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On Rigidity of Analytic Black Holes

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Abstract: We establish global extendibility (to the domain of outer communication) of locally defined isometries of appropriately regular analytic black holes. This allows us to fill a gap in the Hawking–Ellis proof of black–hole rigidity.

1. Introduction

According to Hawking and Ellis [11, Prop. 9.3.6], under appropriate conditions, which include analyticity of all the objects under consideration, the event horizon of a stationary, say electro–vacuum, black hole space–time (M, g) is necessarily a Killing horizon. More precisely, the isometry group of (M, g) should contain an $\mathbb{R}$ subgroup, the orbits of which are normal to the black hole horizon. In order to substantiate their claim the authors of [11] first argue that for each t the map defined as the translation by t along the appropriately parameterized generators of the event horizon extends to an isometry ϕ_t in a neighborhood of the event horizon. Next they assert that for all t one can analytically continue ϕ_t to the whole space–time, to obtain a globally defined one parameter group of isometries. This last claim is wrong, which[1] can be seen as follows: Let (M, g_{ab}) be the extension of the exterior region of the Kerr space–time (with $a^2 < m^2$) consisting of "two type I regions and two type II regions", as described in Sect. 5.6 of [11] (thus (M, g_{ab}) consists of the four uppermost blocks of Fig. 28(i), p. 165 in [11]). Let ϕ_t denote those isometries of (M, g_{ab}) which are time–translations in an asymptotic region $M_{\rm ext}$, and let $\langle\langle M_{\rm ext}\rangle\rangle$ denote the domain of outer communication of (M, g_{ab}) as determined by $M_{\rm ext}$ (*cf.* Eq. (2.1) below; M_{ext} corresponds to one of the blocks "I" of Fig. 28 of [11]). Let Σ be any asymptotically flat Cauchy surface of (M, g_{ab}) (thus Σ has

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[1] The construction that follows is a straightforward adaptation to the problem at hand of a construction in [8, Sect. 5].

two asymptotic regions), and let $\mathcal{E}$ be any embedded two-sided three-dimensional submanifold of $\mathrm{int}\mathcal{D}^+(\Sigma;M)\setminus\langle\langle M_{\rm ext}\rangle\rangle$, invariant under ϕ_t. We shall moreover suppose that $M\setminus\overline{\mathcal{E}}$ is connected, and that $\mathcal{E}$ is *not* invariant under the $U(1)$ factor of the isometry group of (M,g_{ab}). Let (M_a,g_a), $a=1,2$, be two copies of $M\setminus\overline{\mathcal{E}}$ with the metric induced from g. As $\mathcal{E}$ is two-sided, there exists an open neighborhood $\mathcal{O}$ of $\mathcal{E}$ such that $\mathcal{E}$ separates $\mathcal{O}$ into two disjoint open sets $\mathcal{O}_a$, $a=1,2$, with $\overline{\mathcal{O}_1}\cap\overline{\mathcal{O}_2}=\overline{\mathcal{E}}$, $\mathcal{O}_1\cap\mathcal{O}_2=\emptyset$. Let ψ_a denote the natural embedding of $\mathcal{O}_a$ into M_a. Let M_3 be the disjoint union of M_1, M_2 and $\mathcal{O}$, with the following identifications: a point $p\in\mathcal{O}_a\subset\mathcal{O}$ is identified with $\psi_a(p)\in M_a$. It is easily seen that M_3 so defined is a Hausdorff topological space.

We can equip M_3 with the obvious real analytic manifold structure and an obvious metric g_3 coming from (M_1,g_1), (M_2,g_2) and $(\mathcal{O},g|_{\mathcal{O}})$. Note that g_3 is real analytic with respect to this structure. Let finally (M_4,g_4) be any maximal [2] vacuum real analytic extension of (M_3,g_3). Then (M_4,g_4) is a maximal vacuum real analytic stably causal extension of $\langle\langle M_{\rm ext}\rangle\rangle$ which clearly is not isometric to (M,g).

The space–time (M_4,g_4) satisfies all the hypotheses of [11]. It should be clear that all the orbits of the rotation group acting on (M,g_{ab}) meeting $\overline{\mathcal{E}}\setminus\mathcal{E}$ are incomplete in (M_4,g_4), so that the connected component of the identity of the group of isometries drops down from $\mathbb{R}\times U(1)$ (in the case of (M,g_{ab})) to $\mathbb{R}$ (in the case of (M_4,g_4)). We shall not attempt to give a rigorous proof of such a claim in all generality, let us instead give a simple argument showing that at least some orbits cannot be complete whatever (M_4,g_4), under a supplementary condition on $\mathcal{E}$. Suppose, for instance, that for all $p\in\mathcal{E}\subset M$ the standard "rotational" Killing vector, say Z, in (M,g_{ab}) (which is equal to $\partial/\partial\phi$ in the coordinate system used in Eq. (5.31) of [11]) is transverse to $\mathcal{E}$ at p. (A possible example of such set $\mathcal{E}$ is $\mathcal{E}=\{u_+\in\mathbb{R}, r\in(a,b), \theta\in(\frac{\pi}{4},\frac{3\pi}{4}), \phi=0\}$ in the coordinate system used in Eq. (5.31) of [11], where a and b are arbitrary constants satisfying $r_-<a<b<r_+$.) For any point $p\in\mathcal{E}$ the orbit $\phi_t[Z](p)$ of the Killing vector Z starts at p at $t=0$, enters, say, $\mathcal{O}_1$, travels in M to come back at $t=2\pi$ from $\mathcal{O}_2$. Let χ_a be the natural embedding of M_a into M_4, choose a point $p\in\mathcal{E}$, let ϵ be small enough so that $\phi_t[Z](p)\in\mathcal{O}_1$ for $t\in[0,\epsilon)$, set $\hat{p}=\chi_2(\phi_\epsilon[Z](p))$ (here we have implicitly identified the point $\phi_\epsilon[Z](p)$ with the corresponding point in M_2), consider the orbit $\phi_t[\hat{Z}](\hat{p})$ of $\hat{Z}$ through $\hat{p}$, where $\hat{Z}$ is the Killing vector in M_4 which coincides with Z on M. (If no such Killing vector $\hat{Z}$ exists there is nothing to prove.) By construction we have $\phi_{2\pi}[\hat{Z}](\hat{p})\neq\hat{p}$, as the orbit $\phi_t[\hat{Z}](\hat{p})$ leaves M_2 and enters M_1 when crossing $\mathcal{E}\subset\mathcal{O}$ from $\mathcal{O}_2$ into $\mathcal{O}_1$. Thus there exist orbits of $\hat{Z}$ which are not 2π periodic on M_4. Suppose, for contradiction, that all the orbits of $\hat{Z}$ are complete. Choose a domain of outer communication $\langle\langle M_{\rm ext}\rangle\rangle$ of M_4 diffeomorphic to a standard domain of outer communication in the Kerr space–time (M,g_{ab}), we have $\phi_{2\pi}[\hat{Z}](p)=p$ for all $p\in\langle\langle M_{\rm ext}\rangle\rangle$. This, by standard results on one parameter groups of isometries on connected sets implies that $\phi_{2\pi}[\hat{Z}]$ is the identity on M_4 (and, hence, all the orbits of $\hat{Z}$ are 2π periodic), which leads to a contradiction, and establishes incompleteness of some Killing orbits on M_4.

Topological games put aside, the method of proof suggested in [11] of analytically extending ϕ_t faces the problem that ϕ_t might potentially be analytically extendible to a

[2] *Cf., e.g.* [4, Appendix C] for a proof of existence of space–times maximal with respect to some property. It should be pointed out that there is an error in that proof, as the relation $\prec$ defined there is not a partial order. This is however easily corrected by adding the requirement that the isometry Φ considered there restricted to some fixed three–dimensional hypersurface be the identity.

proper subset[3] of the space–time only. One can nevertheless hope that the analyticity of the domain of outer communication and some further conditions, as *e.g.* global hyperbolicity thereof, allow one to extend the locally defined isometries at least to the whole domain of outer communication. The aim of this paper is to show that this is indeed the case. More precisely, we wish to show the following:

Theorem 1.1. *Consider an analytic space–time (M, g_{ab}) with a Killing vector field X with complete orbits. Suppose that M contains an asymptotically flat three–end Σ_{ext} with time-like ADM four–momentum, and with $X(p)$ — time-like for $p \in \Sigma_{\text{ext}}$. (Here asymptotic flatness is defined in the sense of Eq. (2.2) with $\alpha > 1/2$ and $k \geq 3$.) Let $\langle\langle M_{ext}\rangle\rangle$ denote the domain of outer communication associated with Σ_{ext} as defined below, assume that $\langle\langle M_{\text{ext}}\rangle\rangle$ is globally hyperbolic and simply connected. If there exists a Killing vector field Y, which is not a constant multiple of X, defined on an open subset $\mathcal{O}$ of $\langle\langle M_{\text{ext}}\rangle\rangle$, then the isometry group of $\langle\langle M_{\text{ext}}\rangle\rangle$ (with the metric obtained from (M, g_{ab}) by restriction) contains $\mathbb{R} \times U(1)$.*

Remarks. 1. It should be noted that no field equations or energy inequalities are assumed.
2. Simple connectedness of the domain of outer communication necessarily holds when a positivity condition is imposed on the Einstein tensor of g_{ab} [10].[4]
3. When a positivity condition is imposed on the Einstein tensor of g_{ab}, the hypothesis of time-likeness of the ADM momentum can be replaced by that of existence of an appropriately regular Cauchy surface in (M, g_{ab}). See, *e.g.*, [12] and references therein; *cf.* also [2] for a recent discussion.
4. It should be emphasized that no claims about isometries of $M \setminus \langle\langle M_{\text{ext}}\rangle\rangle$ (with the obvious metric) are made.

Theorem 1.1 allows one to give a corrected version of the rigidity theorem, the reader is referred to [7] for a precise statement together with a proof.

It seems of interest to remove the condition of completeness of the Killing orbits of X above. Recall that completeness of those necessarily holds [5] in maximal globally hyperbolic, say vacuum, space–times under various conditions on the Cauchy data. (It was mentioned in [6] that the results of [5] generalize to the electro–vacuum case.) Those conditions are, however, somewhat unsatisfactory in the black hole context for the following reasons: recall that the existing theory of uniqueness of black holes gives only a classification of domains of outer communication $\langle\langle M_{\text{ext}}\rangle\rangle$. Thus in this context one would like to have results which do not make any hypotheses about the global properties of the complement of $\langle\langle M_{\text{ext}}\rangle\rangle$ in M. Moreover the hypotheses of those results of [5] which apply when degenerate Killing horizons are present require further justification. Here we wish to raise the question, whether or not it makes sense to talk about a stationary black hole space–time for space–times for which the Killing orbits are not complete in the asymptotic region. We do not know an answer to that question. It is nevertheless tempting to decree that in "physically reasonable" stationary black hole space–times the

[3] In the physics literature there seem to be misconceptions about existence and uniqueness of analytic extensions of various objects. As a useful example the reader might wish to consider the (both real and complex) analytic function f from, say, the open disc $D(1, 1/2)$ of radius $1/2$ centered at 1 into $\mathbb{C}$, defined as the restriction of the principal branch of $\log z$. Then: 1) There exists no analytic extension of f from $D(1, 1/2)$ to $\mathbb{C}$. 2) There exists no unique maximal subset of $\mathbb{C}$ on which an analytic extension of f is defined.

[4] *Cf.* also [9, 13, 11] for similar but weaker results. Note that in the stationary black hole context, under suitable hypotheses one can use Theorem 1.2 below to obtain completeness of orbits of X in $\langle\langle M_{\text{ext}}\rangle\rangle$, and then use [9] to obtain simple–connectedness of $\langle\langle M_{\text{ext}}\rangle\rangle$.

orbits of the Killing vector field X which is time-like in the asymptotically flat three–end $\Sigma_{\rm ext}$ are complete through points in the asymptotic region $\Sigma_{\rm ext}$. One would then like to be able to derive various desirable global properties of $\langle\langle M_{\rm ext}\rangle\rangle$ using this assumption. Our second result in this paper is the proof that in globally hyperbolic domains of outer communication the orbits of those Killing vector fields which are time-like in $\Sigma_{\rm ext}$ are complete "if and only if" they are so[5] for points $p \in \Sigma_{\rm ext}$ (it should be emphasized that, in contradistinction to [5], no maximality hypotheses are made and no field equations are assumed below; similarly no analyticity or simple connectedness conditions are made here):

Theorem 1.2. *Consider a space–time (M, g_{ab}) with a Killing vector field X and suppose that M contains an asymptotically flat three–end $\Sigma_{\rm ext}$, with X time-like in $\Sigma_{\rm ext}$. (Here the metric is assumed to be twice differentiable, while asymptotic flatness is defined in the sense of Eq. (2.2) with $\alpha > 0$ and $k \geq 0$.) Suppose that the orbits of X are complete through all points $p \in \Sigma_{\rm ext}$. Let $\langle\langle M_{\rm ext}\rangle\rangle$ denote the domain of outer communication associated with $\Sigma_{\rm ext}$ as defined below. If $\langle\langle M_{\rm ext}\rangle\rangle$ is globally hyperbolic, then the orbits of X through points $p \in \langle\langle M_{\rm ext}\rangle\rangle$ are complete.*

In view of the recent classification of orbits of Killing vector field in asymptotically flat space–times of [1] it is of interest to prove the equivalent of Theorem 1.2 for "stationary–rotating" Killing vectors X, as defined in [1]. In Theorem 3.1 below we prove that generalization.

2. Definitions, Proof of Theorem 1.1

Throughout this work all objects under consideration are assumed to be smooth. For a vector field W we denote by $\phi_t[W]$ the (perhaps defined only locally) flow generated by W. Consider a Killing vector field X which is time-like for $p \in \Sigma_{\rm ext}$. If the orbits γ_p of X are complete through points $p \in \Sigma_{\rm ext}$, then we define the asymptotically flat four–end $M_{\rm ext}$ by

$$M_{\rm ext} = \cup_{t\in\mathbb{R}}\phi_t[X](\Sigma_{\rm ext}), \tag{2.1}$$

and the *domain of outer communication* $\langle\langle M_{\rm ext}\rangle\rangle$ by

$$\langle\langle M_{\rm ext}\rangle\rangle = J^-(M_{\rm ext}) \cap J^+(M_{\rm ext}).$$

Let $R > 0$ and let (g_{ij}, K_{ij}) be initial data on $\Sigma_{\rm ext} \equiv \Sigma_R \equiv \mathbb{R}^3 \setminus B(R)$ satisfying

$$g_{ij} - \delta_{ij} = O_k(r^{-\alpha}), \qquad K_{ij} = O_{k-1}(r^{-1-\alpha}), \tag{2.2}$$

with some $k \geq 1$ and some $0 < \alpha < 1$. A set $(\Sigma_{\rm ext}, g_{ij}, K_{ij})$ satisfying the above will be called *an asymptotically flat three–end*. Here a function f, defined on Σ_R, is said to be $O_k(r^\beta)$ if there exists a constant C such that we have

$$0 \leq i \leq k \qquad |\partial^i f| \leq C r^{\beta - i}.$$

We shall need the following result, which is a straightforward consequence[6] of what has been proved in [14]:

[5] The quotation marks here are due to the fact that in our approach the asymptotic four–end $\langle\langle M_{\rm ext}\rangle\rangle$ is not even defined when the orbits of X through $\Sigma_{\rm ext}$ are not complete. In that last case one could make sense of this sentence using Carter's definition of the domain of outer communication [3], involving Scri.

[6] Actually in [14] it is assumed that (M, g_{ab}) is Riemannian. The reader will note that all the assertions and proofs of [14] remain valid word for word when "Riemannian" is replaced by "pseudo–Riemannian".

Theorem 2.1 (Nomizu). *Let (M, g_{ab}) be a (connected) simply connected analytic pseudo–Riemannian manifold, and suppose that there exists a Killing vector field Y defined on an open connected subset $\mathcal{O}$ of M. Then there exists a Killing vector field $\hat{Y}$ defined on M which coincides with Y on $\mathcal{O}$.*

Let us pass to the proof of Theorem 1.1. Without loss of generality we may assume that X is future oriented for $p \in \Sigma_{\text{ext}}$. Simple connectedness and analyticity of $\langle\langle M_{\text{ext}}\rangle\rangle$ together with Theorem 2.1 allow us to conclude that the Killing vector Y can be globally extended to a Killing vector field $\hat{Y}$ defined on $\langle\langle M_{\text{ext}}\rangle\rangle$. The time-likeness of the ADM four–momentum p^μ allows us to use the results in [1] to assert that there exists a linear combination Z (with constant coefficients) of X and $\hat{Y}$ which has complete periodic orbits through all points p in M_{ext} which satisfy $r(p) \geq R$, for some R. (Moreover Z and X commute.) To prove Theorem 1.1 we need to show that the orbits of Z are complete (and periodic) for all $p \in \langle\langle M_{\text{ext}}\rangle\rangle$.

Consider, thus, a point $p \in \langle\langle M_{\text{ext}}\rangle\rangle$. There exist $q_\pm \in M_{\text{ext}}$, with $r(q_\pm) \geq R$, such that $p \in J^-(q_+) \cap J^+(q_-)$. Completeness and periodicity of the orbits $\gamma_{q_\pm}[Z] \equiv \cup_{t\in\mathbb{R}}\phi_t[Z](q_\pm)$ of Z through $q_\pm$ implies that the sets $\gamma_{q_\pm}[Z]$ are compact. Global hyperbolicity of $\langle\langle M_{\text{ext}}\rangle\rangle$ implies then that

$$K \equiv J^-(\gamma_{q_+}[Z]) \cap J^+(\gamma_{q_-}[Z])$$

is compact.

For $q \in \langle\langle M_{\text{ext}}\rangle\rangle$ let $t_\pm(q) \in \mathbb{R} \cup \{\pm\infty\}$ be the forward and backward life time of orbits of Z through q, defined by the requirement that $(t_-(q), t_+(q))$ is the largest connected interval containing 0 such that the solution $\phi_t[Z](q)$ of the equation $d\phi_t[Z](q)/dt = Z \circ \phi_t[Z](q)$ is defined for all $t \in (t_-(q), t_+(q))$. From continuous dependence of solutions of ODE's upon initial values it follows that t_+ is a lower semi–continuous function and t_- is an upper semi–continuous function.

Let $\gamma : [0, 1] \to M$ be any future oriented causal curve such that $\gamma(0) = q_-$, $\gamma(1) = q_+$, and $p \in \gamma$. Set

$$T_+ = \inf_{q\in\gamma} t_+(q), \qquad T_- = \sup_{q\in\gamma} t_-(q). \tag{2.3}$$

Here and elsewhere inf and sup are taken in $\mathbb{R} \cup \{\pm\infty\}$. If $T_\pm = \pm\infty$ we are done, suppose thus that $T_+ \neq \infty$; the case $T_- \neq -\infty$ is analyzed in a similar way. By lower semi–continuity of t_+ and compactness of γ there exists $\tilde{p} \in \gamma$ such that $t_+(\tilde{p}) = T_+$. By global hyperbolicity the family of causal curves $\phi_t[Z](\gamma)$, $t \in [0, T_+)$, accumulates at a causal curve $\tilde{\gamma} \subset K$. Consequently the orbit $\phi_t[Z](\tilde{p})$, $t \in [0, T_+)$, has an accumulation point in K. It follows that $\phi_t[Z](\tilde{p})$ can be extended beyond T_+, which gives a contradiction unless $T_+ = \infty$, and the result follows. □

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Without loss of generality we may suppose that X is future oriented for $p \in \Sigma_{\text{ext}}$. Consider a point $p \in \langle\langle M_{\text{ext}}\rangle\rangle$, there exist $p_\pm \in M_{\text{ext}}$ such that $p \in J^+(p_-) \cap J^-(p_+)$. Let Σ be a Cauchy surface for $\langle\langle M_{\text{ext}}\rangle\rangle$, without loss of generality we may assume that $p_- \in I^-(\Sigma)$ and $p_+ \in I^+(\Sigma)$. Let $t_\pm$ be defined as in the proof of Theorem 1.1, we have $t_-(p_\pm) = -\infty$, $t_+(p_\pm) = \infty$. Let $\gamma : [0, 1] \to \langle\langle M_{\text{ext}}\rangle\rangle$ be any

causal curve such that $\gamma(0) = p_-$, $\gamma(1) = p_+$, and $p \in \gamma$. Define $T_\pm$ by Eq. (2.3). By lower semi–continuity of t_+ there exists $\tilde p \in \gamma$ such that $t_+(\tilde p) = T_+$. Define

$$\tilde\Omega = \{s \in [0, T_+) : \phi_s[X](\tilde p) \in I^-(\Sigma)\} \ .$$

Consider any $s \in \tilde\Omega$. Then the curve obtained by concatenating $\phi_t[X](p_-)$, $t \in [0, s]$, with $\phi_s[X](\gamma)$ is a future directed causal curve which starts at p_- and passes through $\phi_s[X](\tilde p)$, hence

$$s \in \tilde\Omega \quad \Rightarrow \quad \phi_s[X](\tilde p) \in K \equiv J^+(p_-) \cap J^-(\Sigma) \ . \tag{3.1}$$

By global hyperbolicity of $\langle\langle M_{\rm ext}\rangle\rangle$ the set K is compact. If $\tilde\Omega = \emptyset$ set $\omega = 0$, otherwise set

$$\omega = \sup \tilde\Omega \ .$$

Consider any sequence $\omega_i \in \tilde\Omega$ such that $\omega_i \to \omega$. By (3.1) and by compactness of K the sequence $\phi_{\omega_i}[X](\tilde p)$ has an accumulation point in K. It follows that $\omega < T_+$.

By definition of ω we have $\phi_s[X](\tilde p) \in J^+(\Sigma)$ for all $\omega \le s < T_+$. By Lemma 2.5 of [5] it follows that $T_+ = \infty$. As $t_+(p) \ge t_+(\tilde p) = T_+$ we obtain $t_+(p) = \infty$. The equality $t_-(p) = -\infty$ for all $p \in \langle\langle M_{\rm ext}\rangle\rangle$ is obtained similarly by using the time–dual version of Lemma 2.5 of [5]. □

Before presenting a generalization of Theorem 1.2 which covers the case of "stationary–rotating" Killing vectors, as defined in [9, 1], we need to introduce some terminology. Following [9] we shall say that the orbit through p of a Killing vector field Z is time–oriented if there exists $t_p > 0$ such that $\phi_{t_p}[Z](p) \in I^+(p)$. It then follows that for all $\alpha \in R$ and all $z \in \mathbb{N}$ we have $\phi_{\alpha+zt_p}[Z](p) \in I^+(\phi_\alpha[Z](p))$: if γ is a timelike curve from p to $\phi_{t_p}[Z](p)$, one obtains a timelike curve from $\phi_\alpha[Z](p)$ to $\phi_{\alpha+zt_p}[Z](p)$ by concatenating $\phi_\alpha[Z](\gamma)$ with $\phi_{\alpha+t_p}[Z](\gamma)$ with $\phi_{\alpha+2t_p}[Z](\gamma)$, etc.

A trivial example of a Killing vector field with time–oriented orbits is given by a timelike Killing vector field. A more interesting example is that of "stationary–rotating" Killing vector fields, as considered in [9, 1] — loosely speaking, those are Killing vectors which behave like $\alpha\partial/\partial t + \beta\partial/\partial\phi$ in the asymptotic region, with α and β non–vanishing, where ϕ is an angular coordinate. Thus the theorem that follows applies in the "stationary–rotating" case.

Theorem 3.1. *The conclusion of Theorem 1.2 will hold if to its hypotheses one adds the requirement that k in (2.2) is larger than or equal to 2, and if the hypothesis that X is timelike is replaced by the assumption that the orbits of X are* time–oriented *through all $p \in \Sigma_{\rm ext}$.*

Proof. The proof is achieved by a minor modification of the proof of Theorem 1.2, as follows: Let $p_\pm$ be as in that proof, from the asymptotic behavior of Killing vector fields in asymptotically flat space–times (*cf. e.g.* Sect. 2 of [2]) it follows that we can without loss of generality assume that

$$\begin{aligned} &\phi_{2\pi}[X](p_+) \in I^+(p_+), \qquad \phi_{2\pi}[X](p_-) \in I^+(p_-) \ , \\ &\forall s \in [0, 2\pi] \quad \phi_s[X](p_-) \in I^-(\Sigma) \ , \quad \phi_s[X](p_+) \in I^+(\Sigma) \ . \end{aligned}$$

The proof proceeds then as before, up to the definition of the set K, Eq. (3.1). In the present case that definition is replaced by

$$K \equiv J^+(\cup_{s\in[0,2\pi]}\phi_s[X](p_-)) \cap J^-(\Sigma) \ .$$

This set is again compact, in view of global hyperbolicity of $M_{\rm ext}$. The fact that for $s \in \tilde{\Omega}$ we have $\phi_s[X](\tilde{p}) \in K$ follows by considering the causal curve obtained by concatenating a causal curve γ_1 from $\phi_{s-\lfloor s/2\pi \rfloor 2\pi}[X](\tilde{p})$ to $\phi_s[X](\tilde{p})$ with $\phi_s[X](\gamma)$. Here $\lfloor \alpha \rfloor$ denotes the largest integer smaller than or equal to α; the existence of γ_1 is guaranteed by our discussion above. □

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