Jezierski†
Department of Mathematical Methods in Physics, University of Warsaw, ul. Hoża 74, 00-682 Warszawa, Poland

Malcolm A. H. MacCallum‡
School of Mathematical Sciences, Queen Mary and Westfield College, University of London, Mile End Road, London E1 4NS, United Kingdom

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It is shown that the only functionals, within a natural class, which are monotonic in time for all solutions of the vacuum Einstein equations admitting a smooth "piece" of conformal null infinity $\mathscr{I}$, are those depending on the metric only through a specific combination of the Bondi "mass aspect" and other next-to-leading order terms in the metric. Under the extra condition of passive BMS invariance, the unique such functional (up to a multiplicative factor) is the Trautman-Bondi energy. It is also shown that this energy remains well defined for a wide class of "polyhomogeneous" metrics. [S0556-2821(98)05916-5]

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I. INTRODUCTION

Consider a Lagrangian theory of fields ϕ^A defined on a manifold M with a Lagrange function density

$$\mathcal{L}=\mathcal{L}[\phi^A,\partial_\mu\phi^A,\ldots,\partial_{\mu_1}\ldots\partial_{\mu_k}\phi^A], \tag{1.1}$$

for some $k\in\mathbb{N}$, where ∂_μ denotes partial differentiation with respect to x^μ. Suppose further that there exists a function t on M such that M can be decomposed as $\mathbb{R}\times\Sigma$, where $\Sigma\equiv\{t=0\}$ is a hypersurface in M and the vector $\partial/\partial t$ is tangent to the $\mathbb{R}$ factor. The proof of the Noether theorem, as presented, e.g., in [1], Section 10.1, shows that the vector density

$$E^\lambda=X^\mu\sum_{l=0}^{k-1}\phi^A{}_{,\alpha_1\ldots\alpha_l\mu}\sum_{j=0}^{k-l-1}(-1)^j\partial_{\gamma_1}\ldots\partial_{\gamma_j}\times\left(\frac{\partial\mathcal{L}}{\partial\phi^A{}_{,\lambda\alpha_1\ldots\alpha_l\gamma_1\ldots\gamma_j}}\right)-\mathcal{L}X^\lambda \tag{1.2}$$

has vanishing divergence, $E^\lambda{}_{,\lambda}=0$, when the fields ϕ_A are sufficiently smooth and satisfy the variational equations associated with a sufficiently smooth $\mathcal{L}$ (cf. also [2]). [This is in any case easily seen by calculating the divergence of the right-hand side of Eq. (1.2).] Here $\phi^A{}_{,\alpha_1\ldots\alpha_l}=\partial_{\alpha_1}\ldots\partial_{\alpha_l}\phi^A$, and $X^\mu\partial_\mu=\partial_t$. In first order theories, that is theories in which $\mathcal{L}$ depends only upon ϕ^A and its first derivatives, it is customary to define the total energy associated with the hypersurface Σ by the formula

$$E(\Sigma)=\int_\Sigma E^\lambda dS_\lambda, \tag{1.3}$$

with $dS_\lambda=\partial_\lambda\lrcorner dx^0\wedge\ldots dx^3$, where $\lrcorner$ denotes contraction.[1] By extrapolation one can also use (1.3) to define an "energy" for higher order theories. Because of its origin, the right-hand side of Eq. (1.3) will be called the *Noether energy* of Σ, associated with a Lagrange function $\mathcal{L}$ and with the vector field X. Now it is well known that the addition to $\mathcal{L}$ of a functional of the form

$$\partial_\lambda(Y^\lambda[\phi^A,\partial_\alpha\phi^A,\ldots,\partial_{\alpha_1}\ldots\partial_{a_{k-1}}\phi^A]), \tag{1.4}$$

where k is as in (1.1), does not affect the field equations.[2] We show in Appendix E that such a change of the Lagrange function will change $E(\Sigma)$ by a boundary integral:

$$E(\Sigma)\to\hat{E}(\Sigma)=E(\Sigma)+\int_{\partial\Sigma}\Delta E^{\mu\lambda}dS_{\mu\lambda}, \tag{1.5}$$

where $S_{\alpha\beta}=\partial_\alpha\lrcorner\partial_\beta\lrcorner dx^0\wedge\ldots\wedge dx^3$, with $\Delta E^{\mu\lambda}$ given by Eq. (E6). If $\partial\Sigma$ is a "sphere at infinity" the integral over $\partial\Sigma$ has of course to be understood by a limiting process. Unless the boundary conditions at $\partial\Sigma$ force all such boundary integrals to give a zero contribution, if one wants to define energy using this framework one has to have a criterion for choosing a "best" functional, within the class of all functionals ob-

*Email address: Chrusciel@Univ-Tours.Fr
†Email address: jjacekj@fuw.edu.pl
‡Email address: M.A.H.MacCallum@qmw.ac.uk

[1] We use the conventions that $\partial_0\lrcorner dx^0\wedge\ldots dx^3=dx^1\wedge dx^2\wedge dx^3$, $\partial_1\lrcorner dx^1\wedge dx^2\wedge dx^3=dx^2\wedge dx^3$, etc.

[2] Here we adopt the standard point of view, that the field equations are obtained by requiring the action to be stationary with respect to all compactly supported variations (cf. e.g. [3] for a discussion of problems that might arise when this requirement is not enforced).

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tainable in this way. As discussed in more detail in Sec. II, the vanishing of such boundary integrals will not occur in several cases of interest.

Now the concept of energy plays a most important role in the context of fields which are asymptotically flat in lightlike directions. An appropriate mathematical framework here is that of spacelike hypersurfaces which intersect the future null infinity $\mathscr{I}^+$ in a compact cross-section K. For such field configurations it is widely accepted that the ''correct'' definition of energy of a gravitating system is that given by Freud [4], Trautman [5,6], Bondi *et al.* [7], and Sachs [8], which henceforth will be called the Trautman-Bondi (TB) energy. [Because of the difficulty of accessing Refs. [5,6] we have included an Appendix (Appendix A) which describes those results of [5,6] which are related to the problem at hand. This Appendix, together with the date of publication of [5], should make it clear why we are convinced that the name of Trautman should be associated with the notion of mass in the radiating regime in general relativity.] There have been various attempts to exhibit a privileged role of that expression as compared with many alternative ones ([9–20], to quote a few), but the papers known to us have failed, for reasons sometimes closely related to the ones described above, to give a completely unambiguous prescription about how to define energy at $\mathscr{I}$. (We make some more comments about that in Sec. II, cf. also [21].) In this paper we wish to point out that the TB energy is, up to a multiplicative constant $\alpha \in \mathbb{R}$, the *only functional of the gravitational field*, in a certain natural class of functionals, which is *monotonic in time for all vacuum field configurations* which admit a (piece of) a smooth null infinity $\mathscr{I}^+$.

We shall also consider a second, somewhat larger, class of functionals, which contains Hamiltonians that arise in an appropriate symplectic framework. [It will be seen below that the functionals one obtains from the integrals (1.3) are quadratic polynomials of the appropriate Bondi functions and their derivatives; there is no reason for the Hamiltonians to satisfy this restriction.] In that larger class we describe all monotone functionals and then among these the further requirement of passive supertranslation invariance also leads to the TB energy as the unique expression. The symplectic framework which is appropriate in the context of radiating fields will be described elsewhere.

It is natural to ask why the Newman-Penrose constants of motion [22], or the logarithmic constants of motion of [23], do not occur in the conclusions of Theorem IV.1. These quantities are excluded by the hypothesis that the boundary integrand $H^{\alpha\beta}$ which appears in the integrals we consider depends on the coordinates only through the fields. The Newman-Penrose constants could be obtained as integrals of the form (1.2) (cf. e.g. [24]) if explicit r^2 factors were allowed in $H^{\alpha\beta}$. Similarly logarithmic constants could occur as integrals of the form (1.2) if explicit $1/\ln r$ or $r^{+i}\ln^{-j} r$ factors were allowed there.

This paper is organized as follows: In Sec. II we review some results about ''energy expressions'' in general relativity, and comment on nonuniqueness of those. In Sec. III we find all functionals of the fields induced on $\mathscr{I}$ by the metric which are monotonic in retarded time, in a large class of natural functionals. In Sec. IV we analyze those monotonic functionals which are invariant under passive Bondi-Metzner-Sachs (BMS) supertranslations, and prove our claim about uniqueness of the Trautman-Bondi mass. In Sec. V we give a supertranslation-invariant formula for the Trautman-Bondi momentum, for general cuts of $\mathscr{I}$. In Sec. VI we consider the question of convergence of the Freud superpotential to the Trautman-Bondi mass for space-times with a polyhomogeneous $\mathscr{I}$. Remarkably, we find that because of some integral cancellations the Freud integral always converges to a ''generalized Trautman-Bondi'' mass, even for metrics which are polyhomogeneous of order 1 (cf. Sec. VI for definitions). In Sec. VII we briefly discuss the potential extensions of our results to a Hamiltonian setting. An appendix gives a very short review of Trautman's contribution to the notion of energy for radiating metrics, while the remaining four appendices contain some technical results needed in the body of the paper.

II. NONUNIQUENESS OF THE NOETHER ENERGY FOR GRAVITATING SYSTEMS

As an example of applicability of Eq. (1.5), consider a scalar field ϕ in the Minkowski space-time, with $\Sigma=\{t=0\}$. Assume that ϕ satisfies the rather strong fall-off conditions

$$\text{for } (t,x)\in\Sigma \text{ we have } \partial_{\alpha_1}\ldots\partial_{\alpha_j}\phi = o(r^{-2}),$$
$$0\leqslant j\leqslant 2(k-1), \tag{2.1}$$

where k is the integer appearing in (1.1). In this case the boundary integral in (1.5) will vanish for all smooth Y^μ's, as considered in Eq. (1.4). This shows that Eq. (1.3) leads to a well-defined notion of energy on this space of fields (whatever the Lagrange function $\mathcal{L}$), as long as the volume integral there converges. (That will be the case if, e.g., $\mathcal{L}$ has no linear terms in ϕ and its derivatives.)

Consider, next, the same scalar field in Minkowski space-time, with Σ being a hyperboloid, $t=\sqrt{1+x^2+y^2+z^2}$. Suppose further that $\mathcal{L}=\nabla^\mu\phi\nabla_\mu\phi$, so that the field equations read

$$\Box\phi=0. \tag{2.2}$$

In that case the imposition of the boundary condition (2.1) does not seem to be of interest, as such boundary conditions would be incompatible with the asymptotic behavior of those solutions of Eq. (2.2) which are obtained by evolving compactly supported data on $\{t=0\}$. Thus, even for scalar fields in Minkowski space-time, a supplementary condition singling out a preferred E^λ is needed.

Now for various field theories on the Minkowski background, including the scalar field, one can impose some further conditions on E^λ which render it unique [25,13,14]. The extension of that analysis to the gravitational field carried on in [13,14] also leads to a unique E^λ (namely the one obtained from the so-called ''Einstein energy-momentum pseudotensor''), within the class of objects considered. While this is certainly an interesting observation, the hypotheses made in

that last paper are, however, much more restrictive than is desirable. It seems therefore that for gravitating systems another approach is needed. Let us recall how the ''Noether charge'' formalism described in the Introduction works in that case. There exist various variational approaches to general relativity, and depending upon the point of view adopted one finds the following.

(1) Let $\mathcal{L}=\sqrt{|\det g|}R/16\pi$, where the Ricci scalar is considered as a functional of the metric field $g_{\mu\nu}$, a symmetric connection $\Gamma^{\alpha}_{\beta\gamma}$, and its first derivatives. In that case [26] one finds

$$E(\Sigma)=\frac{1}{16\pi}\int_{\Sigma}\nabla_{\mu}\nabla^{[\mu}X^{\lambda]}\sqrt{|\det g|}dS_{\lambda}$$

$$=\frac{1}{8\pi}\int_{\partial\Sigma}\nabla^{[\mu}X^{\lambda]}\sqrt{|\det g|}dS_{\mu\lambda}. \qquad (2.3)$$

This integral is known as the Komar energy, except that (2.3) is actually half of the expression given by Komar [27].

(2) Let $\mathcal{L}=\sqrt{|\det R_{\mu\nu}|}/\lambda$, where the Ricci tensor is considered as a functional of a symmetric connection $\Gamma^{\alpha}_{\beta\gamma}$ and its first derivatives, and λ is a constant. The variational equations for such a theory are the Einstein equations with a cosmological constant [28]. The Noether energy gives again [28] the Komar integral (2.3).

(3) Let $\mathcal{L}=\sqrt{|\det g|}R/16\pi$, where the Ricci scalar is considered as a functional of the metric field $g_{\mu\nu}$ and its first and second derivatives. In that case the value of $E(\Sigma)$ is given again [29] by the Komar integral (2.3) (with a ''wrong'' 1/2 multiplicative factor).

(4) Let $\mathcal{L}=\mathcal{L}(g_{\mu\nu},g_{\mu\nu,\sigma})$ be the Einstein Lagrange function [30], which is obtained by adding an appropriate divergence to the Hilbert Lagrange function $\sqrt{|\det g|}R/16\pi$. In that case one obtains [4]

$$E(\Sigma)=\int_{\partial\Sigma}H^{\mu\nu}dS_{\mu\nu}, \qquad (2.4)$$

where $H^{\mu\nu}$ is the ''Freud superpotential'' for the ''Einstein energy-momentum pseudotensor,'' cf. Eq. (6.1) below.
Yet another approach, leading to a different energy expression, can be found in [31].

Consider first initial data for, say vacuum, Einstein equations satisfying the usual fall-off conditions at spatial infinity;

$$g_{\mu\nu}-\eta_{\mu\nu}=O(r^{-1}),\quad \partial_{\sigma}g_{\mu\nu}=O(r^{-2}). \qquad (2.5)$$

In that case both the integrals (2.3) and (2.4) converge. When the integral over $\partial\Sigma$ in (2.3) is evaluated on a ''two-sphere at infinity'' in Schwarzschild space-time one obtains $m/2$. On the other hand, under the asymptotic conditions (2.5) the integral (2.4) coincides with the standard Arnowitt-Deser-Misner (ADM) expression for energy, and gives m for that same sphere in Schwarzschild space-time.

Under the asymptotic conditions (2.5), a way to obtain a unique expression is given by the symplectic formalism. Namely, one can require that $E(\Sigma)$ be a Hamiltonian on an appropriately-defined phase space (cf. e.g. [32–35]). This requirement, together with the normalization condition that the Hamiltonian vanishes on Minkowski space-time, uniquely singles out the Freud-ADM energy as the ''correct'' global energy for general relativistic initial data sets which satisfy the ''spatial infinity asymptotic flatness conditions.'' Thus the Hamiltonian analysis gives a rather satisfactory way of singling out an energy expression at spatial infinity.

Consider, next, hypersurfaces Σ which extend to $\mathscr{I}$ and intersect $\mathscr{I}$ transversally. There have been attempts to use symplectic methods to define energy in this context [9–11] (see also [36–38]). In particular, the analysis of [9–11] shows that, under appropriate assumptions, the integral of the time-derivative of the TB energy over the retarded time gives a Hamiltonian with respect to a proposed symplectic structure. This does not allow one to extract the integrand itself from the expression for the Hamiltonian in any unambiguous way, for reasons somewhat analogous to those described in the Introduction. Moreover in those papers one has to assume various decay properties of the fields on $\mathscr{I}$ for large absolute values of the retarded time, which have not been established so far. Finally, as the symplectic structure considered in [9–11] has a perhaps less universally accepted status than the one considered on standard asymptotically flat hypersurfaces, one should perhaps also face the question of uniqueness of the symplectic structures involved. For all those reasons we conclude that the framework of [9–11] fails to demonstrate uniqueness of the TB mass.

III. MONOTONIC FUNCTIONALS

From now on, we shall consider metrics $g_{\alpha\beta}$ defined on appropriately large subsets of $\mathbb{R}^4$, but not necessarily globally defined on $\mathbb{R}^4$, and satisfying Einstein's equations near $\mathscr{I}$. We shall examine a class of functionals which includes all the cases discussed in Sec. II, and in particular all functionals differing from the Hilbert Lagrangian by a divergence. These functionals have the form

$$H[u_0,g]=\lim_{\rho\to\infty}\int_{S(t=u_0+\rho,\rho)}H^{\alpha\beta}[g]dS_{\alpha\beta},$$

$$dS_{\alpha\beta}=\partial_{\alpha}\lrcorner\partial_{\beta}\lrcorner dx^0\wedge\ldots\wedge dx^3, \qquad (3.1)$$

where

$$H^{\alpha\beta}[g](x)\equiv H^{\alpha\beta}(g_{\mu\nu}(x),\partial_{\alpha}g_{\mu\nu}(x),\ldots,\partial_{\alpha_1}\ldots\partial_{\alpha_k}g_{\mu\nu}(x)). \qquad (3.2)$$

for some $k\in\mathbb{N}$, and $H^{\alpha\beta}$ is a twice continuously differentiable function of its arguments. Here $S(\tau,\rho)$ denotes a sphere $r\equiv\sqrt{(x^1)^2+(x^2)^2+(x^3)^2}=\rho$, $t\equiv x^0=\tau$. The metrics $g_{\alpha\beta}$ will be assumed to satisfy the standard fall-off conditions corresponding to asymptotic flatness at null infinity. More precisely, consider a space-time (M,g) which admits a conformal completion (which in this section we consider to be smooth) in the following sense: there exists a manifold with boundary $(\bar{M},\bar{g})$, a diffeomorphic embedding $\Phi:M\to\bar{M}\backslash\partial\bar{M}$, and a smooth function Ω on $\bar{M}$ such that

$\Phi^*(\Omega^{-2}\bar{g})=g$. We shall also assume that $\Omega_{|\partial\bar{M}}=0$, that $d\Omega$ is nowhere vanishing on $\partial\bar{M}$, and that $\mathscr{I}\equiv\partial\bar{M}$ is diffeomorphic to $I\times S^2$ where I is an interval (possibly but not necessarily equal to $\mathbb{R}$). By a standard construction we can introduce Bondi coordinates near $\mathscr{I}$ (cf. e.g., [19] or [23]) so that we have

$$ds^2=-\frac{Ve^{2\beta}}{r}du^2-2e^{2\beta}du\,dr+r^2h_{ab}(dx^a-U^adu)(dx^b-U^bdu), \tag{3.3}$$

with $x^a=(\theta,\phi)$. We can introduce quasi-Minkowskian coordinates by setting

$$u=t-r,\quad x=r\sin\theta\cos\phi,\quad y=r\sin\theta\sin\phi,\quad z=r\cos\theta. \tag{3.4}$$

We shall consider only vacuum metrics; recall that this implies the following behavior of h_{ab}, β, U^a, and V ([24,7,8]):

$$\begin{aligned}
h_{ab}&=\breve{h}_{ab}\left(1+\frac{1}{4r^2}\chi^{cd}\chi_{cd}\right)+\frac{\chi_{ab}(v)}{r}+O(r^{-3}),\\
\beta&=-\frac{1}{32}\frac{\breve{h}^{ab}\breve{h}^{cd}\chi_{ac}\chi_{bd}}{r^2}+O(r^{-3}),\\
U^a&=-\frac{1}{2}\frac{\mathcal{D}_b\chi^{ab}}{r^2}+\frac{32N^a(v)+\mathcal{D}^a(\chi^{cd}\chi_{cd})+8\chi^a{}_b\mathcal{D}_c\chi^{bc}}{16r^3}+O(r^{-4}),\\
V&=r-2M(v)+\frac{\chi^{cd}\chi_{cd}+4\mathcal{D}_b\chi^{ab}\mathcal{D}^c\chi_{ac}-16\mathcal{D}_aN^a}{16r}+O(r^{-2}).
\end{aligned} \tag{3.5}$$

Here $(v)\equiv(u,x^a)$ and $\breve{h}_{ab}dx^adx^b=d\theta^2+\sin^2\theta\,d\phi^2$; $\mathcal{D}_a$ is the covariant derivative operator defined by $\breve{h}_{ab}$. Indices a, b, etc., take values 2 and 3, and are raised and lowered with $\breve{h}^{ab}$. The tensor field χ_{ab} satisfies the condition

$$\breve{h}^{ab}\chi_{ab}=0, \tag{3.6}$$

and no other conditions are imposed[3] on $\chi_{ab}(v)$ by the vacuum Einstein equations. The functions M and N^a satisfy the following equations:

$$\begin{aligned}
\frac{\partial M}{\partial u}&=-\tfrac{1}{8}\breve{h}^{ac}\breve{h}^{bd}\dot{\chi}_{ab}\dot{\chi}_{cd}+\tfrac{1}{4}\mathcal{D}_a\mathcal{D}_b\dot{\chi}^{ab},\\
3\frac{\partial N^a}{\partial u}&=-\mathcal{D}^aM+\tfrac{1}{4}\epsilon^{ab}\mathcal{D}_b\lambda-K^a,\\
K^a&\equiv\tfrac{3}{4}\chi^a{}_b\mathcal{D}_c\dot{\chi}^{bc}+\tfrac{1}{4}\dot{\chi}^{cd}\mathcal{D}_d\chi^a{}_c,\\
\lambda&\equiv\breve{h}^{bd}\epsilon^{ac}\mathcal{D}_c\mathcal{D}_b\chi_{da},
\end{aligned} \tag{3.7}$$

where $\dot{\chi}:=\partial_u\chi$. Here $\epsilon_{ab}=\partial_a\lrcorner\partial_b\lrcorner d^2\mu$ where $d^2\mu=\sin\theta\,d\theta\wedge d\phi=\frac{1}{2}\epsilon_{ab}dx^a\wedge dx^b$ is the standard volume form on S^2. If we fix some $u_0\in I$, then the Einstein equations do not impose[3] any restrictions on the function $M(u_0,\theta,\phi)$ and the vector field $N^a(u_0,\theta,\phi)$ on S^2.

Equation (3.5) shows that in the coordinate system (3.4) the metric (3.3) is of the form

$$g_{\mu\nu}=g^0_{\mu\nu}+\frac{g^1_{\mu\nu}(v)}{r}+\frac{g^2_{\mu\nu}(v)}{r^2}+O(r^{-3}), \tag{3.8}$$

with obvious analogous expansions holding for the various derivatives of $g_{\mu\nu}$ when an appropriate expansion for the derivatives of h_{ab} is assumed. Here $g^0_{\mu\nu}=\mathrm{diag}(-1,1,1,1)$. We can now insert a metric of the form (3.8) into a functional of the form (3.1), and as a further restriction we shall require that H has a finite numerical value for all fields $g_{\mu\nu}$ of the type described above. Our hypothesis of differentiability of $H^{\alpha\beta}$ allows us to Taylor expand $H^{\alpha\beta}$ to order 2 in terms of powers of $g_{\mu\nu}-g^0_{\mu\nu}$, $\partial_\sigma(g_{\mu\nu}-g^0_{\mu\nu})=\partial_\sigma g_{\mu\nu}$, etc., about $g_{\mu\nu}=g^0_{\mu\nu}$. Note that by (3.2) the $H^{\alpha\beta}[g^0_{\mu\nu},0,\ldots,0]$ are constants which are either zero or integrate to zero in (3.1) [otherwise the limit in (3.1) would be infinite], so that

$$H[u,g^0_{\mu\nu}]=0\quad\forall u\in I.$$

The $1/r$ terms in $g_{\mu\nu}$ and its v derivatives will give at most a quadratic contribution to H, and the $1/r^2$ terms at most a linear one, while the remainder terms in the Taylor expansion of $H^{\alpha\beta}$ will contribute nothing in the limit $r\to\infty$. It follows that H can be written in the form

$$H=\int_{S^2}h[M,M^{(1)},\ldots,M^{(k)},N^a,N^{a(1)},\ldots,N^{a(k)},\chi_{ab},\chi^{(1)}_{ab},\ldots\chi^{(k)}_{ab},\theta,\phi]d^2\mu. \tag{3.9}$$

Here the addition of a superscript (l) to a quantity denotes the l-th u-derivative of that quantity. The square brackets around the arguments of h are meant to emphasize the fact that h is not a function but a local functional of the fields which is a differentiable function of M, $\partial_\alpha M,\ldots$, $\partial_{\alpha_1}\ldots\partial_{\alpha_k}M$, $M^{(1)}$, $\partial_\alpha M^{(1)},\ldots$, $\partial_{\alpha_1}\ldots\partial_{\alpha_k}M^{(1)},\ldots$, etc., for some finite number of derivatives in directions tangent to S^2. Note that for functionals (3.1) the dependence of H on N^a, $N^{a(1)},\ldots$, etc., as well as on derivatives of N^a, $N^{a(1)},\ldots$, etc., in angular directions, will be linear because N^a comes with a factor r^{-2}, and we shall henceforth only consider such functionals. From a

[3]Note, however, that there may be some restrictions arising from some further global hypotheses if those are made. We emphasize that we do not impose any such global hypotheses here.

symplectic point of view it turns out to be natural not to make the hypothesis that h is a quadratic polynomial of the fields and their derivatives, as would be the case for functionals (3.1), and for this reason we shall allow arbitrary differentiable functions h in (3.9), except for the hypothesis of linearity in N^a with coefficients of linearity independent of the remaining fields.

Assuming that the metric $g_{\mu\nu}$ is vacuum (at least in a neighborhood of $\mathscr{I}$) we can eliminate the u-derivatives of M and N^a in favor of M and u-derivatives of χ_{ab}, using equations (3.7), so that (3.9) can be rewritten as

$$H=\int_{S^2} h[M,N^a,\chi_{ab},\chi^{(1)}_{ab},\ldots\chi^{(k)}_{ab}]d^2\mu, \tag{3.10}$$

where h is still linear in N^a and its derivatives in directions tangent to S^2. By an abuse of notation here, we still denote the integrand of H by h, although it will in general not coincide with the original h of (3.9). Before we proceed further we need the following result based on the work of Friedrich [39,40] as extended by Kannar [41]:

Lemma III.1 Let $M_0(\theta,\phi)$ be a smooth function on S^2, $N^a_0(\theta,\phi)$ be a smooth vector field on S^2, and $\chi^0_{ab}(\theta,\phi)$, $\chi^1_{ab}(\theta,\phi),\ldots,\ \chi^k_{ab}(\theta,\phi),\ldots$ be any sequence of smooth symmetric traceless tensors on S^2. Then there exists $\epsilon>0$ and a vacuum space-time (M,g) with a smooth conformal completion (in the sense described above) $(\bar{M},\bar{g})$ which has a spherical cut $u=u_0$ of $\mathscr{I}^+\supset(u_0-\epsilon,u_0)\times S^2$ such that the Bondi functions $M(u,\theta,\phi)$, $N^a(u,\theta,\phi)$, and $\chi_{ab}(u,\theta,\phi)$ satisfy

$$\lim_{u\to u_0} M(u,\theta,\phi)=M_0(\theta,\phi), \tag{3.11}$$

$$\lim_{u\to u_0} N^a(u,\theta,\phi)=N^a_0(\theta,\phi), \tag{3.12}$$

$$\forall i\in\mathbb{N}\quad \lim_{u\to u_0} \chi^{(i)}_{ab}(u,\theta,\phi)=\chi^i_{ab}(\theta,\phi). \tag{3.13}$$

Remarks. (1) Actually the functions $\chi_{ab}(u,\theta,\phi)$ can be arbitrarily prescribed as functions of (u,θ,ϕ) on an interval $(u_0-\epsilon,u_0]$ for some appropriate ϵ. The above weaker claim is, however, sufficient for our purposes.

(2) The limits $\lim_{u\to u_0}$ in the equations above have been introduced to avoid talking about space-times with boundary.

Proof. For $x\in[0,1]$ and $u_0-1\leqslant u\leqslant u_0$ consider metrics of the form

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu=-Vx^3e^{2\beta}du^2+2e^{2\beta}du\,dx +h_{ab}(dx^a-U^a du)(dx^b-U^b du), \tag{3.14}$$

We wish to show that we can find $\epsilon>0$ and a metric $\bar{g}_{\mu\nu}$ of the form (3.14) defined for $x\in[0,\epsilon]$, $x^a\in S^2$, $u\in(u_0-\epsilon,u_0]$ such that $x^{-2}\bar{g}_{\mu\nu}$ satisfies the Einstein vacuum equations, for which (3.11)–(3.13) hold. We shall construct the appropriate solution backwards in u on $(u_0-\epsilon,u_0]\times[0,\epsilon]\times S^2$ by solving an asymptotic characteristic initial value problem with data given on the null hypersurface $\mathcal{N}=\{u=u_0\}$ and on a piece of $\mathscr{I}=\{x=0,u\in(u_0-1,u_0]\}\times S^2$. This proceeds as follows: For $0\leqslant x\leqslant 1$ let $h^0_{ab}(x,\theta,\phi)$ be any x-dependent family of symmetric non-degenerate tensors on S^2 with $h^0_{ab}(0,\theta,\phi)=\check{h}_{ab}(\theta,\phi)$ (the standard metric on S^2), with $\det h^0_{ab}(x,\theta,\phi)=\det\check{h}_{ab}(\theta,\phi)$, and with

$$\left.\frac{\partial h^0_{ab}}{\partial x}\right|_{x=0}=\chi^0_{ab}. \tag{3.15}$$

Set $h_{ab}(u=u_0,x,\theta,\phi)=h^0_{ab}(x,\theta,\phi)$. Using the Bondi-van der Burg-Sachs prescription in the coordinate system $(r=1/x,x^a)$ [7,8] we can find unique smooth functions $\partial^i_u\beta(u_0,x,\theta,\phi)$, $\partial^i_u U^a(u_0,x,\theta,\phi)$, $\partial^i_u V(u_0,x,\theta,\phi)$, $\partial^i_u h_{ab}(u_0,x,\theta,\phi)$, such that Eqs. (3.11)–(3.13) hold and such that for all $N\in\mathbb{N}$ the metric $g_{\mu\nu}=x^{-2}\bar{g}_{\mu\nu}$ satisfies $R_{\mu\nu}=O[(u-u_0)^N]$, whatever the fields β, V, U, h_{ab} as long as those fields and their derivatives assume the boundary values obtained above. Indeed, the fields appearing at the right hand side of Eqs. (3.11)–(3.13) provide precisely the data needed for the construction of a solution of the hierarchy of equations obtained by u-differentiating the Bondi-van der Burg-Sachs equations. It follows that any geometric quantities, built out of the metric together with an arbitrary finite number of its derivatives, calculated at $u=u_0$ for any two such metrics will coincide at $u=u_0$.

Because χ_{ab} is symmetric and traceless, h_{ab} can be parametrized as

$$\begin{pmatrix} e^{2\gamma}\cosh(2\delta) & \sinh(2\delta)\sin\theta \\ \sinh(2\delta)\sin\theta & e^{-2\gamma}\cosh(2\delta)\sin^2\theta \end{pmatrix}, \tag{3.16}$$

where we write $\gamma=c(v)/r+O(r^{-3})$, $\delta=d(v)/r+O(r^{-3})$. Let $e_{AB'}$ be the following tetrad field:

$$\begin{aligned} e_0&=e_{00'}=-\partial_x,\\ e_1&=e_{11'}=e^{-2\beta}(\partial_u+\tfrac{1}{2}Vx^3\partial_x+U^\theta\partial_\theta+U^\phi\partial_\phi),\\ e_2&=e_{01'}=\frac{1}{\sqrt{2}}[e^{-\gamma}(-\cosh\delta+i\sinh\delta)\partial_\theta\\ &\qquad+e^{\gamma}(\sinh\delta-i\cosh\delta)\operatorname{cosec}\theta\partial_\phi],\\ e_3&=e_{10'}=(e_2)^*, \end{aligned} \tag{3.17}$$

where $(e_2)^*$ denotes the vector whose coordinate components are complex conjugates of those of e_2. From Eqs. (3.16) and (3.17) one can calculate the Newman-Penrose quantity σ ($=\Gamma_{01'00}$ in the notation of [41], and $=\Gamma_{0001'}$ in the notation of [42]) to obtain

$$\sigma|_{x=0}=c+id. \tag{3.18}$$

In [41] the time sense of e_0 and e_1 is unspecified and e_2 is only specified up to rotations in the e_2-e_3 plane at points in the intersection of N and $\mathscr{I}$ (in Kannar's notation). Since they are then parallelly propagated in Kannar's treatment,

these rotations are u and r independent. Note that up to these ambiguities the tetrad (3.17) coincides with that used in [41], time-reversed, at $x=0$, but will in general differ from it at other points. This is irrelevant as far as the value of $\sigma|_{x=0}$ is concerned because σ at $x=0$ is calculated using only derivatives of the tetrad field tangent to the spheres $x=0$, $u=$constant, so that $\sigma|_{x=0}$ calculated for the tetrad (3.17) will coincide with that calculated in the tetrad used in [41] (up to a constant factor of modulus 1). The essential point is that c and d give the requisite data.

Now let $\chi_{ab}(u,\theta,\phi)$ be an arbitrary one-parameter family of symmetric tensor fields on S^2, with $u\in(u_0-1,u_0]$ such that $\partial_u^i\chi_{ab}(u_0,\theta,\phi)=\chi^i_{ab}(\theta,\phi)$; from $\chi_{ab}(u,\theta,\phi)$ we can calculate $\sigma|_{x=0}$. From $\partial_u^i\bar{g}_{\mu\nu}|_{u=u_0}$, $i=0,1,2$ (which we have already calculated previously) we can determine at $u=u_0$ the remaining initial data needed for the Friedrich-Kannar asymptotic initial value problem. The existence of an $\epsilon>0$ (depending on the initial data) and a solution of the vacuum Einstein equations $g_{\mu\nu}$ defined for $(u,x,\theta,\phi)\in(u_0-\epsilon,u_0]\times[0,\epsilon]\times S^2$ assuming those initial data now follows from the main theorem of [41]. The property that the Bondi functions M, N^a, h_{ab}, and $\partial_x h_{ab}$ parametrizing the metric $\bar{g}_{\mu\nu}$ assume the desired values on $\mathscr{I}^+$ follows from the uniqueness theorems of [39]. □

We can now pass to the proof of our main result.

Theorem III.2. Consider any functional of the form

$$
\begin{aligned}
H[g]=\int_{S^2}[&h(M,\mathcal{D}_aM,\ldots,\mathcal{D}_{a_1}\ldots\mathcal{D}_{a_k}M,\\
&\chi_{ab},\mathcal{D}_c\chi_{ab},\ldots,\mathcal{D}_{c_1}\ldots\mathcal{D}_{c_k}\chi_{ab},\\
&\chi^{(1)}_{ab},\mathcal{D}_c\chi^{(1)}_{ab},\ldots,\mathcal{D}_{c_1}\ldots\mathcal{D}_{c_k}\chi^{(1)}_{ab},\\
&\qquad\ldots,\\
&\chi^{(k)}_{ab},\mathcal{D}_c\chi^{(k)}_{ab},\ldots,\mathcal{D}_{c_1}\ldots\mathcal{D}_{c_k}\chi^{(k)}_{ab},x^a)\\
&+\alpha_aN^a+\alpha_{ab}\mathcal{D}^aN^b\\
&+\cdots+\alpha_{a_1\ldots a_kb}\mathcal{D}^{a_1}\ldots\mathcal{D}^{a_k}N^b]d^2\mu,
\end{aligned}
\tag{3.19}
$$

where h is a twice continuously differentiable function of all its arguments, with some, say smooth, tensor fields $\alpha_{a_1\ldots a_kb}$ on S^2. If H is monotone non-increasing in u for all metrics g which satisfy the vacuum Einstein field equations (with M, χ_{ab}, and N interpreted as Bondi functions appearing in g), then H can be rewritten as

$$
H=\int_{S^2}\Psi(M-\tfrac{1}{4}\check{h}^{ac}\check{h}^{bd}\mathcal{D}_a\mathcal{D}_b\chi_{cd},x^a)d^2\mu,
$$

with a differentiable local functional[4] *$\Psi(f)$ whose variational derivative $\delta\Psi/\delta f$ is non-negative.*

[4]That is, $\Psi(f)$ is a differentiable function of x^a, f and a finite number of its derivatives in directions tangent to S^2.

Proof. Note first that the tensor fields $\alpha_{ab},\ldots,\alpha_{a_1\ldots a_kb}$, can be set to zero by integration by parts and a redefinition of α_b, $\alpha_b\to\hat{\alpha}_b$ with an appropriate $\hat{\alpha}_b$. Calculating the u-derivative of (3.19) we obtain

$$
\frac{dH}{du}=\int_{S^2}\left(\frac{\delta h}{\delta M}\dot{M}+\hat{\alpha}_a\dot{N}^a+\frac{\delta h}{\delta\chi_{ab}}\dot{\chi}_{ab}+\cdots+\frac{\delta h}{\delta\chi^{(k)}_{ab}}\chi^{(k+1)}_{ab}\right)d^2\mu. \tag{3.20}
$$

Now $\delta h/\delta\chi^{(k)}_{ab}$ and all the terms in (3.20) except for the last one are independent of $\chi^{(k+1)}(u_0)$. If $\delta h/\delta\chi^{(k)}_{ab}$ were non-zero for some $k\geq1$ we could, by Lemma III.1, find a solution of the vacuum Einstein equations with $\chi^{(k+1)}(u_0)$ so chosen that $dH/du>0$, which shows that $\delta h/\delta\chi^{(k)}_{ab}=0$ for all $k\geq1$. Setting

$$
\begin{aligned}
\hat{h}[M,\chi_{ab},x^a]=h(&M,\mathcal{D}_aM,\ldots,\mathcal{D}_{a_1}\ldots\mathcal{D}_{a_k}M,\\
&\chi_{ab},\mathcal{D}_c\chi_{ab},\ldots,\mathcal{D}_{c_1}\ldots\mathcal{D}_{c_k}\chi_{ab},\\
&\chi^{(1)}_{ab}=0,\mathcal{D}_c\chi^{(1)}_{ab}=0,\ldots,\mathcal{D}_{c_1}\ldots\mathcal{D}_{c_k}\chi^{(1)}_{ab}=0,\\
&\ldots,\chi^{(k)}_{ab}=0,\mathcal{D}_c\chi^{(k)}_{ab}=0,\\
&\qquad\ldots,\mathcal{D}_{c_1}\ldots\mathcal{D}_{c_k}\chi^{(k)}_{ab}=0,x^a),
\end{aligned}
$$

we obtain from $\delta h/\delta\chi^{(k)}_{ab}=0$, $k\geq1$,

$$
\begin{aligned}
&H(g,u)=\int_{S^2}(\hat{h}[M,\chi_{ab}]+\hat{\alpha}_aN^a)d^2\mu,\\
&\frac{dH}{du}=\int_{S^2}\left(\frac{\delta\hat{h}}{\delta M}\dot{M}+\hat{\alpha}_a\dot{N}^a+\frac{\delta\hat{h}}{\delta\chi_{ab}}\dot{\chi}_{ab}\right)d^2\mu.
\end{aligned}
\tag{3.21}
$$

Consider, first, Eq. (3.21) for solutions of the vacuum Einstein equations with $\dot{\chi}_{ab}|_{u=u_0}=0$. Equations (3.21) and (3.7) then yield

$$
\begin{aligned}
\frac{dH}{du}&=\int_{S^2}\left(-\frac{1}{3}\hat{\alpha}_a\left(\mathcal{D}_aM-\frac{1}{4}\epsilon^{ab}\mathcal{D}_b\lambda\right)\right)d^2\mu\\
&=\int_{S^2}\left(\frac{1}{3}M\mathcal{D}^a\hat{\alpha}_a-\frac{1}{12}\lambda\epsilon^{ab}\mathcal{D}_b\hat{\alpha}_a\right)d^2\mu.
\end{aligned}
\tag{3.22}
$$

To proceed further we need to know a little more about λ as defined by (3.7). In Appendix B we show that the image of the operator $\chi_{ab}\to\epsilon_{ac}\mathcal{D}^c\mathcal{D}_b\chi^{ab}$ defined on traceless symmetric tensors consists precisely of functions of the form $P\psi$, where ψ is an arbitrary appropriately differentiable function on S^2 and P is the projection operator defined as

$$
P\psi=\psi-\sum_{i=0}^{3}\Phi_i\int_{S^2}\psi\Phi_i d^2\mu, \tag{3.23}
$$

where the Φ_i form an orthonormal basis of the space of spherical harmonics with $l=0$ (Φ_0) and $l=1$ ($\Phi_i, i=1,2,3$). Consequently λ runs over all smooth functions with no zero or first spherical harmonics as χ_{ab} runs over all smooth symmetric traceless tensors. This, together with Lemma III.1 [note that M in (3.22) is arbitrary] shows that dH/du in (3.22) will have an arbitrary sign unless

$$\mathcal{D}_a\hat{\alpha}^a=0, \quad \epsilon^{ab}\mathcal{D}_a\hat{\alpha}_b=\sum_{i=1}^{3}\alpha_i\Phi_i, \tag{3.24}$$

for some constants α_i. It follows that

$$\hat{\alpha}^a=\frac{1}{2}\,\mathcal{D}_b\left(\epsilon^{ab}\sum_{i=1}^{3}\alpha_i\Phi^i\right) \tag{3.25}$$

(the fact that the above vector field satisfies (3.24) can be checked by a direct calculation; the fact that there is only one such vector field is shown in Appendix B). Returning to Eq. (3.21), we obtain from (3.25) and (3.7)

$$\begin{aligned}\frac{dH}{du}&=\int_{S^2}\left[\frac{\delta\hat{h}}{\delta M}\left(-\frac{1}{8}\,\dot{\chi}_{ab}\dot{\chi}^{ab}+\frac{1}{4}\,\mathcal{D}_a\mathcal{D}_b\dot{\chi}^{ab}\right)\right.\\&\quad\left.-\frac{1}{3}\,\hat{\alpha}_aK^a+\frac{\delta\hat{h}}{\delta\chi_{ab}}\,\dot{\chi}_{ab}\right]d^2\mu\\&=\int_{S^2}\left[\left(\frac{1}{4}\,\mathcal{D}^a\mathcal{D}^b\,\frac{\delta\hat{h}}{\delta M}+\frac{1}{6}\,\hat{\alpha}_c\mathcal{D}^b\chi^{ca}\right.\right.\\&\quad\left.\left.+\frac{1}{4}\,\chi^{ca}\mathcal{D}^b\hat{\alpha}_c+\frac{\delta\hat{h}}{\delta\chi_{ab}}\right)\dot{\chi}_{ab}-\frac{1}{8}\,\frac{\delta\hat{h}}{\delta M}\,\dot{\chi}_{ab}\dot{\chi}^{ab}\right]d^2\mu.\end{aligned} \tag{3.26}$$

Define a new functional $\hat{\Psi}$ by

$$\hat{\Psi}[f,\chi_{ab},x^a]:=\hat{h}[M=f+\tfrac{1}{4}\mathcal{D}_a\mathcal{D}_b\chi^{ab},\chi_{ab},x^a].$$

Equation (3.26) can be rewritten as

$$\begin{aligned}\frac{dH}{du}&=-\frac{1}{8}\int_{S^2}\frac{\delta\hat{\Psi}}{\delta f}\,\dot{\chi}_{ab}\dot{\chi}^{ab}d^2\mu\\&\quad+\int_{S^2}\left[\left(\frac{1}{6}\,\hat{\alpha}_c\mathcal{D}^b\chi^{ca}+\frac{1}{4}\,\chi^{ca}\mathcal{D}^b\hat{\alpha}_c+\frac{\delta\hat{\Psi}}{\delta\chi_{ab}}\right)\dot{\chi}_{ab}\right]d^2\mu.\end{aligned} \tag{3.27}$$

dH/du will be non-positive for all $\dot{\chi}_{ab}$ if and only if $\delta\hat{\Psi}/\delta f$ is non-negative, and the last integral vanishes, which yields

$$\frac{\delta\hat{\Psi}}{\delta\chi_{ab}}=-TS\left\{\left(\frac{1}{6}\,\hat{\alpha}_c\mathcal{D}^b\chi^{ac}+\frac{1}{4}\,\chi^{ca}\mathcal{D}^b\hat{\alpha}_c\right)\right\}, \tag{3.28}$$

where TS denotes the symmetric trace-free part with respect to the indices a,b. We wish to show that $\hat{\alpha}^a$ has to be zero. To do this, fix a smooth f and consider $G_f[\chi_{ab},x^a]=\int\hat{\Psi}[f,\chi_{ab},x^a]d^2\mu$ as a functional of χ_{ab}. Note that if we endow the space of the χ_{ab}'s with a Sobolev space topology $W_{k,2}(S^2)$ with some k large enough, then G_f will be a twice differentiable function on that space, and by (3.28) we have

$$G_f'[\nu]:=-\frac{1}{12}\int_{S^2}(2\hat{\alpha}_c\mathcal{D}^b\chi^{ca}+3\chi^{ac}\mathcal{D}^b\hat{\alpha}_c)\nu_{ab}d^2\mu, \tag{3.29}$$

where $G_f'[\nu]$ denotes the derivative of G_f acting on the symmetric traceless tensor ν. It follows from Schwarz's Lemma that the second derivative G_f'' of G_f satisfies $G_f''[\tau,\nu]=G_f''[\nu,\tau]$, for all smooth symmetric traceless tensor fields τ_{ab} and ν_{ab}. From (3.29) we have

$$G_f''[\tau,\nu]:=-\frac{1}{12}\int_{S^2}(2\hat{\alpha}_c\mathcal{D}^b\tau^{ca}+3\tau^{ac}\mathcal{D}^b\hat{\alpha}_c)\nu_{ab}d^2\mu. \tag{3.30}$$

Letting $F:=\frac{1}{2}\Sigma_{i=1}^{3}\alpha_i\Phi^i$ with some constants α_i, we have $\hat{\alpha}^a=\epsilon^{ab}\mathcal{D}_bF$ [cf. (3.25)]. We also have $\mathcal{D}^a\mathcal{D}_bF=-\delta^a_bF$ (cf. e.g. [43], Lemma 5), so one gets $\mathcal{D}^a\hat{\alpha}^b=\epsilon^{ab}F$. Using those identities, by integration by parts one obtains

$$\begin{aligned}G_f''[\tau,\nu]-G_f''[\nu,\tau]&=-\frac{1}{6}\int_{S^2}(\hat{\alpha}_c\mathcal{D}^b\tau^{ca}+\hat{\alpha}^b\mathcal{D}_c\tau^{ca}\\&\quad+2F\tau^{ac}\epsilon^b{}_c)\nu_{ab}d^2\mu.\end{aligned}$$

Since ν is arbitrary (traceless, symmetric) we obtain

$$TS[\hat{\alpha}_c\mathcal{D}^b\tau^{ca}+\hat{\alpha}^b\mathcal{D}_c\tau^{ca}+2F\tau^{ac}\epsilon^b{}_c]=0, \tag{3.31}$$

for arbitrary τ's. Think of the two-dimensional sphere as a submanifold of $\mathbb{R}^3$. By a rotation of the coordinate axes we can always achieve $F=\lambda\cos\theta$, for some constant λ. Equation (3.31) at a point p_0 lying on the equator, $p_0=(\theta=\pi/2,\phi_0)$, with $a=\theta$, $b=\theta$ reads

$$2\lambda\mathcal{D}_\phi\tau^{\phi\phi}=0. \tag{3.32}$$

Consider the smooth traceless symmetric tensor field $\tau_{ab}dx^adx^b=\rho[(d\theta)^2-\sin^2\theta(d\phi)^2]+2\sigma d\theta\,d\phi$, with ρ and σ smooth functions on S^2, supported near the equator, and satisfying $\rho(p_0)=\sigma(p_0)=0$. Equation (3.32) implies

$$\lambda\partial_\phi\rho(p_0)=0,$$

for all such functions ρ, so clearly $\lambda=0$, and we finally get

$$F=0. \tag{3.33}$$

Define

$$\Psi[f,x^a]=\hat{h}[f+\tfrac{1}{4}\mathcal{D}_a\mathcal{D}_b\chi^{ab},\chi_{ab}=0,x^a].$$

Equations (3.30) and (3.33) give

$$\int_{S^2}\Psi[M-\tfrac{1}{4}\mathcal{D}_a\mathcal{D}_b\chi^{ab}]=\int_{S^2}\hat{h}[M,\chi_{ab}]=H[g],$$

which is what had to be established. □

IV. SUPERTRANSLATION INVARIANCE

Theorem III.2 does not quite lead to the Trautman-Bondi mass as a preferred quantity in the class of functionals considered in that theorem, as it still contains an arbitrary function Ψ of $M-\frac{1}{4}\check{h}^{ac}\check{h}^{bd}\mathcal{D}_a\mathcal{D}_b\chi_{cd}$ and a finite number of its angular derivatives. Let us show that the further requirement of *passive supertranslation invariance* of H can be used to obtain that desired conclusion. Here the qualification "passive" refers to the fact that we use a different Bondi coordinate system but we integrate on the same cut of $\mathscr{I}$. More precisely, consider a functional H as in Theorem III.2. We can calculate the value of H at a cross section S^2 for a metric g, and compare the result with H calculated on the same cross section of $\mathscr{I}$ for the same metric with a different Bondi parametrization, differing by a (finite, or infinitesimal) BMS supertranslation. Let $\mathcal{S}$ denote a given cut of $\mathscr{I}$, which in some Bondi coordinate system (u,θ,ϕ) on $\mathscr{I}$ is given by the equation $u=0$, and set

$$H(\mathcal{S})=\int_{S^2}\Psi(M-\tfrac{1}{4}\check{h}^{ac}\check{h}^{bd}\mathcal{D}_a\mathcal{D}_b\chi_{cd})(u=0,\theta,\phi)d^2\mu. \tag{4.1}$$

Consider another Bondi coordinate system $(\bar{u},\bar{\theta},\bar{\phi})=(u-\alpha(\theta,\phi),\theta,\phi)$, with corresponding functions $\bar{M}$, $\bar{\chi}_{\bar{a}\bar{b}}$, etc. As shown in Appendix C (see also [37]), we have

$$\overline{[4M-\chi^{ab}{}_{\|ab}]}(u,\theta,\phi)$$
$$=[4M-\chi^{ab}{}_{\|ab}+\Delta_2(\Delta_2+2)\alpha](u,\theta,\phi). \tag{4.2}$$

The overbar in the left hand side of the last equation denotes the quantity $4M-\chi^{ab}{}_{\|ab}$ calculated in the barred Bondi frame, using the barred Bondi functions $\bar{M}$, etc. The requirement that $H(\mathcal{S})$, calculated in the unbarred Bondi coordinate system, coincides with $H(\mathcal{S})$, calculated in the barred Bondi coordinate system, gives thus the equation

$$\forall\,\alpha\quad \int_{S^2}\Psi[M-\tfrac{1}{4}\mathcal{D}_a\mathcal{D}_b\chi^{ab}]d^2\mu=\int_{S^2}\Psi[M-\tfrac{1}{4}\mathcal{D}_a\mathcal{D}_b\chi^{ab}$$
$$+\tfrac{1}{4}(\mathcal{D}_a\mathcal{D}^a+2)\mathcal{D}_b\mathcal{D}^b\alpha]d^2\mu. \tag{4.3}$$

[It should be emphasized that $\mathcal{S}$ is not given by the equation $\bar{u}=0$. We are *not* requiring that the value $H(\bar{\mathcal{S}})$ of H, calculated on the cut $\bar{\mathcal{S}}=\{\bar{u}=0\}$, coincides with that of $H(\mathcal{S})$. That last condition would be the requirement that the value of H does not depend on the cut under consideration, which is of course absurd in the radiating regime.] Now, elementary considerations using spherical harmonics show that $\chi=(\mathcal{D}_a\mathcal{D}^a+2)\mathcal{D}_b\mathcal{D}^b\alpha$ is an arbitrary function such that $P\chi=\chi$, where P is the projection operator introduced in equation (3.23). If we replace α by $t\alpha$ in Eq. (4.3), differentiate with respect to t, and set $t=0$, we obtain thus $P(\delta\Psi/\delta f)=0$. It follows that there exist constants w^μ, $\mu=0,1,2,3$, such that $\delta\Psi/\delta f=(w^0+w^k n_k)/4\pi$, $n_k:=x_k/r$ being an orthogonal (but not orthonormal) basis in the space SH^1 of the $l=1$ spherical harmonics. The condition that $\delta\Psi/\delta f$ be nonnegative gives $w^0+w^k n_k\geqslant 0$ for all $n_k\in S^2$. That will hold if and only if $w^0\geqslant|w|$, where $|w|=\sqrt{\delta^{kl}w_k w_l}$, so that one may think of w^μ as of a future timelike vector. We have thus obtained $\Psi(f)=(w^0+w^k n_k)f/4\pi$, and finally

$$H=\frac{1}{4\pi}\int_{S^2}(w^0+w^k n_k)(M-\tfrac{1}{4}\chi^{ab}{}_{\|ab})d^2\mu$$
$$=\frac{1}{4\pi}\int_{S^2}(w^0+w^k n_k)Md^2\mu. \tag{4.4}$$

Equation (4.4) has the clear interpretation that H is the Trautman-Bondi mass as measured with respect to a frame with time-like four-velocity vector (w^0,w^i), which can be checked from the transformation properties of Bondi coordinate systems under (passive) Lorentz transformations. For completeness we analyze that question in Appendix D.

The results of this section and Theorem III.2 imply the following.

Theorem IV.1. Let H be a functional of the form

$$H[g,u]=\int_{S^2(u)}H^{\alpha\beta}(g_{\mu\nu},g_{\mu\nu,\sigma},\ldots,g_{\mu\nu,\sigma_1\ldots\sigma_k})dS_{\alpha\beta}, \tag{4.5}$$

where the $H^{\alpha\beta}$ are twice differentiable functions of their arguments, and the integral over $S^2(u)$ is understood as a limit as ρ goes to infinity of integrals over the spheres $t=u+\rho$, $r=\rho$. Suppose that H is finite and monotonic in u for all vacuum metrics $g_{\mu\nu}$ satisfying

$$g_{\mu\nu}=\eta_{\mu\nu}+\frac{h^1_{\mu\nu}(u,\theta,\phi)}{r}+\frac{h^2_{\mu\nu}(u,\theta,\phi)}{r^2}+o(r^{-2}),$$

$$\partial_{\sigma_1}\ldots\partial_{\sigma_i}\left(g_{\mu\nu}-\frac{h^1_{\mu\nu}(u,\theta,\phi)}{r}-\frac{h^2_{\mu\nu}(u,\theta,\phi)}{r^2}\right)=o(r^{-2}), \tag{4.6}$$

with $1\leqslant i\leqslant k$, for some C^k functions $h^a_{\mu\nu}(u,\theta,\phi)$, $a=1,2$. If H is invariant under passive BMS supertranslations, then the numerical value of H equals (up to a proportionality constant) the Trautman-Bondi mass.

Proof. If H is monotonic for all such metrics, then it is monotonic for Bondi-Sachs type metrics (3.3) for which a quasi-Minkowskian coordinate system (3.4) has been introduced. As discussed at the beginning of Sec. III, for such metrics (4.5) can be written as a quadratic polynomial in the relevant fields, linear in N^a, so that Theorem III.2 applies. Now the asymptotic behavior of the functions appearing in the metric (3.3) shows that any quadratic terms in M that could possibly survive in the limit $r\to\infty$ come with no angular derivatives acting on M. The definiteness of the variational derivative of Ψ, where Ψ is given by Theorem III.2, together with Lemma III.1, implies then that Ψ is necessarily linear, and the result follows from the argument leading to (4.4). □

Note that the trivial monotone functional, namely $H \equiv 0$, is contained in the result above, the relevant constant of proportionality being zero.

V. GENERAL CUTS OF $\mathscr{I}$

So far we have been considering the TB mass of those cuts of $\mathscr{I}$ which are given by the equation $u=0$. Consider now a cut $\mathcal{S}$ of $\mathscr{I}$ which, in Bondi coordinates, is given by the equation

$$\mathcal{S}=\{u=s(\theta,\phi)\},$$

for some, say smooth, function s on S^2. Theorem III.2, together with the discussion of the previous section, suggests that it is natural to define

$$m_{TB}(\mathcal{S}) := \frac{1}{16\pi}\int_{S^2}(4M - \chi^{ab}{}_{\|ab})(u=s(\theta,\phi),\theta,\phi)\sin\theta\, d\theta\, d\phi \tag{5.1}$$

$$p^k(\mathcal{S}) := \frac{1}{16\pi}\int_{S^2}(4M - \chi^{ab}{}_{\|ab})(u=s(\theta,\phi),\theta,\phi)n^k\sin\theta\, d\theta\, d\phi, \tag{5.2}$$

where n^k, $k=1,2,3$ denotes the functions $\sin\theta\cos\phi$, $\sin\theta\sin\phi$ and $\cos\theta$, in that order. We have the following.

(1) As observed in Sec. IV [cf. Eq. (4.4)], Eq. (5.1) reduces to the standard Trautman-Bondi-Sachs definition when $s\equiv 0$.

(2) It also follows from what is said in the previous section that the quantities (5.1) and (5.2) are invariant under passive BMS supertranslations.

(3) Equation (4.4) together with passive supertranslation invariance and the discussion of Appendix D imply that the quantities $(p^\mu)=(m_{TB},p^k)$ transform as a Lorentz vector under those boosts which map $\mathcal{S}$ into itself.

(4) The definitions (5.1) and (5.2) allow us to define a flux of energy-momentum through a subset of $\mathscr{I}^+$ bounded by two cross-sections thereof. More precisely, let $\mathcal{S}_i$, $i=1,2$ be two cross-sections of $\mathscr{I}^+$ which are graphs over the cut $u=0$:

$$\mathcal{S}_i=\{u=s_i(\theta,\phi)\},$$

and let $N\subset\mathscr{I}^+$ be such that $\partial N = s_2(S^2)\cup s_1(S^2)$. From the definition (D6) and the relation (3.7) we have

$$m_{TB}(\mathcal{S}_2)-m_{TB}(\mathcal{S}_1) = \frac{1}{16\pi}\int_{\partial N}(4M-\chi^{ab}{}_{\|ab})\sin\theta\, d\theta\, d\phi = -\frac{1}{32\pi}\int_N \chi_{ab,u}\chi^{ab}{}_{,u}\sin\theta\, du\, d\theta\, d\phi, \tag{5.3}$$

which can be thought of as a flux of energy through N. A similar formula holds for the space-momentum p^k defined by (D7):

$$p^k(\mathcal{S}_2)-p^k(\mathcal{S}_1) = \frac{1}{16\pi}\int_{\partial N}(4M-\chi^{ab}{}_{\|ab})n^k\sin\theta\, d\theta\, d\phi = -\frac{1}{32\pi}\int_N \chi_{ab,u}\chi^{ab}{}_{,u}n^k\sin\theta\, du\, d\theta\, d\phi. \tag{5.4}$$

We note that the existence of a flux formula is a rather trivial property, since one can always take the u derivative of any integrand to obtain a flux. The interest of the above formulas stems from the fact that $\chi_{ab,u}$ is invariant under (passive) supertranslations, so that the fluxes (5.3) and (5.4) also share this property.

(5) Passive supertranslation invariance together with the flux formulas (5.3) and (5.4) imply that in a stationary spacetime the four-momentum p^μ defined by (5.1) and (5.2) is $\mathcal{S}$ independent. In particular p^μ vanishes in Minkowski spacetime, independently of the cut $\mathcal{S}$.

VI. POLYHOMOGENEOUS METRICS

Having established the preferred role played by the Trautman-Bondi mass, it is of interest to enquire under what weaker asymptotic conditions one can still obtain a definition of mass which is finite and monotonic in u. Recall that in [23] an *ad hoc* definition of mass was given for all Bondi-type metrics with a "polyhomogeneous $\mathscr{I}$", and that mass was shown there to be monotonic. Similarly it was checked in [44] that for a class of asymptotically flat asymptotically vacuum space-times[5] the energy expression defined in [18] converges to an appropriately defined Bondi mass. From a field theoretic point of view it is natural to define mass in terms of an integral, as considered in Theorem IV.1, using, e.g., the Freud potential, where the $H^{\alpha\beta}$ of equation (3.1) is given by the expression (cf. e.g., [45])

$$H^{\mu\nu}=\mathfrak{U}^{\mu\nu}{}_\alpha X^\alpha, \tag{6.1}$$

$$\mathfrak{U}^{\mu\nu}{}_\alpha = \frac{1}{16\pi\sqrt{|\det g_{\rho\sigma}|}}\times g_{\alpha\beta}(|\det g_{\rho\sigma}|g^{\beta[\mu}g^{\nu]\lambda})_{,\lambda}, \tag{6.2}$$

with $X^\mu=\delta^\mu_0$. Inserting the metric (3.14) into (6.2), with $X^\alpha\partial_\alpha=\partial_u$ and with h_{ab} parametrized as in (3.16), one obtains via a SHEEP [46] calculation

[5]The class of metric considered in [44] includes the metrics polyhomogeneous of order 2 (see [23] and below for definitions).

$$\int_{u=u_0,r=r_0} \mathfrak{U}^{\mu\nu}{}_{\alpha} X^{\alpha} dS_{\mu\nu}$$

$$= \frac{1}{16\pi} \int_{S^2} \Big\{ -2V + 2re^{2\beta}\cosh(2\gamma)\cosh(2\delta) - r^4 e^{-2\beta} \Big[\frac{\partial U^{\theta}}{\partial r} \left(U^{\theta} e^{2\gamma}\cosh(2\delta) + U^{\phi}\sinh(2\delta)\sin(\theta)\right) + \sin(\theta) \frac{\partial U^{\phi}}{\partial r} \left(U^{\phi} e^{-2\gamma}\cosh(2\delta)\sin(\theta) + U^{\theta}\sinh(2\delta)\right) \Big] + r^2 \mathcal{D}_a U^a \Big\} \Big|_{u=u_0,r=r_0} \sin(\theta) d\theta\, d\phi \tag{6.3}$$

$$= \frac{1}{16\pi} \int_{S^2} \Big\{ 2(r-V) + 2r(1 - e^{2\beta}\cosh(2\gamma)\cosh(2\delta)) - r^4 e^{-2\beta} h_{ab} \frac{\partial U^a}{\partial r} U^b + r^2 \mathcal{D}_a U^a \Big\} \Big|_{u=u_0,r=r_0} \sin(\theta) d\theta\, d\phi. \tag{6.4}$$

More precisely, this formula is obtained by ''covariantizing'' (as described in [34]) Eq. (6.2) with the following flat background metric η:

$$\eta_{\mu\nu} dx^{\mu} dx^{\nu} = -du^2 - 2du\, dr + r^2(d\theta^2 + \sin^2(\theta) d\phi^2).$$

Equation (6.4) is exact; no hypotheses about the asymptotic behavior of the quantities involved have been made. Note that the last term in Eq. (6.4) integrates out to zero. We shall say that a metric is *polyhomogeneous of order k* if in the Bondi coordinates (3.3) the functions h_{ab} have a polyhomogeneous expansion (see [23] for definitions) in which the $\ln r$ terms start at a power r^{-k}:

$$h_{ab} = \breve{h}_{ab} + \frac{h^1_{ab}}{r} + \cdots + \frac{h^{k,n}_{ab} \ln^n r}{r^k} + \frac{h^{k,(n-1)}_{ab} \ln^{(n-1)} r}{r^k} + \cdots .$$

Consider first metrics which are polyhomogeneous of order 2. We have then $\gamma = O(r^{-1})$, $\delta = O(r^{-1})$ and it follows from the Einstein equations as written out, e.g., in [23], Appendix C[6] that $\beta = O(r^{-2})$, $U^a = O(r^{-2})$, $\partial U^a/\partial r = O(r^{-3})$ and $r - V = O(1)$. Equations (3.1) and (6.4) then give

$$H[u_0, g] = \lim_{r\to\infty} \frac{1}{8\pi} \int_{S^2} (r - V)\sin(\theta) d\theta\, d\phi, \tag{6.5}$$

which is the standard Bondi integral. Consider, next, metrics which are polyhomogeneous of order 1. In that case one has $\gamma = O(r^{-1}\ln^N r)$, $\delta = O(r^{-1}\ln^N r)$ for some N. The Einstein equations imply (see the proof of Prop. 2.1 in [23]) that $\beta = O(r^{-2}\ln^{2N} r)$, $U^a = O(r^{-2}\ln^N r)$, $\partial U^a/\partial r = O(r^{-3}\ln^N r)$ and $r - V = O(\ln^{N_V} r)$ for some N_V. Equations (3.1) and (6.4) lead again to (6.5). At first sight it appears that the integral at the right hand side of (6.5) might diverge for some vacuum metrics which are polyhomogeneous of order 1. However, careful study of the leading terms in the Einstein equations shows that those terms in V which are linear combinations of $\ln^i r$ are divergences, so that their integral over a sphere vanishes. Thus the Freud integral always converges to the monotonic mass expression considered in [23]. Remarkably, the polyhomogeneous case of order $k \geqslant 1$ always has a finite energy.

Let us mention that for metrics which are polyhomogeneous of order $k \geqslant 2$ the Freud integral can be given a Hamiltonian interpretation—this will be discussed elsewhere.

VII. CLOSING REMARKS

We have shown that every functional of the fields which is monotonic in time in a certain class of functionals for all metrics ''having a piece of $\mathscr{I}$'' is proportional to the Trautman-Bondi mass. The key ingredient of our proof was the Friedrich-Kannar construction of space-times ''having a piece of $\mathscr{I}$.'' Now in general the space-times we have constructed in the proof above will not have any reasonable global properties. For example, in Lemma III.1 the function M could be chosen to be negative. In such a case one expects, from the positive TB mass conjecture, that the space-time constructed in Lemma III.1 will have no extension with complete Cauchy surfaces. Now the property of having such Cauchy surfaces is a starting point of any standard Hamiltonian analysis, and for this reason it would be rather useful to have an equivalent of Lemma III.1 in which well behaved space-times are constructed. We expect that a result of that kind can be proved, under some mild (yet to be determined) restrictions on the function M (such as, e.g., positivity), and we are planning to investigate this problem in the future.

Let us finally mention that using similar ideas to those presented here one can prove related results for other field theories, such as, e.g., Maxwell theory, or for scalar fields. More precisely, for a scalar field one has the following:

Theorem VII.1 The only functional F, in the class of functionals defined in the Introduction, of a scalar field ϕ on Minkowski space-time, which is monotonic in retarded time for all solutions of the massless linear wave equation, and which is a Hamiltonian for the dynamics on a hyperboloid Σ, is the integral H of the standard energy-momentum tensor over Σ.

[6]There are unfortunately some misprints in Appendix C of [23]: (1) The last term in Eq. (C4), $\frac{1}{2}r^2\text{cosec}\,\theta(W_{13} + 4W_3)$, should be replaced by $\frac{1}{2}r\,\text{cosec}\,\theta(rW_{13} + 4W_3)$; (2) in the 8th line of Eq. (C6) the factor 4 in front of the term $4\gamma_1\gamma_2 U$ should be replaced by 2.

To prove this one uses an equivalent of Lemma III.1 which, for a scalar field on Minkowski space-time, can be easily modified to obtain globally defined solutions. The question of how to define a symplectic structure for dynamics on hyperboloids will be discussed elsewhere [47]. The requirement that the functional considered is a Hamiltonian leads to the conclusion that F differs from H by a boundary integral. Using arguments similar to the ones presented in this paper (and actually rather simpler, as the corresponding equations on $\mathscr{T}$ are much simpler in the case of a scalar field) one then proves [48] that all the boundary integrands, in the case of the scalar field, which have the right monotonicity properties, have to integrate out to zero. Minkowski space-time above can be replaced by any Lorentzian manifold which has sufficiently regular conformal completions.

Let us finally mention that one can set up a Hamiltonian framework in which some of the problems related to the Ashtekar-Streubel or Ashtekar-Bombelli-Reula approaches, listed in Sec. II, are avoided [47]. Unsurprisingly, the Hamiltonians one obtains in such a formalism are again not unique, but the nonuniqueness can be controlled in a very precise way. The Trautman-Bondi mass turns out to be a Hamiltonian, and an appropriate version of the uniqueness Theorem III.2 proved above can be used to single out the TB mass amongst the family of all possible Hamiltonians.

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APPENDIX A: TRAUTMAN'S DEFINITION OF MASS IN THE RADIATION REGIME

In [5][7] Trautman considers gravitational fields for which a coordinate system exists in which the metric can be written in the form

$$g_{\mu\nu}=\eta_{\mu\nu}+O(r^{-1}),\quad g_{\mu\nu,\rho}=h_{\mu\nu}k_\rho+O(r^{-2}),\tag{A1}$$

$$(h_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}h_{\rho\sigma})k^\nu=O(r^{-2}).\tag{A2}$$

Here the functions $h_{\mu\nu}$ satisfy $h_{\mu\nu}=O(r^{-1})$, while the null vector field k_ν is defined as follows: Let σ be a spacelike hypersurface, and define n^μ to be a unit spacelike vector lying in σ perpendicular to the sphere $r=$const, and pointing outside it. Trautman defines k^ν to be $n^\nu+t^\nu$, where t denotes a unit timelike vector normal to σ, such that $t^0>0$.

Trautman shows that under the conditions (A1) and (A2) the integral at the right-hand side of the equation

$$P_\mu[\sigma]=\oint_S \mathfrak{U}^{\nu\lambda}{}_\mu dS_{\nu\lambda}\tag{A3}$$

exists[8] and is finite because of cancellations among the divergent terms. Here $\mathfrak{U}^{\nu\lambda}{}_\mu$ is the Freud potential given in Eq. (6.2). Next, Trautman shows that $P_\mu[\sigma]$ is coordinate independent in the following sense: Let a new coordinate system x'^ν be given by the equations

$$x^\nu\to x'^\nu=x^\nu+a^\nu,\tag{A4}$$

with a^ν satisfying

$$a^\nu=o(r),\quad a_{\nu,\mu}=b_\nu k_\mu+O(r^{-2}),\tag{A5}$$

where

$$a_\nu=\eta_{\nu\mu}a^\mu,\quad b_\nu=O(r^{-1}),$$

and

$$a_{\nu,\mu\rho}=b_{\nu,\mu}k_\rho+O(r^{-2}),\quad b_{\nu,\rho}=O(r^{-1}).\tag{A6}$$

Those coordinate transformations preserve the boundary conditions introduced above. Trautman notices that under those transformations the integrand in (A3) changes by terms which are $O(r^{-3})$, so that $P_\mu[\sigma]$ itself remains unchanged.

In Sec. 4 of [5] Trautman gives the formula for the total energy and momentum, which he calls p_μ, radiated between two hypersurfaces σ and σ',

$$p_\mu=P_\mu[\sigma]-P_\mu[\sigma']=\int_\Sigma \mathfrak{t}_\mu{}^\nu dS_\nu,\tag{A7}$$

under the hypothesis that the energy-momentum tensor of matter fields gives no contribution on Σ. Here

$$\mathfrak{t}_\mu{}^\nu=\tau k_\mu k^\nu+O(r^{-3}),\tag{A8}$$

where

$$4\kappa\tau=h^{\mu\nu}(h_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}h_{\rho\sigma}),\tag{A9}$$

and κ is the constant of proportionality between the Einstein tensor and the energy-momentum tensor, and it is clear that the integral over Σ in Eq. (A7) is defined by a limiting

[7]The first chapter of [6] is a slightly expanded version of [5].

[8]It is clear that S in Eq. (A3) is understood as "a boundary of σ at infinity," defined as far as integration is concerned by a limiting process. In the section in which he talks about radiating fields Trautman does not give a precise definition of what S is.

process.[9] He emphasizes that τ is invariant with respect to the transformations (A4) and is *non-negative* by virtue of (A2), so that $p_0 \geqslant 0$.

For our purposes we need to change the definition of k_μ given above: we require k_μ to be a null vector field satisfying

(1) k_μ is normal to the spheres $r=\{\text{const}\}$, future pointing and outwards pointing;

(2) k_μ satisfies the following asymptotic conditions:

$$k^0 - 1 = O(r^{-1}), \quad k^i - \frac{x^i}{r} = O(r^{-1}).$$

(This is compatible with Trautman's definition if one takes σ to be the hypersurface $\{x^0=\text{const}\}$ in the coordinate system in which (A1) and (A2) hold. However, the hypersurfaces we consider here are *not* of this form.) With this modification Eqs. (A7) and (A8) together with positivity of τ are the fundamental statement that on hypersurfaces which, in modern terminology, "intersect $\mathscr{I}^+$" the energy can only be radiated away. It should be emphasized that this is a more general statement than that discussed by Bondi *et al.* and by Sachs four years later [7,8], as the boundary conditions (A1) and (A2) are *weaker* than those of [7,8]. Indeed, consider a Bondi-Sachs type metric (3.3), with all the functions appearing there satisfying the fall-off requirements of [7,8]. If quasi-Minkowskian coordinates are introduced via the Eq. (3.4), one finds that Trautman's conditions (A1) and (A2) hold with $k_\mu = u_{,\mu}$. If σ is taken to be the null hypersurface $\{u=u_0\}$ (note that with our minor modification of the definition of what k_μ is, the hypothesis that σ is spacelike is not needed any more in the above formalism) the four-momentum $P_\mu[\sigma]$ defined by Eq. (A3) gives the Bondi mass as defined in [7,8]. If σ' is taken to be another such null hypersurface, Eq. (A7) yields the Bondi mass loss formula (integrated in u). Further, the coordinate transformations (A4) comprise the BMS "supertranslations": a supertranslation given by Eqs. (C1)–(C3) below corresponds to a transformation (A4) with $a^\mu = \phi^\mu(\theta,\phi) + O(1/r)$, for some appropriate functions $\phi^\mu(\theta,\phi)$, so that b_μ in (A5) vanishes.

It should be pointed out that, as discussed in Sec. VI above, the fall-off conditions (A1) and (A2) allow for a large class of metrics with polyhomogeneous asymptotics. Last but not least, using the framework of [5] reduces the computational complexity of the proof of positivity of mass loss, as compared to several other frameworks, e.g., the Bondi-Sachs one.

APPENDIX B: ON SOME OPERATORS ON S^2

Let us denote by Δ_2 the Laplace-Beltrami operator associated with the standard metric on S^2, $\Delta_2 = \mathcal{D}^a\mathcal{D}_a$. Let SH^l denote the space of spherical harmonics of degree l ($g \in SH^l \Leftrightarrow \Delta_2 g = -l(l+1)g$). Consider the following sequence

$$V^0 \oplus V^0 \xrightarrow{i_{01}} V^1 \xrightarrow{i_{12}} V^2 \xrightarrow{i_{21}} V^1 \xrightarrow{i_{10}} V^0 \oplus V^0.$$

Here V^0 is the space of, say, smooth functions on S^2, V^1—that of smooth covectors on S^2, and V^2—that of symmetric traceless tensors on S^2. The various mappings above are defined as follows:

$$i_{01}(f,g) = f_{\|a} + \varepsilon_a{}^b g_{\|b},$$

$$i_{12}(v) = v_{a\|b} + v_{b\|a} - \check{h}_{ab} v^c_{\|c},$$

$$i_{21}(\chi) = \chi_a{}^b{}_{\|b},$$

$$i_{10}(v) = (v^a{}_{\|a}, \varepsilon^{ab} v_{a\|b}),$$

where $\|$ is used to denote the covariant derivative with respect to the Levi-Civita connection of the standard metric $\check{h}_{ab}$ on S^2. The following equality holds

$$i_{10} \circ i_{21} \circ i_{12} \circ i_{01} = [\Delta_2(\Delta_2+2)] \oplus [\Delta_2(\Delta_2+2)]. \quad \text{(B1)}$$

Note that we have $i_{10} \circ i_{21}(\chi) = (\chi^{ab}{}_{\|ab}, \varepsilon^{bc}\chi_b{}^a{}_{\|ac})$. Consider the space $\bar{V}^0 := [SH^0 \oplus SH^1]^\perp$, where $\perp$ denotes L^2 orthogonality in $L^2(S^2) \cap C^\infty(S^2)$. Now the operator $\Delta_2(\Delta_2+2)$ is surjective from V^0 to $\bar{V}^0$, so that for any $\lambda \in \bar{V}^0$ there exists $f \in V^0$ such that $\Delta_2(\Delta_2+2)f = \lambda$. Consider the tensor field $\chi = i_{12} \circ i_{01}((f,0))$, then (B1) shows that $\chi^{ab}{}_{\|ab} = \lambda$, which establishes surjectivity of the double divergence operator, from the space of symmetric traceless tensors to that of functions on the sphere which have no zero and first harmonics. Similarly the tensor field $\chi = i_{12} \circ i_{01}((0,g))$ shows that the map $V^2 \ni \chi_{ab} \to \varepsilon^c{}_b \chi^{ab}{}_{\|ac} \in [SH^0 \oplus SH^1]^\perp$ is surjective.

To justify our claim, that the vector field $\hat{\alpha}$ given by Eq. (3.25) is the unique solution of Eq. (3.24), consider the sequence

$$V^0 \oplus V^0 \xrightarrow{i_{01}} V^1 \xrightarrow{i_{10}} V^0 \oplus V^0.$$

It is easy to check that

$$i_{10} \circ i_{01} = \Delta_2 \oplus \Delta_2,$$

so if $\alpha_{a\|b}\varepsilon^{ab}$, $\alpha^a{}_{\|a} \in (SH^0)^\perp$ then there exist $f,g \in (SH^0)^\perp$ such that $i_{01}(f,g) = \alpha$, and they are the unique solutions in $(SH^0)^\perp$ of the equations:

$$\Delta_2 f = \alpha^a{}_{\|a}, \quad \Delta_2 g = \alpha_{a\|b}\varepsilon^{ab}.$$

Our claim follows immediately from this observation.

APPENDIX C: SUPERTRANSLATIONS

As in Appendix B we use the notation $f_{\|a} \equiv \mathcal{D}_a f$, $\Delta_2 \equiv \mathcal{D}^a\mathcal{D}_a$.

Consider a supertranslation which in an appropriate coordinate system on $\mathscr{I}$ reduces to a transformation u→u

[9] In the section in which he talks about radiating fields Trautman does not give a precise definition of what Σ is. In a preceding section of [5] where boundary conditions appropriate for spatial infinity are considered he uses the same equation to show that P_μ is conserved, and in that case he defines Σ as "a timelike "cylindrical" hypersurface at spatial infinity."

$-\alpha(\theta,\phi)$, for some, say smooth, function α on S^2, with the angular coordinate being left invariant. The supertranslation can be extended from $\mathscr{I}$ to a neighborhood thereof in the physical space-time using Bondi coordinates [cf. Eq. (3.3)]. This leads to the following asymptotic expansions (see also [8], p. 119):

$$\bar{x}^a = x^a + \frac{1}{r}\,\alpha^{\|a} - \frac{1}{2r^2}\,(\chi^{ab}\alpha_{\|b} - 2\alpha^{\|ab}\alpha_{\|b} + \breve{\Gamma}_{bc}{}^{a}\alpha^{\|b}\alpha^{\|c}) + \cdots, \tag{C1}$$

$$\bar{u} = u - \alpha - \frac{1}{2r}\,\alpha^{\|a}\alpha_{\|a} + \frac{1}{4r^2}\,[\chi^{ab}\alpha_{\|a}\alpha_{\|b} - \alpha^{\|a}(\alpha_{\|b}\alpha^{\|b})_{\|a}] + \cdots, \tag{C2}$$

$$\bar{r} = r - \frac{1}{2}\,\Delta_2\alpha + \frac{1}{2r}\left[\chi^{ab}{}_{\|b}\alpha_{\|a} + \frac{1}{2}\,\chi^{ab}\alpha_{\|ab} + \frac{1}{2}\,\chi^{ab}{}_{,u}\alpha_{\|a}\alpha_{\|b} - \frac{1}{2}\,\alpha^{\|ab}\alpha_{\|ab} - \alpha_{\|a}\alpha^{\|a} + \frac{1}{4}\,(\Delta_2\alpha)^2 - (\Delta_2\alpha)^{\|a}\alpha_{\|a}\right] + \cdots, \tag{C3}$$

where $\breve{\Gamma}_{bc}{}^{a}$ is the connection defined by the metric $\breve{h}_{ab}$. From those formulas we obtain the transformation laws for χ and M:

$$\bar{M}(\bar{u} = u - \alpha(x^a),\, x^a) = \left[M + \frac{1}{2}\,\chi^{ab}_{,u\|b}\alpha_{\|a} + \frac{1}{4}\,\chi^{ab}_{,u}\alpha_{\|ab} + \frac{1}{4}\,\chi^{ab}_{,uu}\alpha_{\|a}\alpha_{\|b}\right](u,x^a),$$

$$\bar{\chi}_{ab}(\bar{u} = u - \alpha(x^c), x^c) = [\chi_{ab} - 2\alpha_{\|ab} + \breve{h}_{ab}\Delta_2\alpha](u,x^c).$$

Consider the quantity $\bar{\chi}^{ab}{}_{\|\bar{a}\bar{b}}$, where $\|\bar{a}\bar{b}$ denotes covariant derivatives with respect to the transformed coordinates, $\partial_{\bar{a}} = \partial_a + \alpha_{,a}\partial_0$. Note that the occurrence of u derivatives in $\partial_{\bar{a}}$ will introduce u derivatives of χ_{ab} in the transformation formula for this quantity, and one finds that the combination $4M - \chi^{ab}{}_{\|ab}$ has a simple transformation law with respect to the supertranslations:

$$\overline{[4M - \chi^{ab}{}_{\|ab}]}(\bar{u} = u - \alpha(\theta,\phi), \theta, \phi) = [4M - \chi^{ab}{}_{\|ab} + \Delta_2(\Delta_2 + 2)\alpha](u,\theta,\phi).$$

The overbar in the left hand side of the last equation denotes the corresponding quantity calculated in the new Bondi frame. Note that while the equations (C1)–(C3) had only an asymptotic character in $1/r$, the last three equations are exact; in particular no smallness conditions on α have been imposed.

APPENDIX D: BOOST TRANSFORMATIONS AND P^μ

Let Λ be a boost-transformation with boost parameter ν; by an appropriate choice of space-coordinates we can choose it to act along the z axis. In coordinates (3.4) on Minkowski space-time one has

$$\bar{u} = \frac{u}{\cosh\,\nu - \sinh\,\nu\,\cos\,\theta} + O\left(\frac{u}{r}\right),$$

$$\tan\frac{\bar{\theta}}{2} = e^{\nu}\tan\frac{\theta}{2} + O\left(\frac{u}{r}\right),$$

with ϕ remaining unchanged. It follows that on $\mathscr{I}$ the boost Λ reduces to the transformation

$$\bar{u} = \frac{u}{\cosh\,\nu - \sinh\,\nu\,\cos\,\theta}, \quad \tan\frac{\bar{\theta}}{2} = e^{\nu}\tan\frac{\theta}{2}, \quad \bar{\phi} = \phi. \tag{D1}$$

It is natural to interpret (D1) as the definition of the action of the Lorentz boost Λ on $\mathscr{I}$ for general space-times admitting a $\mathscr{I}$.

Equation (D1) leads to the following transformation laws

$$\partial_{\bar{u}} = (\cosh\,\nu - \sinh\,\nu\,\cos\,\theta)\partial_u,$$

$$\partial_{\bar{\theta}} = u\,\sinh\,\nu\,\sin\,\theta\partial_u + (\cosh\,\nu - \sinh\,\nu\,\cos\,\theta)\partial_\theta,$$

$$\sin\,\bar{\theta} = \frac{\sin\,\theta}{\cosh\,\nu - \sinh\,\nu\,\cos\,\theta},$$

$$d\bar{\theta} = \frac{d\theta}{\cosh\,\nu - \sinh\,\nu\,\cos\,\theta},$$

$$\cos\,\bar{\theta} = \frac{\cosh\,\nu\,\cos\,\theta - \sinh\,\nu}{\cosh\,\nu - \sinh\,\nu\,\cos\,\theta}. \tag{D2}$$

From Eq. (D2) one obtains the well known statement, that boosts induce conformal transformations of "spheres at infinity": if we denote by ψ the transformation which takes (θ,ϕ) to $(\bar{\theta},\bar{\phi})$, then

$$\psi^*\breve{h}_{ab} = \varphi^{-2}\breve{h}_{ab}, \tag{D3}$$

with

$$\varphi = \cosh\,\nu - \sinh\,\nu\,\cos\,\theta.$$

We note that φ is a linear combination of $l=0$ and $l=1$ spherical harmonics. Set

$$\bar{r} = \varphi r. \tag{D4}$$

The coordinate transformation (D1), (D4) preserves the leading order behavior of all the components of the metric (3.3).

It follows from [19] (compare also [23]) that (D1), (D4) can be extended to a neighborhood of $\mathscr{I}$ while preserving the Bondi form of the metric (3.3), the hypersurface $u=0$ being mapped into the hypersurface $\bar{u}=0$. From (D1), (D4), and (3.3) at $u=0$ one immediately obtains

$$\bar{M}=\varphi^3 M, \tag{D5}$$

so that

$$\int_{S^2}\bar{M}\,\sin\bar{\theta}\,d\bar{\theta}\,d\bar{\phi}$$

$$=\int_{S^2}M(\cosh\nu-\sinh\nu\,\cos\theta)\sin\theta\,d\theta\,d\phi.$$

It follows that the knowledge of the $l=0$ harmonics of M is not sufficient to determine the $l=0$ harmonics of $\bar{M}$. Let us set

$$m_{TB}|_{u=0}:=\frac{1}{4\pi}\int_{S^2}M|_{u=0}\sin\theta\,d\theta\,d\phi, \tag{D6}$$

$$p^k|_{u=0}:=\frac{1}{4\pi}\int_{S^2}M|_{u=0}n^k\sin\theta\,d\theta\,d\phi, \tag{D7}$$

where n^k, $k=1,2,3$ denotes the functions $\sin\theta\cos\phi$, $\sin\theta\sin\phi$ and $\cos\theta$, in that order. Equations (D1), (D4), and (D5) also yield

$$\bar{M}\,\cos\bar{\theta}\,\sin\bar{\theta}\,d\bar{\theta}=M(\cosh\nu\,\cos\theta-\sinh\nu)\sin\theta\,d\theta.$$

Consequently we obtain the transformation law

$$\bar{m}_{TB}=m_{TB}\cosh\nu-p^z\sinh\nu, \tag{D8}$$

$$\bar{p}^z=p^z\cosh\nu-m_{TB}\sinh\nu. \tag{D9}$$

As the choice of the axis along which Λ acts was arbitrary, the set of numbers $(p^\mu)=(m_{TB},p^k)$ transforms as a (contravariant) four-vector under the *passive* action of the Lorentz group on $\mathscr{I}$. It is therefore natural to interpret m_{TB} as the time component, and the p^k's as space-components of an energy-momentum four-vector p^μ. We use the qualification "passive" above to emphasize the fact that such a simple transformation property holds only for those Lorentz transformations which map a chosen cross-section of $\mathscr{I}$ into itself.

APPENDIX E: CHANGES OF THE NOETHER CHARGE INDUCED BY CHANGES OF THE LAGRANGE FUNCTION

In this Appendix we wish to derive the transformation rule of the "Noether charge" (1.5), when the Lagrange function is changed by the addition of a term of the form (1.4),

$$\mathcal{L}\to\hat{\mathcal{L}}=\mathcal{L}+R,\quad R\equiv\partial_\lambda Y^\lambda, \tag{E1}$$

with Y^μ being a smooth function of the fields and their derivatives up to order $k-1$. Letting Ω be an arbitrary domain of $\mathbb{R}^n$ with smooth boundary and compact closure, we have

$$\int_{\partial\Omega}Y^\mu dS_\mu=\int_\Omega R\;d^nx.$$

Integration by parts gives

$$\int_{\partial\Omega}\sum_{i=0}^{k-1}\left(\frac{\partial Y^\mu}{\partial\phi^A_{\alpha_1\ldots\alpha_i}}-\sum_{j=0}^{k-i-1}(-1)^j\partial_{\beta_1}\ldots\partial_{\beta_j}\right.$$

$$\left.\times\left(\frac{\partial R}{\partial\phi^A_{\mu\alpha_1\ldots\alpha_i\beta_1\ldots\beta_j}}\right)\right)\delta\phi^A_{\alpha_1\ldots\alpha_i}dS_\mu$$

$$=\int_\Omega\frac{\delta R}{\delta\phi^A}\,\delta\phi^A d^nx, \tag{E2}$$

where $\delta R/\delta\phi^A$ is the variational derivative of R, for any smooth fields $\delta\phi^A$. Equation (E2) still holds with $\Omega=\mathbb{R}^n$ if the $\delta\phi^A$'s are compactly supported. In that case arbitrariness of the $\delta\phi^A$'s implies

$$\frac{\delta R}{\delta\phi^A}=0,$$

which expresses the well known fact that the field equations are unchanged by the above transformation of the Lagrange function. It follows that

$$\int_{\partial\Omega}\sum_{i=0}^{k-1}\left(\frac{\partial Y^\mu}{\partial\phi^A_{\alpha_1\ldots\alpha_i}}-\sum_{j=0}^{k-i-1}(-1)^j\partial_{\beta_1}\ldots\partial_{\beta_j}\right.$$

$$\left.\times\left(\frac{\partial R}{\partial\phi^A_{\mu\alpha_1\ldots\alpha_i\beta_1\ldots\beta_j}}\right)\right)\delta\phi^A_{\alpha_1\ldots\alpha_i}dS_\mu=0. \tag{E3}$$

It is convenient to choose a coordinate system $(x^\mu)=(x^1,v^a)$ such that $\partial\Omega$ is given by the equation $x^1=0$, the v^a's, $a=1,\ldots,n-1$ being coordinates on $\partial\Omega$. Define

$$\phi^{A,m}_{a_1\ldots a_l}=\phi^A_{\underbrace{1\ldots1}_{m\ \text{times}}a_1\ldots a_l},$$

$$R^{a_1\ldots a_l}_{A,m}=\sum_{j=0}^{k-l-m}(-1)^j\partial_{\beta_1}\ldots\partial_{\beta_j}\left(\frac{\partial R}{\partial\phi^{A,m}_{a_1\ldots a_l\beta_1\ldots\beta_j}}\right).$$

Integration by parts in (E3) yields

$$\int_{\partial\Omega}\sum_{m=0}^{k-1}\sum_{i=0}^{k-m-1}(-1)^i\partial_{a_1}\ldots\partial_{a_i}\times\left(\frac{\partial Y^1}{\partial\phi^{A,m}_{a_1\ldots a_i}}-R^{a_1\ldots a_i}_{A,m+1}\right)\delta\phi^{A,m}d^{n-1}v=0. \quad \text{(E4)}$$

As the $\delta\phi^{A,m}$'s are arbitrary we conclude that

$$\sum_{i=0}^{k-m-1}(-1)^i\partial_{a_1}\ldots\partial_{a_i}\left(\frac{\partial Y^1}{\partial\phi^{A,m}_{a_1\ldots a_i}}-R^{a_1\ldots a_i}_{A,m+1}\right)=0. \quad \text{(E5)}$$

Let $\hat{E}^\lambda$ be the Noether current (1.2) corresponding to the Lagrange function $\hat{\mathcal{L}}$, as in (E1). For our purposes it is sufficient to consider vector fields X^λ which are transverse to Σ. We can choose a coordinate system in a neighborhood of Σ so that Σ is given by the equation $x^1=0$, and moreover $X^\lambda\partial_\lambda=\partial_1$. From the definition of $\hat{E}^\lambda$ and E^λ we obtain

$$\begin{aligned}\hat{E}^1&=E^1+\sum_{m=0}^{k-1}\sum_{i=0}^{k-m-1}R^{a_1\ldots a_i}_{A,m+1}\phi^{A,m+1}_{a_1\ldots a_i}-\partial_\mu Y^\mu\\&=E^1+\sum_{m=0}^{k-1}\sum_{i=0}^{k-m-1}\left(R^{a_1\ldots a_i}_{A,m+1}-\frac{\partial Y^1}{\partial\phi^{A,m}_{a_1\ldots a_i}}\right)\phi^{A,m+1}_{a_1\ldots a_i}-\partial_a Y^a.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_\Sigma\hat{E}^\lambda dS_\lambda-\int_\Sigma E^\lambda dS_\lambda\\&\quad=\int_\Sigma\sum_{m=0}^{k-1}\sum_{i=0}^{k-m-1}(-1)^i\partial_{a_1}\ldots\partial_{a_i}\times\left(R^{a_1\ldots a_i}_{A,m+1}-\frac{\partial Y^1}{\partial\phi^{A,m}_{a_1\ldots a_i}}\right)\phi^{A,m+1}d^{n-1}v\\&\quad\quad+\int_{\partial\Sigma}\left(Y^a-\sum_{m=0}^{k-1}\sum_{i=0}^{k-m-2}\phi^{A,m+1}_{a_1\ldots a_i}\times\sum_{l=0}^{k-m-i-2}(-1)^l\partial_{b_1}\ldots\partial_{b_l}\left(R^{a_1\ldots a_i b_1\ldots b_l a}_{A,m+1}-\frac{\partial Y^1}{\partial\phi^{A,m}_{a_1\ldots a_i b_1\ldots b_l a}}\right)\right)X^\lambda dS_{\lambda a}.\end{aligned} \quad \text{(E6)}$$

The integral over Σ in the right hand side of this last equation vanishes by (E5), which establishes our claim that the Noether charge of Σ, defined as $\int_\Sigma E^\lambda dS_\lambda$, changes by a boundary integral under the change (E1) of the Lagrange function.

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