ANNALS OF PHYSICS 237, 322–354 (1995)

On "Asymptotically Flat" Space-Times with G_2-Invariant Cauchy Surfaces

B. K. BERGER*, P. T. CHRUŚCIEL†, AND V. MONCRIEF‡

Institute for Theoretical Physics, University of California, Santa Barbara, California 93106–4030

Received April 5, 1994

In this paper we study space-times which evolve out of Cauchy data $(\Sigma, {}^3g, K)$ invariant under the action of a two-dimensional commutative Lie group. Moreover, $(\Sigma, {}^3g, K)$ are assumed to satisfy certain completeness and asymptotic flatness conditions in spacelike directions. We show that asymptotic flatness and energy conditions exclude all topologies and group actions except for a cylindrically symmetric $\mathbb{R}^3$, or a periodic identification thereof along the z-axis. We prove that asymptotic flatness, energy conditions, and cylindrical symmetry exclude the existence of compact trapped surfaces. Finally, we show that the recent results of Christodoulou and Tahvildar-Zadeh concerning global existence of a class of wave-maps imply that strong cosmic censorship holds in the class of asymptotically flat cylindrically symmetric electro-vacuum space-times.

1. INTRODUCTION

It is widely believed that an important question in classical general relativity is that of *strong cosmic censorship*, due to Penrose [34]. A mathematical formulation thereof, essentially due to Moncrief and Eardley [29] (cf. also [8, 9]), is the following:

> Consider the collection of initial data for, say, vacuum or electro-vacuum space-times, with the initial data surface Σ being compact, or with the initial data $(\Sigma, {}^3g, K)$—asymptotically flat. For generic such data the maximal globally hyperbolic development thereof is inextendible.

* Permanent address: Department of Physics, Oakland University, Rochester, MI 48309. Supported in part by NSF Grant PHY 89-04035 to the ITP, UCSB and NSF Grants PHY-9107162 and PHY-9305599 to Oakland University. e-mail: berger@vela.acs.oakland.edu.

† On leave from the Institute of Mathematics of the Polish Academy of Sciences. Supported in part by KBN Grant 2 1047 9101, NSF Grant PHY 89-04035 to the ITP, UCSB, and the Alexander von Humboldt Foundation. e-mail: piotr@mpa-garching.mpg.de. Present adress: Max Planck Institut für Astrophysik, Karl Schwarzschild Str. 1, D 85740 Garching bei München, Germany.

‡ Permanent address: Department of Physics, Yale University, P.O. Box 6666, New Haven, CT 06511-8167. Supported in part by NSF Grant PHY 89-04035 to the ITP, UCSB, and PHY-9201196 to Yale University. e-mail: moncrief@yalph2.physics.yale.edu.

0003-4916/95 $9.00

The failure of the above would mean a serious lack of predictability of Einstein's equations, an unacceptable feature of a physical theory.

Because of the difficulty of the strong cosmic censorship problem, a full understanding of the issues which arise in this context seems to be completely out of reach at this stage. For this reason there is some interest in trying to understand that question under various restrictive hypotheses, e.g., under symmetry hypotheses. Such a program has been undertaken by one of us (V.M.) in [29, 26], and some further results in the spatially compact case have been obtained in [11, 19, 8, 7]. Here we consider the question of strong cosmic censorship in the space of initial data $(\Sigma, {}^3g, K, A, E)$ for the electro-vacuum Einstein equations which are invariant under the action of a two-dimensional commutative Lie group G_2, and which satisfy some completeness and asymptotic flatness conditions.

Clearly it would be desirable to analyze the strong cosmic censorship problem with the minimal amount of restrictive conditions imposed, for example assuming the existence of only *one* Killing vector. This problem seems still out of reach at this stage. The next "smallest" isometry group possible G_2 is two-dimensional. As discussed in Section 2, non-commutative G_2's are incompatible with (our notion of) asymptotic flatness. This leads us to the commutative groups considered in this paper. Let us also mention that an isometry group G_n with $n = \dim G_n \geqslant 3$ and asymptotic flatness seem to be compatible only with metrics which, locally, are isometric to the Schwarzschild metric in the vacuum case and to the Reissner–Nordström metric in the electro-vacuum case.

Thus, in this paper we address the question of global properties of maximal globally hyperbolic electro-vacuum space-times with complete Cauchy surfaces Σ and with "asymptotically flat" (in a sense to be made precise below) Cauchy data invariant under the effective, proper action of a commutative connected two-dimensional Lie group G_2. We list all the possible topologies of Σ and actions of G_2 (Section 2). In that same section we show that the constraint equations and some energy conditions exclude all but the standard cylindrically symmetric model: $G_2 = \mathbb{R} \times U(1)$ with an action on $\Sigma \approx \mathbb{R}^3$ by translations in z and rotations in the planes $z = \text{const}$ (or a periodic identification thereof along the z-axis). (For these results we do not actually need to assume that the matter is of electromagnetic nature; all we need are some appropriate energy conditions.) We also show that trapped surfaces which are either compact or invariant under G_2 are not allowed when asymptotic flatness conditions and energy conditions are imposed. Now it is folklore knowledge that electro-vacuum equations and cylindrical symmetry reduce to a wave-map equation on $(2+1)$-dimensional Minkowski space-time. This is, however, not true without some further conditions and the main point of the considerations in that section (and in fact one of the main points of this paper) is to prove that our global hypotheses on the Cauchy data do indeed lead to such a reduction. In Section 3 we briefly analyze some global properties of cylindrically symmetric asymptotically flat models: we prove the "cylindrically symmetric positive energy theorem"; we note the non-existence of vacuum or electro-vacuum "spinning" solutions. In Section 4 we discuss the reduced electro-vacuum equations;

here again, the local form of the final result is well known, but our emphasis is to take into account the global aspects of the problem. In Section 5 we use some recent results of Christoudoulou and Tahvildar-Zadeh [5, 6] to prove that strong cosmic censorship holds in the class of electro-vacuum cylindrically symmetric space-times considered here. This is the main result of this paper. On one hand, it should be clear that at the heart of this assertion lie the deep and difficult theorems of Christoudoulou and Tahvildar-Zadeh. On the other, the global aspects of the reduction of the strong cosmic censorship question to the corresponding wave-map problem have never been considered in the literature in our context, and we believe that several of the results presented here are new from this point of view.

Let us close this Introduction with some bibliographical remarks. The class of metrics analyzed here seems to have been first considered by Kompaneets [20]; their "polarized" counterpart has been first studied by Beck [2]. Significant steps in the "reduction program" have been done by Papapetrou [32] and, independently, by Kundt and Trumper [22]. The final reduction of the equations to a "wave-map" problem is essentially due to Ernst [12, 13]. A description of cylindrically symmetric metrics using tools completely different from ours can be found in [38].

2. Some General Properties of Space-Times with Two Commuting Killing Vectors Tangent to a Cauchy Surface

Let $(\Sigma, {}^3g, K)$ be Cauchy data for Einstein equations [4] (perhaps with matter satisfying some well-behaved equations, in which case the appropriate data for the matter fields should also be given), thus Σ is a three-dimensional manifold (which we assume throughout to be smooth, connected, paracompact, Hausdorff; Σ will also be assumed to be orientable unless explicitly indicated otherwise), ${}^3g$ is a Riemannian metric on Σ and K is a symmetric tensor field on Σ. $({}^3g, K)$ are assumed to satisfy the general relativistic constraint equations [4]. We will, moreover, assume throughout that $(\Sigma, {}^3g)$ is geodesically complete and that there exist two linearly independent vector fields X_a, $a = 1, 2$, on Σ such that

$$\mathcal{L}_{X_a} {}^3g = \mathcal{L}_{X_a} K = 0,$$

where $\mathcal{L}$ denotes a Lie derivative. It is well known that geodesic completeness of $(\Sigma, {}^3g)$ implies that the orbits of X_a are complete (cf., e.g., [30]); hence, assuming that there are no more linearly independent Killing vectors, there exists a two-dimensional Lie group G which acts effectively and properly on $(\Sigma, {}^3g)$ by isometries.

Let us start by showing that an appropriate notion[1] of asymptotic flatness implies[2] that the Killing vectors have to commute. For the purpose of the discussion here we shall say that $(\Sigma, {}^3g)$ is asymptotically flat if: (1) $(\Sigma, {}^3g)$ is geodesically complete with Σ, Σ/G-*not* compact; (2) there exists a G-invariant subset $\mathscr{K}$ of Σ such that ${}^3g$ is flat on $\Sigma\backslash\mathscr{K}$, with (3) $\mathscr{K}/G$-compact. Under these conditions, the Lie algebra of a group G of isometries of ${}^3g$ has to be a subalgebra of the Lie algebra of isometries of a flat $\mathbb{R}^3$. Now it is easily seen that such algebras are commutative when $\dim G = 2$ is assumed. This shows that a non-commutative two-dimensional G is incompatible with the notion of asymptotic flatness defined above. In the remainder of this paper we shall assume that G is abelian, so that the Killing vectors X_a commute.

Let (M, g) be a globally hyperbolic development of the initial data on which G acts by isometries (such a development always exists if one solves the vacuum equations, cf., e.g., [10 or 8, Section 2.1]; more generally, this result still holds if the matter fields satisfy some well-behaved equations of hyperbolic type). Define

$$\tilde{M} = \{p \in M : \det \lambda_{ab} \neq 0\},$$

where

$$\lambda_{ab} \equiv g(X_a, X_b).$$

By well-known properties[3] of Killing vectors and group actions, $\tilde{M}$ is an open dense subset of M diffeomorphic to ${}^2\tilde{M} \times G$, for some two-dimensional manifold ${}^2\tilde{M}$. On $\tilde{M}$ (passing to an appropriate subset of $\tilde{M}$ if necessary) we can choose coordinates (t, ρ, x^a) so that we have

$$X_a^\mu \partial_\mu = \frac{\partial}{\partial x^a}, \qquad x^a = z, \theta,$$

and we can always parametrize the metric as

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = h_{AB}\, dx^A\, dx^B + \lambda_{ab}(dx^a + M^a_A\, dx^A)(dx^b + M^b_B\, dx^B), \tag{2.1}$$

$$h_{AB}\, dx^A\, dx^B = e^{2(\nu-\gamma)}(-dt^2 + d\rho^2), \tag{2.2}$$

$$\lambda_{ab}\, dx^a\, dx^b = e^{2\gamma}(dz + a\, d\theta)^2 + R^2 e^{-2\gamma}\, d\theta^2, \tag{2.3}$$

[1] The notion of asymptotic flatness described here coincides with that used in our strong cosmic censorship theorems for cylindrically symmetric electro-vacuum space-times, Section 5. The main justification for the "reasonableness" of this definition is that it is compatible with a large class of nontrivial geometries. On the other hand it does not allow for the Schwarzschild geometry, or for initial data of "hyperboloidal type" which are asymptotic to "Scri" rather than to "i^0". The reader should, however, note that several results in this Section are proved assuming various considerably weaker notions of asymptotic flatness.

[2] We are grateful to Bernd Schmidt for useful discussions concerning this point.

[3] Alternatively, this result follows directly from our list of topologies and actions given below.

with some functions ν, γ, a, R, M^a_A, which do not depend upon x^a. Note that we have

$$R^2 = \det \lambda_{ab} \tag{2.4}$$

so that R is the area density of the orbits of the isometry group. The constraint equations imply that

$$D_j(X^i_a P^j_i) = -T_{i\mu} n^\mu X^i_a, \tag{2.5}$$

where n^μ is the future pointing normal to Σ, D_i is the covariant derivative of $g_{ij} = {}^3g_{ij}$, with

$$P_{ij} = {}^3g^{kl} K_{kl}\, {}^3g_{ij} - K_{ij}.$$

(Here we have absorbed the usual [4] constant $8\pi G/c^4$ in the definition of $T_{\alpha\beta}$.) The Einstein equations for this class of metrics can be found in Appendix C [31]. From the constraint equations (C.31)–(C.32) one derives

$$\frac{\partial R_\pm}{\partial \rho} = R_\pm \nu_\pm - h_\pm, \tag{2.6}$$

$$h_\pm = R\left[\gamma_\pm^2 + \frac{e^{4\gamma}}{4R^2} a_\pm^2\right] + Re^{2(\nu-\gamma)} T_{\mu\nu} n^\mu (n^\mu \pm m^\mu)$$
$$+ \frac{e^{2(\nu-\gamma)}}{4R} \lambda^{ab} c_a c_b, \tag{2.7}$$

with

$$f_\pm \equiv \partial_\pm f \equiv \partial_\rho f \pm \partial_t f,$$
$$c_a \equiv \varepsilon_{\mu\nu\rho\sigma} X^\mu_1 X^\nu_2 \nabla^\rho X^\sigma_a = 2RK_{ij} m^i X^j_a, \tag{2.8}$$

and λ^{ab} is the matrix inverse to λ_{ab}. Here $m^\mu\, \partial_\mu = m^i\, \partial_i$ denotes the field of unit vectors tangent to Σ and normal to the orbits of G, and ∇_μ is the covariant derivative of the space-time metric $g_{\mu\nu}$.

The case of compact Σ's has been discussed in some detail in [18, 7, 26] (cf. also [19, 11, 8]). It has been pointed out to us by H. J. Seifert[4] that the following list[5] exhausts all the smooth, effective, proper[6] actions by isometries of a commutative, connected, two-dimensional Lie group G on a connected, smooth, Hausdorff,

[4] H. J. Seifert, private communication. We are grateful to H. J. Seifert for several discussions concerning this point.

[5] The following manifolds and actions are rather obvious. The point of the list given below is to emphasize that no other possibilities occur.

[6] The hypothesis that the action is proper will be automatically satisfied if we assume that the group G here is the connected component of the group of *all* the isometries of $(\Sigma, {}^3g)$.

three-dimensional *non-compact* Riemannian manifold $(\Sigma, {}^3g)$ (up to automorphism of G and diffeomorphism of Σ):

1. $G = \mathbb{R} \times U(1)$.

(a) $\Sigma = \mathbb{R}^3$: $G \times \Sigma \ni (g = (a, e^{i\psi}), p = (\rho e^{i\theta}, z)) \to \phi_g(p) = (\rho e^{i(\psi+\theta)}, z + a)$; here and below, whenever convenient a point $(x, y) \in \mathbb{R}^2$ is represented in a standard way as $x + iy = \rho e^{i\theta}$. When used without further qualifications, the notion of *cylindrical symmetry* will refer to this model.

(b) $\Sigma = \mathbb{R}^2 \times S^1$: $G \times \Sigma \ni (g = (a, e^{i\psi}), p = (\rho, e^{i\theta}, z)) \to \phi_g(p) = (\rho, e^{i(\psi+\theta)}, z + a)$; this model will be referred to as the *cylindrically symmetric wormhole*.

(c) $\Sigma = \{[1, \infty) \times S^1 \times \mathbb{R}\}/\sim$, where the equivalence relation $\sim$ identifies $(\rho = 1, e^{i\theta}, z)$ with $(\rho = 1, e^{i(\theta+\pi)}, z)$. It should be pointed out, however, that the manifold Σ here is *not* orientable. The action $\phi_g(p)$ is the same as the one in point 1(a).

(d) $\Sigma = S^2 \times \mathbb{R}$; the action here consists of translations of the $\mathbb{R}$ factor of Σ and of rotations of S^2 around some fixed axis, when S^2 is identified as a subset of $\mathbb{R}^3$ in the standard way.

(e) $\Sigma = \{B^2 \times \mathbb{R}\}/\sim$, where B^2 is the closed two-dimensional ball of radius 1, $B^2 = \{\rho e^{i\theta}, 0 \leqslant \rho \leqslant 1\} \subset \mathbb{R}^2$, and where the equivalence relation $\sim$ identifies the points $\rho e^{i\theta}$, $\rho = 1$ with $\rho e^{i(\theta+\pi)}$, $\rho = 1$. This manifold is *not* orientable. The action here is the same as in point 1(a) above.

(f) $\Sigma = S^1 \times \mathbb{R} \times S^1$. The action here is the same as in point 1(b) above, except for a supplementary S^1 identification of the $\mathbb{R} = \{\rho\}$ factor of Σ in 1(b).

(g) $\Sigma = \{[1, 2] \times S^1 \times \mathbb{R}\}/\sim$, where the equivalence relation $\sim$ identifies $(\rho = 1, e^{i\theta}, z)$ with $(\rho = 1, e^{i(\theta+\pi)}, z)$, and $(\rho = 2, e^{i\theta}, z)$ with $(\rho = 2, e^{i(\theta+\pi)}, z)$. Similarly to the points 1(c) and 1(d) above, the manifold Σ here is *not* orientable. The action $\phi_g(p)$ is the same as the one in point 1(a).

2. $G = \mathbb{R}^2$.

(a) $\Sigma = \mathbb{R}^3$; this is the standard action of $\mathbb{R}^2$ on $\mathbb{R}^3$ by translations.

(b) $\Sigma = S^1 \times \mathbb{R}^2$; the action is by translations in the $\mathbb{R}^2$ factor of Σ.

3. $G = U(1) \times U(1)$. Here the manifolds and actions are as in points 1(a)–1(c) above, except for a supplementary S^1 identification in the $\mathbb{R} = \{z\}$ factor of the manifolds listed there. (The manifolds and actions listed in points 1(d)–1(g) drop out, as a supplementary S^1 identification in the $\mathbb{R} = \{z\}$ factor would lead to compact models.)

It is natural to assume that the initial data manifold Σ is orientable, and that (Σ, g) is complete. We shall show now that these requirements together with some asymptotic flatness conditions and positivity conditions on the energy-momentum tensor exclude essentially all cases above, except the cylindrically symmetric model (or the periodic identification thereof along the z-axis). Let us first note that the geometries of point 3 differ from those of point 1 by trivial identifications and, in this sense, do not require separate considerations. Next, note that any G-invariant complete metric of the geometries of points 1(d)–1(g) and of point 2(b) above defines naturally a complete metric on the corresponding compact model (in which the appropriate $\mathbb{R}$ factors have been compactified to S^1). It is clear that no such model can be termed as asymptotically flat in any sense; indeed, it seems that a reasonable prerequisite for a definition of asymptotic regions is to require that the quotient manifold Σ/G be *non-compact*. Keeping in mind the requirement of orientability of Σ, it then suffices to discuss the geometries 1(a), 1(b), and 2(a).

We shall start with the cylindrically symmetric wormhole, case 1(b) above: By way of example, consider the following metric on $M = \mathbb{R}^3 \times S^1$:

$$ds^2 = -dt^2 + dz^2 + d\rho^2 + (1+\rho^2)\, d\theta^2, \qquad t, \rho, z \in (-\infty, \infty),\ \theta \in S^1 \approx [0, 2\pi]|_{\mathrm{mod}\, 2\pi}. \tag{2.9}$$

Clearly $X_1 = \partial/\partial z$ and $X_2 = \partial/\partial\theta$ generate an action of $G = \mathbb{R} \times S^1$ on $(M, {}^3g)$ by isometries. M has two asymptotically flat ends connected by a throat, and one can replace $(1+\rho^2)$ by some function which will preserve the overall features of the metric (2.9) with the metric being exactly flat outside a set $\mathscr{U}$ the projection of which onto the orbit set M/G is compact. The area function $R \equiv \sqrt{\det g(X_a, X_b)} = \sqrt{1+\rho^2}$ satisfies

$$\partial_t R = 0, \tag{2.10}$$

$$\partial_\rho R \to \begin{cases} -1, & \rho \to -\infty, \\ 1, & \rho \to \infty \end{cases}. \tag{2.11}$$

We shall try to mimic this behavior as follows: let $G = \mathbb{R} \times U(1)$ act by translations in z and rotations in θ on $\Sigma = \{(\rho, z, \theta) \in \mathbb{R}^2 \times S^1\}$, and let $({}^3g, K)$ be G-invariant initial data for a space-time metric of the form (2.1)–(2.3). Suppose, moreover, that there exist constants C, $\varepsilon > 0$ such that on Σ it holds that

$$R \geqslant \varepsilon, \tag{2.12}$$

$$\rho \leqslant -C\colon\ (m^\mu \pm n^\mu)\, \partial_\mu R \leqslant -\varepsilon, \tag{2.13}$$

$$\rho \geqslant C\colon\ (m^\mu \pm n^\mu)\, \partial_\mu R \geqslant \varepsilon. \tag{2.14}$$

A metric satisfying (2.12)–(2.14) will be called "asymptotically of wormhole type." Comparing with (2.10)–(2.11), the conditions (2.12)–(2.14) do not seem to be overly stringent. Lemma 4.1 of [7] implies the following immediately.

PROPOSITION 2.1. *No C^2 solutions of the constraint equations satisfying (2.12)–(2.14) exist when the energy-momentum tensor of the matter fields satisfies the inequality*

$$T_{\mu\nu} X^\mu Y^\nu \geqslant 0 \text{ for } X^\mu \text{ timelike future pointing, } Y^\mu \text{ null future pointing.} \tag{2.15}$$

Although the point 1(c) has been excluded from our considerations by the requirement of orientability, let us nevertheless point out that this geometry can also be excluded using the constraint equations: This follows immediately from the fact that the Cauchy surface for the $(2+1)$-dimensional wormhole discussed above provides the universal covering space for the manifold Σ described in point 1(c). (This example is a $(2+1)$-dimensional analogue of the $\mathbb{R}P^2$ identified Schwarzschild throat, discussed in [16].)

Consider next the case described in point 2(a) of our classification, which is that of *pp*-wave metrics. Thus let $G=\mathbb{R}^2$ act on $\mathbb{R}^3=\{z, \rho, \theta \in \mathbb{R}\}$ by translations in z and θ and suppose that $({}^3g, K)$ are G-invariant initial data such that

1. ${}^3g$ is flat outside of a set $\mathscr{K}$ such that $\mathscr{K}/G$ is compact;
2. $(\Sigma, {}^3g)$ is a complete Riemannian manifold; it then follows that in the coordinates of (2.1)–(2.3) ρ covers the whole range $(-\infty, \infty)$;
3. $\lim_{|\rho| \to \infty} \partial R/\partial n = 0$, where $\partial R/\partial n$ is the derivative of R in the direction normal to Σ.

Such initial data will be called *initial data for a localized pp-wave*. We have the following result which is somewhat reminiscent of a result of Penrose [33].

Proposition 2.2. *Let $(\Sigma, {}^3g, K)$ be C^2 initial data for a localized pp-wave. If* (2.15) *holds, then we must have $T_{\mu\nu}n^\mu(n^\mu \pm m^\mu)|_\Sigma \equiv 0$, and $(\Sigma, {}^3g, K)$ must be initial data for Minkowski space-time.*

Proof. It is an easy exercise to find the most general form of ${}^3g$ on $\Sigma \backslash \mathscr{K}$. One finds in particular that we must have

$$\lim_{\rho \to -\infty} \frac{\partial R}{\partial \rho} \leqslant 0, \qquad \lim_{\rho \to \infty} \frac{\partial R}{\partial \rho} \geqslant 0,$$

and that the functions $f_\pm$ appearing in Eq. (4.1) of [7] satisfy $f_\pm \in L^1(\mathbb{R})$. The result follows now from the arguments of the proof of Lemma 4.1 of [7]. ∎

Let us note, that both in Propositions 2.1 and 2.2 we can allow initial data of C^1 differentiability and with a distributional component in $T_{\mu\nu}$ (in which case we need to assume that (2.15) holds in a distributional sense). The asymptotic flatness conditions in Proposition 2.2 can be considerably weakened; this will be discussed elsewhere.

In the remainder of this paper we shall be concerned with cylindrical symmetry: Namely, let $G=\mathbb{R} \times U(1)$ act on $\Sigma=\mathbb{R}^3$ by translations in z and rotations in the planes $z=\text{const}$. Initial data $({}^3g, K)$ on $\Sigma=\mathbb{R}^3$ that is invariant under this action of G will be called cylindrically symmetric. It is well known that this topology of Σ and this action of G are compatible with initial data $(\Sigma, {}^3g, K)$ such that $({}^3g, K)$ are data for Minkowski space-time outside a set $\mathscr{K}$ such that $\mathscr{K}/G$ is compact. We have the following generalization of Corollary 5.2 of [7] (cf. also [36, 18]), which follows immediately from Eqs. (2.6)–(2.7) and from the arguments of the proof of Corollary 5.2 of [7].

PROPOSITION 2.3. *Let* $(\Sigma, {}^3g, K)$ *be* C^2 *cylindrically symmetric initial data and suppose that* (2.15) *holds. Suppose, moreover, that there exists* $C > 0$ *such that*

$$t = 0, \qquad \rho \geqslant C\colon (m^\mu \pm n^\mu)\, \nabla_\mu R > 0. \tag{2.16}$$

(*Recall that* n^μ *is a field of unit normals to* Σ *and* m^μ *is a unit vector field that is tangent to* Σ *and orthogonal* (*outwards pointing*) *to the orbits of* G.) *Then*

1. $\nabla^\mu R$ *is spacelike on* Σ,

2. *There are no trapped surfaces* $\mathscr{T} \subset \Sigma$ *which are either compact, or invariant under* G.

Note that the vector fields $m^\mu \pm n^\mu$ are null and orthogonal to the orbits of G. For cylindrically symmetric metrics we shall always choose m^μ to be outwards directed in the obvious sense; then the above null vector fields will also be called outwards directed.

Let us mention that (2.16) can be thought of as a rather mild asymptotic flatness condition (compare Eqs. (2.10), (2.11)). Let us also note, that if a globally hyperbolic development (M, g) of $(\Sigma, {}^3g, K)$ can be foliated by Cauchy surfaces on which (2.15) and (2.16) hold, then $\nabla^\mu R$ will be globally spacelike, and there will be no trapped surfaces of the kind considered above in M. (It should be clear from the analysis below that if $T_{\mu\nu}(n^\mu n^\nu - m^\mu m^\nu) \equiv 0$ holds, then these last conclusions will hold when (2.16) holds on one single Cauchy surface, as long as dynamics preserves (2.16). This is indeed the case for electro-vacuum space-times satisfying appropriate asymptotic conditions; cf. Theorem 5.2 below.)

To proceed further, let us rewrite somewhat more explicitly the metric (2.1) in the form

$$\begin{aligned} g_{\mu\nu}\, dx^\mu\, dx^\nu = e^{2(\nu-\gamma)}(-dt^2 + d\rho^2) + \lambda_{ab}(dx^a + M^a\, dt + g^a\, d\rho) \\ \times (dx^b + M^b\, dt + g^b\, d\rho). \end{aligned} \tag{2.17}$$

Replacing x^a by $x^a + \int_0^\rho g^a(t, s)\, ds$ we can achieve

$$g^a \equiv 0. \tag{2.18}$$

(It is not too difficult to show, using, e.g., the methods of [7, Appendix C], that the above is a smooth coordinate transformation if the metric $g_{\mu\nu}$ is smooth.) A MATHEMATICA calculation (using MATHTENSOR [31]) gives the equations (cf. (C.21)–(C.22))

$$\frac{\partial M^z}{\partial \rho} = -a\frac{e^{2\nu}}{R^3}(ac_1 - c_2) - \frac{e^{2\nu - 4\gamma}}{R}c_1, \tag{2.19}$$

$$\frac{\partial M^\theta}{\partial \rho} = \frac{e^{2\nu}}{R^3}(ac_1 - c_2). \tag{2.20}$$

Let us also assume that

$$T_{\mu\nu} n^\mu X_a^\nu \equiv 0. \tag{2.21}$$

Equation (2.5) implies that $\partial c_a / \partial \rho = 0$. By (2.8) the c_a's vanish on the axis of symmetry; hence it follows that (cf. also [22, 17])

$$c_a \equiv 0. \tag{2.22}$$

Equations (2.19)–(2.20) now give $\partial M^a / \partial \rho \equiv 0$. We have $M^\theta |_{\rho=0} = 0$ by regularity of $g_{\mu\nu}$ on the symmetry axis, and replacing z by $z + \int_0^t M^z(s)\, ds$ leads to

$$M^a \equiv 0. \tag{2.23}$$

A MATHEMATICA calculation shows that one also has (cf. (C.29))

$$\frac{\partial^2 R}{\partial t^2} - \frac{\partial^2 R}{\partial \rho^2} = \frac{e^{2\nu}}{2R} \left[e^{-4\gamma} c_1^2 + \frac{(c_2 - a c_1)^2}{R^2} \right] + e^{2(\nu - \gamma)} R T_{\mu\nu} (n^\mu n^\nu - m^\mu m^\nu). \tag{2.24}$$

Let us suppose that (2.22) holds and that

$$T_{\mu\nu} n^\mu n^\nu = T_{\mu\nu} m^\mu m^\nu. \tag{2.25}$$

It follows that the right-hand side of (2.24) vanishes; hence there exist functions f, g such that

$$R = f(\rho + t) + g(\rho - t).$$

Define

$$\rho' = f(\rho + t) + g(\rho - t) \tag{2.26}$$

$$t' = f(\rho + t) - g(\rho - t). \tag{2.27}$$

Equations (C.18)–(C.20) show that we have

$$\det \frac{\partial(\rho', t')}{\partial(\rho, t)} = 4 f'(\rho + t)\, g'(\rho - t) = e^{2(\nu - \gamma)} g^{\mu\nu} R_{,\mu} R_{,\nu}. \tag{2.28}$$

If we assume now that (2.15) and (2.16) hold, point 1 of Proposition 2.3 shows that (2.26)–(2.27) define a diffeomorphism in a neighborhood of Σ in M (the axes of symmetry, again, can be taken care of by the methods of Appendix C of [7]).

Dropping primes, in the new coordinates we have $R \equiv \rho$, so that we can put the metric in the Kompaneets form [20]

$$ds^2 = e^{2(\nu-\gamma)}(-dt^2 + d\rho^2) + \lambda_{ab}\, dx^a\, dx^b, \tag{2.29}$$

$$\det \lambda_{ab} = \rho^2. \tag{2.30}$$

To summarize, we have proved the following.

THEOREM 2.4. *Let* (M, g) *be a cylindrically symmetric globally hyperbolic space-time with Killing vectors* X_a, $a = 1, 2$, *with* $M \approx (-T, T) \times \Sigma$ *for some* $0 < T < \infty$, *and with* $\Sigma \approx \mathbb{R}^3$. *Suppose that the energy-momentum tensor of g satisfies*

$$(g^{\mu\nu} - \lambda^{ab} X^\mu_a X^\nu_b)\, X^\rho_c T_{\nu\rho} = 0, \tag{2.31}$$

$$(g^{\mu\nu} - \lambda^{ab} X^\mu_a X^\nu_b)\, T_{\mu\nu} = 0 \tag{2.32}$$

(recall that $\lambda_{ab} = g(X_a X_b)$, *and that* λ^{ab} *is the matrix inverse to the matrix* λ_{ab}*) and*

$$T_{\mu\nu} X^\mu Y^\nu \geqslant 0 \tag{2.33}$$

for all X^ν*—null,* Y^ν*—timelike, consistently time-oriented. Suppose, moreover, that there exists* $C > 0$ *such that*

$$\rho \geqslant C: \qquad Y^\mu \nabla_\mu R > 0, \tag{2.34}$$

where Y^μ *is any* outwards directed *null vector orthogonal to the orbits of G and* ρ *is any coordinate labelling the orbits of the groups on the hypersurfaces* $\{\tau\} \times \Sigma$. *Then there exists a global coordinate system on M such that* (M, g) *is isometric to a subset of* $\mathbb{R}^4$ *with a metric of the form* (2.29)–(2.30).

Strictly speaking, in Theorem 2.4 the condition (2.31) can be replaced by the weaker condition (2.21). Condition (2.31) seems to be somewhat more elegant, as it does not explicitly use the foliation-dependent vector field m^μ.

As discussed in detail below, the hypotheses (2.31)–(2.33) on the energy-momentum tensor are satisfied by cylindrically symmetric electro-vacuum space-times (cf. Theorem 4.1 below). It should be noted that in general $i(\Sigma)$ will *not* be given by the equation $t = \text{const}$. Let us also mention that the tensor

$$h^{\mu\nu} = g^{\mu\nu} - \lambda^{ab} X^\mu_a X^\nu_b$$

which appears in (2.31)–(2.32) is a projection operator on the space orthogonal to the orbits of the isometry group G.

Choosing (M, g) in Theorem 2.4 to be (perhaps an appropriate subset of) the maximal globally hyperbolic [4] development of $(\Sigma, {}^3g, K)$ one obtains

THEOREM 2.5. *Let $(\Sigma, {}^3g, K)$ be cylindrically symmetric initial data for vacuum Einstein equations. Suppose, moreover, that there exists $C > 0$ such that*

$$x^3 \geqslant C: \qquad Y^\mu \nabla_\mu R|_\Sigma > 0, \tag{2.35}$$

where Y^μ is any outwards directed *null vector orthogonal to the orbits of G, and $x^3 \geqslant 0$ is any coordinate parametrizing the orbits of the group on Σ. Then there exists a globally hyperbolic vacuum space-time (M, g) with a metric of the form* (2.29)–(2.30) *and an isometric embedding $i: \Sigma \to M$ such that $i(\Sigma)$ is a Cauchy surface for (M, g).*

It will be seen in Section 5 that, under the conditions of Theorem 5.2, the coordinates in which the metric takes the form (2.31)–(2.32) are global on the maximal globally hyperbolic development of the data; cf. Corollary 5.3.

3. Asymptotically Flat Cylindrically Symmetric Geometries

Let us consider a cylindrically symmetric Riemannian metric ${}^3g$ on $\Sigma = \mathbb{R}^3$, parametrized as in (2.1)–(2.3):

$${}^3g_{ij}\, dx^i\, dx^j = e^{2(v-\gamma)}\, d\rho^2 + \lambda_{ab}(dx^a + g^a\, d\rho)\,(dx^b + g^b\, d\rho), \tag{3.1}$$

$$\lambda_{ab}\, dx^a\, dx^b = e^{2\gamma}(dz + a\, d\theta)^2 + R^2 e^{-2\gamma}\, d\theta^2. \tag{3.2}$$

Without loss of generality, rescaling ρ if necessary, we may assume that

$$e^{2v}|_{\rho=0} = 1. \tag{3.3}$$

The regularity of ${}^3g$ at the axis $\rho = R = 0$ requires that the limits

$$\lim_{\rho \to 0} \frac{g^\theta}{\rho^2},\ \lim_{\rho \to 0} \frac{a}{\rho^2} \text{ exist, and are finite;} \tag{3.4}$$

$$\lim_{\rho \to 0} \frac{R}{\rho} = 1. \tag{3.5}$$

For the purpose of this section we shall say that the initial data $({}^3g, K)$ are asymptotically flat if the limits

$$\lim_{\rho \to \infty} v,\ \lim_{\rho \to \infty} \gamma \text{ exist, and are finite;} \tag{3.6}$$

$$\lim_{\rho \to \infty} R_\rho \text{ exists and is (strictly) positive;} \tag{3.7}$$

$$\lim_{\rho \to \infty} \rho g^\theta = \lim_{\rho \to \infty} g^z = \lim_{\rho \to \infty} R_t = 0. \tag{3.8}$$

(Clearly, the notion of asymptotic flatness used here is compatible with (and stronger than) the one used in the previous Section.) Assuming that the energy condition (2.15) holds, it follows from Proposition 2.3 that

$$R_{\pm} > 0, \tag{3.9}$$

so that, using (3.3), we can solve (2.6) for v to obtain

$$v = \frac{1}{2}\ln(R_{+}R_{-}) + \frac{1}{2}\int_{0}^{\rho}\left(\frac{h_{+}}{R_{+}} + \frac{h_{-}}{R_{-}}\right), \tag{3.10}$$

with $h_{\pm}$ given by (2.7) (we have $\ln(R_{+}R_{-})|_{\rho=0} = 0$ because of (3.5) and because $R_{,t}|_{\rho=0} = 0$ by regularity of the metric on the symmetry axis). If the energy condition (2.15) holds, (3.6)–(3.8) and (3.10) imply that

$$\frac{h_{+}}{R_{+}}, \frac{h_{-}}{R_{-}} \in L^{1}([0, \infty)).$$

From (2.7) it now follows:

PROPOSITION 3.1. *Let* $(\Sigma, {}^{3}g, K)$ *be asymptotically flat cylindrically symmetric initial data (in the sense of* (3.6)–(3.8)*) with matter satisfying the energy condition* (2.15)*. Then we must necessarily have*

$$\begin{gathered} \sqrt{R}\,\gamma_{\pm},\ R^{-1/2}a_{\pm} \in L^{2}([0, \infty)), \\ R^{-1}\lambda^{ab}c_{a}c_{b},\ RT_{\mu\nu}n^{\mu}(n^{\mu} \pm m^{\mu}) \in L^{1}([0, \infty)). \end{gathered} \tag{3.11}$$

Under the hypotheses of asymptotic flatness we may without loss of generality assume, rescaling z if necessary, that

$$\lim_{\rho \to \infty} \gamma = 0. \tag{3.12}$$

Moreover, (3.9) shows that we can redefine ρ so that $R = \rho$ on Σ. (Note, however, that we do *not* assume at this stage that this will hold at later times, with a metric of the form (2.1)–(2.3). It will nevertheless be seen below that such a hypothesis could be assumed without loss of generality for electro-vacuum metrics.) When (3.11) holds it can be shown that $a = o(\rho)$ for large ρ, so that with those normalizations it is easily seen that when ρ tends to infinity the geometry approaches a flat conical geometry, with opening angle equal to

$$\theta_{0} = 2\pi \exp\left\{-\frac{1}{2}\int_{0}^{\infty}\left(\frac{h_{+}}{R_{+}} + \frac{h_{-}}{R_{-}}\right)d\rho\right\}. \tag{3.13}$$

We thus obtain the well-known "positive energy theorem" for cylindrically symmetric initial data sets.

PROPOSITION 3.2. *Under the hypotheses of Proposition* 3.1, *the deficit angle*

$$\begin{aligned}\Delta\theta &= 2\pi - \theta_0 \\ &= 2\pi\left(1 - \exp\left\{-\frac{1}{2}\int_0^\infty \left(\frac{h_+}{R_+} + \frac{h_-}{R_-}\right) d\rho\right\}\right)\end{aligned}$$

satisfies

$$0 \leqslant \Delta\theta < 2\pi.$$

Moreover, $\Delta\theta = 0$ *if and only if* $(\Sigma, {}^3g, K)$ *are initial data for Minkowski space-time.*

A (u-dependent) quantity analogous to the opening angle θ_0 defined above can also be defined in the radiation regime[7] at $\mathscr{I}$, cf. Eq. (5.20); for vacuum metric this is briefly discussed at the end of Section 5.

Let us finally mention that the quantity

$$\lim_{\rho \to \infty} g_{t\theta}$$

is usually associated with global rotation ("spinning strings, etc.), whenever it exists. Theorem 2.4 shows that this quantity necessarily vanishes, when the energy-momentum tensor satisfies the conditions of this theorem and when the asymptotic flatness condition (2.35) holds. In particular there are no "spinning" purely vacuum (cf. Theorem 2.5) or electrovacuum (cf. Theorem 4.1 below) asymptotically flat space-times.[8]

4. ELECTROVACUUM SPACE-TIMES

In this section we shall suppose that the metric satisfies the Einstein–Maxwell equations. Let us start with a short discussion of the vacuum case.

4.1. *Reduced Vacuum Field Equations*

Consider any cylindrically symmetric vacuum initial data $(\Sigma, {}^3g, K)$ such that (2.35) holds. Theorem 2.5 shows that we can assume that the resulting space-time metric takes the form (2.29)–(2.30), with the Cauchy surface Σ given by an equation $t = i(\rho)$, for some smooth function i: $[0, \infty) \to \mathbb{R}$. Parametrizing λ_{ab} as in (2.3) we then have

$$ds^2 = e^{2(\nu-\gamma)}(-dt^2 + d\rho^2) + e^{2\gamma}(dz + a\,d\theta)^2 + \rho^2 e^{-2\gamma}\,d\theta^2. \tag{4.1}$$

[7] Here and elsewhere, when talking about $\mathscr{I}$ we mean the conformal boundary at future null infinity of (2 + 1)-dimensional Minkowski space-time. Indeed from what is said in this paper it follows that the coordinates in which the metric takes the form (2.29)–(2.30) provide a natural identification of $M/\mathbb{R}$, where $\mathbb{R}$ here refers to the orbits of the Killing vector $\partial/\partial z$, with $\mathbb{R}^{2,1}$ (this is, of course, *not* an isometry).

[8] We are grateful to P. Tod for pointing out this implication to us.

Equation (C.28) specialized to a vacuum implies that we can introduce the Geroch–Ernst potential ω [17, 12], in terms of which we have

$$\partial_t a = -\rho e^{-4\gamma}\omega_{,\rho}, \tag{4.2}$$

$$\partial_\rho a = -\rho e^{-4\gamma}\omega_{,t}. \tag{4.3}$$

The vacuum Einstein equations then yield a wave-map equation for a map $\phi(t, \rho) = (\gamma(t, \rho), \omega(t, \rho))$ from $(2+1)$-dimensional Minkowski space $(\mathbb{R}^{2,1}, \eta)$,

$$\mathbb{R}^{2,1} \approx \mathbb{R}^3, \qquad \eta = -dt^2 + d\rho^2 + \rho^2\, d\theta^2,$$

to the two-dimensional hyperbolic space $({}^2\mathscr{H}, h)$:

$${}^2\mathscr{H} \approx \mathbb{R}^2,\ h_{ab}\, dx^a\, dx^b = d\gamma^2 + \frac{e^{-4\gamma}}{4}\, d\omega^2. \tag{4.4}$$

Moreover, ϕ is invariant under rotations in the $t = \text{const}$ surfaces in $\mathbb{R}^{2,1}$. The equations satisfied by ϕ are the variational equations for the action

$$I = \int_{\mathbb{R}^3} \eta^{\mu\nu} h_{ab} \frac{\partial\phi^a}{\partial x^\mu}\frac{\partial\phi^b}{\partial x^\nu}\, dt\, dx\, dy,$$

and we write them symbolically in the form

$$\mathscr{D}^\mu \phi_{,\mu} = 0. \tag{4.5}$$

Given a solution of (4.5), a can be obtained from (4.3) using the regularity condition

$$a(t, \rho = 0) = 0. \tag{4.6}$$

v is then given by (3.10), and we actually have

$$v_\pm = \rho\left[\gamma_\pm^2 + \frac{e^{4\gamma}}{4\rho^2} a_\pm^2\right] = \rho\left[\gamma_\pm^2 + \frac{e^{-4\gamma}}{4}\omega_\pm^2\right]. \tag{4.7}$$

Let $f = \phi|_\Sigma$, $\dot f = \partial\phi/\partial t|_\Sigma$, be the Cauchy data for (4.5). We can find non-trivial space-times with $(f, \dot f)$—compactly supported, by which we mean that f is constant outside of a compact set $\mathscr{C}$ in $\mathbb{R}^2$, and $\dot f$ vanishes in $\mathbb{R}^3 \backslash \mathscr{C}$. From the hyperbolic character of the semi-linear equation (4.5) it follows that compactness of the support of ϕ is preserved by evolution; consequently there exists a constant $C > 0$ such that

$$\left.\frac{\partial\gamma}{\partial x^\mu}\right|_{\rho \geq C + |t|} = \left.\frac{\partial a}{\partial x^\mu}\right|_{\rho \geq C + |t|} = \left.\frac{\partial v}{\partial x^\mu}\right|_{\rho \geq C + |t|} = 0. \tag{4.8}$$

It follows that there exist constants γ_{i^0}, a_{i^0}, v_{i^0} such that

$$a|_{\rho \geq C+|t|} = a_{i^0} \equiv -\int_0^\infty \left(\rho e^{-4\gamma}\frac{\partial\omega}{\partial t}\right)\Big|_\Sigma d\rho, \tag{4.9}$$

$$v|_{\rho \geq C+|t|} = v_{i^0} \equiv \int_0^\infty \rho\left[\gamma_t^2+\gamma_\rho^2+\frac{e^{-4\gamma}}{4}(\omega_t^2+\omega_\rho^2)\right]\Big|_\Sigma d\rho, \tag{4.10}$$

$$\gamma|_{\rho \geq C+|t|} = \gamma_{i^0} = 0, \tag{4.11}$$

where the last equality in (4.11) has been achieved by a rescaling of z. It follows that for $\rho \geq C+|t|$ the metric takes the form

$$ds^2|_{\rho\geq C+|t|} = e^{2v_{i^0}}(-dt^2+d\rho^2)+(dz+a_{i^0}\,d\theta)^2+\rho^2\,d\theta^2, \tag{4.12}$$

which is easily seen to be flat. This metric is isometric to the one induced from the standard Minkowski metric on the quotient of the set $M_C = \{x^\mu \in \mathbb{R}^4 : x^2+y^2 \geq C+|t|\}$ by the equivalence relation $\sim$, where $\sim$ is defined as

$$(t, z, \rho, \theta) \sim (t, z+a_{i^0}\theta_{i^0}, \rho, \theta+\theta_{i^0}).$$

Here $\theta_{i^0} = \theta_0$ is the opening angle given by (3.13),

$$0 < \theta_{i^0} = 2\pi e^{-2v_{i^0}} \leq 2\pi. \tag{4.13}$$

The quantities (4.9) and (4.10) have a direct interpretation in terms of the wave-map equation: (4.10) is, up to a factor $(2\pi)^{-1}$, the (conserved) energy of the wave-map. On the other hand, a_{i^0} is a conserved quantity for (4.5) which is obtained as follows: Let $Z^a(\partial/\partial y^a)$ be a Killing vector for the metric h_{ab} given by (4.4). It is well known (and in any case easily checked) that the quantity

$$Q(Z, \Sigma) = \int_\Sigma \eta^{\mu\nu}h_{ab}\frac{\partial\phi^a}{\partial x^\mu}Z^b\,d\Sigma_\mu \tag{4.14}$$

(where $d\Sigma_\mu = \partial_\mu \vee (dt \wedge dx \wedge dy)$, and $\vee$ denotes contraction) is (under appropriate asymptotic conditions which are satisfied here) independent of the choice of the asymptotically flat Cauchy surface $\Sigma \subset \mathbb{R}^{2,1}$. Taking $Z^a(\partial/\partial y^a) = \partial/\partial\omega$ and $\Sigma = \{t=0\}$ one finds that

$$Q\left(\frac{\partial}{\partial\omega}, \Sigma\right) = -2\pi a_{i^0}.$$

Let us now describe how the above results generalize to the electro-vacuum case.

4.2. *Electro-Vacuum Cylindrically Symmetric Initial Data*

Consider any globally hyperbolic electro-vacuum space-time $(\tilde{M}, g)$ with a Cauchy surface $\Sigma \approx \mathbb{R}^3$. Let $G = \mathbb{R} \times U(1)$ act on $\tilde{M}$ by isometries of g, the action on Σ being that by translations in z and rotations in the planes $z = \text{const}$. Consider a source-free electromagnetic field F invariant w.r.t. the Killing fields $\partial/\partial\theta$, $\partial/\partial z$, i.e.,

$$\mathscr{L}_{X_a} F = 0, \qquad a = 1, 2$$

(as before, $\mathscr{L}$ denotes a Lie derivative). Let x^0 be any coordinate defined in a neighborhood of Σ such that $\Sigma = \{x^0 = 0\}$. We can decompose F into its electric and magnetic parts with respect to the surfaces $\{x^0 = \text{const}\}$. Let x^3 be any coordinate on Σ which is constant on the orbits of the isometry group, extend x^3 to a coordinate on some neighborhood of Σ in $\tilde{M}$ in any way, preserving, however, the property of x^3 being invariant under G. Now, the electric and magnetic vector densities $\mathscr{E}^i$ and $\mathscr{B}^i$ obey (cf., e.g., [25, Chap. 21]) $\mathscr{E}^i{}_{,i} = \mathscr{B}^i{}_{,i} = 0$ so, in the present case, we have $\mathscr{E}^3{}_{,3} = \mathscr{B}^3{}_{,3} = 0$. Regularity at $x^3 = 0$ implies that $\mathscr{E}^3|_{x^3=0} = \mathscr{B}^3|_{x^3=0} = 0$ so that we get

$$\mathscr{E}^3 = \mathscr{B}^3 = 0. \tag{4.15}$$

(Equation (4.15) actually holds on that open connected subset $\tilde{M}'$ of $\tilde{M}$ which contains complete rays $\{(x^0, x^a, sx^3), s \in [0, 1]\}$. Replacing $\tilde{M}$ by $\tilde{M}'$ we can assume that (4.15) holds on $\tilde{M}$.) Equation (4.15) shows that (2.31)–(2.32) hold (cf. also [22]), and, taking (M, g) to be (perhaps an appropriate subset[9]) the maximal globally hyperbolic development of the data, from Theorem 2.4 one obtains the following.

THEOREM 4.1. *Let $(\Sigma, {}^3g, K, A, E)$ be cylindrically symmetric initial data for electro-vacuum Einstein equations. Suppose, moreover, that there exists $C > 0$ such that*

$$x^3 \geqslant C: \qquad Y^\mu \nabla_\mu R|_\Sigma > 0, \tag{4.16}$$

where Y^μ is any outwards directed *null vector orthogonal to the orbits of G and $x^3 \geqslant 0$ is any coordinate parametrizing the orbits of the group on Σ. Then there exists a globally hyperbolic electro-vacuum space-time (M, g) with a metric of the form (2.29)–(2.30) and an isometric embedding $i: \Sigma \to M$ such that $i(\Sigma)$ is a Cauchy surface for (M, g).*

4.3. *The Electromagnetic Gauge Conditions*

In the coordinate system in which (2.29) holds the hypersurface $i(\Sigma)$, where i is given by Theorem 4.1, is a graph of a function $t = f(\rho)$. Define new coordinates x^μ

[9] It will be seen in Section 5 that, under the conditions of Theorem 5.2, the coordinates in which the metric takes the form (2.31)–(2.32) are global on the maximal globally hyperbolic development of the data; cf. Corollary 5.3.

by $x^0 = t - f(\rho)$, $x^3 = \rho$; note that in the coordinate system (x^0, x^a, x^3) as defined here the metric does *not* take the form (2.29), but it is *block-diagonal* with respect to the pairs of variables (x^0, x^3) and x^a, $a = 1, 2$. The equation $dF = 0$ on the simply connected manifold M, together with the Poincaré lemma show that $F = dA$. We can always impose the *gauge condition* $A_0 = 0$. The equations $\mathscr{E}^3 = 0 = A_0$ imply that $A_{3,0} = 0$. Moreover, the equation $\mathscr{B}^3 = 0$ yields $A_{\theta,z} - A_{z,\theta} = 0$. Let

$$s = -\int_0^\rho A_\rho(\rho', z, \theta)\, d\rho'.$$

As A_ρ is independent of t, the gauge transformation $A_\mu \to A_\mu + s_{,\mu}$ leaves $A_0 = 0$ and transforms A_ρ to zero. In the new gauge we have

$$A_0 \equiv 0, \qquad A_\rho \equiv 0,$$

so that

$$F_{\rho\theta} = -A_{\theta,\rho}, \qquad F_{t\theta} = -A_{\theta,t}, \qquad F_{\rho z} = -A_{z,\rho}, \qquad F_{tz} = -A_{z,t},$$
$$F_{t\rho} \equiv 0, \qquad F_{\theta z} = 0 = \partial_\theta A_z - \partial_z A_\theta.$$

On the initial surface Σ we choose

$$A_\theta|_\Sigma = -\int_0^\rho F_{\rho\theta}|_\Sigma\, d\rho', \qquad A_z|_\Sigma = -\int_0^\rho F_{\rho z}|_\Sigma\, d\rho',$$
$$A_{\theta,t}|_\Sigma = -F_{t\theta}, \qquad A_{z,t}|_\Sigma = -F_{tz},$$

which yields $\partial_\theta A_\theta|_\Sigma = \partial_\theta A_z|_\Sigma = \partial_z A_z|_\Sigma = 0$ and thus $(\partial_\theta A_z - \partial_z A_\theta)|_\Sigma = F_{\theta z}|_\Sigma = 0$. However, since in the $A_0 = A_\rho = 0$ gauge we have $A_{\theta,t} = -F_{t\theta}$, $A_{z,t} = -F_{tz}$ it is clear that A_θ, A_z remain invariant with respect to $\partial/\partial\theta$, $\partial/\partial z$ off the initial surface. Thus we may assume without loss of generality that

$$A = A_\theta\, d\theta + A_z\, dz, \tag{4.17}$$

where

$$\partial_\theta A_\mu = \partial_z A_\mu = 0. \tag{4.18}$$

Note that (4.17)–(4.18) are invariant under changes of coordinates in the (t, ρ) plane. We can thus go to the coordinates in which the metric takes the form (2.29), with (4.17)–(4.18) still holding.

4.4. *Reduced Electro-Vacuum Field Equations*

It is convenient to introduce the gravitational and electromagnetic twist potentials $\{\omega, \eta\}$ satisfying [12, 21, 23, 27, 28]:

$$a_{,t} = -\rho e^{-4\gamma}(\omega_{,\rho} + A_z\eta_{,\rho}), \qquad a_{,\rho} = -\rho e^{-4\gamma}(\omega_{,t} + A_z\eta_{,t}),$$
$$A_{\theta,\rho} - aA_{z,\rho} = \rho e^{-2\gamma}\eta_{,t}, \qquad A_{\theta,t} - aA_{z,t} = \rho e^{2\gamma}\eta_{,\rho}.$$

This corresponds to solving the two divergence constraints given in Eq. (2.6) of [27], i.e., $\tilde{f}^a{}_{,a}=0$, $\mathscr{E}^a{}_{,a}=0$ by $\tilde{f}^a=\varepsilon^{ab}\omega_{,b}$, $\mathscr{E}^a=\varepsilon^{ab}\eta_{,b}$. Now, define $\lambda := A_z$ and consider the map, taking values in $\mathbb{R}^4$, defined by $\{\gamma, \omega, \lambda, \eta\}$. One can show, after introducing η as in [27] (setting to zero the Higgs field Φ of [27]) that the electrovacuum field equations yield a wave-map for a map $\phi=\{\gamma, \omega, \lambda, \eta\}$ from $(2+1)$-dimensional Minkowski space $(\mathbb{R}^{2,1}, \eta)$ to the four-dimensional Riemannian manifold $(\mathbb{R}^4, h)$:

$$h_{ab}\,dx^a\,dx^b=4(d\gamma)^2+e^{-2\gamma}[(d\lambda)^2+(d\eta)^2]+e^{-4\gamma}(d\omega+\lambda\,d\eta)^2. \tag{4.19}$$

Moreover ϕ is invariant under rotations generated by $\partial/\partial\theta$ in $\mathbb{R}^{2,1}$. The equations satisfied by ϕ are the variational equations for the action

$$I=\int_{\mathbb{R}^3}\eta^{\nu\mu}h_{ab}\frac{\partial\phi^a}{\partial x^\nu}\frac{\partial\phi^b}{\partial x^\mu}\sqrt{-\det\eta}\;dt\,dx\,dy$$

and we write them symbolically in the form

$$\mathscr{D}^\mu\phi_{,\mu}=0.$$

Given a solution, we can recover a and A_θ from the equations above defining the twist potentials (upon imposing regularity $a(t, \rho=0)=A_\theta(t, \rho=0)=0$). Note that $A_z=\lambda$ so the only remaining unknown is the metric function v which is determined from

$$v_{,\rho}\pm v_{,t}=\frac{\rho}{4}h_{ab}(\phi^a{}_{,\rho}\pm\phi^a{}_{,t})(\phi^b{}_{,\rho}\pm\phi^b{}_{,t}),$$

where $\{\phi^a\}=\{\gamma, \omega, \lambda, \eta\}$ and h_{ab} is given above, Eq. (4.19), i.e.,

$$v_\pm=\rho\left\{(\gamma_\pm)^2+\frac{e^{-2\gamma}}{4}[(\lambda_\pm)^2+(\eta_\pm)^2]+\frac{e^{-4\gamma}}{4}(\omega_\pm+\lambda\eta_\pm)^2\right\}.$$

The discussion after Eq. (4.7) goes through as before with Eq. (4.9) modified to read

$$a|_{\rho\geqslant C+|t|}=a_{i^0}\equiv-\int_0^\infty\rho e^{-4\gamma}(\omega_{,t}+\lambda\eta_{,t})|_\Sigma\,d\rho,$$

and

$$\begin{aligned}v|_{\rho\geqslant C+|t|}=v_{i^0}\equiv\int_0^\infty\rho&\left\{(\gamma_{,\rho})^2+\frac{e^{-2\gamma}}{4}[(\lambda_{,\rho})^2+(\eta_{,\rho})^2]\right.\\&+\frac{e^{-4\gamma}}{4}(\omega_{,\rho}+\lambda\eta_{,\rho})^2+(\gamma_{,t})^2\\&\left.+\frac{e^{-2\gamma}}{4}[(\lambda_{,t})^2+(\eta_{,t})^2]+\frac{e^{-4\gamma}}{4}(\omega_{,t}+\lambda\eta_{,t})^2\right\}d\rho.\end{aligned}$$

Note that again up to a constant factor v_{i° is the conserved energy of the wave map; similarly, a_{i° is again conserved because $\partial/\partial\omega$ is a Killing vector field of the metric h_{ab} (cf. the discussion around Eq. (4.14)). In fact, there are a total of eight conserved quantities as defined by (4.14) corresponding to the $SU(2, 1)$ isometry group of the metric h_{ab} (cf. [21, 23, 28]). A subset of these Killing fields given by

$$Z_{(1)} = \frac{\partial}{\partial\omega}, \qquad Z_{(2)} = \frac{\partial}{\partial\eta}, \qquad Z_{(3)} = \frac{\partial}{\partial\lambda} - \eta\frac{\partial}{\partial\omega},$$
$$Z_{(4)} = \left(\frac{\partial}{\partial\gamma} + \eta\frac{\partial}{\partial\eta} + \lambda\frac{\partial}{\partial\lambda} + 2\omega\frac{\partial}{\partial\omega}\right)$$

defines a basis to the tangent space at every point of $(\mathbb{R}^4, h)$ and in fact generates the action of a transitive subgroup of $SU(2, 1)$. One sees this, for example, by integrating the flow generated by $Z = \sum_{i=1}^4 \alpha^i Z_{(i)}$, α^i constant and showing that the resulting action is transitive explicitly. That, in any case, $SU(2, 1)$ acts transitively on $(\mathbb{R}^4, h)$ follows from its structure as a coset space (cf. [23, 21]). The Lie subalgebra generated by the $Z_{(i)}$'s is

$$[Z_{(1)}, Z_{(2)}] = [Z_{(1)}, Z_{(3)}] = 0,$$
$$[Z_{(1)}, Z_{(4)}] = 2Z_{(1)}, \qquad [Z_{(2)}, Z_{(3)}] = -Z_{(1)},$$
$$[Z_{(2)}, Z_{(4)}] = Z_{(2)}, \qquad [Z_{(3)}, Z_{(4)}] = Z_{(3)}.$$

The metric h admits an orthonormal frame $\{X_{(i)}\}$ which determines a Lie algebra isomorphic to that of the $Z_{(i)}$'s. Defining

$$X_{(1)} = \frac{1}{2}\frac{\partial}{\partial\gamma}, \qquad X_{(2)} = e^{\gamma}\frac{\partial}{\partial\lambda},$$
$$X_{(3)} = e^{2\gamma}\frac{\partial}{\partial\omega}, \qquad X_{(4)} = e^{\gamma}\frac{\partial}{\partial\eta} - \lambda e^{\gamma}\frac{\partial}{\partial\omega},$$

one finds that $h(X_{(i)}, X_{(j)}) = \delta_{ij}$ and that

$$[X_{(1)}, X_{(2)}] = \tfrac{1}{2}X_{(2)}, \qquad [X_{(1)}, X_{(3)}] = X_{(3)}, \qquad [X_{(1)}, X_{(4)}] = \tfrac{1}{2}X_{(4)},$$
$$[X_{(2)}, X_{(3)}] = 0, \qquad [X_{(2)}, X_{(4)}] = -X_{(3)}, \qquad [X_{(3)}, X_{(4)}] = 0.$$

The isomorphism of Lie algebras is seen by making the correspondence $X_{(1)} \to -Z_{(4)}/2$, $X_{(2)} \to Z_{(2)}$, $X_{(3)} \to Z_{(1)}$, and $X_{(4)} \to Z_{(3)}$. Note however, that the group action generated by the $X_{(i)}$'s is not in general an isometry of h. As will be seen in the next section, the existence of an orthonormal frame with bounded structure functions (in this case constants) plays a key note in the global existence theorem of Christodoulou and Tahvildar-Zadeh.

Another key element in the application of the Christodoulou–Tahvildar-Zadeh theorem involves verifying that for any point p in the manifold $(\mathbb{R}^4, h)$ the geodesic sphere $\Sigma(p, s)$ of radius s centered at p has a second fundamental form k_{ab} whose eigenvalues obey certain bounds. Since $(\mathbb{R}^4, h)$ is a homogeneous space (cf. [23] or the explicit argument above) it suffices to verify the conditions on $\Sigma(p, s)$ for any fixed point p. Introducing two complex coordinates $\{w^1, w^2\}$ via the definitions

$$\frac{1-w^1}{1+w^1} = e^{2\gamma} + \frac{1}{4}(\eta^2 - \lambda^2) + i\left(\omega + \frac{1}{2}\eta\lambda\right),$$
$$\frac{w^2}{1+w^1} = \frac{1}{2}(\eta + i\lambda),$$

one finds (à la Mazur) that the metric h_{ab} becomes

$$dh^2 = \frac{4}{(1 - |w^1|^2 - |w^2|^2)^2}\{dw^{1*}\, dw^1 + dw^{2*}\, dw^2 - (w^2\, dw^1 - w^1\, dw^2)(w^{2*}\, dw^{1*} - w^{1*}\, dw^{2*})\}.$$

Taking $w^1 = x + iy$, $w^2 = u + iv$ and introducing the "spherical" coordinates $\{r, \theta, \varphi, \chi\}$ defined by

$$x = r\cos(\varphi)\sin(\chi)\sin(\theta), \qquad y = r\sin(\varphi)\sin(\chi)\sin(\theta),$$
$$u = r\sin(\chi)\cos(\theta), \qquad v = r\cos(\chi),$$

and re-expressing h_{ab}, once again we obtain a convenient coordinate system for the computation of the extrinsic curvature of the geodesic spheres centered at the origin $(r = 0)$ of the new coordinate system. The result of a lengthy calculation (done using MATHEMATICA) shows that the three eigenvalues of k_{ab} are in fact independent of direction and given explicitly by

$$k_1 = \coth(s), \qquad k_2 = k_3 = \frac{1}{2}\coth\left(\frac{s}{2}\right).$$

Note that there exists constants c, $C > 0$ such that the smallest eigenvalue λ $(= k_2 = k_3)$ satisfies $s\lambda \geqslant c$ and such that the largest eigenvalue Λ $(= k_1)$ satisfies $s\Lambda \leqslant C(1 + s)$.

5. Global Existence, Strong Cosmic Censorship

In the previous sections we have seen how to reduce the dynamics of cylindrically symmetric electro-vacuum space-times to a wave-map problem. Global existence and asymptotic properties of this last problem have been studied recently by Christodoulou and Tahvildar-Zadeh [5, 6]. These authors consider rotation-invariant maps $\phi: (\mathbb{R}^{2,1}, \eta) \to (N, h)$, where (N, h) is a complete Riemannian manifold. Passing to the universal cover of N if necessary we may assume that N is simply connected. Christodoulou and Tahvildar-Zadeh, moreover, assume that

C1. There exists an orthonormal frame of vector fields Ω_A on N such that the functions e^A_{BC} defined by

$$[\Omega_B, \Omega_C] = \sum_A e^A_{BC} \Omega_A \tag{5.1}$$

are uniformly bounded on N;

C2. For each $p \in N$ let $\Sigma(p, s)$ denote the geodesic sphere of radius s centered at p, and let k_{AB} be its extrinsic curvature. There exists a constant $c > 0$ such that the smallest eigenvalue λ, respectively, the largest eigenvalue Λ, of k_{AB} (with respect to h), satisfies

$$s\lambda \geqslant c^{-1}, \qquad s\Lambda \leqslant c(1+s). \tag{5.2}$$

Let us mention that (5.2) implies that the geodesic spheres are differentiable spheres (recall that we have excluded cut points by assuming that N is simply connected). It follows that N is topologically trivial. Let us also mention that the Eschenburg comparison theorem [14, Theorem 3.2] shows that C2 will hold when the sectional curvatures κ of (N, h) satisfy

$$-C \leqslant \kappa \leqslant 0,$$

for some constant $C \geqslant 0$ and that this criterion is satisfied by the target space which occurs in the electro-vacuum case [24].

Under C1 and C2 we have the following (by definition, ϕ is rotation invariant if ϕ depends only upon ρ and t).

THEOREM 5.1 [5, 6]. *Let Σ be a rotation-invariant Cauchy surface in $\mathbb{R}^{2,1}$ and let ϕ_0, $\dot\phi_0$ be any smooth rotation-invariant Cauchy data for the wave map equation for a map $\phi\colon (\mathbb{R}^{2,1}, \eta) \to (N, h)$, where (N, h) satisfies the hypotheses* C1 *and* C2. *Then*

1. *There exists a unique smooth-map $\phi\colon \mathbb{R}^{2,1} \to N$ satisfying the wave-map equation and assuming the Cauchy data ϕ_0, $\dot\phi_0$;*

2. *Suppose, moreover, that $\dot\phi_0$ has support in a compact set $\mathscr{C} \subset \Sigma$, while ϕ_0 maps $\Sigma \backslash \mathscr{C}$ into a point. For $t \geqslant 0$ we have the following pointwise estimates for the derivatives of ϕ,*

$$\|\partial_- \phi\| \equiv \|(\partial_\rho - \partial_t)\phi\| \leqslant \frac{C}{(1+t+\rho)^{1/2}(1+|t-\rho|)}, \tag{5.3}$$

$$\|\partial_+ \phi\| \equiv \|(\partial_\rho + \partial_t)\phi\| \leqslant \frac{C}{(1+t+\rho)^{3/2}}, \tag{5.4}$$

$$\|\partial_\mu \partial_\nu \phi\| \leqslant C, \tag{5.5}$$

for some constant C. Moreover, if we denote by C^+_u the interior of the future light cone with vertex at $(t = u,\ \rho = 0)$, then there exists a constant $C_1 \geqslant 0$ such that for $t \geqslant \rho$ we have

$$\operatorname{diam}(\phi(C^+_{t-\rho})) \leqslant \frac{C_1}{(1+|t-\rho|)^{1/2}}. \tag{5.6}$$

Here the norm $\|\partial_\pm \phi\|$ *is taken with respect to the metric* h *on* N, *and* $\operatorname{diam}(\Omega)$ *denotes the diameter of a set* Ω.

(The second derivatives estimates (5.5) are proved in Section 4 of [6]. We have been informed by Christodoulou[10] that under the hypotheses above for $t \geqslant 0$ one can actually prove the following decay estimates

$$\forall n, m \in \mathbb{N} \cup \{0\},\ n+m>0: \qquad \|\partial_+^n \partial_-^m \phi\| \leqslant C(1+t+\rho)^{-1/2-n}(1+|t-\rho|)^{-m}, \tag{5.7}$$

for some constant C.)

For $t \leqslant 0$ estimates analogous to (5.3)–(5.6) immediately follow from (5.3)–(5.6) and from time-reversal invariance of the wave-map equation.

Let us first consider the vacuum Beck [2] space-times ("polarized" metrics), $a \equiv 0$. Equation (4.5) reduces to a linear scalar wave-equation on three-dimensional Minkowski space-time, so that the global existence immediately follows. Moreover, the asymptotic estimates (5.3)–(5.4) readily follow from the explicit representation of solutions of the scalar wave-equation on three-dimensional Minkowski space-time; an exhaustive analysis of this case can be found in [1].

As discussed in Section 4.4, the hypotheses of Theorem 5.1 are satisfied by the target manifold (N, h) of the wave-map associated with the electro-vacuum Einstein equations, so that Theorem 5.1 implies global existence on $\mathbb{R}^4$ of cylindrically symmetric electro-vacuum space-times. The main result of this paper is the following.

THEOREM 5.2. *Let* $(\Sigma, {}^3g, K, A, E)$ *be smooth cylindrically symmetric Cauchy data for electro-vacuum Einstein equations such that the initial data for the corresponding wave-map are supported in some compact set* $\mathscr{C} \subset \Sigma/\mathbb{R}$. *Then the maximal globally hyperbolic development* (M, g) *of* $(\Sigma, {}^3g, K, A, E)$ *is causally geodesically complete, hence inextendible. In particular, strong cosmic censorship holds in this class of space-times.*

Proof. The proof of Theorem 5.2 consists in obtaining estimates on the geometry of (M, g) which will allow us to show causal geodesic completeness of (M, g). By Theorems 2.4 and 5.1 it is sufficient to consider metrics on $\mathbb{R}^4$ of the form (2.29)–(2.30). We shall prove the result in vacuum, the analysis of the electro-vacuum case is essentially a repetition of the arguments below. Replacing t by $t+T$ if necessary, for some appropriately chosen constant T, we may without loss of generality assume that $\mathscr{C} \subset C_0^+ \cap \Sigma$. By time-reversal invariance of the equations it is sufficient to prove causal future geodesic completeness, and henceforth we shall only consider $M^+ = \{p \in \mathbb{R}^4 : t(p) \geqslant 0\}$. It follows that (4.9)–(4.11) hold in M^+ with $C=0$, and to understand the geometry of (M, g) we will have to analyze the

[10] D. Christodoulou, private communication.

behavior of a and v in the set $\{t \geq \rho, t \geq T\}$. By (5.3)–(5.6) there exist constants C, γ_{i^+}, and ω_{i^-} such that we have

$$|\partial_+ \gamma| + |\partial_+ \omega| \leq \frac{C}{(1+t+\rho)^{3/2}}, \tag{5.8}$$

$$|\partial_- \gamma| + |\partial_- \omega| \leq \frac{C}{(1+|t-\rho|)(1+t+\rho)^{1/2}}, \tag{5.9}$$

$$|\gamma - \gamma_{i^+}| + |\omega - \omega_{i^+}| \leq \frac{C}{(1+|t-\rho|)^{1/2}}, \tag{5.10}$$

$$|\partial_\mu \partial_\nu \gamma| + |\partial_\mu \partial_\nu \omega| \leq C. \tag{5.11}$$

In what follows the letter C denotes a positive constant the value of which may vary from line to line. From (4.2)–(4.3), (4.7), and (5.8)–(5.10) it follows that we have the estimates (recall that $\partial_\pm = \partial_\rho \pm \partial_t$)

$$|\partial_+ a| \leq \frac{C\rho}{(1+t+\rho)^{3/2}}, \tag{5.12}$$

$$|\partial_- a| \leq \frac{C\rho}{(1+|t-\rho|)(1+t+\rho)^{1/2}}, \tag{5.13}$$

$$|\partial_+ v| \leq \frac{C\rho}{(1+t+\rho)^3}, \tag{5.14}$$

$$|\partial_- v| \leq \frac{C\rho}{(1+|t-\rho|)^2 (1+t+\rho)}. \tag{5.15}$$

By direct integration of (5.14) along the null geodesics $u = \text{const}$ one finds that the function

$$v_{\mathscr{I}^+}(u) \equiv \lim_{\rho \to \infty} v(t = u + \rho, \rho) \tag{5.16}$$

exists, and we have

$$|v_{\mathscr{I}^+}(u)| \leq \frac{C}{1+u}, \tag{5.17}$$

$$|v(t, \rho)| \leq C\left(\frac{1}{1+|t-\rho|} + \frac{1+t+3\rho}{(1+t+\rho)^2}\right); \tag{5.18}$$

in particular v is uniformly bounded on M^+. From Eq. (5.12) we also find

$$\begin{aligned} \left|\frac{a}{\rho}\right| &\leq \frac{1}{\rho} \int_0^\rho |\partial_+ a|\,(t-\rho+s, s)\, ds \\ &\leq C\left(\frac{2}{(1+t+\rho)^{1/2} + (1+t-\rho)^{1/2}} - \frac{1}{(1+t+\rho)^{1/2}}\right). \end{aligned} \tag{5.19}$$

Equation (5.19) shows that $|a/\rho|$ is uniformly bounded on M^+, and uniformly decays to zero as $t \to \infty$. It is now easy to check that for $\rho \geqslant 1$ the inequalities (A.4) of Appendix A are satisfied. By rotational symmetry the derivatives $\partial\omega/\partial\rho$ and $\partial\gamma/\partial\rho$ vanish at $\rho = 0$, and from the second derivative estimate (5.11) it follows that

$$\rho \leqslant 1: \qquad |\partial_\rho \gamma| + |\partial_\rho \omega| \leqslant C\rho.$$

It is easily seen now that (A.4) holds for all ρ, and our claims follow by Proposition A.1 of the Appendix A. ∎

From Theorems 4.1, 5.1, and 5.2 we obtain the following.

COROLLARY 5.3. *For initial data as in Theorem* 5.2, *the corresponding maximal globally hyperbolic development is isometrically diffeomorphic to* $\mathbb{R}^4$ *with a metric of the form* (2.29)–(2.30).

It is of some interest to enquire about completeness of affinely parametized trajectories of charged test particles. Using estimates similar to the ones given in the proof of Theorem 5.2 one can check that the inequalities (B.5) are satisfied; in particular one finds that there exists a constant C such that

$$|v| + \left|\frac{a}{\rho}\right| + \left|\frac{A_0}{\rho}\right| \leqslant C.$$

(Actually the somewhat sharper estimates for v and a given in the proof above still hold; moreover, A_0/ρ satisfies an estimate of the form (5.19).) It then follows from the arguments of Appendix B that for all space-time as considered in Theorem 5.2 the trajectories of non-tachyonic charged test particles are affinely complete.

Let us finally note that the angle

$$\theta(u) = 2\pi e^{-v_{\mathscr{I}^+}(u)} \tag{5.20}$$

can be thought of as the "instantaneous conical angle[11] of the geometry at $\{u = \text{const}\} \cap \mathscr{I}$" (cf. the discussion at the end of Section 3). In particular, Eq. (5.17) gives a bound on the rate at which $\theta(u)$ approaches 2π as u tends to infinity. Now under the hypotheses of Theorem 5.2 and assuming vacuum field equations for simplicity, it is an easy exercise to show from the estimates (5.7) that the limit

$$\lim_{\rho \to \infty} \rho \left[\gamma_-^2 + \frac{e^{-4\gamma}}{4} \omega^2 \right] (u + \rho, \rho)$$

exists and is a smooth function of u. This and (4.7) imply then that $v_{\mathscr{I}^+}(u)$ is a smooth, monotonously decreasing function of u; indeed the limit as $\rho \to \infty$, $t = u + \rho$, $u = \text{const}$, of Eq. (4.7) gives a Fock–Trautman–Bondi–Sachs-type mass-loss formula [3, 15, 35, 37] for cylindrically symmetric waves.

[11] Here we are referring to the fact that the space-time geometry near the $(2+1)$-dimensional Scri (cf. footnote 7) "looks conical"; no singularity of the geometry of Scri is implied.

APPENDIX A: CAUSAL GEODESIC COMPLETENESS OF SOME METRICS

In this appendix we shall prove causal geodesic completeness of a family of metrics on $M = \mathbb{R}^4$ with a cylindrically symmetric metric of the form

$$ds^2 = h_{AB}\, dx^A\, dx^B + \lambda_{ab}(dx^a + M^a_A\, dx^A)(dx^b + M^b_B\, dx^B), \tag{A.1}$$

$$h_{AB} = e^{2\psi}\eta_{AB}, \qquad \eta_{AB} = \mathrm{diag}(-1, 1), \tag{A.2}$$

and all the functions above depend only upon x^A, $A = 0, 1$. (It should be clear from the arguments below that the proof goes through for more general topologies and actions of the symmetry group, under suitably modified assumptions. More precisely, the arguments below apply to $(m+2)$-dimensional space-times (M, g) with m Killing vectors.) For the electro-vacuum space-times it would have been sufficient to assume that $M^a_A = 0$, $\det \lambda_{ab} = \rho^2$. These assumption, however, are unnecessary in the argument below.

To avoid ambiguities, we shall say that a metric of the form (A.1) is of differentiability class C^k if the components of the metric tensor in coordinates (t, x, y, z) defined by

$$t, x = \rho\cos\theta, \qquad y = \rho\sin\theta, z$$

are, locally, of C^k differentiability class (no uniform bounds are implied). We have the following.

PROPOSITION A.1. *Let $M = \mathbb{R}^4$, consider a cylindrically symmetric C^2 metric of the form* (A.1)–(A.2), *with*

$$\lambda_{ab}\, dx^a\, dx^b = e^{2\gamma}(dz + a\, d\theta)^2 + R^2 e^{-2\gamma}\, d\theta^2. \tag{A.3}$$

Suppose that there exists a constant C such that

$$\begin{aligned}|\gamma| + \left|\frac{\partial\gamma}{\partial t}\right| + |\psi| + \left|\frac{\partial\psi}{\partial t}\right| + \frac{1}{R^2}\left|\frac{\partial\gamma}{\partial t} + \frac{\partial\psi}{\partial t} - \frac{1}{R}\frac{\partial R}{\partial t}\right| (1 + |a|^2) \\ + \frac{1}{R^2}\left|\frac{\partial a}{\partial t}\right|(1 + |a|) + \left|\frac{\partial M^a_A}{\partial t} - \frac{\partial M^a_t}{\partial x^A}\right| + |M^z_A| \leqslant C.\end{aligned} \tag{A.4}$$

Then (M, g) is globally hyperbolic, the level sets of $x^0 = t$ are Cauchy surfaces, and (M, g) is causally geodesically complete.

Remark. Let us mention that the regularity of the metric at the axis $\rho = R = 0$ implies, among other things, that

$$\lim_{\rho\to 0}\left(\frac{\partial\gamma}{\partial t} + \frac{\partial\psi}{\partial t} - \frac{1}{R}\frac{\partial R}{\partial t}\right) = 0.$$

Proof. The equations for affinely parametrized geodesics are easily found from the variational principle for the action

$$I=\int \{h_{AB}\dot{x}^A\dot{x}^B+\lambda_{ab}(\dot{x}^a+M^a_A\dot{x}^A)(\dot{x}^b+M^b_B\dot{x}^B)\}\,ds, \tag{A.5}$$

where a dot over a quantity denotes a derivative with respect to the affine parameter s. The variation of (A.5) with respect to $\dot{x}^a$ gives the conservation equations $dp_a/ds=0$, where

$$p_a=\lambda_{ab}(\dot{x}^b+M^b_A\dot{x}^A)\Rightarrow\dot{x}^a=\lambda^{ab}p_b-M^a_A\dot{x}^A \tag{A.6}$$

and where λ^{ab} is the matrix inverse to λ_{ab}. Equation (A.6) and the affine parametrization condition give

$$\left(\frac{dt}{ds}\right)^2=\left(\frac{d\rho}{ds}\right)^2+e^{-2\psi}(\varepsilon+\lambda^{ab}p_ap_b), \tag{A.7}$$

with $\varepsilon=0$ for null geodesics and $\varepsilon=1$ for timelike ones. Varying (A.5) with respect to $\dot{x}^A$ one obtains

$$\begin{aligned}\frac{d}{ds}(e^{2\psi}\eta_{AB}\dot{x}^A)=&-\frac{1}{2}\frac{\partial\lambda^{ab}}{\partial x^B}p_ap_b-(\varepsilon+\lambda^{ab}p_ap_b)\frac{\partial\psi}{\partial x^B}\\&+p_a\left(\frac{\partial M^a_A}{\partial x^B}-\frac{\partial M^a_B}{\partial x^A}\right)\dot{x}^A.\end{aligned} \tag{A.8}$$

We have

$$\lambda^{ab}p_ap_b=\frac{e^{2\gamma}}{R^2}(p_\theta-ap_z)^2+e^{-2\gamma}p_z^2, \tag{A.9}$$

so that (A.8) with $B=0$ gives

$$\begin{aligned}\frac{d}{ds}\left(e^{2\psi}\frac{dt}{ds}\right)=&\frac{e^{2\gamma}}{R^2}\left(\frac{\partial\gamma}{\partial t}+\frac{\partial\psi}{\partial t}-\frac{1}{R}\frac{\partial R}{\partial t}\right)(p_\theta-ap_z)^2\\&-\frac{e^{2\gamma}}{R^2}\frac{\partial a}{\partial t}(p_\theta-ap_z)+e^{-2\gamma}\left(\frac{\partial\psi}{\partial t}-\frac{\partial\gamma}{\partial t}\right)p_z^2\\&+\varepsilon\frac{\partial\psi}{\partial t}+p_a\left(\frac{\partial M^a_0}{\partial x^A}-\frac{\partial M^a_A}{\partial t}\right)\dot{x}^A.\end{aligned} \tag{A.10}$$

Equation (A.7) gives $|d\rho/ds|\leqslant|dt/ds|$, and assuming that $dt/ds>0$ from (A.4) and (A.10), one obtains

$$\frac{d}{ds}\left(e^{2\psi}\frac{dt}{ds}\right)\leqslant C\left(1+e^{2\psi}\frac{dt}{ds}\right), \tag{A.11}$$

for some constant C. Equation (A.11) and the Gronwall lemma show that dt/ds cannot blow up in finite affine time s; consequently t cannot tend to infinity in finite time, and causal geodesic completeness will follow if we can show that the level sets of t are Cauchy surfaces.

By (A.7) clearly t is monotonous along causal geodesics. From what has been said it also follows that both $d\rho/ds$ and ρ cannot blow up in finite time. The regularity of the metric implies that on every compact subset $\mathscr{K}$ in the $\rho - t$ plane there exists a constant $C = C(\mathscr{K})$ such that

$$\left|\frac{a}{\rho^2}\right| \leqslant C, \qquad C^{-1} \leqslant \frac{R}{\rho} \leqslant C. \tag{A.12}$$

By (A.7) $\lambda^{ab} p_a p_b$ cannot blow up in finite time, which together with (A.12) shows that the right-hand side of the second equality in (A.6) remains finite for finite times, so that z, dz/ds remain finite for finite times. Our claim follows now by standard extendability results for causal curves. ∎

APPENDIX B: Motion of Charged Test Particles

Throughout this section we shall assume that the metric takes the form (A.1)–(A.2). The equations of motion for charged particles can be derived from the variational principle for the action

$$\tilde{I} = \int \left\{ h_{AB} \dot{x}^A \dot{x}^B + \lambda_{ab} (\dot{x}^a + M^a_A \dot{x}^A)(\dot{x}^b + M^b_B \dot{x}^B) + 2 \frac{e}{m} A_\mu \dot{x}^\mu \right\} ds. \tag{B.1}$$

We shall assume that all the functions appearing in (B.1) depend only upon the x^A's. This leads to the conservation equations $d\tilde{p}_a/ds = 0$, where

$$\tilde{p}_a = \lambda_{ab}(\dot{x}^b + M^b{}_A \dot{x}^A) + \frac{e}{m} A_a$$

$$\Rightarrow \dot{x}^a = \lambda^{ab}\left(\tilde{p}_b - \frac{e}{m} A_b\right) - M^a{}_A \dot{x}^A, \tag{B.2}$$

λ^{ab} being the inverse matrix to λ_{ab}. Assume that the trajectories are affinely parametrized; from (B.2) one obtains

$$\left(\frac{dt}{ds}\right)^2 = \left(\frac{d\rho}{ds}\right)^2 + e^{-2\psi}\left\{\varepsilon + \lambda^{ab}\left(\tilde{p}_a - \frac{e}{m} A_a\right)\left(\tilde{p}_b - \frac{e}{m} A_b\right)\right\}. \tag{B.3}$$

The variation of $\tilde{I}$ with respect to x^A gives

$$\begin{aligned}\frac{d}{ds}(e^{2\psi}\eta_{AB}\dot{x}^A) = &\left\{\left(\tilde{p}_a - \frac{e}{m}A_a\right)\left(\frac{\partial M^a{}_A}{\partial x^B} - \frac{\partial M^a{}_B}{\partial x^A}\right)\right.\\ &\left.+\frac{e}{m}\left[F_{AB} + F_{aA}M^a{}_B - F_{aB}M^a{}_A\right]\right\}\dot{x}^A\\ &-\left(\frac{\partial\psi}{\partial x^B}\lambda^{ab} + \frac{1}{2}\frac{\partial\lambda^{ab}}{\partial x^B}\right)\left(\tilde{p}_a - \frac{e}{m}A_a\right)\left(\tilde{p}_b - \frac{e}{m}A_b\right)\\ &-\varepsilon\frac{\partial\psi}{\partial x^B} + \frac{e}{m}F_{aB}\lambda^{ab}\left(\tilde{p}_b - \frac{e}{m}A_b\right).\end{aligned} \tag{B.4}$$

Here we have, as usual

$$F_{\mu\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}.$$

If we assume that the following inequalities hold,

$$\begin{aligned}&|\gamma| + \left|\frac{\partial\gamma}{\partial t}\right| + |\psi| + \left|\frac{\partial\psi}{\partial t}\right| + (1 + |A_\theta|)\left|\frac{\partial M^a{}_A}{\partial t} - \frac{\partial M^a{}_t}{\partial x^A}\right|\\ &+\frac{1}{R^2}\left|\frac{\partial a}{\partial t}\right|(1 + |a| + |A_\theta|) + \frac{1}{R^2}\left|\frac{\partial\psi}{\partial t} + \frac{\partial\gamma}{\partial t} - \frac{1}{R}\frac{\partial R}{\partial t}\right|\\ &\times(1 + |a|^2 + |A_\theta|^2) + |A_z| + |M^z{}_A| + |F_{t\rho}|\\ &+|F_{aA}M^a{}_t - F_{at}M^a{}_A| + |\lambda^{ab}F_{at}|(1 + |A_\theta|) \leqslant C\end{aligned} \tag{B.5}$$

for some constant C, then the argument of the proof of Proposition A.1 can be repeated to conclude that causal trajectories of charged particles are affinely complete.

APPENDIX C: EINSTEIN'S EQUATIONS

This appendix [31] uses the coordinates $x^1 = z$, $x^2 = \theta$, $x^3 = \rho$, $x^4 = t$. (The reader should be warned that the ordering here differs from the ordering in the main body of the paper, which is (t, ρ, z, θ).) The metric $g_{\alpha\beta}$ is expressed in terms of the variables $\gamma(\rho, t)$, $v(\rho, t)$, $a(\rho, t)$, and $R(\rho, t)$ with shifts $M^b(\rho, t)$ and twists

$g^b(t)$ with $b = 1, 2$, consistently with (2.17) and (3.1)–(3.2). The distinct components of the metric are

$$g_{11} = e^{2\gamma}, \tag{C.1}$$

$$g_{12} = e^{2\gamma} a, \tag{C.2}$$

$$g_{13} = e^{2\gamma}(g^1 + a g^2), \tag{C.3}$$

$$g_{14} = e^{2\gamma}(M^1 + a M^2), \tag{C.4}$$

$$g_{22} = e^{2\gamma} a^2 + e^{-2\gamma} R^2, \tag{C.5}$$

$$g_{23} = e^{2\gamma} a (g^1 + a g^2) + e^{-2\gamma} g^2 R^2, \tag{C.6}$$

$$g_{24} = e^{2\gamma} a (M^1 + a M^2) + e^{-2\gamma} M^2 R^2, \tag{C.7}$$

$$g_{33} = e^{2(\nu-\gamma)} + e^{2\gamma}(g^1 + a g^2)^2 + e^{-2\gamma}(g^2)^2 R^2, \tag{C.8}$$

$$g_{34} = e^{2\gamma}(M^1 + a M^2)(g^1 + a g^2) + e^{-2\gamma} g^2 M^2 R^2, \tag{C.9}$$

$$g_{44} = -e^{2(\nu-\gamma)} + e^{2\gamma}(M^1 + a M^2)^2 + e^{-2\gamma}(M^2)^2 R^2, \tag{C.10}$$

with corresponding inverse metric components

$$g^{11} = e^{-2\gamma} + e^{-2\nu+2\gamma}((g^1)^2 - (M^1)^2) + \frac{e^{2\gamma} a^2}{R^2}, \tag{C.11}$$

$$g^{12} = e^{-2\nu+2\gamma}(g^1 g^2 - M^1 M^2) - \frac{e^{2\gamma} a}{R^2}, \tag{C.12}$$

$$g^{13} = -e^{-2\nu+2\gamma} g^1, \tag{C.13}$$

$$g^{14} = e^{-2\nu+2\gamma} M^1, \tag{C.14}$$

$$g^{22} = e^{-2\nu+2\gamma}((g^2)^2 - (M^2)^2) + \frac{e^{2\gamma}}{R^2}, \tag{C.15}$$

$$g^{23} = -e^{-2\nu+2\gamma} g^2, \tag{C.16}$$

$$g^{24} = e^{-2\nu+2\gamma} M^2, \tag{C.17}$$

$$g^{33} = e^{-2\nu+2\gamma}, \tag{C.18}$$

$$g^{34} = 0, \tag{C.19}$$

$$g^{44} = -e^{-2\nu+2\gamma}. \tag{C.20}$$

Einstein's equations $G^\beta_\alpha = T^\beta_\alpha$ are given for a general matter source T^β_α. (Here we have absorbed the usual [4] constant $8\pi G/c^4$ in the definition of T^β_α.) To simplify the equations, parameters c_b for $b = 1, 2$ are defined through (where $'$ is d/dt)

$$M^1{}_{,\rho} - g^{1'} = -\frac{e^{2\nu-4\gamma}}{R} c_1 - \frac{e^{2\nu} a}{R^3}(a c_1 - c_2) \tag{C.21}$$

and

$$M^2{}_{,\rho} - g^{2'} = \frac{e^{2\nu}}{R^3}(ac_1 - c_2). \tag{C.22}$$

The c_b's are strictly constant if certain off-diagonal matter terms vanish. These are identified through

$$c_{1,\rho} = -2e^{2\nu-2\gamma}RT_1^4 = 2Re^{\nu-\gamma}T_{\mu\nu}n^\mu X_1^\nu, \tag{C.23}$$

$$c_{1,t} = 2e^{2\nu-2\gamma}RT_1^3 = 2Re^{\nu-\gamma}T_{\mu\nu}m^\mu X_1^\nu, \tag{C.24}$$

$$c_{2,\rho} = -2e^{2\nu-2\gamma}RT_2^4 = 2Re^{\nu-\gamma}T_{\mu\nu}n^\mu X_2^\nu, \tag{C.25}$$

$$c_{2,t} = 2e^{2\nu-2\gamma}RT_2^3 = 2Re^{\nu-\gamma}T_{\mu\nu}m^\mu X_2^\nu, \tag{C.26}$$

which are the surviving terms in the Einstein equations with indices indicated by the T's. Here, as elsewhere, n^ν denotes the field of unit, future directed vectors which are normal to Σ and m^μ denotes the field of unit vectors which are tangent to Σ and orthogonal to the orbits of the isometry group. The dynamical Einstein's equations are combined to yield wave equations for γ, a, R, and ν:

$$\gamma_{,tt} - \gamma_{,\rho\rho} + \frac{R_{,t}}{R}\gamma_{,t} - \frac{R_{,\rho}}{R}\gamma_{,\rho}$$

$$= \frac{e^{4\gamma}}{2R^2}(a_{,t}{}^2 - a_{,\rho}{}^2) + \frac{e^{2\nu-4\gamma}}{2R^2}c_1^2 - \frac{e^{2(\nu-\gamma)}}{2}T + e^{2\nu-4\gamma}T_{11}, \tag{C.27}$$

$$a_{,tt} - a_{,\rho\rho} - \frac{R_{,t}}{R}a_{,t} + 4a_{,t}\gamma_{,t} + \frac{R_{,\rho}}{R}a_{,\rho} - 4a_{,\rho}\gamma_{,\rho}$$

$$= -\frac{e^{2\nu-4\gamma}}{R^2}c_1(ac_1 - c_2) + 2e^{2\nu-6\gamma}R^2(T_1^2 + g^2T_1^3 + M^2T_1^4), \tag{C.28}$$

$$\begin{aligned} R_{,tt} - R_{,\rho\rho} &= \frac{e^{2\nu}}{2R^3}(ac_1 - c_2)^2 + \frac{e^{2\nu-4\gamma}}{2R}c_1^2 \\ &\quad - e^{2(\nu-\gamma)}R(T_4^4 + T_3^3 - g^2T_2^3 - M^2T_2^4 - g^1T_1^3 - M^1T_1^4) \\ &= \frac{e^{2\nu}}{2R^3}(ac_1 - c_2)^2 + \frac{e^{2\nu-4\gamma}}{2R}c_1^2 + e^{2(\nu-\gamma)}RT_{\mu\nu}(n^\mu n^\nu - m^\mu m^\nu), \end{aligned} \tag{C.29}$$

$$\begin{aligned} \nu_{,tt} - \nu_{,\rho\rho} &= \frac{e^{4\gamma}}{4R^2}(a_{,t}{}^2 - a_{,\rho}{}^2) - \gamma_{,t}{}^2 + \gamma_{,\rho}{}^2 - \frac{3e^{2\nu}}{4R^4}(ac_1 - c_2)^2 - \frac{e^{2\nu-4\gamma}}{4R^2}c_1^2 \\ &\quad - e^{2(\nu-\gamma)}(T_2^2 + g^2T_2^3 + M^2T_2^4 - aT_1^2 - ag^2T_1^3 - aM^2T_1^4). \end{aligned} \tag{C.30}$$

Above we have have used the notation $T = T^{\alpha}{}_{\alpha}$. The wave equation for ν can in fact be obtained from the other wave equations and the constraints. The Hamiltonian constraint equation is

$$0 = \frac{e^{-2\nu+6\gamma}}{4R^2}(a_{,t}{}^2 + a_{,\rho}{}^2) - \frac{e^{2(-\nu+\gamma)}}{R}(\nu_{,t}R_{,t} + \nu_{,\rho}R_{,\rho} - R_{,\rho\rho})$$
$$+ e^{2(-\nu+\gamma)}(\gamma_{,t}{}^2 + \gamma_{,\rho}{}^2) + \frac{e^{2\gamma}}{4R^4}(ac_1 - c_2)^2 + \frac{e^{-2\gamma}}{4R^2}c_1^2 + T_{\mu\nu}n^\mu n^\nu, \qquad \text{(C.31)}$$

with the remaining momentum constraint equation

$$0 = \frac{e^{-2\nu+6\gamma}}{2R^2}a_{,t}a_{,\rho} - \frac{e^{2(-\nu+\gamma)}}{R}(R_{,t}\nu_{,\rho} + \nu_{,t}R_{,\rho} - R_{,t\rho})$$
$$+ 2e^{2(-\nu+\gamma)}\gamma_{,t}\gamma_{,\rho} + T_{\mu\nu}n^\mu m^\nu. \qquad \text{(C.32)}$$

Acknowledgments

B.K.B., P.T.C., and V.M. are grateful to the ITP, UCSB for hospitality and financial support during part of work on this paper. Useful comments on a previous version of this paper by Hans-Jürgen Seifert and Bernd Schmidt are acknowledged. B.K.B. thanks Dr. Ed Seidel for invaluable assistance and the Institute of Geophysics and Planetary Physics at Lawrence Livermore National Laboratory for hospitality.

References

1. A. Ashtekar, J. Bičák, and B. Schmidt, Generalized cylindrical gravitational waves in 3 + 1 dimensions or 2 + 1-dimensional gravity with scalar fields: Asymptotics, preprint.
2. G. Beck, *Z. Phys.* **33** (1925), 713–728.
3. H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, *Proc. R. Soc. London A* **269** (1962), 21–52.
4. Y. Choquet-Bruhat and J. York, The Cauchy problem, *in* "General Relativity and Gravitation" (A. Held, Ed.), Plenum, New York, 1980.
5. D. Christodoulou and A. S. Tahvildar-Zadeh, *Commun. Pure Appl. Math.* **46** (1993), 1041–1091.
6. D. Christodoulou and A. S. Tahvildar-Zadeh, *Duke Math. J.* **71** (1993), 31–69.
7. P. T. Chruściel, *Ann. Phys. (N.Y.)* **202** (1990), 100–150.
8. P. T. Chruściel, "On Uniqueness in the Large of Solutions of Einstein Equations ('Strong Cosmic Censorship')," Australian Univ. Press, Canberra, 1991.
9. P. T. Chruściel, *Contemp. Math.* **132** (1992), 235–273.
10. P. T. Chruściel, "On Completeness of Orbits of Killing Vector Fields," Preprint NSF-ITP-93-44; gr-gc/9304029; *Class. Quantum Grav.*, in press.
11. P. T. Chruściel, J. Isenberg, and V. Moncrief, *Class. Quantum Grav.* **7** (1990), 1671–1680.
12. F. J. Ernst, *Phys. Rev.* **167** (1968), 1175–1179.
13. F. J. Ernst, *Phys. Rev.* **168** (1968) 1415–1417; *Phys. Rev.* **172** (1968), 1850.
14. J. H. Eschenburg, *Manuscr. Math.* **59** (1987), 295–323.
15. V. Fock, "Theorie von Raum-Zeit und Gravitation," Akademie Verlag, Berlin, 1960.

16. J. L. Friedman, K. Schleich, and D. M. Witt, Topological censorship, *Phys. Rev. Lett.* **71** (1993), 1486.
17. R. Geroch, *J. Math. Phys.* **13** (1972), 394–404.
18. R. Gowdy, *Ann. Phys. (N.Y.)* **83** (1974), 203-241.
19. J. Isenberg and V. Moncrief, *Ann. Phys. (N.Y.)* **199** (1990), 84–122.
20. A. S. Kompaneets, *Sov. Phys. JETP* **7** (1958), 659–660.
21. W. Kinnersley, *J. Math. Phys.* **14** (1973), 651–653.
22. W. Kundt and M. Trumper, *Z. Phys.* **192** (1966), 419–422.
23. P. O. Mazur, *Acta. Phys. Pol. B* **14** (1983), 219–234.
24. P. O. Mazur, Black hole uniqueness theorems, *in* "General Relativity and Gravitation, Proceedings, GRG 11, Stockholm, July 1986," (M. A. H. MacCallum, Ed.), Cambridge Univ. Press, Cambridge, 1987.
25. C. Misner, K. Thorne, and J. Wheeler, "Gravitation," Freeman, San Fransisco, 1973.
26. V. Moncrief, *Ann. Phys. (N.Y.)* **132** (1981), 87–107.
27. V. Moncrief, *Phys. Rev. D* **39** (1989), 429–433.
28. V. Moncrief, *Class. Quant. Grav.* **7** (1990), 329–352.
29. V. Moncrief and D. M. Eardley, *Gen. Rel. Grav.* **13** (1981), 887–892.
30. B. O'Neil, "Semi-Riemannian Geometry," Academic Press, New York, 1983.
31. Einstein's equations were generated using L. Parker, S. M. Christensen, MathTensor (Version 2.1.5) (MathSolutions, 1993) within Mathematica, Wolfram Research, 1988.
32. A. P. Papapetrou, *Ann. Inst. H. Poincaré A* **4** (1966), 83–105.
33. R. Penrose, *Rev. Mod. Phys.* **37** (1965), 215–220.
34. R. Penrose, *Riv. Nuovo Cimento* **1** (1969), 252–276.
35. R. K. Sachs, *Proc. R. Soc. London A* **264** (1962), 309–338.
36. K. Thorne, Nonspherical gravitational collapse: A short review, *in* "Magic without Magic" (J. R. Klauder, Ed.), Freeman, San Francisco, 1972.
37. A. Trautman, *Bull. Acad. Pol. Sci., Sér. Sci. Math., Astron. Phys.* **6** (1958), 407–412.
38. N. M. J. Woodhouse, *Class. Quantum Grav.* **6** (1989), 933–943.