Killing vectors in asymptotically flat space–times. I. Asymptotically translational Killing vectors and the rigid positive energy theorem

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We study Killing vector fields in asymptotically flat space–times. We prove the following result, implicitly assumed in the uniqueness theory of stationary black holes. If the conditions of the rigidity part of the positive energy theorem are met, then in such space–times there are no asymptotically null Killing vector fields, except if the initial data set can be embedded in Minkowski space–time. We also give a proof of the nonexistence of nonsingular (in an appropriate sense) asymptotically flat space–times that satisfy an energy condition and that have a null ADM four-momentum, under conditions weaker than previously considered. © *1996 American Institute of Physics*. [S0022-2488(96)01604-1]

I. INTRODUCTION

A prerequisite for an analysis of stationary black holes is the understanding of properties of Killing vector fields in asymptotically flat space–times. There exist various papers analyzing properties of Killing vector fields in asymptotically flat space–times.[1–4] These papers do not, however, seem to give answers to the questions asked here. Moreover, the asymptotic conditions here are considerably weaker than considered in those references. Consider an asymptotically flat partial Cauchy surface Σ in a space–time $(M,g_{\mu\nu})$ with a Killing vector field X^μ. In the case of a stationary black hole one is interested in situations where X^μ is time-like in the asymptotic regions. [Here we say that an asymptotically flat space–time $(M,g_{\mu\nu})$ with a Killing vector field X^μ is stationary if X^μ is time-like in the asymptotic regions of M.] A natural question to ask is, how does X^μ behave in the asymptotic regions? Now it is easily seen from the equations

$$\nabla_\mu\nabla_\nu X_\alpha = R^\lambda{}_{\mu\nu\alpha}X_\lambda \tag{1.1}$$

(which are a well-known consequence of the Killing equations) and from the asymptotic flatness conditions (cf. Propositions 2.1, Sec. II, for a precise description of the asymptotic conditions needed here) that there exist constants A^μ such that every Killing vector field X^μ that is time-like for $r\geqslant R$ for some R satisfies

$$\begin{aligned} &X^\mu - A^\mu \to_{r\to\infty} 0,\\ &\eta_{\alpha\beta}A^\alpha A^\beta \leqslant 0. \end{aligned} \tag{1.2}$$

Here $\eta_{\alpha\beta}$ is the Minkowski metric, and we use the signature $(-,+,+,+)$. It should be emphasized that the requirement of time-likeness of X^μ for large r does *not* exclude the possibility that $\eta_{\alpha\beta}A^\alpha A^\beta$ vanishes. Indeed, an explicit example of a metric (not satisfying any reasonable field

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equations) with an everywhere time-like Killing vector that is asymptotically null can be found in Ref. 5 (cf. the Remark preceding Theorem A.1, Appendix A of Ref. 5. (Let us point out that by a null vector we mean a nonzero vector of zero Lorentzian length.) Now in the uniqueness theory of black holes it is customary to assume that $A^\mu = \delta^\mu_0$ in an asymptotically flat coordinate system in which Σ is given by an equation $x^0 = 0$. If the orbits of the Killing vector field X^μ are complete (at least in the asymptotic region) and if A^μ is time-like, then Σ can be deformed (''boosted'') to a new partial Cauchy surface for which $A^\mu = \delta^\mu_0$ (in an appropriately redefined asymptotically flat coordinate system). If, however, X^μ is asymptotically null [by which we mean that the vector A^μ appearing in (1.2) is null], then no such deformation is possible and we are faced with the intriguing possibility of existence of stationary space–times in which the Killing vector cannot be reduced to a standard form, where the flat background metric is diagonal and the vector A^μ of (1.2) equals δ^μ_0. As has been argued in Ref. 6, the existence of such Killing vector fields does not seem to be compatible with the rigidity part of the positive energy theorems. Here we make the arguments of Ref. 6 precise and show the following (the reader is referred to Theorem 3.4 for a more precise formulation).

Theorem 1.1: *Let $(M,g_{\mu\nu})$ be a space–time with a Killing vector field that is asymptotically null along an (appropriately regular) asymptotically flat space-like hypersurface Σ. Then the ADM energy-momentum vector of Σ vanishes.*

To say more about space–times considered in Theorem 1.1, one can use the positive energy theorem. In Sec. IV below we prove the following.

Theorem 1.2 (''Time-like future-pointing four-momentum theorem''): *Under the conditions of Theorems 4.1 and 4.2 below, suppose that the initial data (Σ,g_{ij},K_{ij}) are not initial data for Minkowski space–time. Then the ADM energy-momentum vector p^μ of Σ satisfies*

$$p^0 > \sqrt{\sum_{i=1}^{3} (p^i)^2}.$$

(*Various variants of Theorem 1.2 are, of course, well known; cf. Sec. IV for a detailed discussion.*)

Theorem 1.1 can be used together with Theorem 1.2 to obtain the following.

Theorem 1.3: *Let $(M,g_{\mu\nu})$ be a maximal globally hyperbolic space–time with a Cauchy surface satisfying the requirements of Theorems 4.1 and 4.2. Let X^μ be a Killing vector field on M that is asymptotically null along an asymptotically flat Cauchy surface. Then X^μ is everywhere null and $(M,g_{\mu\nu})$ is the Minkowski space–time.*

Theorem 1.3 and the results of Ref. 7 [cf. also Ref. 6 (Theorem 1.7)] show that there is no loss of generality in assuming that $A^\mu = \delta^\mu_0$ in, say electrovacuum, maximal globally hyperbolic space-times with an appropriately regular asymptotically flat Cauchy surface. Let us mention that the results here settle in the positive Conjecture 1.8 of Ref. 6.

This paper is organized as follows. In Sec. II we discuss some general properties of Killing vector fields in asymptotically flat space–times. In order to minimize the number of assumptions, we adopt a 3+1-dimensional point of view; the various advantages for doing that are explained at the beginning of Sec. II. The main result there is Proposition 2.1, which establishes the asymptotic behavior of Killing vectors along asymptotically flat space-like surfaces. In that section we also introduce the notion of *Killing development*, which turns out to be very useful in our analysis. In Sec. III we study the relationship between the ADM four-momentum and the asymptotic behavior of the Killing vector. The results there can be summarized as follows: If $X^\mu \to_{r\to\infty} A^\mu$ along an asymptotically flat space-like surface Σ, then the ADM four-momentum is proportional to A^μ. The proportionality constant is zero when A^μ is not time-like. Let us point out that some similar results can be found in Ref. 1. However, in that reference the possibility of asymptotically null Killing vector fields is not taken into consideration. Also, in Ref. 1, rather strong asymptotic conditions are imposed. In a sense most of the work here consists in showing that the asymptotic conditions needed to be able to obtain the desired conclusions can actually be derived from the decay

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conditions on the matter sources and from the hypothesis of existence of Killing vector fields. In Sec. IV we prove a positive energy theorem with hypotheses and asymptotic conditions appropriate for our purposes. In Theorems 4.1 and 4.2 there are improvements of known results; cf. the beginning of Sec. IV for a more detailed discussion.

II. KILLING VECTORS AND SPACE-LIKE HYPERSURFACES

Consider a space–time $(M,g_{\mu\nu})$ with a Killing vector field X^μ,

$$\nabla_\mu X_\nu+\nabla_\nu X_\mu=0, \tag{2.1}$$

where ∇_μ is the covariant derivative operator of the metric $g_{\mu\nu}$. Let Σ be a space-like hypersurface in M and suppose that on Σ the field of unit normals n^μ can be defined; this will be the case, e.g., if $(M,g_{\mu\nu})$ is time orientable. On Σ define a scalar field N and a vector field Y^i by the equations

$$N=-n_\mu X^\mu, \tag{2.2}$$

$$g_{ij}Y^i\,dx^j=i^*(g_{\mu\nu}X^\mu\,dx^\nu), \tag{2.3}$$

where i denotes the embedding of Σ into M. We use the symbol g_{ij} to denote the pull-back metric $i^*g_{\mu\nu}$. Equation (2.1) with $\mu=i$ and $\nu=j$ reads as

$$2NK_{ij}+\mathcal{L}_Y g_{ij}=0, \tag{2.4}$$

where $\mathcal{L}$ denotes the Lie derivative and K_{ij} is the extrinsic curvature tensor of $i(\Sigma)$ in $(M,g_{\mu\nu})$, defined as the pull-back to Σ of $\nabla_\mu n_\nu$. (K_{ij} as defined here is $-K_{ij}$ as in Ref. 8; however, J^i as defined here coincides with J^i as defined there.) Set

$$\Sigma_{N>0}=\{p\in\Sigma: N\neq 0\}.$$

In a neighborhood of $\Sigma_{N>0}$ we can introduce a coordinate system (u,x^i) in which $X^\mu\partial_\mu=\partial_u$ and in which $\Sigma_{N>0}$ is given by the equation $u=0$. The metric on this neighborhood takes the form

$$g_{\mu\nu}\,dx^\mu\,dx^\nu=-N^2\,du^2+g_{ij}(dx^i+Y^i\,du)(dx^j+Y^j\,du), \tag{2.5}$$

with some functions that do not depend upon u. Let $G_{\mu\nu}$ be the Einstein tensor of $g_{\mu\nu}$, that is, $G_{\mu\nu}=R_{\mu\nu}-(g^{\alpha\beta}R_{\alpha\beta}/2)g_{\mu\nu}$, where $R_{\mu\nu}$ is the Ricci tensor of $g_{\mu\nu}$. We have the 3+1 decomposition formulas (cf. e.g., Ref. 8),

$$2G_{\mu\nu}n^\mu n^\nu={}^3R+K^2-K^{ij}K_{ij}, \tag{2.6}$$

$$G_{i\mu}n^\mu=D_j(K^{ij}-g^{kl}K_{kl}g^{ij}), \tag{2.7}$$

$$G_{ij}-\tfrac{1}{2}g^{kl}G_{kl}g_{ij}={}^3R_{ij}+KK_{ij}-2K_{ik}K^k,\\ -N^{-1}(\mathcal{L}_Y K_{ij}+D_i\,D_j N)-\tfrac{1}{2}G_{\mu\nu}n^\mu n^\nu g_{ij}. \tag{2.8}$$

Here g^{ij} is the tensor inverse to g_{ij}, $K=g^{kl}K_{kl}$, ${}^3R_{ij}$ is the Ricci tensor of the metric g_{ij}, and ${}^3R={}^3R_{ij}\,g^{ij}$. All the above is, of course, well known; we have written it down in detail to fix the notation and to spell out the conditions needed for the definition of the fields N and Y^i. In particular, we wish to emphasize that we did not need to assume completeness of the orbits of X^μ, we did not need to assume that the orbits of X^μ intersect Σ only once, etc. It is, however, the case that those last properties are needed for several arguments, e.g., in the uniqueness theory of black holes (cf., e.g., Ref. 6). By way of example, consider a maximal globally hyperbolic space–time

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$(M,g_{\mu\nu})$ with an asymptotically flat Cauchy surface with compact interior, with a metric satisfying the Einstein–Yang–Mills–Higgs equations, and with a Killing vector field X^μ. While one expects the orbits of X^μ to be complete (cf., e.g., Ref. 7 for an analysis in the vacuum case), no proof of such a result has been established so far. It is therefore of interest to establish various properties of space–times $(M,g_{\mu\nu})$ with Killing vectors with a minimal amount of global assumptions on M. As one is often interested in globally hyperbolic space–times it does not seem to be overly restrictive to assume the existence in M of a space-like hypersurface Σ satisfying the hypotheses spelled out at the beginning of this section. The construction above yields then a scalar field N and a vector field Y^i defined on Σ, such that Eqs. (2.4)–(2.8) hold. Consider then a set $(\Sigma,g_{ij},K_{ij},N,Y^i)$. We shall call the *Killing development of* $(\Sigma,g_{ij},K_{ij},N,Y^i)$ the space–time $(\hat{M},\hat{g}_{\mu\nu})$, where

$$\hat{M}=\mathbf{R}\times\Sigma_{N>0},$$

and where $\hat{g}_{\mu\nu}$ is given by the equation

$$\begin{aligned}&\hat{g}_{\mu\nu}\,dx^\mu\,dx^\nu=-\hat{N}^2\,du^2+\hat{g}_{ij}(dx^i+\hat{Y}^i\,du)(dx^j+\hat{Y}^j\,du),\\&\hat{N}(u,x^i)=N(x^i),\quad \hat{g}_{ij}(u,x^i)=g_{ij}(x^i),\quad \hat{Y}^i(u,x^i)=Y^i(x^i).\end{aligned}\tag{2.9}$$

Here the u coordinate runs over the $\mathbf{R}$ factor in $\mathbf{R}\times\Sigma_{N>0}$. Clearly the vector field $X^\mu\partial_\mu=\partial_u$ is a Killing vector, so that

$$\hat{\nabla}_\mu X_\nu+\hat{\nabla}_\nu X_\mu=0,\tag{2.10}$$

where $\hat{\nabla}_\mu$ is the covariant derivative operator of the metric $\hat{g}_{\mu\nu}$. Note that

$$X_i|_{u=0}=Y_i,\quad \hat{N}|_{u=0}=N.\tag{2.11}$$

Consider the extrinsic curvature tensor $\hat{K}_{ij}$ of the slices $u=$const. In general, $\hat{K}_{ij}$ will have nothing to do with the tensor field K_{ij}. Suppose, however, that (2.4) holds. Equation (2.10), with $i=\mu$ and $\nu=j$; Eqs. (2.11) and (2.4) give, then, at $u=0$,

$$\hat{K}_{ij}=K_{ij}.\tag{2.12}$$

Since $\hat{K}_{ij}$ is u independent, it follows that this last relation holds throughout $\hat{M}$. One also notices that (2.12) will hold if and only if (2.4) holds.

Consider, next, the Einstein tensor $\hat{G}_{\mu\nu}$ of the metric $\hat{g}_{\mu\nu}$. It is given by the hatted equivalent of Eqs. (2.6)–(2.8). Given the set $(\Sigma,g_{ij},K_{ij},N,Y^i)$, one can define on $\Sigma_{N>0}$ a scalar field ρ, a vector field J^i, and a tensor field τ_{ij} via the equations

$$2\rho={}^3R+K^2-K^{ij}K_{ij},\tag{2.13}$$

$$J^i=D_j(-K^{ij}+Kg^{ij}),\tag{2.14}$$

$$\begin{aligned}\tau_{ij}-\tfrac{1}{2}g^{kl}\tau_{kl}g_{ij}={}&{}^3R_{ij}+KK_{ij}-2K_{ik}K^k{}_j\\&-N^{-1}(\mathcal{L}_Y K_{ij}+D_i\,D_j N)-(\rho/2)g_{ij}.\end{aligned}\tag{2.15}$$

If Eq. (2.4) holds, it follows from (2.11)–(2.12) that we will have

$$\hat{G}_{\mu\nu}\hat{n}^\mu\hat{n}^\nu(u,x^l)=\rho(x^l),\quad \hat{G}_{i\nu}\hat{n}^\nu(u,x^l)=-J_i(x^l),\quad G_{ij}(u,x^l)=\tau_{ij}(x^l),\tag{2.16}$$

where $\hat{n}_\mu$ is the unit normal to the slices $u=$const.

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It is of interest to consider the case of covariantly constant Killing vector fields. In that case on a hypersurface Σ as at the beginning of this section, we will have

$$NK_{ij}+D_iY_j=0, \tag{2.17}$$

$$K_{ij}Y^j+D_iN=0. \tag{2.18}$$

Let us show that if (2.17)–(2.18) hold, then the vector field $X^\mu\partial_\mu=\partial_\mu$ on the Killing development $(\hat{M},\hat{g}_{\mu\nu})$ of $(\Sigma,g_{ij},K_{ij},N,Y^i)$ will be covariantly constant. To see that, note that Eqs. (2.17), (2.11), and (2.12) imply

$$\hat{\nabla}_iX_j=0,$$

at $u=0$, hence, throughout $\hat{M}$. Equation (2.18) similarly gives

$$\hat{\nabla}_iX_0=0.$$

As X^μ satisfies (2.10), the equations $\hat{\nabla}_\mu X_\nu=0$ readily follow.

When considering space–times with symmetries, it is essential to have precise information on the behavior of Killing vector fields in the asymptotic regions. The following is a straightforward consequence of Eqs. (2.13) and (2.15) (cf. also Ref. 9, Theorem 3.3 and Proposition 3.2). The notation O_k is defined in Appendix A. An outline of the proof is given in Appendix C.

Proposition 2.1: Let $R>0$ and let (g_{ij},K_{ij}) be initial data on $\Sigma_R\equiv\mathbf{R}^3\backslash B(R)$, satisfying

$$g_{ij}-\delta_{ij}=O_k(r^{-\alpha}),\quad K_{ij}=O_{k-1}(r^{-1-\alpha}), \tag{2.19}$$

with some $k>1$ and some $0<\alpha<1$. Let N be a C^2 scalar field and Y^i a C^2 vector field on Σ_R, such that Eqs. (2.4), (2.13), and the equation

$$N(\tau_{ij}-\tfrac{1}{2}g^{kl}\tau_{kl}g_{ij})=N({}^3R_{ij}+KK_{ij}-2K_{ik}K^k{}_j)-\mathcal{L}_YK_{ij}-D_i\,D_jN-N(\rho/2)g_{ij},$$

hold, with some ρ and τ_{ij} satisfying

$$|\rho|+|\tau_{ij}|\leqslant C(1+r)^{-2-\alpha}. \tag{2.20}$$

Then there exists numbers $\Lambda_{\mu\nu}=\Lambda_{[\mu\nu]}$, such that we have

$$Y^i-\Lambda_{ij}x^j=O_k(r^{1-\alpha}),\quad N+\Lambda_{0i}x^i=O_k(r^{1-\alpha}). \tag{2.21}$$

If $\Lambda_{\mu\nu}=0$, then there exist numbers A^μ, such that we have

$$Y^i-A^i=O_k(r^{-\alpha}),\quad N-A^0=O_k(r^{-\alpha}). \tag{2.22}$$

If $\Lambda_{\mu\nu}=A^\mu=0$, then $Y^i\equiv N\equiv 0$.

Let us remark that if $\alpha=1$, then Proposition 2.1 holds with the function $r^{1-\alpha}$ in the right-hand side of Eq. (2.21) replaced by $1+|\log r|$; similarly in (2.22), $r^{-\alpha}$ has to be replaced by $r^{-1}(1+|\log r|)$.

A Killing vector field for which $\Lambda_{\mu\nu}=0$ will be called *asymptotically translational*.

For further use let us mention the following: Consider (g_{ij},K_{ij}), such that (2.19) holds, and suppose that (N,Y^i) satisfy (2.22) with some $A^0\neq 0$. Suppose finally that (2.4) is weakened to

$$2NK_{ij}+\mathcal{L}_Yg_{ij}=O_{k-1}(r^{-\beta}), \tag{2.23}$$

with some $\beta\geqslant 1$. In that case (2.16) will be replaced by

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$$\hat{G}_{\mu\nu}\hat{n}^{\mu}\hat{n}^{\nu}-\rho=O_{k-1}(r^{-\min(1+\alpha,\beta)-\beta}),\quad \hat{G}_{i\nu}\hat{n}^{\nu}+J_i=O_{k-2}(r^{-\beta-1}),$$
$$\hat{G}_{ij}-\tau_{ij}=O_{k-2}(r^{-\beta-1}). \tag{2.24}$$

III. ADM FOUR-MOMENTUM IN SPACE–TIMES WITH ASYMPTOTICALLY TRANSLATIONAL KILLING VECTORS

In this section we address the following question: Consider a space–time with an asymptotically flat space-like surface Σ, and with a translational Killing vector field X^{μ}, that is, there exist constants A^{μ} such that $X^{\mu}\to_{r\to\infty}A^{\mu}$ along Σ. Then, is it true that

(1) If $A^{\mu}A_{\mu}\geqslant 0$, then the ADM four-momentum p^{μ} vanishes.

(2) If $A^{\mu}A_{\mu}<0$, then p^{μ} is proportional to A^{μ}.

We shall show that this is indeed the case, using the three-dimensional framework discussed in Sec. II. Proposition 2.1 in that section justifies our fall-off conditions on the fields N and Y^i. The results here are actually slightly more general than stated above, in that we allow for fields that satisfy the relevant Killing equations up to terms that decay at an appropriate rate; cf. below for the precise conditions.

Proposition 3.1: Let $R>0$ and let (g_{ij},K_{ij}) be initial data on $\Sigma_R\equiv\mathbf{R}^3\backslash B(R)$, satisfying

$$g_{ij}-\delta_{ij}=O_2(r^{-\alpha}),\quad K_{ij}=O_1(r^{-1-\alpha}),\quad \alpha>\tfrac{1}{2}, \tag{3.1}$$

$$J^i=O(r^{-3-\epsilon}),\quad \rho=O(r^{-3-\epsilon}),\quad \epsilon>0. \tag{3.2}$$

Let N be a C^1 scalar field and Y^i a C^1 vector field on Σ_R, such that

$$N-A^0=O_1(r^{-\alpha}),\quad Y^i\to_{r\to\infty}A^i, \tag{3.3}$$

for some set of constants $(A^{\mu})\neq 0$. Suppose further that

$$2NK_{ij}+\mathcal{L}_Y g_{ij}=O_1(r^{-2-\epsilon}). \tag{3.4}$$

Let p^{μ} be the ADM four-momentum of Σ_R. Then

(1) If $A^0=0$, then $p^0=0$.

(2) If $A^0\neq 0$, then p^{μ} is proportional to A^{μ}.

Remark: The pointwise decay estimates assumed above can be weakened to weighted Sobolev spaces conditions. To avoid a tedious discussion of technicalities we shall, however, not consider such fields here.

Proof: Without loss of generality we may assume that both α and ϵ are strictly smaller than 1. Equation (3.3) and a simple analysis of Eq. (3.4) (cf., e.g., the proof of Proposition 2.1, Appendix C) show that

$$Y^i-A^i=O_2(r^{-\alpha}). \tag{3.5}$$

By our asymptotic conditions, Eq. (3.4) can be rewritten as

$$g_{ij,k}A^k+Y^i{}_{,j}+Y^j{}_{,i}=-2A^0K_{ij}+O_1(r^{-2-\epsilon}), \tag{3.6}$$

and we have redefined ϵ to be $\min(\epsilon,2\alpha-1)>0$. The momentum-constraint equation reads as

$$\partial_i K_{ij}=\partial_j K+O(r^{-3-\epsilon}), \tag{3.7}$$

where $K=g^{ij}K_{ij}$. Taking the divergence of (3.6) and using (3.7) gives

$$g_{ij,kj}A^k+\Delta_{\delta}Y^i+\partial_i(Y^j{}_{,j})=-2A^0K_{,i}+O(r^{-3-\epsilon}). \tag{3.8}$$

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Here $\Delta_\delta=\Sigma_i\partial_i\partial_i$. Contracting i with j in (3.6) allows us to eliminate $\partial_j Y^j$ in (3.8) in terms of $K_{,i}$, so that (3.8) leads to

$$\Delta_\delta Y^i=-A^0K_{,i}-(g_{ij,j}-\tfrac{1}{2}g_{jj,i})_{,k}A^{,k}+O(r^{-3-\epsilon}).$$

In what follows we shall freely make use of properties of harmonic functions on Σ_R that were established in, e.g., (Refs. 10–12). Increasing R if necessary we may choose harmonic coordinates on Σ_R,

$$\partial_i(g^{ij}\sqrt{\det g})=0,$$

with

$$g_{ij}-\delta_{ij}=O_1(r^{-\alpha}).$$

[There arises a slight difficulty here, related to the fact that the metric might not satisfy the conditions (3.1) in harmonic coordinates due to a loss of classical differentiability. All the details of the proof as written here can be justified if a Hölder differentiability index λ is added in Eqs. (3.1)–(3.2). We wish to stress that the statement of our result is correct as stated. This can be verified by keeping track of weighted Sobolev differentiability of various error terms that arise in our equations, making use of the estimates of Ref. 11. In order to make the argument more transparent we have chosen to present our proof without the introduction of weighted Sobolev spaces.]

If $A^0=0$, define φ to be identically zero; otherwise let $\varphi=O_1(r^{1-\alpha})$ be a solution of

$$\Delta_\delta\varphi=-A^0K. \tag{3.9}$$

Setting $Z^i=Y^i-A^i-\varphi^{,i}$, one is led to

$$\Delta_\delta Z^i=O(r^{-3-\epsilon}),$$

so that there exist numbers $\alpha^i\in\mathbf{R}$ such that

$$Z^i=\frac{\alpha^i}{r}+O_1(r^{-1-\epsilon}).$$

A contraction over i and j in (3.6) gives

$$Z^i{}_{,i}=-\frac{\alpha^ix^i}{r^3}+O(r^{-2-\epsilon})=-\frac{1}{2}\,g_{ii,k}A^k+O(r^{-2-\epsilon}). \tag{3.10}$$

The scalar constraint equation in harmonic coordinates gives

$$\Delta_\delta g_{ii}=O(r^{-3-\epsilon})\Rightarrow g_{ii}=3+\frac{\beta}{r}+O_1(r^{-1-\epsilon}), \tag{3.11}$$

for some constant β. Equation (3.11) inserted in the formula for the ADM mass yields

$$m=\frac{1}{16\pi}\int_{S_\infty}(g_{ij,j}-g_{jj,i})\,dS_i=-\frac{1}{32\pi}\int g_{jj,i}\,dS_i=\frac{\beta}{8}. \tag{3.12}$$

Inserting this in (3.10), one is led to

$$\alpha^i=-4mA^i,$$

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so that one finally obtains

$$Y^i = A^i\left(1-\frac{4m}{r}\right)+\varphi_{,i}+O_1(r^{-1-\epsilon}). \tag{3.13}$$

Suppose first that $A^0=0$. In this case we necessarily have $A^i\not\equiv 0$, and rescaling $X^\mu\partial_\mu$ if necessary, we can choose coordinates so that $A^i=\delta^i_z$. Equation (3.6) now reads as

$$g_{AB,z}=O(r^{-2-\epsilon}), \tag{3.14}$$

$$(g_{zz}+2Y^z)_{,z}=O(r^{-2-\epsilon}), \tag{3.15}$$

$$g_{zA,z}=\left(\frac{4m}{r}\right)_{,A}+O(r^{-2-\epsilon}). \tag{3.16}$$

Let $\rho^2=x^2+y^2$. For $\rho\geq R$, Eq. (3.16) gives

$$0=x^A\int_{-\infty}^{\infty} g_{zA,z}\,dz=-4m\int_{-\infty}^{\infty}\frac{dz}{(1+z^2)^{3/2}}+\int_{-\infty}^{\infty}O(r^{-2-\epsilon})dz.$$

To estimate the second integral, it is convenient to consider separately the integrals $\int_{-\infty}^{-\rho}$, $\int_{-\rho}^{\rho}$, and $\int_{\rho}^{\infty}$. Elementary estimates then show that this integral is $O(\rho^{-\epsilon})$; passing to the limit $\rho\to\infty$, one subsequently obtains $m=0$, which establishes point 1. To establish point 2, suppose that $A^0\neq 0$. After a rescaling of X^μ if necessary, we can, without loss of generality, assume that $A^0=1$. Eq. (3.6) thus gives

$$K_{ij}=-\tfrac{1}{2}\{Y^i{}_{,j}+Y^j{}_{,i}+g_{ij,k}A^k\}+O_1(r^{-1-2\alpha})=-\tfrac{1}{2}\{Z^i{}_{,j}+Z^j{}_{,i}+2\varphi_{,ij}+g_{ij,k}A^k\}+O_1(r^{-1-2\alpha}). \tag{3.17}$$

Consider the ADM momentum p_i [the unusual sign in Eq. (3.18) is due to our convention on K_{ij}; cf. the remark in parentheses after Eq. (2.4)]:

$$p_i=-\frac{1}{8\pi}\int_{S_\infty}(K_i{}^j-K\delta_i{}^j)dS_j. \tag{3.18}$$

After insertion of (3.17) in (3.18), one finds

$$p_i=\frac{1}{16\pi}\int_{S_\infty}(Z^i{}_{,j}+Z^j{}_{,i}+A_j g_{ik,k})dS_j. \tag{3.19}$$

Here the φ contribution drops out because of the following calculation:

$$\int_{S_\infty}(\Delta_\delta\varphi\,\delta_{ij}-\partial_i\,\partial_j\varphi)dS_j=\int_{S_\infty}(\partial_k\varphi\;\delta_{ij}-\partial_j\varphi\;\delta_{ki})_{,k}\,dS_j=0. \tag{3.20}$$

We have also used the identities

$$g_{ij,k}A^k=(g_{ij}A^k-g_{ik}A^j)_k+g_{ik,k}A^j,$$

and integration by parts to rearrange the $g_{ij,k}A^k$ terms. Inserting (3.13) in (3.19) and using the harmonic coordinates condition, one obtains

$$p_i=mA_i,$$

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which had to be established. □

Point 1 of Proposition 3.1 suggests strongly that the ADM four-momentum must vanish when A^μ is space-like. We can show that if we assume some further asymptotic conditions on the fields under consideration. A similar result has been established previously in Ref. 1 under rather stronger asymptotic and global conditions.

Proposition 3.2: Under the hypotheses of Proposition 3.1, suppose further that N is C^2 and that

$$N\tau_{ij}=O(r^{-3-\epsilon}). \tag{3.21}$$

If

$$(A^0)^2<\sum_i A^iA^i, \tag{3.22}$$

then p^μ vanishes.

Proof: It follows from Eqs. (3.3), (3.4), and (3.21) that

$$Y^i-A^i=O_2(r^{-\alpha}), \quad N-A^0=O_2(r^{-\alpha}). \tag{3.23}$$

Consider first the case $A^0=0$; by Proposition 3.1 we have $p^0=0$. Let ψ be any function on Σ_R such that $\psi_{,z}=N$. Equation (3.21) gives

$$(K_{ij}-\partial_i\,\partial_j\psi)_{,z}=O(r^{-3-\epsilon}),$$

so that by z integration one obtains

$$K_{ij}-\partial_i\,\partial_j\psi=O(r^{-2-\epsilon}).$$

Inserting this in Eq. (3.18), one obtains

$$\begin{aligned} p_i&=-\frac{1}{8\pi}\int_{S_\infty}(\Delta_\delta\psi\delta_{ij}-\partial_i\,\partial_j\psi)dS_j \\ &=-\frac{1}{8\pi}\int_{S_\infty}(\partial_k\psi\delta_{ij}-\partial_j\psi\;\delta_{ki})_{,k}\,dS_j \\ &=0. \end{aligned} \tag{3.24}$$

Consider, next, the case $A^0\neq 0$; let $(\hat{M},\hat{g}_{\mu\nu})$ be the Killing development of $(\Sigma_R,g_{ij},K_{ij},N,Y^i)$ as constructed in Sec. II. As discussed in the paragraph preceding Eq. (2.24), Eqs. (3.2) and (3.21) imply that the Einstein tensor $\hat{G}_{\mu\nu}$ of $\hat{g}_{\mu\nu}$ will satisfy the fall-off condition

$$\hat{G}_{\mu\nu}=O(r^{-3-\epsilon}). \tag{3.25}$$

Let $\Lambda^\mu{}_\nu$ be the matrix of a Lorentz transformation such that $\Lambda^0{}_\nu A^\nu=0$. Further, let $\Lambda\Sigma$ be the image under $\Lambda^\mu{}_\nu$ of $\Sigma_R\cap\hat{M}$ in $\hat{M}$. On $\Lambda\Sigma$ the Killing vector X^μ satisfies $X^0\to_{r\to\infty}0$. Equation (3.25) shows that we can apply the previous analysis to conclude that the ADM four-momentum of $\Lambda\Sigma$ vanishes. Moreover, the decay condition (3.25) ensures (cf. e.g., Ref. 13) that p^μ transforms as a Lorentz vector under Lorentz transformations of hypersurfaces, so that the ADM four-momentum of Σ_R vanishes as well.

It is of interest to consider Killing vector fields that are covariantly constant. As discussed in Sec. II, in such a case Eqs. (3.26)–(3.27) below will hold (with 0 on the right-hand sides). We have the following result, which does not cover asymptotically null Killing vectors.

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Proposition 3.3: Under the hypotheses of Proposition 3.1, assume, moreover, that N is C^2; that Eq. (3.21) holds and that

$$NK_{ij}+D_iY_j=O_1(r^{-2-\epsilon}), \tag{3.26}$$

$$K_{ij}Y^j+D_iN=O_1(r^{-2-\epsilon}), \quad A^\mu A_\mu\neq 0. \tag{3.27}$$

Then the ADM four-momentum p^μ vanishes.

Proof: Let $(\hat{M},\hat{g}_{\mu\nu})$ be the Killing development of $(\Sigma_R,g_{ij},K_{ij},N,Y^i)$, as constructed in Sec. II. From what is said in that section [cf. the discussion following Eqs. (2.17)–(2.18)], it follows that $X^\mu\partial_\mu=\partial_u$ will satisfy

$$\hat{\nabla}_\mu X_\nu=O_1(r^{-2-\epsilon}). \tag{3.28}$$

As is well known,[14,1] we have

$$p_\mu A^\mu=\lim_{r\to\infty}\frac{1}{8\pi}\int \hat{\nabla}^{[\mu}X^{\nu]}\,dS_{\mu\nu} \tag{3.29}$$

(cf., e.g., Ref. 13 for a proof under the present asymptotic conditions). By (3.28) we have $p_\mu A^\mu=0$. Now, by Proposition 3.1, p_μ is proportional to A_μ, so if $A^\mu A_\mu\neq 0$ the result follows.□

The main result of this section addresses the case of asymptotically null Killing vectors. Unfortunately the proof below requires more asymptotic regularity than one would wish to have. It would be of some interest to find out whether or not the result below is sharp, in the sense that decay conditions on three derivatives of the metric and two derivatives of the extrinsic curvature are necessary.

Theorem 3.4: *Let $R>0$ and let (g_{ij},K_{ij}) be initial data on $\Sigma_R=\mathbf{R}^3\backslash B(R)$, satisfying*

$$g_{ij}-\delta_{ij}=O_{3+\lambda}(r^{-\alpha}), \quad K_{ij}=O_{2+\lambda}(r^{-1-\alpha}), \tag{3.30}$$

$$J^i=O_{1+\lambda}(r^{-3-\epsilon}), \quad \rho=O_{1+\lambda}(r^{-3-\epsilon}),$$
$$\alpha>\tfrac{1}{2}, \quad \epsilon>0, \quad 0<\lambda<1. \tag{3.31}$$

Let N be a scalar field and Y^i a vector field on Σ_R such that

$$N\to_{r\to\infty}A^0, \quad Y^i\to_{r\to\infty}A^i, \quad A^\mu A_\mu=0,$$

for some constants $A^\mu\not\equiv 0$. Suppose further that

$$2NK_{ij}+\mathscr{L}_Y g_{ij}=O_{3+\lambda}(r^{-2-\epsilon}), \tag{3.32}$$

$$\tau_{ij}=O_{1+\lambda}(r^{-3-\epsilon}), \tag{3.33}$$

where τ_{ij} is defined by the equation

$$N(\tau_{ij}-\tfrac{1}{2}g^{kl}\tau_{kl}g_{ij})=N({}^3R_{ij}+KK_{ij}-2K_{ik}K^k{}_j)-\mathscr{L}_Y K_{ij}+D_iD_jN-(\rho/2)Ng_{ij}. \tag{3.34}$$

Then the ADM four-momentum of Σ_R vanishes.

Remark: There is little doubt that the result is still true with $\lambda=0$. To prove that one would however, need to extend the weighted-Sobolev estimates of Ref. 11 to the case dim $M=2$, a task that lies beyond the scope of this paper.

Proof: Arguments similar to the proof of Proposition 2.1, Appendix C, show that

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$$N-A^0=O_{3+\lambda}(r^{-\alpha}),\quad Y^i-A^i=O_{3+\lambda}(r^{-\alpha}).$$

Rescaling A^μ if necessary, we can choose the coordinate system so that $A^0=1$, $A^i=\delta^i_z$. Replacing ϵ by any number smaller than one if necessary, we can assume that $\epsilon<1$ and $\epsilon\leqslant 2\alpha-1$. Taking the trace of Eq. (3.33) and using the scalar constraint equation, we find

$$\Delta_\delta N+K_{,z}=O_{1+\lambda}(r^{-3-\epsilon}).$$

Here, as before, $\Delta_\delta=\partial_x^2+\partial_y^2+\partial_z^2$. Let φ be as in Eq. (3.9), we obtain

$$\Delta_\delta(N-\varphi_{,z})=O_{1+\lambda}(r^{-3-\epsilon});$$

hence there exists a constant D such that

$$N-\varphi_{,z}=1+\frac{D}{r}+O_{3+\lambda}(r^{-1-\epsilon}). \tag{3.35}$$

In harmonic coordinates, Eqs. (3.4), (3.13), (3.33), and (3.35) give

$$-\tfrac{1}{2}\Delta_2 g_{ij}=\chi_{ij}+\Psi_{ij}, \tag{3.36}$$

$$\chi_{ij}=-2m\partial_z\left[\delta^j_z\partial_i\frac{1}{r}+\delta^i_z\partial_j\frac{1}{r}\right]+\partial_i\partial_j\frac{D}{r}, \tag{3.37}$$

$$\Psi_{ij}=O_{1+\lambda}(r^{-3-\epsilon}). \tag{3.38}$$

Here $\Delta_2=\partial_x^2+\partial_y^2$. In what follows the indices A, B, etc. take values in the set {1,2}. Consider the equation (3.36) with $i=z$, $j=A$. We have

$$\Delta_2 g_{zA}=(8m-2D)\partial_A\partial_z\frac{1}{r}+O(r^{-3-\epsilon}). \tag{3.39}$$

It follows from Refs. 12 and 10 that for every fixed value of z the functions g_{zA} have the asymptotic expansion,

$$g_{zA}=C_{AB}(z)\partial_B\ln\rho+O_{(1)}(\rho^{-1-\epsilon}\ln\rho). \tag{3.40}$$

Here $\rho^2=x^2+y^2$, the functions $C_{AB}(z)$ are functions of z only, and we write

$$f=O_{(1)}(\rho^{-\alpha}\ln^\beta\rho),\quad \text{if } |f|+\rho|\partial_A f|\leqslant C(1+\rho)^{-\alpha}[1+\ln(1+\rho)]^\beta, \tag{3.41}$$

for some constant C that may depend upon z. Let us define $S(\rho,a)$ to be a circle of radius ρ centered at $x=y=0$ lying in the plane $z=a$. Equation (3.40) shows that for any fixed value of z the limits

$$\lim_{\rho\to\infty}\int_{S(\rho,z)}g_{zB}\,dx^C,\quad \lim_{\rho\to\infty}\int_{S(\rho,z)}x^D\,\partial_A g_{zB}\,dx^C,$$

exist. It also follows from our asymptotic conditions on g_{ij}, Eq. (3.30), that these limits are z independent. Set

$$\Omega=\lim_{\rho\to\infty}\int_{S(\rho,z)}(x^A\,\partial_C g_{zA}-g_{zC})dx^C. \tag{3.42}$$

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For $|z|>R$ by the Stokes theorem we have

$$\Omega=\int_{\mathbf{R}^2} x^A\ \Delta_2 g_{Az}=(1)+(2),$$

$$(1)=(8m-2D)\int_{\mathbf{R}^2} x^A\ \partial_z\partial_A\ \frac{1}{r},$$

$$(2)=\int_{\mathbf{R}^2} x^A\Psi_{Az}$$

with Ψ_{Az} as in (3.36). The first integral is easily calculated and equals

$$8\pi(4m-D)\operatorname{sgn} z, \tag{3.43}$$

where $\operatorname{sgn} z$ denotes the sign of z. To estimate the second integral, it is convenient to split the region of integration into the sets $\rho\leqslant|z|$ and $\rho\geqslant|z|$. One then finds

$$|(2)|\leqslant C|z|^{-\epsilon},\quad \text{for } |z|>R, \tag{3.44}$$

with a constant C that does *not* depend upon z. Equations (3.43)–(3.44) are consistent with $\partial\Omega/\partial z=0$ if and only if

$$4m=D. \tag{3.45}$$

Consider now Eq. (3.36) with $i=A$, $j=B$. Differentiating this equation with respect to z, one obtains

$$\Delta_2\ \frac{\partial g_{AB}}{\partial z}=-2D\partial_A\ \partial_B\partial_Z\ \frac{1}{r}+O(r^{-4-\epsilon}). \tag{3.46}$$

By hypothesis we have $\partial g_{ij}/\partial z=O(r^{-1-\epsilon})$, and the estimates of Refs. 12 or 10 show that there exist functions $D_{ABCD}(z)$ such that for any fixed value of z, we have

$$\frac{\partial g_{AB}}{\partial z}=D_{ABCD}\ \partial_C\partial_D\ \ln\rho+O_{(1)}(\rho^{-2-\epsilon}\ \ln\rho). \tag{3.47}$$

Let us set

$$\Omega'=\lim_{\rho\to\infty}\int_{S(\rho,z)}(2x^Ax^B\partial_C\ \partial_z g_{AB}-x^Ax^A\partial_C\ \partial_z g_{BB}+2x_C\ \partial_z g_{AB}-4x^B\ \partial_z g_{CB})dx^C.$$

(3.47) shows that Ω' is well defined, while (3.30) implies that Ω' is z independent. For $|z|>R$ we again use the Stokes theorem to obtain

$$\Omega'=\int_{\mathbf{R}^2}(2x^Ax^B\Delta_2\ \partial_z g_{AB}-x^Ax^A\Delta_2\ \partial_z g_{BB}).$$

A calculation as above leads to

$$\Omega'=16\pi D\ \operatorname{sgn} z+O(|z|^{-\epsilon}),\quad |z|>R.$$

Hence $D=m=0$ [cf. Eq. (3.45)], which together with Proposition 3.1, establishes our claims. □

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IV. A POSITIVE ENERGY THEOREM

In this section we shall prove a "future-pointing-time-like-or-vanishing-energy-momentum theorem," under conditions weaker than previously considered. The main two issues we wish to address are (1) the impossibility of a null ADM four-momentum and (2) a result that invokes hypotheses concerning only the fields g_{ij} and K_{ij}.

Let us start with an example of a metric with "null ADM four-momentum." Recall that in Ref. 15, Aichelburg and Sexl consider a sequence of Schwarzschild space–times with energy-momentum vector $(m,0,0,0)$. After applying a "boost" transformation to the Schwarzschild space–time, one obtains an energy-momentum vector $(\gamma m,\gamma v m,0,0)$. Then one takes the limit $v\to 1$, keeping γm equal to a fixed constant p. The resulting space–time has a distributional metric, and it is not clear if it is asymptotically flat. Nevertheless, it seems reasonable to assign to the Aichelburg–Sexl solutions a null energy-momentum vector $(p,p,0,0)$. So, in this sense, there exist space–times with a null energy-momentum vector.

The Aichelburg–Sexl metrics are plane-fronted waves, and it is of interest to inquire whether any asymptotically flat plane-fronted wave metrics exist. Recall that the usual approach in defining asymptotic flatness is to introduce coordinate systems on $(\mathbf{R}^3\backslash B(R))$. Thus, consider a plane-fronted wave metric on $\mathbf{R}\times(\mathbf{R}^3\backslash B(R))$,

$$ds^2=-2\,du\,dz+\alpha\,dz^2+dx^2+dy^2. \tag{4.1}$$

As is well known (cf., e.g., Refs. 16, 17), the metric (4.1) is vacuum if and only if $\alpha=\alpha(x,y,z)$, with

$$(\partial_x^2+\partial_y^2)\alpha=0. \tag{4.2}$$

Then, let α be any solution of (4.2) such that $\alpha=1$ for $|z|\geqslant R$, but $\alpha\not\equiv 1$. Such solutions are easily found, and for any finite l we can choose α to satisfy

$$0\leqslant k\leqslant l,\quad |\partial_{A_1}\cdots\partial_{A_k}(\alpha-1)|\leqslant Cr^{-k-1}.$$

An example is given by the function

$$\alpha=1+\phi(z)C^{A_1\cdots A_l}\partial_{A_1}\cdots\partial_{A_l}\ln\rho, \tag{4.3}$$

where $\phi(z)$ is a smooth compactly supported function and $C^{A_1\cdots A_l}$ is a totally symmetric tensor with constant coefficients. We have the following.

(1) If $l=1$, the metric (4.1) with α given by (4.3) will not satisfy the fall-off requirements of the positive energy theorem; cf. Theorem 4.1 below, because the z derivatives of the metric do not vanish fast enough as r tends to infinity. This fall-off of the metric is not known to be sufficient for a well-defined notion of ADM mass (compare Refs. 11, 13, and 18). However, one can calculate the ADM integral (3.12) in the coordinate system (x,y,z) as above and find that this integral vanishes.

(2) For all $l\geqslant 2$ the hypersurfaces $u=$const will have a well-defined vanishing ADM mass. This does, however, not follow from Theorem 3.4 unless $l\geqslant 3$. (Strictly speaking, we would need to have $l\geqslant 4$ to be able to apply Theorem 3.4 as is; cf., however, the remark following that theorem. When we know *a priori* that the metric is a plane-fronted wave, we can use independent arguments to get rid of the Hölder differentiability index λ in Theorem 3.4; no details will be given.)

Nevertheless, this example shows that nontrivial, vacuum, asymptotically flat plane-fronted waves exist (with $p^\mu=0$), as long as no further global conditions are imposed.

With those examples in mind, let us briefly recall what is known about the nonexistence of appropriately regular space–times with null energy momentum. In Ref. 19 an argument was given

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to support the expectation that the ADM momentum cannot be null for vacuum or electrovacuum space–times, the general case being left open. In Ref. 20 this case has been excluded under rather strong global hypotheses on the space–time and under stringent asymptotic conditions. In Ref. 21 a proof was given, assuming only hypotheses on the initial data. However, the proof there is rather more complicated than ours. Moreover, the asymptotic conditions of Ref. 21 are more restrictive than ours.

We wish next to emphasize the following issue: The statement that the ADM mass m is non-negative requires only the inequality $\rho \geqslant \sqrt{J_i J^i}$, where ρ and J^i are quantities that can be purely defined in terms of the fields g_{ij} and K_{ij}; cf. Eqs. (4.5)–(4.6) below. Now the published Witten-type proofs that the vanishing of m implies, loosely speaking, flatness of the resulting space–time, involve the full dominant energy condition ($T_{\mu\nu}X^\mu Y^\nu \geqslant 0$ for all time-like consistently time-oriented vectors X^μ and Y^ν) (cf., e.g., Ref. 22). Recall that the corresponding statement of Schoen and Yau[23] does not involve any supplementary field $T_{\mu\nu}$. (Their proof, however, requires rather strong asymptotic conditions on the fields. Moreover, Schoen and Yau require the trace of the extrinsic curvature to fall-off at least as r^{-3}. In general, this can be justified by applying a ''logarithmic supertranslation'' in time to the initial data surface, and requires the supplementary hypothesis that the associated space–time is large enough. Finally, to guarantee that all the required hypotheses hold on the deformed hypersurface, one needs again the full dominant energy condition.) Similarly, both the proof in Ref. 20 and the proof in Ref. 21 that exclude a null ADM energy momentum assume the full dominant energy condition. A result involving only conditions on g_{ij} and K_{ij} seems to be much more satisfactory from a conceptual point of view, and it seems reasonable to expect that the desired conclusion could be obtained in the Witten-type setting without imposing conditions on fields other than g_{ij} and K_{ij}. We show below that this is indeed the case.

Before passing to the statement of our results, in addition to the papers already quoted, let us mention the papers,[11,24–35] where proofs or arguments relevant to the positive energy theorem have been given. The review paper[36] contains some further references.

We have the following.

Theorem 4.1 [(Rigid) positive energy theorem]: *Consider a data set* (Σ, g_{ij}, K_{ij}), *with* Σ *of the form* $\Sigma = \Sigma_{int} \cup_{i=1}^{I} \Sigma_i$, *for some* $I < \infty$. *Here we assume that* Σ_{int} *is compact, and that each of the ends* Σ_i *is diffeomorphic to* $\mathbf{R}^3 \backslash B(R_i)$ *for some* $R_i > 0$, *with* $B(R_i)$*—coordinate ball of radius* R_i. *In each of the ends* Σ_i *the fields* (g,K) *are assumed to satisfy the following inequalities*:

$$|g_{ij} - \delta_{ij}| + |r\ \partial_k g_{ij}| + |rK_{ij}| \leqslant Cr^{-\alpha}, \tag{4.4}$$

for some constants $C > 0$ *and* $\alpha > \frac{1}{2}$, *with* $r = \sqrt{\Sigma (x^i)^2}$. *Suppose, moreover, that the quantities* ρ *and* J,

$$2\rho := {}^3R + K^2 - K^{ij}K_{ij}, \tag{4.5}$$

$$J^k := D_l(-K^{kl} + Kg^{kl}), \tag{4.6}$$

are well defined (perhaps in a distributional sense), and satisfy

$$\sqrt{g_{ij}J^iJ^j} \leqslant \rho \leqslant C(1+r)^{-3-\epsilon}, \quad \epsilon > 0. \tag{4.7}$$

Then the ADM four-momentum (m, p^i) *of any of the asymptotic ends of* Σ *satisfies* $m \geqslant \sqrt{p_j p^i}$. *If* $m = 0$, *then* $\rho \equiv J^i \equiv 0$, *and there exists an isometric embedding* i *of* Σ *into Minkowski space–time* $(\mathbf{R}^4, \eta_{\mu\nu})$ *such that* K_{ij} *represents the extrinsic curvature of* $i(\Sigma)$ *in* $(\mathbf{R}^4, \eta_{\mu\nu})$. *Moreover,* $i(\Sigma)$ *is an asymptotically flat Cauchy surface in* $(\mathbf{R}^4, \eta_{\mu\nu})$.

Proof: Under the conditions here the ADM four-momentum of each of the asymptotic regions of Σ is finite and well defined.[18,11] As discussed, e.g., in Ref. 13, under the present boundary

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conditions the Witten boundary integral reproduces correctly the ADM four-momentum. The arguments of any of the references[11,31,13] show that one can find solutions to the Witten equation that asymptote to a constant nonzero spinor in one of the asymptotic ends, and to zero in all the other ones. Witten's identity subsequently implies that the ADM momentum of each of the ends is non-space-like.

Suppose that in one of the ends m vanishes. Then for each $\vec{n}\in\mathbf{R}^3$ there exists a spinor field $\lambda_M(\vec{n})$ defined on Σ satisfying Eq. (B7), such that the corresponding vector field $Y^j(\vec{n})$ defined via Eq. (B8), and the scalar field $N(\vec{n})$ defined by Eq. (B9), satisfy

$$Y^j(\vec{n})\to_{r\to\infty}\vec{n}^j,\quad N(\vec{n})\to_{r\mathring{a}\to\infty}|\vec{n}|_\delta.$$

Here $|\vec{n}|_\delta$ is the norm of $\vec{n}$ in the flat metric on $\mathbf{R}^3$. As shown in Appendix B, the fields $N(\vec{n})$ and $Y^i(\vec{n})$ satisfy the linear system of equations [cf. Eqs. (B11) and (B.12)],

$$D_iY_j+NK_{ij}=0, \tag{4.8}$$

$$D_iN+K_{ij}Y^j=0. \tag{4.9}$$

Consider the fields

$$Y_j=Y_j((\tfrac{1}{2},\tfrac{1}{2},0))-Y_j((-\tfrac{1}{2},\tfrac{1}{2},0))-Y_j((1,0,0)), \tag{4.10}$$

$$N=N((\tfrac{1}{2},\tfrac{1}{2},0))-N((-\tfrac{1}{2},\tfrac{1}{2},0))-N((1,0,0)). \tag{4.11}$$

The fields Y_j and N satisfy Eqs. (4.8)–(4.9) by linearity of those equations. Moreover, we have

$$Y^j\to_{r\to\infty}0,\quad N\to_{r\to\infty}1. \tag{4.12}$$

Let $(\hat{M},\hat{g}_{\mu\nu})$ be the Killing development of $(\Sigma,g_{ij},K_{ij},N,Y_i)$. As discussed in Sec. II, it follows from Eqs. (4.8)–(4.9) that the vector field $X^\mu\partial_\mu=\partial_u$ is covariantly constant on $\hat{M}$; (4.12) implies then

$$\hat{g}_{\mu\nu}X^\mu X^\nu=-1\Rightarrow N^2-g_{ij}Y^iY^j=1. \tag{4.13}$$

By Proposition 3.1 of Ref. 5, Σ is a Cauchy surface for $(\hat{M},\hat{g}_{\mu\nu})$. We wish to show that $(\hat{M},\hat{g}_{\mu\nu})$ is geodesically complete. Consider, then, an affinely parametrized geodesic $x^\mu(s)$, and let p denote the constant of motion associated with the Killing vector X^μ:

$$p=\hat{g}_{\mu\nu}\dot{x}^\mu X^\nu=-\dot{u}+Y_i\dot{x}^i. \tag{4.14}$$

Here Eqs. (2.9) and (4.13) have been taken into account; a dot over a quantity means differentiation with respect to s. Since s is an affine parameter we have, with $\epsilon=0,\pm1$,

$$-\dot{u}^2+2Y_i\dot{x}^i\dot{u}+g_{ij}\dot{x}^i\dot{x}^j=\epsilon. \tag{4.15}$$

Equations (4.14) and (4.15) give

$$(g_{ij}+Y_iY_j)\dot{x}^i\dot{x}^j=\epsilon+p^2. \tag{4.16}$$

(4.16) and (4.15) imply that there exists a function $C(p)$, such that

$$|\dot{x}|_g+|\dot{u}|\leqslant C(p). \tag{4.17}$$

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Choose $p \in \mathbf{R}$ and consider the set Ω_p of maximally extended affinely parametrized geodesics with that value of p, with $x^\mu(0) \in \Sigma$. We can, without loss of generality, assume that $\alpha<1$; an analysis of Eqs. (4.8)–(4.9) along the lines of Appendix C shows that $\hat{g}_{\mu\nu} - \eta_{\mu\nu} = O_1(r^{-\alpha})$. By asymptotic flatness of $\hat{g}_{\mu\nu}$ (cf. Proposition 2.1) and the interior compactness condition on Σ, there exists $\delta>0$ such that all geodesics in Ω_p are defined for $s \in (-\delta,\delta)$. Equation (4.17) shows that in that affine time the value of $|u|$ can change at most by $C(p)\delta$, similarly for the value of $r(s) \equiv (x^2(s)+y^2(s)+z^2(s))^{1/2}$ in the asymptotic regions. One can now invoke the fact that the u translations are isometries to conclude that all geodesics in Ω_p are complete, and the result follows.

Let us show now that $(\hat{M},\hat{g})$ is flat. Let $Y^i_k = Y^i(\vec{e}_k)$, where $Y^i(\vec{n})$ is as at the beginning of this proof and where the $\vec{e}_k$'s, $k=1,2,3$, form an orthonormal basis of $\mathbf{R}^3$. Let $N_{(k)} = N(\vec{e}_k)$ be the corresponding lapse functions. On $\hat{M}$ define the fields $X^\mu_{(k)}$ by the equation

$$
\begin{aligned}
&X^\mu_{(k)}\partial_\mu = \hat{N}_{(k)} n^\mu \partial_\mu + \hat{Y}^i_{(k)}\partial_i ,\\
&\hat{Y}^i_{(k)}(u,x^i) = Y^i_{(k)}(x^i), \quad \hat{N}_{(k)}(u,x^i) = N_{(k)}(x^i).
\end{aligned}
\tag{4.18}
$$

Here n^μ is the field of unit normals to the slices $\{u=\text{const}\}$. By Eqs. (B11) and (B12), we have

$$
\hat{\mathbf{\nabla}}_j X^\mu_{(k)} = 0. \tag{4.19}
$$

By construction of $(\hat{M},\hat{g}_{\mu\nu})$, it also holds that

$$
\hat{\mathbf{\nabla}}_\mu X^\nu = \hat{\Gamma}^\nu_{\mu\lambda} X^\lambda = \hat{\Gamma}^\nu_{\mu u} = 0. \tag{4.20}
$$

As the components of $X^\mu_{(i)}$ are u independent by (4.18), Eq. (4.20) gives

$$
\hat{\mathbf{\nabla}}_u X^\mu_{(k)} = \partial_u X^\mu_{(k)} + \hat{\Gamma}^\mu_{\lambda u} X^\lambda_{(k)} = 0. \tag{4.21}
$$

Consequently,

$$
\hat{\mathbf{\nabla}}_\mu X^\nu_{(k)} = 0. \tag{4.22}
$$

Differentiating (4.22), one obtains

$$
\hat{R}_{\mu\nu\rho\sigma} X^\sigma_{(i)} = 0. \tag{4.23}
$$

As the vector fields $X^\sigma_{(i)}$ are everywhere null and linearly independent, standard algebra gives

$$
\hat{R}_{\mu\nu\rho\sigma} \equiv 0. \tag{4.24}
$$

Consider, next, the universal covering space $\widetilde{\Sigma}$ of Σ with fields $(\widetilde{g}_{ij},\widetilde{K}_{ij},\widetilde{Y}_i,\widetilde{N})$ obtained by pull-back. Let $(\overline{M},\overline{g}_{\mu\nu})$ be the Killing development of $\widetilde{\Sigma},\widetilde{g}_{ij},\widetilde{K}_{ij},\widetilde{Y}_j,\widetilde{N}$. Clearly, $\overline{M}$ is the universal covering space of $\hat{M}$ with $\overline{g}_{\mu\nu}$ being the pull-back of $\hat{g}_{\mu\nu}$. It is easily seen that $(\overline{M},\overline{g})$ inherits from $(\hat{M},\hat{g})$ the following properties:

(1) $(\overline{M},\overline{g}_{\mu\nu})$ is globally hyperbolic with Cauchy surface $\widetilde{\Sigma}$;

(2) $(\overline{M},\overline{g}_{\mu\nu})$ is geodesically complete; and

(3) $(\overline{M},\overline{g}_{\mu\nu})$ is flat.

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As $\bar{M}$ is simply connected, it follows, e.g., from Ref. 37. Theorem 2.4.9 that $(\bar{M},\bar{g}_{\mu\nu})$ is the Minkowski space–time $(\mathbf{R}^4,\eta_{\mu\nu})$. As $\tilde{\Sigma}$ is a Cauchy surface for $\bar{M}$, it is necessarily a graph over a space-like plane $t=0$ in $(\mathbf{R}^4,\eta_{\mu\nu})$. In particular, $\tilde{\Sigma}$ has only one asymptotically flat end (compare also [Ref. 38, Lemma 2]). If Σ had been nonsimply connected, then $\tilde{\Sigma}$ would have had more than one asymptotic end. It follows that $\Sigma = \tilde{\Sigma}$, $\hat{M}=\mathbf{R}^4$, and our claims follow. □

To exclude the case of a null ADM four-momentum, we need to assume some further asymptotic regularity conditions.

Theorem 4.2: *Under the hypotheses of Theorem 4.1, suppose, moreover, that in some of the asymptotic ends it holds that*

$$g_{ij}-\delta_{ij}=O_{3+\lambda}(r^{-\alpha}), \quad K_{ij}=O_{2+\lambda}(r^{-1-\alpha}), \tag{4.25}$$

$$\rho=O_{1+\lambda}(r^{-3-\epsilon}), \tag{4.26}$$

with some $0<\lambda<1$. *Then the ADM four-momentum of that end cannot be null.*

Remark: It can be shown by rather different techniques that the result is still true with $\lambda=0$; we shall, however, not discuss that here.

Proof: Consider an asymptotic end Σ_1 in which Eqs. (4.25)–(4.26) hold and that has a null ADM four-momentum p^μ. As discussed in the proof of Theorem 4.1 and in Appendix B, the hypotheses of Proposition B.1 and Corollary B.2 are satisfied. We can thus apply Theorem 3.4 to conclude that the ADM four-momentum of the end under consideration vanishes, and the result follows from Theorem 4.1. □

Let us close this section by proving Theorem 1.3: By the arguments given above, ρ and J^i vanish on Σ. It follows from a result of Hawking and Ellis (Ref. 39, Chap. 4, Sec. 4.3) that $(M,g_{\mu\nu})$ must be a vacuum. By uniqueness of the maximal globally hyperbolic vacuum developments, it follows that the Killing development constructed in the proof of Theorem 4.2 (cf. Appendix B) coincides with the maximal globally hyperbolic development of (Σ,g_{ij},K_{ij}), and Theorem 1.3 follows.

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APPENDIX A: DEFINITIONS AND CONVENTIONS

We say that $(M,g_{\mu\nu})$ is a C^k space–time if M is a paracompact, connected, Hausdorff, orientable manifold of C^k differentiability class, with a C^{k-1} Lorentzian metric. We use the signature $(-,+,+,+)$.

Consider a function f defined on $\Sigma_R\equiv\mathbf{R}^3\backslash B(R)$, where $B(R)$ is a closed ball of radius $R>0$. We shall write $f=O_k(r^\beta)$ if there exists a constant C, such that we have

$$0\leqslant i\leqslant k, \quad |\partial^i f|\leqslant Cr^{\beta-i}.$$

For $\sigma\in(0,1)$ we shall write $f=O_{k+\sigma}(r^\beta)$ if $f=O_k(r^\beta)$ and if there exists a constant C, such that we have

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$$|y-x|\leqslant r(x)/2 \Rightarrow |\partial^k f(x)-\partial^k f(y)|\leqslant C|x-y|^{\sigma} r^{\beta-k-\sigma}.$$

Let us note that $f=O_{k+1}(r^{\beta})$ implies $f=O_{k+\sigma}(r^{\beta})$ for all $\sigma\in(0,1)$, so that the reader unfamiliar with Hölder-type spaces might wish to simply replace, in the hypotheses of our theorems, the $k+\sigma$ by $k+1$ wherever convenient.

APPENDIX B: COVARIANTLY CONSTANT SPINORS

In this appendix we prove a differential geometric proposition on initial data sets (Σ, g_{ij}, K_{ij}) having a nowhere vanishing spinor field that is covariantly constant on Σ with respect to the "Sen-connection"[40] [cf. Eq. (B7) below]. This result forms the local input of the rigidity part of the positive-mass theorem. Similar results in the literature we are aware of implicitly or explicitly use Cauchy developments $(M, g_{\mu\nu}, \phi^A)$ of $(\Sigma, g_{ij}, K_{ij}, \psi^A)$ for some fields ϕ^A with Cauchy data ψ^A, with energy-momentum tensor $T_{\mu\nu}$ satisfying the *full* dominant energy condition (cf. the discussion at the beginning of Sec. IV). For our results below, neither the existence of such a Cauchy evolution nor, in fact, the DEC for the given triple (Σ, g_{ij}, K_{ij}) (i.e., $\sqrt{g_{ij}J^iJ^j} \leqslant \rho$) is required.

In the case of a "bad" matter model—such as, e.g., dust as a source for the Einstein equations—an evolution is not known to exist. Similarly, even for "good" models, such as vacuum Einstein equations, the differentiability hypotheses on the initial data in Theorems 4.1 and 4.2 are not known to guarantee existence of a Cauchy development.

To motivate our three-dimensional discussion, we shall, as before, start with the four-dimensional picture. Consider thus a space–time $(M, g_{\mu\nu})$ with $g_{\mu\nu}$ in C^2 and a nowhere zero C^2 spinor field λ_M on M, satisfying

$$\nabla_{\mu}\lambda_N=0 \Leftrightarrow \nabla_{MM'}\lambda_N=0, \tag{B1}$$

i.e., λ_M is covariantly constant. We use capital letters in the second half of the alphabet to denote spinor indices. Since the considerations in this appendix are purely local, there is no question of the existence of a spinor structure. The spinorial Ricci identities (cf. Ref. 41, Vol. 1, pp. 242–244) immediately imply that the Ricci scalar $R_{\mu}{}^{\mu}$ of $g_{\mu\nu}$ is zero, and that the spinor equivalent of $S_{\mu\nu}:=R_{\mu\nu}-\frac{1}{48}g_{\mu\nu}R_{\lambda}{}^{\lambda}$, namely the Hermitian spinor $\phi_{MNM'N'}$ satisfies

$$\lambda^M\phi_{MNM'N'}=0$$

$$\Rightarrow \epsilon_{MN}\lambda^P\phi_{PRR'(M'}\bar{\lambda}_{N')}+\lambda_{(M}\phi_{N)RP'R'}\bar{\lambda}^{P'}\epsilon_{M'N'}=0. \tag{B2}$$

This last equation, in tensor terms, says that

$$X_{[\mu}S_{\nu]\lambda}=0, \tag{B3}$$

where X^{μ} is the null vector corresponding to $\lambda_M\bar{\lambda}_{M'}$. Consequently,

$$R_{\mu\nu}=\sigma X_{\mu}X_{\nu}, \tag{B4}$$

for some function σ on M. By Eq. (B1), X^{μ} is covariantly constant, i.e.,

$$\nabla_{\mu}X_{\nu}=0, \quad \text{with} \quad g_{\mu\nu}X^{\mu}X^{\nu}=0. \tag{B5}$$

According to one of several equivalent definitions (cf., e.g., Ref. 17), Eqs. (B4)–(B5) imply that $(M, g_{\mu\nu})$ is a pp space–time. We have recovered the well-known fact (cf., e.g., Refs. 42, 43, 17) that a space–time admitting a covariantly constant spinor describes a pp wave.

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Next, let Σ be a space-like hypersurface of $(M,g_{\mu\nu})$ with unit-normal n_μ. With $n_{MM'}$ being the spinor equivalent of n_μ, Eq. (B1) implies that

$$n_{(M}{}^{M'}\nabla_{N)M'}\lambda_P=0\Leftrightarrow n_{[\mu}\nabla_{\nu]}\lambda_P=0. \tag{B6}$$

Equation (B6) contains only derivatives tangential to Σ. When λ_M is interpreted as a SU(2)-spinor on (Σ,g_{ij},K_{ij}), (B6) can be written as (we use the conventions of Appendix A of Ref. 44),

$$D_{MN}\lambda_P+\frac{i}{\sqrt{2}}K_{MNPQ}\lambda^Q=0, \tag{B7}$$

where K_{MNPQ} is the $SU(2)$-spinor version of K_{ij} and D_{MN} the covariant derivative on Σ associated with g_{ij}.

Let us turn to the three-dimensional formulation of the problem. Suppose that we are given (Σ,g_{ij},K_{ij}) with g_{ij} in C^k, for some $k\geqslant 1$, K_{ij} in C^{k-1}, and a C^k-spinor λ_M satisfying Eq. (B7). We want to embed Σ into some Lorentz manifold $(M,g_{\mu\nu})$ in which λ_M extends to a spinor field obeying Eq. (B1).

Denote by M_i the complex-valued null vector field on Σ associated with $\lambda_M\lambda_N$ and define a real vector Y_i by

$$Y_i=\frac{i}{\sqrt{g_{jk}M^j\bar{M}^k}}\epsilon_i{}^{jk}M_j\bar{M}_k, \tag{B8}$$

and a real positive scalar N by

$$N=\sqrt{g_{ij}M^i\bar{M}^j}=\sqrt{g_{ij}Y^iY^j}. \tag{B9}$$

By, e.g., Ref. 22, Lemma 4.3, λ_N is nowhere zero; hence N is nowhere vanishing. From (B7), M_i satisfies

$$D_iM_j=-i\epsilon^{lm}{}_jK_{il}M_m, \tag{B10}$$

which, after some calculation, implies

$$D_iY_j+NK_{ij}=0. \tag{B11}$$

We also note, for use in the body of the paper, the equation

$$D_iN+K_{ij}Y^j=0, \tag{B12}$$

which follows from (B9) and (B11). Now define $(\hat{M},\hat{g}_{\mu\nu})$ to be the Killing development $(\mathbf{R}\times\Sigma,\hat{g}_{\mu\nu})$ of (Σ,g_{ij},K_{ij}) based on (N,Y^i), i.e.,

$$\hat{g}_{\mu\nu}\,dx^\mu\,dx^\nu=-N^2(x^l)du^2+g_{ij}(x^l)[dx^i+Y^i(x^l)du][dx^j+Y^j(x^l)du]. \tag{B13}$$

This, as shown in Sec. II, has $X=\partial/\partial u$ as a covariantly constant null vector, the induced metric on $u=0$ coincides with g_{ij}, and the extrinsic curvature is K_{ij}. The field of unit normals n_μ to the hypersurfaces $\{u=\text{const}\}$ is Lie derived by this Killing, vector field,

$$\mathcal{L}_Xn_\mu=0, \tag{B14}$$

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which can be seen as follows: By construction $X(u)=1$. Since Lie derivation and exterior differentiation commute, we have that $\mathcal{L}_X\, du=0$. By the Killing property of X, $\mathcal{L}_X(du,du)$ is also zero, and Eq. (B14) follows. But, by the covariant constancy of X, i.e.,

$$\hat{\nabla}_\mu X_\nu=0, \tag{B15}$$

this implies that

$$X^\nu\,\hat{\nabla}_\nu n_\mu=0. \tag{B16}$$

Now extend λ_M off $u=0$ to a spinor field $\hat{\lambda}_{\mathrm{M}}$ on $(M,\hat{g}_{\mu\nu})$ by requiring

$$X^\mu\,\hat{\nabla}_\mu\hat{\lambda}_M=0. \tag{B17}$$

Consider the expression

$$U_{MNP}=n_{(M}{}^{M'}\hat{\nabla}_{N)M'}\hat{\lambda}_P. \tag{B18}$$

By Eqs. (B6)–(B7), U_{MNP} vanishes for $u=0$. Now compute

$$X^\mu\,\hat{\nabla}_\mu U_{MNP}=n_{(M}{}^{M'}X^\mu\hat{\nabla}_{|\mu|}\hat{\nabla}_{N)M'}\hat{\lambda}_P, \tag{B19}$$

where we have used (B16). Since X is covariantly constant, $X^\mu\hat{\nabla}_\mu$ commutes with covariant differentiation. Applying this on the right-hand side of (B19) and using (B16), we infer

$$X^\mu\,\hat{\nabla}_\mu U_{MNP}=0. \tag{B20}$$

Thus

$$n_{(M}{}^{M'}\,\hat{\nabla}_{N)M'}\hat{\lambda}_P=0\Leftrightarrow n_{[\mu}\,\hat{\nabla}_{\nu]}\hat{\lambda}_P=0. \tag{B21}$$

By (B17) we also have that

$$(Nn^\mu\hat{\nabla}_\mu+Y^i\hat{\nabla}_i)\hat{\lambda}_M=0. \tag{B22}$$

Due to (B21) the second term in (B22) is zero. As N is nowhere vanishing, we obtain

$$n^\mu\,\hat{\nabla}_\mu\hat{\lambda}_P=0. \tag{B23}$$

Since n^μ is time-like and again using (B21) we get

$$\hat{\nabla}_\mu\hat{\lambda}_P=0, \tag{B24}$$

as promised. (Strictly speaking, the above calculations require $k\geqslant 2$. One can use a slightly different argument to show that Proposition B.1 is correct as stated.) Combining the above calculation with Eq. (B4), we obtain the following.

Proposition B.1: Let $k\geqslant 1$ and let (Σ,g_{ij},K_{ij}), $g_{ij}\in C^k$, $K_{ij}\in C^{k-1}$ be such that there exists a C^k spinor field satisfying Eq. (B7). Then there exists a nowhere zero vector field Y_i in C^k, such that

$$D_iY_j+NK_{ij}=0, \tag{B25}$$

where $N:=\sqrt{g_{ij}Y^iY^j}$. If, moreover, $k\geqslant 2$, then the fields (ρ,J_i,τ_{kl}) defined in Eqs. (2.13)–(2.15) satisfy

$$NJ_i=\rho Y_i,\quad N^2\tau_{ij}=\rho Y_iY_j. \tag{B26}$$

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In the case where the ADM four-momentum p^μ is null, the Witten argument gives rise to a spinor field on Σ obeying Eq. (B7) (cf. the discussion and the references in the proof of Theorem 4.1). Proposition B.1 and an analysis of Eqs. (B11)–(B12) similar to that of Appendix C lead to the following.

Corollary B.2: Let (Σ, g_{ij}, K_{ij}) satisfy the hypotheses of Theorem 4.1 and let p^μ be null. Then there exists a nowhere zero C^1 field Y_i with $Y^i - A^i = O_1(r^{-\alpha})$ for some constants A^i, so that Eq. (B25) holds. If, moreover, the hypotheses of Theorem 4.2 are satisfied, then $Y^i - A^i = O_3(r^{-\alpha})$, and (B26) holds.

APPENDIX C: PROOF OF PROPOSITION 2.1

Equation (2.4) gives the equation

$$D_i D_j Y_k = R_{mijk} Y^m + D_k(NK_{ij}) - D_i(NK_{jk}) - D_j(NK_{ki}). \tag{C1}$$

Here R_{mijk} is the curvature tensor of the metric g_{ij}. Consider the system of equations,

$$\frac{\partial N}{\partial r} = \frac{x^i}{r}\, \partial_i N, \tag{C2}$$

$$\partial_r Y_i = \frac{x^j}{r}\, (D_j Y_i + \Gamma^k_{ij} Y_k), \tag{C3}$$

$$\partial_r D_i N = \frac{x^k}{r}\, (D_k D_i N + \Gamma^j_{ki} D_j N), \tag{C4}$$

$$\partial_r D_i Y_j = \frac{x^k}{r}\, (D_k D_i Y_j + \Gamma^l_{ki} D_l Y_j + \Gamma^l_{kj} D_i Y_l). \tag{C5}$$

Here we are implicitly assuming that, in (C4) and (C5), the terms $D_k D_i N$ and $D_k D_i Y_j$ have been eliminated using (2.15) and (C1). Set $f = (f^A) = (N, r D_i N, Y_j, r D_i Y_j)$, $g = \Sigma_A f^A f^A$. We have

$$\left|\frac{\partial g}{\partial r}\right| \leqslant \frac{Cg}{r}, \tag{C6}$$

and by r integration one finds

$$|f| \leqslant C(1 + r^\beta), \tag{C7}$$

for some constants C, β. Suppose that $\beta > 2$, using (C7) and (C2)–(C5), one finds by r integration $|f| \leqslant C(1 + r^{\beta - \alpha})$, so that (C7) has been improved by α. Iterating this process, one obtains (2.21) and (2.22); cf. also Ref. 38. Appendix A, Lemma. Suppose finally that $A^\mu = \Lambda_{\mu\nu} = 0$. Iterating further, one finds

$$|f| \leqslant C r^{-\sigma}, \quad \text{for any } \sigma > 0. \tag{C8}$$

Note that if $g(r_0) = 0$, at some r_0, then by (C6) we will have $g \equiv 0$. Suppose thus that for all r there holds $g(r) \neq 0$. For $r_1 \geqslant r_0$ we then have, by (C6),

$$\frac{\partial g}{\partial r} \geqslant -\frac{Cg}{r} \Rightarrow \ln(g(r_1) r_1^C) \geqslant \ln(g(r_0) r_0^C).$$

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Passing with r_1 to infinity from (C8), we obtain $g(r_0)=0$, which gives a contradiction, and the result follows. □

[1] A. Ashtekar and A. Magnon-Ashtekar, "On conserved quantities in general relativity," J. Math. Phys. **20**, 793–800 (1979).
[2] A. Ashtekar and B. Schmidt, "Null infinity and Killing fields," J. Math. Phys. **21**, 862–867 (1978).
[3] A. Ashtekar and B. C. Xanthopoulos, "Isometries compatible with asymptotic flatness at null infinity: A complete description," J. Math. Phys. **19**, 2216–2222 (1978).
[4] J. Bičák and B. Schmidt, "Isometries compatible with gravitational radiation," J. Math. Phys. **25**, 600–606 (1984).
[5] P. T. Chruściel and R. Wald, "Maximal hypersurfaces in stationary asymptotically flat space–times," gr-qc/9304009, Commun. Math. Phys. **163**, 561–604 (1994).
[6] P. T. Chruściel, "'No hair' theorems—Folklore, conjectures, results," gr-qc/9402032, Proceedings of the Joint AMS CMS Conference on Mathematical Physics and Differential Geometry, August 1993, Vancouver, edited by J. Beem and K. L. Duggal, Contemporary Math. 1994, pp. 23–49.
[7] P. T. Chruściel, "On completeness of orbits of Killing vector fields," gr-qc/9304029, Class. Quantum Grav. **10**, 2091–2101 (1993).
[8] J. W. York, Jr., "Kinematics and dynamics of general relativity," in *Sources of Gravitational Radiation*, edited by L. L. Smarr (Cambridge University, Cambridge 1979).
[9] D. Christodoulou and N. O'Murchadha, "The boost problem in general relativity," Commun. Math. Phys. **80**, 271–300 (1981).
[10] N. Meyers, "An expansion about infinity for solutions of linear elliptic equations," J. Math. Mech. **12**, 247–264 (1963).
[11] R. Bartnik, "The mass of an asymptotically flat manifold," Commun. Pure Appl. Math. **39**, 661–693 (1986).
[12] P. T. Chruściel, "*A-priori* estimates in weighted Hoelder spaces for a class of scale-covariant second order elliptic operators," Ann. Fac. Sci. Toulouse **XI**, 21–37 (1990).
[13] P. T. Chruściel, "A remark on the positive energy theorem," Class. Quantum Grav. **3**, L115–L121 (1986); *Erratum* **4**, 1049 (1987).
[14] R. Beig, "Arnowitt–Deser–Misner energy and g_{00}," Phys. Lett. A **69**, 153–155 (1978).
[15] P. C. Aichelburg and R. U. Sexl, "On the gravitational field of a massless particle," Gen. Relat. Grav. 303–312 (1971).
[16] H. W. Brinkmann, "Einstein spaces which are mapped conformally on each other," Math. Ann. **94**, 119–145 (1925).
[17] R. Schimming, "Riemannsche Räume mit ebenfrontiger und mit ebener Symmetrie," Math. Nachr. **59**, 129–162 (1974).
[18] P. T. Chruściel, "Boundary conditions at spatial infinity from a Hamiltonian point of view," in *Topological Properties and Global Structure of Space-Time*, edited by P. Bergmann and V. de Sabbata (Plenum, New York, 1986), pp. 49–59.
[19] E. Witten, "A new proof of the positive energy theorem," Commun. Math. Phys. **80**, 381–402 (1981).
[20] A. Ashtekar and G. T. Horowitz, "Energy-momentum of isolated systems cannot be null," Phys. Lett. A **89**, 181–184 (1982).
[21] P. F. Yip, "A strictly-positive mass theorem," Commun. Math. Phys. **108**, 653–665 (1987).
[22] T. Parker and C. H. Taubes, "On Witten's proof of the positive energy theorem," Commun. Math. Phys. **84**, 223–238 (1982).
[23] R. Schoen and S. T. Yau, "Proof of the positive mass conjecture. II," Commun. Math. Phys. **79**, 231–260 (1981).
[24] R. Schoen and S. T. Yau, "Proof that the Bondi mass is positive," Phys. Rev. Lett. **48**, 369–374 (1982).
[25] M. Ludvigsen and J. A. G. Vickers, "An inequality relating mass and electric charge in general relativity," J. Phys. A **16**, 1169–1174 (1983).
[26] J. Jezierski, "Positivity of mass for certain spacetimes with horizons," Class. Quantum Grav. **6**, 1535–1539 (1989).
[27] J. Jezierski and J. Kijowski, "A simple argument for positivity of gravitational energy," in *Differential Geometric Methods in Theoretical Physics*, Proceedings of the 15th International Conference, Clausthal (World Scientific, Singapore, 1987).
[28] J. Jezierski and J. Kijowski, "Positivity of total energy in general relativity," Phys. Rev. D **36**, 1041–1044 (1987).
[29] G. W. Gibbons, S. W. Hawking, G. T. Horowitz, and M. J. Perry, "Positive mass theorem for black holes," Commun. Math. Phys. **99**, 285–308 (1983).
[30] Y. Choquet-Bruhat, "Positive energy theorems," in *Relativity, Groups and Topology II*, Proceedings of the Les Houches Summer School, June–August 1983, edited by B. S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984).
[31] O. Reula, "Existence theorem for solutions of Witten's equation and non-negativity of total mass," J. Math. Phys. **23**, 810–814 (1982).
[32] O. Reula and P. Tod, "Positivity of the Bondi energy," J. Math. Phys. **25**, 1004–1008 (1984).
[33] P. Bizon and E. Malec, "On Witten's positive-energy proof for weakly asymptotically flat space–times," Class. Quantum Grav. **3**, L123–128 (1986).
[34] R. Penrose, "Light rays near i^0: A new mass-positivity theorem," Twistor Newslett. **30**, 1–5 (1990).
[35] R. Penrose, R. D. Sorkin, and E. Woolgar, "Positive mass theorem based on the focusing and retardation of null geodesics," Cambridge preprint, 1993, gr-qc/9301015.
[36] G. T. Horowitz, "The positive energy theorem and its extensions," in *Asymptotic Behavior of Mass and Spacetime Geometry*, edited by F. Flaherty, Springer Lecture Notes in Physics, 1984, p. 202.
[37] J. A. Wolf, *Spaces of Constant Curvature* (McGraw-Hill, New York, 1967).
[38] P. T. Chruściel, "On the invariant mass conjecture in general relativity," Commun. Math. Phys. **120**, 233–248 (1988).
[39] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University, Cambridge, 1973).

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[40]A. Sen, ''On the existence of neutrino 'zero-modes' in vacuum spacetimes,'' J. Math. Phys. **22**, 1781–1786 (1981).
[41]R. Penrose and W. Rindler, *Spinors and Spacetime* (Cambridge University, Cambridge, 1984), Vol. 1.
[42]K. P. Tod, ''All metrics admitting super-covariantly constant spinors,'' Phys. Lett. B **121**, 241–244 (1983).
[43]A. Taub, ''Space-times admitting a covariantly constant spinor field,'' Ann. Inst. Henri Poincaré **41**, 227–236 (1984).
[44]A. Ashtekar, *Lectures on Non-Perturbative Canonical Gravity* (World Scientific, Singapore, 1991).

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