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Global Foliations of Vacuum Spacetimes with T^2 Isometry

Beverly K. Berger*

Department of Physics, Oakland University, Rochester, Michigan 48309

Piotr T. Chruściel†

Département de Mathématiques, Faculté des Sciences, Parc de Grandmont, F 37200 Tours, France

James Isenberg‡

Department of Mathematics and Institute for Theoretical Science, University of Oregon, Eugene, Oregon 97403

and

Vincent Moncrief§

Department of Physics, Yale University, P.O. Box 208120, New Haven, Connecticut 06520-8120

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We prove a global existence theorem (with respect to a geometrically defined time) for globally hyperbolic solutions of the vacuum Einstein equations which admit a T^2 isometry group with two-dimensional spacelike orbits, acting on T^3 spacelike surfaces.

1. INTRODUCTION

One of the ways in which to begin to study the behavior of solutions of complicated partial differential equation systems like the Einstein equations is to focus on families of solutions with some prescribed symmetry. This has long been a practice in general relativity, and hence, much has been learned about solutions

* E-mail: berger@vela.acs.oakland.edu.
† E-mail: chrusciel@univ-tours.fr.
‡ Visiting scientist: Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut) Schlaatzweg 1, 14473 Potsdam, Germany. E-mail: jim@newton.uoregon.edu.
§ E-mail: moncrief@hepvms.physics.yale.edu.

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of the Einstein equations which are spherically symmetric [4], which are spatially homogeneous [10, 19], which are axisymmetric and stationary [20], or have various other prescribed isometries [6, 13].

One of the more extensively studied families of solutions are the Gowdy spacetimes [11]. Long time existence (in an appropriate sense) has been proven for them [15], strong cosmic censorship has been proven for Gowdy spacetimes which are polarized [9], and much is known about the small (but infinite dimensional) set of Gowdy spacetimes which admit extension across a Cauchy horizon [16, 8].

The Gowdy spacetimes are characterized primarily by their admittance of a spatially-acting T^2-isometry group, but certain other restrictions are imposed as well (see Section 2) [11]. In this paper, we examine spacetimes which retain the spatial T^2-isometry, but which do not satisfy these other restrictions [5]. Our main result is that, for this wider class of spacetimes, long-time existence (in a sense similar to that proven for Gowdy T^3 spacetimes) holds.

While the notion of global long-time existence is clear for partial differential equations formulated on a given background spacetime, its meaning is less evident when the spacetime itself is the object of study as in general relativity. Here what one means by long–time, or global existence, is a theorem that characterizes the maximal Cauchy development of given Cauchy data sets. To prove such a theorem one often needs to choose a time function which extends to the maximal development of every solution under consideration. To accomplish this one tries to find a geometrically natural time function. For the Gowdy spacetimes, as considered in [15], there is such a geometric time choice, defined (up to proportionality constant) to be the area R of the orbits of the T^2 isometry group. Now the spacetimes considered in [15] form only a small subset of the set of T^2-symmetric spacetimes with T^3 spatial topology. Here we show that for all such spacetimes the following holds: Except for some Kasner spacetimes in which the T^2 subgroup of the full T^3 isometry group has been chosen in a rather inconvenient way, there is always a global $R = \lambda t$ foliation, with $R \in (R_0, \infty)$ for some $R_0 \geqslant 0$ and some constant $\lambda > 0$. (We note that for the spacetimes considered in [15] we have $R_0 = 0$; that will not be true in general for the spacetimes considered in this paper, as can be seen by considering some Kasner solutions.)

The bulk of this paper is devoted to the proof of this result, which we state as Theorem 1 in Section 3. Before stating this theorem, we define (in Section 2) our family of spacetimes and write the general parametrization for the metrics as well as the corresponding expressions for the field equations in two different coordinate choices which we find useful. We divide our discussion of the proof into three parts. In the first part (carried out in Section 4), we focus on the "contracting ($R \searrow 0$) direction." We establish the necessary estimates and prove a global existence result for this direction in terms of an auxiliary choice of time—"conformal coordinate" time. In the second part (Section 5), we prove a number of geometric results which relate R to the maximal globally hyperbolic region of a spacetime and to the conformal coordinate representation. Using these results, we show that we have an $R = t$ foliation covering the contracting direction. Then in the third part (Section 6),

we focus on the "expanding ($R \to \infty$) direction." Working directly with $R = t$ coordinates, we establish some estimates and then combine these with results from Section 4 to prove the existence of a global $R = t$ foliation in the expanding direction. This completes the proof of our main result. We conclude in Section 7 with a discussion of questions concerning these spacetimes which we plan to explore further.

2. DESCRIPTION AND FIELD EQUATIONS FOR THE T^2-SYMMETRIC SPACETIMES

The Gowdy spacetimes all admit a T^2 isometry group with spacelike orbits. The additional condition which characterizes the Gowdy spacetimes is that the "twists" associated to the T^2 isometry group must vanish. More specifically, the twist quantities

$$\underset{(X)}{K} := \varepsilon_{\mu\nu\rho\lambda} X^\mu Y^\nu \nabla^\rho X^\lambda \qquad \text{and} \qquad \underset{(Y)}{K} := \varepsilon_{\mu\nu\rho\lambda} X^\mu Y^\nu \nabla^\rho Y^\lambda, \tag{2.1}$$

where X and Y are any pair of Killing vector fields generating the T^2 group action, must both be zero (as a defining characteristic) in a Gowdy spacetime.

In this paper we wish to analyze spacetimes in which this condition on the twists is relaxed: we shall assume that at least one of the quantities $\underset{(X)}{K}$ or $\underset{(Y)}{K}$ is nonzero. It is interesting and important to note that in any spacetime with a T^2 isometry, one can *always* replace X and Y by constant linear combinations of themselves, $\hat{X}$ and $\hat{Y}$, and thereby cause one or the other of the twist quantities $\underset{(\hat{X})}{K}$ or $\underset{(\hat{Y})}{K}$ to be zero. The vanishing of *both* twist quantities is, however, independent of such mixing of Killing vector fields, so the Gowdy spacetimes are unambiguously distinguished. It is also useful to note that, while the Gowdy spacetimes are compatible with $S^3 \times \mathbb{R}^1$, $S^2 \times S^1 \times \mathbb{R}^1$, as well as $T^3 \times \mathbb{R}^1$ spacetime manifolds, if one of the twists is nonvanishing, then the manifold must be $T^3 \times \mathbb{R}^1$. Hence, we shall restrict our attention to T^2 symmetric spacetimes on $T^3 \times \mathbb{R}^1$.

There are two choices of coordinates we will find useful for specifying the form of the metric and the form of the field equations for these spacetimes.[1] Both sets of coordinates are chosen to be compatible with the Killing vector fields in the sense that $X = a(\partial/\partial x) + b(\partial/\partial y)$ and $Y = c(\partial/\partial x) + d(\partial/\partial y)$, for a set of constants a, b, c, d with $\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \neq 0$, and both use $\theta \in S^1$ to label the remaining spatial coordinate. The two sets differ mainly in the choice of time slicing. In the first set, which we call the "areal coordinate system," we set $t = R$, where R is proportional to the geometric area function of the orbits of the isometry group. In terms of these coordinates (x, y, θ, t), we have the following general form for the metric and for the vacuum field equations.

[1] The field equations were obtained with the assistance of S. Christensen and L. Parker's, MATHTENSOR package in Mathematica (Wolfram Research, Inc.).

Areal Coordinates

Metric

$$g = e^{2(v-U)}(-\alpha\, dt^2 + d\theta^2) + \lambda e^{2U}[dx + A\, dy + (\underset{1}{G} + A\underset{2}{G})\, d\theta + (\underset{1}{M} + A\underset{2}{M})\, dt]^2$$
$$+ \lambda e^{-2U} t^2 [dy + \underset{2}{G}\, d\theta + \underset{2}{M}\, dt]^2. \tag{2.3}$$

(Here $U, A, v, \alpha, \underset{1}{G}, \underset{2}{G}, \underset{1}{M}$, and $\underset{2}{M}$ are functions of $\theta \in S^1$ and $t \in \mathbb{R}^+$, and λ is a strictly positive constant.)

Einstein Constraint Equations

$$v_t = t\left[(U_t^2 + \alpha U_\theta^2) + \frac{e^{4U}}{4t^2}(A_t^2 + \alpha A_\theta^2)\right] + \lambda \frac{e^{2v}}{4t^3}\alpha K^2, \tag{2.4a}$$

$$v_\theta = t\left[2U_t U_\theta + \frac{e^{4U}}{2t^2} A_t A_\theta\right] - \frac{\alpha_\theta}{2\alpha}, \tag{2.4b}$$

$$\alpha_t = -\lambda \frac{e^{2v}}{t^3}(\alpha K)^2, \tag{2.4c}$$

$$K_\theta = 0, \tag{2.4d}$$

$$K_t = 0. \tag{2.4e}$$

Einstein Evolution Equations

$$U_{tt} - \alpha U_{\theta\theta} = -\frac{U_t}{t} + \frac{\alpha_\theta U_\theta}{2} + \frac{\alpha_t U_t}{2\alpha} + \frac{e^{4U}}{2t^2}(A_t^2 - \alpha A_\theta^2), \tag{2.5a}$$

$$A_{tt} - \alpha A_{\theta\theta} = \frac{A_t}{t} + \frac{\alpha_\theta A_\theta}{2} + \frac{\alpha_t A_t}{2\alpha} - 4A_t U_t + 4\alpha A_\theta U_\theta, \tag{2.5b}$$

$$v_{tt} - \alpha v_{\theta\theta} = \frac{\alpha_\theta v_\theta}{2} + \frac{\alpha_t v_t}{2\alpha} - \frac{\alpha_\theta^2}{4\alpha} + \frac{\alpha_{\theta\theta}}{2} - U_t^2 + \alpha U_\theta^2$$
$$+ \frac{e^{4U}}{4t^2}(A_t^2 - \alpha A_\theta^2) - \lambda \frac{3e^{2v}\alpha}{4t^4} K^2. \tag{2.5c}$$

Auxiliary Equations

$$\underset{1}{M}_\theta = \underset{1}{G}_t + A(\underset{2}{G}_t - \underset{2}{M}_\theta), \tag{2.6a}$$

$$\underset{2}{M}_\theta = \underset{2}{G}_t - \frac{e^{2v}}{t^3}\alpha^{1/2} K. \tag{2.6b}$$

(Note that here and below, we use t and θ as subscripts on U, A, v, α, etc. exclusively to indicate partial derivatives of these functions. Note also that, without loss of generality, we have set the twist quantity $\underset{(X)}{K}$ equal to zero, and we have used "K" to label the remaining twist quantity $\underset{(Y)}{K}$.)

The other coordinate system we use here chooses t and θ so that, if we forget the "shift"-type metric components $\underset{1}{G}$, $\underset{2}{G}$, $\underset{1}{M}$, and $\underset{2}{M}$, then the induced Lorentz metric on the (θ, t)-labeled space of orbits of the T^2 isometry group is conformally flat. Doing this allows us to remove the function $\alpha(\theta, t)$ from the metric, but it requires that we let the orbital area R be a function of θ and t. So in this system of coordinates, which we call "conformal coordinates," the metric and the field equations take the following form.

Conformal Coordinates

Metric

$$g = e^{2(v-U)}(-dt^2 + d\theta^2) + \lambda e^{2U}[dx + A\,dy + (\underset{1}{G} + A\underset{2}{G})\,d\theta + (\underset{1}{M} + A\underset{2}{M})\,dt]^2$$
$$+ \lambda R^2 e^{-2U}[dy + \underset{2}{G}\,d\theta + \underset{2}{M}\,dt]^2. \tag{2.7}$$

(Here, as before, U, A, v, R, $\underset{1}{M}$, $\underset{1}{G}$, $\underset{2}{G}$, and $\underset{2}{M}$ are functions of θ and t.)

Einstein Constraint Equations

$$0 = U_t^2 + U_\theta^2 + \frac{e^{4U}}{4R^2}(A_t^2 + A_\theta^2) + \frac{R_{\theta\theta}}{R} - \frac{v_t R_t}{R} - \frac{v_\theta R_\theta}{R} + \lambda \frac{e^{2v}}{4R^4} K^2, \tag{2.8a}$$

$$0 = 2U_t U_\theta + \frac{e^{4U}}{2R^2} A_t A_\theta + \frac{R_{\theta t}}{R} - \frac{v_\theta R_t}{R} - \frac{v_t R_\theta}{R}, \tag{2.8b}$$

$$K_\theta = 0, \tag{2.8c}$$

$$K_t = 0. \tag{2.8d}$$

Einstein Evolution Equations

$$U_{tt} - U_{\theta\theta} = \frac{R_\theta U_\theta}{R} - \frac{R_t U_t}{R} + \frac{e^{4U}}{2R^2}(A_t^2 - A_\theta^2), \tag{2.9a}$$

$$A_{tt} - A_{\theta\theta} = \frac{R_t A_t}{R} - \frac{R_\theta A_\theta}{R} + 4(A_\theta U_\theta - A_t U_t), \tag{2.9b}$$

$$R_{tt} - R_{\theta\theta} = \lambda \frac{e^{2v}}{2R^3} K^2, \tag{2.9c}$$

$$v_{tt} - v_{\theta\theta} = U_\theta^2 - U_t^2 + \frac{e^{4U}}{4R^2}(A_t^2 - A_\theta^2) - \lambda \frac{3e^{2v}}{4R^4} K^2. \tag{2.9d}$$

Auxiliary Equations

$$\underset{1}{M}_\theta = \underset{1}{G}_t + A(\underset{2}{G}_t - \underset{2}{M}_\theta), \tag{2.10a}$$

$$\underset{2}{M}_\theta = \underset{2}{G}_t - \frac{e^{2v}}{R^3} K. \tag{2.10b}$$

(Our conventions regarding subscripts and our handling of the twist quantities are the same here as above.)

It has been proven in earlier work [5] that, at least locally, any globally hyperbolic T^2-symmetric spacetime on $T^3 \times \mathbb{R}^1$ admits each of these coordinate forms (unless the spacetime is flat). Since our notion of long-time existence is tied to the orbital area function R, the statement of our main result focuses on R and the areal coordinate system (with $R = t$). Indeed it establishes that areal coordinates cover any globally hyperbolic T^2-symmetric spacetime. To prove this in the expanding direction, areal coordinates are used directly. However to prove this in the contracting direction, we find that the conformal coordinate form is the most useful for carrying out the analysis.

3. THE LONG-TIME EXISTENCE THEOREM

Let us call (γ, π) *T^2-symmetric initial data on T^3* if (a) γ is a Riemannian metric on T^3, invariant under an effective T^2 action; (b) π is a symmetric 2-tensor on T^3, also invariant under the same T^2 group action; and (c) (γ, π) together satisfy the Einstein constraint equations.

To avoid unnecessary details, we will assume that (γ, π) are smooth (C^∞) on T^3. Our result holds for weaker differentiability conditions imposed on (γ, π), but we will not state those conditions here. Our main result is the following

THEOREM 1. *Let (γ, π) be a set of smooth T^2 symmetric initial data on T^3. For some nonnegative constant c there exists a globally hyperbolic spacetime (M^4, g) such that*

(i) $M^4 = T^3 \times (c, \infty)$,

(ii) *g satisfies the vacuum Einstein equations,*

(iii) *M^4 is covered by areal coordinates (x, y, θ, t), with $t \in (c, \infty)$, so the metric globally takes the form* (2.3),

(iv) *(M^4, g) is isometrically diffeomorphic to the maximal globally hyperbolic development of the initial data (γ, π).*

Outline of Proof. As noted in the Introduction, we carry out our proof of Theorem 1 in Sections 4, 5, and 6. The logic of the proof is as follows: First (in Section 4), working in terms of conformal coordinates, we look at the evolution of a solution from the data (γ, π) toward the contracting direction. (Without loss of generality, we may choose the time-orientation so that this is towards the past.) Using the field equations and light cone arguments to establish a number of estimates for various components of the metric, we go on to prove (using results from, e.g., [14]) a global existence result in the following sense: So long as R stays bounded away from zero, the past maximal development of (γ, π) in terms of conformal coordinates—let us label it $D^-_{\text{conf}}(\gamma, \pi)$—has $t \to -\infty$. The evolution (in terms of conformal coordinates) stops only if R approaches zero.

Next (in Section 5) we prove a number of geometric results, most of which concern the behavior of the orbital area function R in a globally hyperbolic spacetime such as $D^-_{\text{conf}}(\gamma, \pi)$. The first of these, a Killing vector argument, shows that R is positive everywhere in the globally hyperbolic region of a T^2 symmetric spacetime. The next one shows that along any past inextendible timelike path in $D^-_{\text{conf}}(\gamma, \pi)$, R approaches a limit $R_0 \geqslant 0$ (to be identified with "c" in Theorem 1). Moreover, one has the same limit along all such paths. Combining this result with another (proven in Section 5) which shows that for any $\rho \in (R_0, R_1)$—where R_1 is the minimum value of R on the initial surface with data (γ, π)—the $R = \rho$ level set in $D^-_{\text{conf}}(\gamma, \pi)$ is a Cauchy surface, we can argue that $D^-_{\text{conf}}(\gamma, \pi)$ admits areal coordinates, at least to the past of the hypersurface with constant $R = R_1$. To show that $D^-_{\text{conf}}(\gamma, \pi)$ is isometrically diffeomorphic to the maximal [2] globally hyperbolic past development $D^-(\gamma, \pi)$ of (γ, π) on T^3, we need two further geometric results. We show that if $t \to -\infty$ in $D^-_{\text{conf}}(\gamma, \pi)$, then $D^-_{\text{conf}}(\gamma, \pi) \approx D^-(\gamma, \pi)$, and we show that if $R \to 0$ in $D^-_{\text{conf}}(\gamma, \pi)$, then $D^-_{\text{conf}}(\gamma, \pi) \approx D^-(\gamma, \pi)$. This completes the argument that to the past of the hypersurface with $R = R_1$, $D^-(\gamma, \pi)$ can be covered by areal coordinates. Note, that we also prove in Section 5 that if $R \to \infty$ in any future development of (γ, π), then that future development is maximal.

Our proof is more direct for the expanding, future direction. Based on results from Section 5, we have an $R =$ constant Cauchy surface—say $R = R_2$—to the past of our original Cauchy surface. Let us call the data on this surface (γ_1, π_1). Then working in areal coordinates, we can use the field equations and light cone arguments to establish a collection of estimates, and from these (and, again, results from [14]) we prove a global existence result which says that the future maximal development of (γ_1, π_1) in terms of areal coordinates—$D^+_{\text{areal}}(\gamma, \pi)$—has $t = R \to \infty$. As shown in Section 5, it follows that $D^+_{\text{areal}}(\gamma_1, \pi_1) \approx D^+(\gamma_1, \pi_1)$. This completes the proof of Theorem 1.

Note that since Theorem 1 is known to be true in the Gowdy case [15] (cf. also [5]), we will henceforth presume that the twist quantity K is *not* zero. Note also that in both areal and conformal coordinates, the constraint equations require that K be constant on spacetimes.

4. ANALYSIS IN THE CONTRACTING DIRECTION

As noted above in the outline of the proof of Theorem 1, our goal in this section is to show that, so long as the orbit area function R stays bounded away from zero, the past (contracting direction) development of (γ, π) in terms of conformal coordinates—the spacetime region we call $D^-_{\text{conf}}(\gamma, \pi)$—has $t \to -\infty$. Our argument for this begins by recalling the local existence result for T^2-symmetric solutions of the Einstein equations in conformal coordinate form. This result (see Lemma 4.2 in [5]) tells us that for any T^2-symmetric initial data (γ, π) on T^3, we can always find an interval (t_1, t_2) and real valued C^∞ functions $R, U, A, v, \underset{1}{G}, \underset{2}{G}, \underset{1}{M}$, and $\underset{2}{M}$ on $T^3 \times (t_1, t_2)$ such that (1) these functions satisfy the Einstein equations (2.8)–(2.10) in conformal coordinate form; and (2) for some $t_0 \in (t_1, t_2)$, the spacetime metric g constructed from $(R, U, A, v, \underset{1}{G}, \underset{2}{G}, \underset{1}{M}, \underset{2}{M})$ according to Eq. (2.7) induces initial data on the t_0-slice which is smoothly spatially diffeomorphic to (γ, π).

With this established, it follows as a consequence of standard long-time existence theorems from PDE theory (see, e.g., Theorems 2.1 and 2.2, and Corollaries 1 and 2 in Chapter 2 of [14]), that to show that these fields extend to $t \to -\infty$ as a solution of (2.8)–(2.10), it is sufficient to verify the following: For any globally hyperbolic subset of the (t, θ) cylinder on which they exist as a solution to (2.8)–(2.9), the functions (R, U, A, v) and their first and second derivatives are uniformly bounded.

So that is our task here: to establish these C^2 bounds. We do this in a series of steps:

Step 1 (Monotonicity of R and bounds on its first derivatives). A key first step here is the verification that the function R and its first derivatives are controlled. The argument for this control starts with Theorem 4.1 of [5], where it is shown that it follows from the constraint equations in conformal coordinate form (2.8) that ∇R is timelike (i.e., $g(\nabla R, \nabla R) < 0$). Note that if ∇R is timelike everywhere, one must have R_t nonzero everywhere. One could have $R_t > 0$ or $R_t < 0$, but our choice of time-orientation—the past corresponds to contracting T^2 orbits—dictates $R_t > 0$. Thus, along any past directed causal path, R monotonically decreases.

To show that the first derivatives of R are bounded along any such path, one uses the wave equation (2.9c) for R, which takes the form

$$\partial_\lambda R_\xi = \lambda \frac{e^{2v}}{4R^3} K^2, \tag{4.1a}$$

or equivalently,

$$\partial_\xi R_\lambda = \lambda \frac{e^{2v}}{4R^3} K^2, \tag{4.1b}$$

in terms of null[2] coordinates

$$\lambda = \frac{1}{\sqrt{2}}(t+\theta), \qquad \xi = \frac{1}{\sqrt{2}}(t-\theta). \tag{4.2}$$

Since the right-hand side of (4.1a) is positive, it follows that if we start at any point (θ_0, t_0) on the initial surface and follow back along the path $(\theta_0 + s,\ t_0 - s)$ generated by ∂_λ, then for any $s > 0$, we have

$$R_\xi(\theta_0 + s, t_0 - s) < R_\xi(\theta_0, t_0). \tag{4.3a}$$

Similarly, we obtain (from (4.1b))

$$R_\lambda(\theta - s, t_0 - s) < R_\lambda(\theta_0, t_0). \tag{4.3b}$$

While it does not follow that R_ξ and R_λ decrease with decreasing t for fixed θ_0, it *does* follow from (4.3) that R_ξ and R_λ are bounded into the past in the sense

$$R_\xi(\hat\theta, \hat t) < \max_{\theta \in S^1} R_\xi(\theta, t_0), \qquad R_\lambda(\hat\theta, \hat t) < \max_{\theta \in S^1} R_\lambda(\theta, t_0) \tag{4.4}$$

for any $\hat t < t_0$ and for any $\hat\theta$. Since $R_t = (1/\sqrt{2})(R_\xi + R_\lambda)$, we therefore have

$$R_t(\hat\theta, \hat t) < \max_{\theta \in S^1} (R_\xi + R_\lambda)(\theta, t_0) \tag{4.5}$$

for any $\hat t < t_0$ and for any $\hat\theta$.

There is no equivalent result for R_θ. But since ∇R is timelike, we have $|R_\theta| < |R_t|$ everywhere. Thus we find that $|R_\theta|$, as well as R_t, is bounded into the past, and we conclude that R is uniformly C^1-bounded to the past of the initial data surface.

Step 2 (Bounds on U, U_θ, T_t, and A, A_θ, A_t). The method we use to argue that quantities like U and A—which satisfy nonlinear coupled wave-type evolution equations—are controlled to the past of the initial data surface for all $t \in (\tau, t_0)$ is called the "light cone estimate" method [15]. The basic idea of this method is to first

[2] These are null coordinates for the two-dimensional base spacetime $(S^1 \times \mathbb{R}^+, {}^2g)$ with the conformally flat metric $g = e^{2(v-U)}(-dt^2 + d\theta^2)$.

show that the evolution equations for the quantities of interest—say U and A—imply that U and A satisfy equations of the form

$$n(E+P)=J, \tag{4.6a}$$

$$l(E-P)=L, \tag{4.6b}$$

where E and P are quadratic functions in the first derivatives of U and A, where J and L are composed of functions which are bounded on the region of interest, and possibly of first derivatives of U and A as well, and where n and l are a pair of independent null tangent vectors. One then formally integrates these Eqs. (4.6) along null paths generated by n and l; using Gromwall's lemma as in [5], one thereby derives estimates for the first derivatives of U and A for any value of $t \in (\tau, t_0)$ in terms of their values at t_0.

To obtain equations of the form (4.6) for U and A, it is useful to first rewrite their evolution equations in terms of a *wave map*. So we consider a base Lorentzian manifold $({}^2M_\tau, {}^2\eta)$ with the two-dimensional manifold ${}^2M_\tau$ corresponding to the past conformal coordinate development of (γ, π)—with coordinates (θ, t) in $S^1 \times \mathbb{R}$, or a subset thereof—and with the metric

$${}^2\eta := -dt^2 + d\theta^2; \tag{4.7a}$$

and we consider a family of target Riemannian manifolds $(\mathbb{R}, h_{(\theta, t)})$ with

$$h_{(\theta, t)} := R(\theta, t)\, dU^2 + \frac{e^{4U}}{4R(\theta, t)}\, dA^2. \tag{4.7b}$$

(Note the explicit (θ, t)-dependence of the target manifold metric; this makes our setup slightly different from a standard wave map, but the difference is easily handled.) The maps we consider take the form

$$\begin{aligned} &\Phi\colon\ {}^2M \to \mathbb{R}^2, \\ &(\theta, t) \mapsto \Phi(\theta, t) = (U(\theta, t)\ A(\theta, t)) \end{aligned} \tag{4.8}$$

with $h_{(\theta, t)}$ providing an inner product on their tangents; e.g.,

$$\langle \Phi_\theta, \Phi_t \rangle = h_{ab} \Phi^a_\theta \Phi^b_t = R U_\theta U_t + \frac{e^{4U}}{4R} A_\theta A_t. \tag{4.9}$$

There is a covariant derivative D compatible with ${}^2\eta$ and semi-compatible with $h(\theta, t)$. Using Greek indices for the base $(\mu \leftrightarrow \theta, t)$ and Latin indices for the target $(a \leftrightarrow U, A)$, we express the action of D as

$$D_\nu \Phi^a_\mu = \partial_\nu \Phi^a_\mu + \Gamma^a{}_{bc} \Phi^b_\mu \Phi^c_\nu - \Gamma^\lambda{}_{\mu\nu} \Phi^a_\lambda, \tag{4.10}$$

with the base Christoffel coefficients $\Gamma^{\lambda}{}_{\mu\nu}$ vanishing (in Section 6, we will work with a wave map for which these are not zero) and with the target Christoffel coefficients taking the values

$$\begin{aligned} &\Gamma^{U}{}_{UU}=0, \qquad \Gamma^{U}{}_{UA}=0, \qquad \Gamma^{U}{}_{AA}=-\frac{e^{4U}}{2R^2} \\ &\Gamma^{A}{}_{AA}=0, \qquad \Gamma^{A}{}_{AU}=2, \qquad \Gamma^{A}{}_{UU}=0. \end{aligned} \tag{4.11}$$

As noted, D is compatible with the flat metric ${}^2g$, but not with $h_{(\theta,t)}$, because of the explicit θ and t dependence (through R); we have

$$D_\mu h_{ab}=R_\mu\left(\delta^U_a\delta^U_b-\frac{e^{4U}}{4R^2}\,\delta^A_a\delta^A_b\right). \tag{4.12}$$

The covariant derivative D defines a wave operator $\Box := g^{\mu\nu}\,D_\mu\,D_\nu$ on our maps. Using this operator, the evolution equations (2.9a)–(2.9b) for U and A take the form

$$\Box U=\frac{U_tR_t}{R}-\frac{U_\theta R_\theta}{R}, \qquad \Box A=-\frac{A_tR_t}{R}+\frac{A_\theta R_\theta}{R} \tag{4.13}$$

which we may write jointly as

$$\Box\Phi^a=\psi^a, \tag{4.14a}$$

where

$$\psi^a=\begin{pmatrix} \dfrac{U_tR_t}{R}-\dfrac{U_\theta R_\theta}{R} \\ -\dfrac{A_tR_t}{R}+\dfrac{A_\theta R_\theta}{R} \end{pmatrix}. \tag{4.14b}$$

To derive equations of the form (4.6) from the wave equations (4.14), we now define an "energy-momentum tensor" for the maps Φ:

$$\begin{aligned} T_{\mu\nu} &:= \langle\Phi_\mu,\Phi_\nu\rangle-\frac{1}{2}\,g_{\mu\nu}g^{\alpha\beta}\langle\Phi_\alpha,\Phi_\beta\rangle \\ &=RU_\mu U_\nu+\frac{e^{4U}}{4R}\,A_\mu A_\nu+\frac{1}{2}\,g_{\mu\nu}\left[R(U_t^2-U_\theta^2)+\frac{e^{4U}}{4R}\,(A_t^2-A_\theta^2)\right]. \end{aligned} \tag{4.15}$$

One notes the components of $T_{\mu\nu}$,

$$T_{tt} = \frac{1}{2} R(U_t^2 + U_\theta^2) + \frac{e^{4U}}{8R}(A_t^2 + A_\theta^2), \tag{4.16a}$$

$$T_{\theta t} = RU_t U_\theta + \frac{e^{4U}}{4R} A_t A_\theta, \tag{4.16b}$$

$$T_{\theta\theta} = T_{tt}. \tag{4.16c}$$

T_{tt} will be our quantity E in (4.6) while $T_{t\theta}$ will be P. One also notes the formula for the covariant divergence of $T_{\mu\nu}$,

$$D_\nu T_\mu{}^\nu = \langle \partial_\mu \Phi, \psi \rangle + U_\mu U^\nu(\partial_\nu R) - \frac{1}{2} U^\alpha U_\alpha(\partial_\mu R) + \frac{e^{4U}}{4R}\left(\frac{1}{2} A^\alpha A_\alpha(\partial_\mu R) - A_\mu A^\nu(\partial_\nu R)\right), \tag{4.17}$$

where ψ is defined in (4.14b) and where all but the first term on the right-hand side of (4.17) appear because of the θ and t dependence of $h_{(\theta, t)}$ (see Eq. (4.12)).

Since the base metric ${}^2\eta$ is flat, one readily identifies null vectors and the corresponding null coordinates: We use λ and ξ as in (4.2) for the coordinates and $n = \partial_\xi$ and $l = \partial_\lambda$ for the corresponding vectors. It is now straightforward to show that

$$D_\nu T^\nu{}_n = -D_l T_{nn}, \qquad D_\nu T^\nu{}_l = -D_n T_{ll}. \tag{4.18}$$

Then combining (4.17) and (4.18), we obtain the formulas

$$\partial_\xi T_{\lambda\lambda} = n(T_{\lambda\lambda}) = -\frac{R_\lambda}{2\sqrt{2}}\left[(U_t^2 - U_\theta^2) + \frac{e^{4U}}{4R^2}(-A_t^2 + A_\theta^2)\right], \tag{4.19a}$$

$$\partial_\lambda T_{\xi\xi} = l(T_{\lambda\lambda}) = -\frac{R_\xi}{2\sqrt{2}}\left[(U_t^2 - U_\theta^2) + \frac{e^{4U}}{4R^2}(-A_t^2 + A_\theta^2)\right]. \tag{4.19b}$$

Now, since we note that $T_{\lambda\lambda} = T_{tt} + T_{t\theta}$ and $T_{\xi\xi} = T_{tt} - T_{t\theta}$ and since we note that the expressions on the right-hand side of (4.19) involve only terms quadratic in the first derivations of U and A, along with quantities (first derivatives of R) which are bounded in ${}^2M$, we see that if we set $E = T_{tt}$, $P = T_{t\theta}$, and $J =$ (the right-hand side of (4.19a)) and $L =$ (the right-hand side of (4.19b)), then (4.19) matches (4.6).

We now describe how one uses Eqs. (4.19) to obtain estimates for $E = T_{tt}$ at any $(\hat{\theta}, \hat{t}) \in {}^2M_\tau$, in terms of E and other controlled quantities on the initial surface. Such estimates control the derivatives of U and A in terms of the data at t_0.

We start by formally integrating both Eqs. (4.19)—in the form $\partial_\xi(E+P)=J$ and $\partial_\lambda(E-P)=L$ of (4.6)—along null paths which start at $(\hat\theta, \hat t)$ and end back on the initial data surface. Adding the results of these integrations, we have

$$\begin{aligned} E(\hat\theta, \hat t) = {} & E(\hat\theta + (t_0 - \hat t), t_0) + E(\hat\theta - (t_0 - \hat t), t_0) \\ & + P(\hat\theta + (t_0 - \hat t), t_0) - P(\hat\theta - (t_0 - \hat t), t_0) \\ & + \int_{t_0}^{\hat t} [\tau(\hat\theta + (s - \hat t), s) + L(\hat\theta - (s - \hat t), s)]\, ds. \end{aligned} \tag{4.20}$$

We next take supremums[3] over all values of the space coordinate θ on both sides of (4.20). Carefully noting the explicit forms of J and L (see Eqs. (4.19)) and using various straightforward inequalities, we obtain

$$\begin{aligned} \sup_\theta E(\theta, \hat t) \leqslant {} & 2 \sup_\theta E(\theta, t_0) + 2 \sup_\theta P(\theta, t_0) \\ & + \int_{t_0}^{\hat t} \beta(s) \sup_\theta E(\theta, s)\, ds, \end{aligned} \tag{4.21}$$

where $\beta(s)$ is a bounded function (related to the bounds of R_t and R_θ). We now apply Gromwall's lemma (see, e.g., Lemma 3.23 in [5]) to (4.21); we get the inequality

$$\sup_\theta E(\theta, \hat t) \leqslant [2 \sup_\theta E(\theta, t_0) + 2 \sup_\theta P(\theta, t_0)] \exp\left(\int_{t_0}^{\hat t} \beta(s)\, ds\right). \tag{4.22}$$

Since $\beta(t)$ and therefore $\exp(\int_{t_0}^{\hat t} \beta(s)\, ds)$ are bounded, (4.22) provides the desired bounds on $|U_\theta|$, $|U_t|$, $|(e^{2U}/2R)\, A_\theta|$ and $|(e^{2U}/2R)\, A_t|$ for all $t \in (\tau, t_0]$.

Once we have bounds on the first derivatives of U, it immediately follows (by integration over appropriate paths in ${}^2M_\tau$) that U is bounded for $t \in (\tau, t_0]$ as well. Then, so long as R is bounded away from zero on ${}^2M_\tau$, we obtain bounds on A_θ and A_t and consequently on A. We thus have uniform C^1 bounds on U and A in ${}^2M_\tau$, so long as R is bounded away from zero.

Step 3 (Bounds on v, v_θ, v_t). With the first derivatives of U and A bounded as argued in Step 2, it appears as if the evolution equation (2.9d) for v might imply that v and its first derivatives can be controlled by applying a light cone estimate. This cannot be done directly, however, because the last term on the right-hand side of Eq. (2.9d)—$\lambda(e^{2v}/4R^4)\, K^2$—is not controlled by previous estimates.

To handle this, we consider the quantity

$$\beta := v + \tfrac{3}{2} \ln R. \tag{4.23}$$

[3] The supremums over θ include all values for which the solution exists for a given value of t. For t_0, we have $\theta \in S^1$. For smaller values of t, we generally have only a portion of S^1.

Using the evolution equations for μ and R, we readily derive one for β:

$$\beta_{tt} - \beta_{\theta\theta} = U_\theta^2 - U_t^2 + \frac{e^{4U}}{4R^2}(A_t^2 - A_\theta^2) - \frac{3}{2R^2}(R_t^2 - R_\theta^2). \tag{4.24}$$

We could apply a light cone estimate argument (as in step 2) to this equation, but since the right-hand side of (4.24) contains nothing but controlled quantities (and does not contain β or its derivatives), we argue more simply as follows: We first use the null coordinates ξ and λ to rewrite (4.24) in the form

$$\partial_\lambda \beta_\xi = Z, \tag{4.25}$$

where Z is just the right-hand side of (4.24). Choosing an arbitrary point $(\hat{\theta}, \hat{t}) \in {}^2M_\tau$, we obtain $\beta_\xi(\hat{\theta}, \hat{t})$ by integrating up along the null path $(\hat{\theta} - s, \hat{t} + s)$ which ends at $(\hat{\theta} + \hat{t} - t_0, t_0)$. Since Z is bounded in ${}^2M_\tau$ (so long as R is bounded away from zero), β_ξ is bounded in ${}^2M_\tau$. Similarly we argue from the equation $\partial_\xi \beta_\lambda = Z$ that β_λ is bounded in ${}^2M_\tau$. Since $\beta_t = (1/\sqrt{2})(\beta_\lambda + \beta_\xi)$ and $\beta_\theta = (1/\sqrt{2})(\beta_\lambda - \beta_\xi)$, it follows that β_t and β_θ are bounded as well. But $v_t = \beta_t - (3/2)(R_t/R)$ and $v_\theta = \beta_\theta - (3/2)(R_\theta/R)$, so we have shown that the first derivatives of v—and consequently v itself—must be uniformly bounded for $t \in (\tau, t_0]$. We now have uniform C^1 bounds on all of the principal quantities.

Step 4 (Bounds on second derivatives of R). The constraint equations (2.8a) and (2.8b) allow one to express $R_{\theta\theta}$, as well as $R_{\theta t}$, algebraically as functions of R, U, A, and v, and their first derivatives. It follows immediately that $R_{\theta\theta}$ and $R_{\theta t}$ are bounded, so long as R is bounded away from zero. It then follows from the wave equation (2.9c) for R that $R_{tt} = R_{\theta\theta} + \lambda(e^{2v}/2R^3)\, K^2$ is also bounded, so long as R is bounded away from zero.

Step 5 (Bounds on second derivatives of U, A, v). It should be clear from the forms of the evolution equations (2.9a) and (2.9b) for U and A that if we take time derivatives of both of these equations we obtain evolution equations for U_t and A_t which are of the appropriate form for applying light cone estimates. We thereby obtain uniform bounds on U_{tt}, $U_{t\theta}$, A_{tt}, and $A_{t\theta}$. Bounds on $U_{\theta\theta}$ and $A_{\theta\theta}$ then follow from (2.9a) and (2.9b) directly.

Similarly, if we take the time derivative of Eq. (2.9c), the evolution equation for v, we obtain a wave equation for v_t to which the analysis described in Step 3 can be applied, giving us bounds on v_{tt} and $v_{t\theta}$ (note that for these C^2 bounds on v, we can work directly with v rather having to work with β). We then bound $v_{\theta\theta}$ using Eq. (2.9d).

We could continue to obtain bounds on higher derivatives of these quantities (along with R); however, to apply the global existence theorem cited in [14], C^2 bounds are sufficient. Thus we have shown that so long as R stays bounded away from zero, the functions R, U, A, and v extend (as solutions of Eqs. (2.8)–(2.9)) to $t \to -\infty$.

Step 6 (Extension of the shift functions). We still need to show that the "shift" functions $\underset{1}{G}$, $\underset{2}{G}$, $\underset{1}{M}$, and $\underset{2}{M}$ extend to $D^-_{\text{conf}}(\gamma, \pi)$ (which we have defined to be the maximal subset of $T^3 \times \mathbb{R}$ on which the functions (R, U, A, v) are solutions of (2.8)–(2.9), with the corresponding spacetime metric assuming initial data (γ, π)). The only equations in which they appear are the auxiliary constraint equations (2.10). Recall that in the Cauchy problem for the Einstein equations, the "shift" functions have a gauge character corresponding to the freedom of propagating the coordinate system from the initial value hypersurface to the spacetime. In the gauges presented above, that freedom of propagating the coordinates x^a still persists. So for an arbitrary choice throughout $D^-_{\text{conf}}(\gamma, \pi)$ of the "shift" functions $\underset{1}{M}(\theta, t)$ and $\underset{2}{M}(\theta, t)$, and for any initial data for $\underset{1}{G}$ and $\underset{2}{G}$, we can integrate (2.10) in time to obtain

$$\underset{1}{G}(\theta, t_1) = \int_{t_0}^{t_1} \left[-\frac{Ae^{2v}K}{R^3} + \underset{1}{M}_\theta \right] (\theta, \tau)\, d\tau + \underset{1}{G}(\theta, t_0), \tag{4.26a}$$

$$\underset{2}{G}(\theta, t_1) = \int_{t_0}^{t_1} \left[\frac{e^{2v}K}{R^3} + \underset{2}{M}_\theta \right] (\theta, \tau)\, d\tau + \underset{2}{G}(\theta, t_0). \tag{4.26b}$$

We now have a solution of the full set of Einstein Eqs. (2.8)–(2.10) throughout $D^-_{\text{conf}}(\gamma, \pi)$. We shall show in Section 5 that this set is the maximal past development of our chosen T^2-symmetric initial data.

5. SOME GEOMETRIC RESULTS CONCERNING R

In Section 4, we have shown that if R stays bounded away from zero, then $D^-_{\text{conf}}(\gamma, \pi)$ has $t \to -\infty$. In this section, we prove a collection of geometric results which allow us to conclude that, whether or not R stays bounded away from zero, $D^-_{\text{conf}}(\gamma, \pi)$ admits a foliation by areal coordinates and also covers the past maximal globally hyperbolic development $D^-(\gamma, \pi)$ of (γ, π). The first of these results concerns zeroes of R.

PROPOSITION 1. *Let (M, g) be a globally hyperbolic development of T^2-symmetric initial data. The T^2 orbital area function R is positive everywhere in (M, g).*

Proof. The proof of this proposition depends upon two lemmas, which have wider application than our T^2-symmetric case.

LEMMA 1.1. *Let (M, g) be a globally hyperbolic development of initial data (Σ^3, γ, π). Assume that the initial data is invariant under the action of a compact group G. Let X be any Killing vector field on (M, g) which is generated by the action of G on (M, g). Then wherever X is nonvanishing, it is spacelike.*

Proof of Lemma 1.1. In [7], it is shown that, indeed, the action of the group G extends from the initial data hypersurface to the spacetime. So (M, g) is G-invariant, and at least the initial hypersurface is invariant under the action. We now construct a foliation of (M, g) by spacelike hypersurfaces which is also G-invariant.

Let t be any time function on the globally hyperbolic spacetime (M, g). Then we may define a new function on M

$$\hat{t} := \int_G t \cdot \phi_g \, d\mu_g, \tag{5.1}$$

where ϕ_g denotes the action of G on M and $d\mu_g$ is the Haar measure. Since the spacetime metric is invariant under G, we verify that $\hat{t}$ is also a time function and its level hypersurfaces $\Sigma_{\hat{t}}$ are Cauchy surfaces. It follows from the definition of $\hat{t}$ that the hypersurfaces $\Sigma_{\hat{t}}$ are invariant under G. Hence, the Killing vector fields generated by G are tangent to the spacelike hypersurfaces $\Sigma_{\hat{t}}$. The result follows. ∎

LEMMA 1.2. *Let (M, g) be a globally hyperbolic spacetime, and let ϕ_λ be a one-parameter group of isometries of (M, g) which leaves a particular Cauchy surface Σ_0 invariant. Let $X = (d/d\lambda)\, \phi_\lambda|_{\lambda=0}$ be the Killing vector field which generates ϕ_λ. If for some point $p \in \Sigma_0$ one has $X(p) = 0$, then on every Cauchy surface $\hat{\Sigma}$ in M there exists a point $\hat{p} \in \hat{\Sigma}$ such that $X(\hat{p}) = 0$.*

Proof of Lemma 1.2. Let $\Gamma(s)$ be a maximally extended, affinely parametrized timelike geodesic such that $\Gamma(0) = p$ and $(d/ds)\, \Gamma(0)$ is a unit-length vector normal to Σ_0. Since Σ_0 is invariant under ϕ_t, the unit-length vector field $e_\perp(0)$ normal to Σ_0 is also invariant under ϕ_t; it follows that $\phi_\lambda(\Gamma) = \Gamma$. Since the affine parameter is invariant under ϕ_λ, we further have $\phi_\lambda(\Gamma(s)) = \Gamma(s)$ for all s and λ.

We now pick a Cauchy surface $\hat{\Sigma}$. Since $\Gamma(s)$ is an inextendible timelike path, it must intersect $\hat{\Sigma}$ at some point $\hat{p} = \Gamma(\hat{s})$. But

$$\phi_\lambda(\hat{p}) = \phi_\lambda(\Gamma(\hat{s})) = \Gamma(\hat{s}) = \hat{p}, \tag{5.2}$$

so we have $X(\hat{p}) = 0$. ∎

Note that Lemma 1.2 tells us that if a spacetime Killing field tangent to a particular Cauchy surface in a globally hyperbolic spacetime (M, g) has no zeroes on that surface, then it has no zeroes anywhere.

Proof of Proposition 1. Let (Σ_0, γ, π) be the given initial surface and initial data for the spacetime Killing vector fields which generate the T^2 group action on Σ_0. By definition, X and Y are nonzero everywhere on Σ_0, and nonparallel everywhere on Σ_0 as well. It also follows from the definition of areal coordinates and the orbital area function R that we have

$$\begin{aligned} R^2 &= \lambda^{-2} \det \begin{pmatrix} g(X, X) & g(X, Y) \\ g(Y, X) & g(Y, Y) \end{pmatrix} \\ &= \lambda^{-2}(g(X, X)\, g(Y, Y) - g^2(X, Y)). \end{aligned} \tag{5.3}$$

Note that on Σ_0,

$$R^2|_{\Sigma_0} = \lambda^{-2}(\gamma(X, X)\,\gamma(Y, Y) - \gamma^2(X, Y)), \tag{5.4}$$

and, since X and Y have no zeroes and are nowhere parallel on Σ_0, R^2 has no zeroes on Σ_0.

Let us suppose that for some point $p \in M$, $R(p) = 0$. It follows from Lemma 1.1 that there exists a Cauchy surface Σ_p such that X and Y are tangent to Σ_p, and so we have

$$0 = R^2(p) = \gamma_p(X, X)\,\gamma_p(Y, Y) - \gamma_p^2(X, Y), \tag{5.5}$$

where γ_p is the induced spatial metric on Σ_p. From (5.5) we see that it must be true that either X or Y is zero at p, or X is parallel to Y at p. By Lemma 1.2, the first possibility cannot be true. The second possibility is also ruled out by Lemma 1.2; we argue this as follows: If X were parallel to Y at p, then we would have $X(p) = \beta Y(p)$. We now consider the Killing fields X and $Z = X - \beta Y$. On Σ_0, X and Z have no zeroes, and are nonparallel. But at p, $Z(p) = X(p) - \beta Y(p) = 0$, which contradicts Lemma 1.2.

We conclude that R^2 and, hence, R have no zeroes. Since R is continuous and since we choose the convention $R = +\sqrt{\det g}$, we have $R > 0$ everywhere. ∎

Our next result concerns limits of R along past directed paths in $D^-_{\text{conf}}(\gamma, \pi)$, which we recall is the maximal globally hyperbolic past development of (γ, π), in which the conformal components of the metric $(R, U, A, v, \underset{1}{M}, \underset{2}{M}, \underset{1}{G}, \underset{2}{G})$ exist as a solution of the Einstein equations (2.8)–(2.10).

PROPOSITION 2. *For any choice of T^2-symmetric initial data (γ, π), there exists a unique nonnegative number R_0 such that every past inextendible causal path $\Gamma\colon (-\infty, s_0) \to D^-_{\text{conf}}(\gamma, \pi)$ satisfies*

$$\lim_{s \to -\infty} R \circ \Gamma(s) = R_0. \tag{5.6}$$

Proof. There are two cases to consider, depending upon whether $D^-_{\text{conf}}(\gamma, \pi) \backslash \Sigma_0$ has a past boundary or not.

We first assume that there is no such boundary. It then follows from the discussion of Section 4 that $D^-_{\text{conf}}(\gamma, \pi) = S^1 \times T^2 \times (-\infty, t_0]$, with the conformal coordinates (θ, x, y, t) covering $D^-_{\text{conf}}(\gamma, \pi)$.

Now consider the level sets of R in $D^-_{\text{conf}}(\gamma, \pi)$:

$$\Sigma_\rho := \{(\theta, x, y, t) \mid R(\theta, t) = \rho, \text{ with } \rho < \inf_{\Sigma_0} R\}. \tag{5.7}$$

We wish to show:

LEMMA 2.1. *The sets Σ_ρ, if nonempty, are Cauchy surfaces.*

Proof of Lemma 2.1. Since the function R is smooth and since ∇R is timelike (see Step 1 of Section 4), it follows that the sets Σ_ρ are smooth, embedded, achronal, spacelike submanifolds of $D^-_{\text{conf}}(\gamma, \pi)$. Since $\rho < \inf_{\Sigma_0} R$, the sets Σ_ρ have no boundary. Thus if we can show that each set Σ_ρ is compact, then it follows from [1] that the Σ_ρ's are Cauchy surfaces.

Since they are closed in $D^-_{\text{conf}}(\gamma, \pi)$, it suffices to show that each Σ_ρ is bounded. To do this, we note that, since a given Σ_ρ is spacelike, we can describe it as the graph of a function:

$$\Sigma_\rho = \{(\theta, x, y, f(\theta)) \mid \theta \in S^1, (x, y) \in T^2\}. \tag{5.8}$$

Now every vector tangent to Σ_ρ must be spacelike. Considering the tangent vector

$$W = f'\partial_t + \partial_\theta + W_x \partial_x + W_y \partial_y, \tag{5.9a}$$

where

$$W_x = -\underset{1}{G} - \underset{1}{M} f', \qquad W_y = -\underset{2}{G} - \underset{2}{M} f', \tag{5.9b}$$

we see that W is spacelike—i.e., $g(W, W) > 0$ for g from (2.7)—only if $|f'| < 1$. But if $|f'| < 1$, then since the range of θ is 2π, we find that the range of f is less than or equal to π; that is,

$$\sup_{\Sigma_\rho} t - \inf_{\Sigma_\rho} t \leqslant \pi. \tag{5.10}$$

This condition bounds Σ_ρ, so it must be compact and, consequently, it must be a Cauchy surface. ∎

Now consider a past inextendible causal path $\Gamma(s)$ with s, say, in $(-\infty, 0]$. Since R monotonically decreases with decreasing t and since $R > 0$, it follows that $\lim_{s \to -\infty} R \circ \Gamma(s)$ exists; we shall call it R_0. If we consider another such path $\hat{\Gamma}$, we also have $\lim_{s \to -\infty} R \circ \hat{\Gamma}(s)$ existing; we call it $\hat{R}_0$.

Let us assume that $R_0 \neq \hat{R}_0$, without loss of generality, say $R_0 < \hat{R}_0$. Then for some $p \in D^-_{\text{conf}}$, we have $R(p) = \frac{1}{2}(R_0 + \hat{R}_0) < \hat{R}_0$. It follows from Lemma 2.1 that $\Sigma_{1/2(R_0 + \hat{R}_0)}$ is a Cauchy surface. But then $\hat{\Gamma}$ must intersect $\Sigma_{1/2(R_0 + \hat{R}_0)}$. This is a contradiction, since we have $R \circ \hat{\Gamma}(s) > \hat{R}_0$ for all s. So we must have $R_0 = \hat{R}_0$. This shows that, in this first case, R has the same limit along all past inextendible causal paths.

We now consider the case in which $D^-_{\text{conf}}(\gamma, \pi)$ corresponds to a proper subset of $S^1 \times T^2 \times (-\infty, t_0]$. Since the spacetime D^-_{conf} is globally hyperbolic, we know from [17] that the boundary, which we shall call Ω, is a closed achronal set. We can also

argue that Ω is bounded and, hence, compact. To argue this, we construct a smooth, T^2-invariant time function τ which goes to zero as one approaches the boundary. (Such a time function exists, since we know from [12] that one can construct a smooth time function $\hat{\tau}$ on D^-_{conf} which approaches $-\infty$ near Ω; one can then average $\hat{\tau}$ along T^2 and set $\tau = \exp(\hat{\tau})$.) So Ω is the limit of $\tau = \text{constant}$ Cauchy surfaces in D^-_{conf}. Now arguing as in Lemma 2.1, we can show that for any $\tau = \text{constant}$ Cauchy surface Σ_τ, one has a result equivalent to (5.10):

$$\sup_{\Sigma_\tau} t - \inf_{\Sigma_\tau} t \leqslant \pi, \tag{5.11}$$

where t is the conformal coordinate time. It follows by continuity in τ that Ω must be bounded; compactness follows.

The Whitney extension theorem allows one to extend the function R past D^-_{conf}; in particular, it guarantees the existence of a Lipschitz continuous function $\bar{R}$ on $S^1 \times T^2 \times (-\infty, t_0)$ for which $\bar{R}|_{D^-_{\text{conf}}} = R$. (Alternatively, uniform continuity of R in a neighbourhood of the boundary implies that R has a continuous extension to that boundary.) Now let us set $\sup_\Omega \bar{R} = R_0$. The compactness of Ω guarantees that R_0 is attained on Ω. If $R_0 = 0$, then it follows from Proposition 1 and continuity that $\bar{R}|_\Omega = 0$. Hence, since all past inextendible paths in D^-_{conf} must approach Ω, R must approach the same value—zero—along every such path.

It remains to show that $R_0 \neq 0$ cannot occur. We will show this by arguing that if it did occur, then one could extend the conformal coordinate solution of (2.8)–(2.10) into the past of D^-_{conf}, which by presumption cannot be done.

So we presume that $R_0 \neq 0$, and we consider the set of points

$$I := \{(\theta, x, y, t) \in \Omega \mid \bar{R}(\theta, x, y, t) = R_0\} \tag{5.12}$$

which is nonempty since Ω is compact. Letting ξ denote the sup of the time function t on I, we locate a point $p \in I$ with $t(p) = \xi$. Such a point exists since I is compact, and we label its coordinates $(\hat{\theta}, \hat{x}, \hat{y}, \hat{\xi}) \leftrightarrow p$.

Now it follows from the definition of p that for sufficiently small $\varepsilon > 0$, the set

$$S_{(p,\varepsilon)} := \{(\theta, x, y, \xi) \mid \hat{\theta} - \varepsilon < \theta < \hat{\theta} + \varepsilon,\ (x, y) \in T^2\} \tag{5.13}$$

is a well-defined subset of $D^-_{\text{conf}}(\gamma, \pi) \cap \Omega$ which forms a spacelike submanifold and has $R \geqslant (R_0/2)$. Thus, based on our estimates in Section 4 with R bounded away from zero, we can smoothly extend all of the fields (R, U, A, v, $\underset{1}{G}$, $\underset{2}{G}$, $\underset{1}{M}$, and $\underset{2}{M}$) to $S_{(p,\varepsilon)}$ and set up a well-posed initial value problem on $S_{(p,\varepsilon)}$. Standard existence results (see, e.g., [3]) then allow us to evolve the fields to the past of p. This contradicts the definition of D^-_{conf}, and thus tells us that $R_0 = 0$, completing the proof of Proposition 2. ∎

With the uniform limits of the function R along all past inextendible causal paths in $D^-_{\text{conf}}(\gamma, \pi)$ established, one easily proves the following key part of the proof of Theorem 1.

PROPOSITION 3. *Let (Σ_0, γ, π) be T^2-symmetric initial data. Define R_1 as $\inf_{\Sigma_0} R$, and R_0 as the past limit of R along past inextendible paths in $D^-_{\text{conf}}(\gamma, \pi)$. For every $\rho \in (R_0, R_1)$, the $R = \rho$ level set Σ_ρ (see Eq. (5.7)) is a Cauchy surface, and these Σ_ρ foliate the spacetime region $D^-_{\text{conf}}(\gamma, \pi) \cap I^-(\Sigma_{R_1})$. Further, this spacetime region admits areal coordinates.*

Proof. The proof that $\Sigma_\rho \subset D^-_{\text{conf}}(\gamma, \pi) \cap I^-(\Sigma_{R_1})$ is a Cauchy surface is essentially that given in Lemma 2.1 for a slightly more restricted situation. The generalization here is that $D^-_{\text{conf}}(\gamma, \pi)$ may have a boundary. However, as shown in Proposition 2, if the boundary Ω exists, then all past inextendible causal paths must approach it, with R approaching 0 on Ω. Hence for $R = \rho > 0$ in this case, Σ_ρ is bounded away from the boundary. Compactness then follows; hence as a consequence of [1], Σ_ρ is a Cauchy surface.

Since R is smooth and monotonically decreasing along past causal paths, all values of R between R_0 and R_1 are realized in order. Thus we verify that $D^-_{\text{conf}}(\gamma, \pi) \cap I^-(\Sigma_{R_1})$ is foliated by the Σ_R.

Once we have an $R = \text{const}$ foliation, it readily follows from arguments of the form surrounding equation (4.15) in [5] that the region $D^-_{\text{conf}}(\gamma, \pi) \cap I^-(\Sigma_{R_1})$ admits areal coordinates with the metric taking the form (2.3) and satisfying (2.4)–(2.6). ∎

We now know that $D^-_{\text{conf}}(\gamma, \pi) \cap I^-(\Sigma_{R_1})$ admits an R foliation and areal coordinates. To finish the proof of Theorem 1 for the contracting direction, it remains to show that $D^-_{\text{conf}}(\gamma, \pi)$ covers the past maximal globally hyperbolic development $D^-(\gamma, \pi)$ of the initial data. We show this separately for two different cases: First, in the case that $t \to -\infty$ (and $R_0 \neq 0$), and then in the case that $R_0 = 0$.

PROPOSITION 4. *If $D^-_{\text{conf}}(\gamma, \pi)$ has $t \to -\infty$, then $D^-_{\text{conf}}(\gamma, \pi)$ is diffeomorphic to $D^-(\gamma, \pi)$.*

Proof. Let us for convenience designate the conformal coordinate spacetime region $D^-_{\text{conf}}(\gamma, \pi)$ as (M, g), and the maximal past development as $(\hat{M}, \hat{g})$. We now suppose that (M, g) is *not* diffeomorphic to $(\hat{M}, \hat{g})$; then there must be an isometric embedding

$$\psi : M \to \hat{M} \tag{5.14}$$

and $\psi(M)$ must have a nonempty boundary $\partial\psi(M)$ in M. Note that it follows from our choice of time convention that $\partial\psi(M)$ is a past boundary of $\psi(M)$ in M.

Since the spacetimes M and $\hat{M}$, as well as the embedding ψ, are all T^2-invariant, those orbits of the action of T^2 which intersect $\psi(M)$ are contained in $\psi(M)$. We consider a point $p \in \partial\psi(M)$ and its corresponding T^2 orbit $\mathcal{O}_p \subset \partial\psi(M)$. The main idea of the proof is to obtain a contradiction regarding the causal future $J^+(\mathcal{O}_p)$ of this orbit and its intersection with nearby Cauchy surfaces in $\psi(M)$.

We work with two families of Cauchy surfaces. One family is Σ_t, where the Σ_t's are level sets of the conformal time coordinate t in $\psi(M)$. The other family is given by $\Sigma_{\hat{t}}$, where the $\Sigma_{\hat{t}}$'s are level sets of any T^2-invariant time coordinate $\hat{t}$ in $\hat{M}$ (see Lemma 1.1). Note that since both time functions are T^2-invariant, we have $t(r)=t(p)$ and $\hat{t}(r)=\hat{t}(p)$ for all $r \in \mathcal{O}_p$, so that we can refer unambiguously to $t(\mathcal{O}_p)$ and $\hat{t}(\mathcal{O}_p)$.

We use the $\hat{t}$ time function to prove the following.

LEMMA 4.1. (a) *There exists a a sufficiently small* $\varepsilon > 0$ *such that for* $|\hat{t}-\hat{t}(\mathcal{O}_p)| < \varepsilon$ *we have* $J^+(\mathcal{O}_p) \cap \Sigma_{\hat{t}} \neq \Sigma_{\hat{t}}$.

(b) *For any Cauchy surface* $\Sigma \subset J^-(\Sigma_{\hat{t}_1}) \cap \psi(M)$ *such that* $|\hat{t}_1 - \hat{t}(\mathcal{O}_p)| < \varepsilon$, *one has* $J^+(\mathcal{O}_p) \cap \Sigma \neq \Sigma$.

Proof of Lemma 4.1. Consider a point w which is contained in the Cauchy surface $\Sigma_{\hat{t}(\mathcal{O}_p)}$, but is not contained in the orbit $\mathcal{O}_p$. It follows from T^2-invariance that $\mathcal{O}_p \cap \mathcal{O}_w$ is empty, and it follows from the acausality of $\Sigma_{\hat{t}(\mathcal{O}_p)}$ that no path between a point in $\mathcal{O}_p$ and a point in $\mathcal{O}_w$ is causal.

Using the $\hat{t}$ function we can construct a neighbourhood of $\Sigma_{\hat{t}(\mathcal{O}_p)}$ which is diffeomorphic to $\Sigma_{\hat{t}(\mathcal{O}_p)} \times (t(\mathcal{O}_p)-\delta,\ t(\mathcal{O}_p)+\delta)$ for some $\delta > 0$. Let t_k be a monotonically decreasing sequence converging to $t(\mathcal{O}_p)$ and let $w_k = (w, t_k)$. If the lemma were false, then for every value of k there would be a causal path γ_k from w_k to a point in $\mathcal{O}_p$. By global hyperbolicity there exists a causal path γ at which a subsequence of the γ_k's accumulates. It follows that γ is a causal path between $\mathcal{O}_p$ and $\mathcal{O}_w$, so we have a contradiction, and point (a) follows.

To establish point (b), let us fix a value $\hat{t}_1$ such that $J^+(\mathcal{O}_p) \cap \Sigma_{\hat{t}_1} \neq \Sigma_{\hat{t}_1}$, and let Σ be any Cauchy surface for which $\Sigma \subset J^-(\Sigma_{\hat{t}_1}) \cap \psi(M)$. Through every point $r \in \Sigma_{\hat{t}_1}$ there is a past–directed causal path which intersects Σ. Hence, if $J^+(\mathcal{O}_p) \cap \Sigma = \Sigma$, then there would be a past-directed causal path from every point in $\Sigma_{\hat{t}_1}$ to a point in $\mathcal{O}_p$. But we know this is false, so we must have $J^+(\mathcal{O}_p) \cap \Sigma \neq \Sigma$. ∎

We now use the conformal time t to show that for all Cauchy surfaces Σ contained in $J^-(\Sigma_{\hat{t}_1}) \cap \psi(M)$ we actually have $J^+(\mathcal{O}_p) \cap \Sigma = \Sigma$, thus contradicting Lemma 4.1 if we assume that M is extendible. To do this, we first note that for any Cauchy surface Σ which intersects $\psi(M)$, it must be true that Σ is entirely contained in $\psi(M)$, and also one must have

$$\sup_{\Sigma} t - \inf_{\Sigma} t \leqslant \pi, \tag{5.15}$$

where t is the conformal coordinate. The proof of these two claims is essentially a local version of the proof of Lemma 2.1. That is, starting at a point p in the intersection of Σ with $\psi(M)$, we write Σ locally as a graph of a function $f(\theta)$ (see Eq. (5.8)), and then note that the spacelike character of Σ implies that $|f'| < 1$. Since the conformal coordinate time t has a finite value at p and since $t \to -\infty$ in $\psi(M)$, it follows from $|f'| < 1$ and from the conformal coordinate metric form (2.7)

that t is bounded on Σ and therefore Σ cannot escape $\psi(M)$. Then choosing the graph representation (5.8) of Σ globally, one verifies (5.15) as in Lemma 2.1.

We now consider a point q contained in the past set $J^-(\Sigma, \psi(M))$ such that

$$\sup_{\Sigma} t - t(q) \geqslant \pi. \tag{5.16a}$$

Such a point exists, since the compact set Σ has finite t and $t \to -\infty$ as one approaches $\partial\psi(M)$. Moreover, since $t(\mathcal{O}_q) = t(q)$, we have

$$\sup_{\Sigma} t - t(\mathcal{O}_q) \geqslant \pi. \tag{5.16b}$$

We claim that for any q and corresponding $\mathcal{O}_q$, the set $J^+(\mathcal{O}_q) \cap \Sigma$ is equal to Σ. To show this, let us label the coordinates of q as $(\theta_0, x_0, y_0, t_0)$, so that (θ_0, t_0) labels the orbits $\mathcal{O}_q$ as we vary q, and let us consider the one-parameter family, labeled by α, of paths, labeled by s, of T^2 orbits $\{\underset{[\alpha]}{\Gamma}(s) \mid \alpha \in [-1, +1], s > 0\}$ given by

$$\underset{[\alpha]}{\Gamma}(s) = \{(\theta_0 + \alpha s, x, y, t_0 + s),\ x, y \in T^2\}. \tag{5.17a}$$

We note two important features of this family of paths of T^2 orbits: First, the set of all points in these orbits is contained in the future set $J^+(\mathcal{O}_q)$. Second,

$$\begin{aligned} \underset{[\alpha]}{S} &= \{\underset{[\alpha]}{\Gamma}(\pi) \mid \alpha \in [-1, +1]\} \\ &= \{\underset{[\alpha]}{\theta} = \theta_0 + \alpha\pi, (x, y) \in T^2, \underset{[\alpha]}{t} = t_0 + \pi \mid \alpha \in [-1, +1]\} \\ &= \Sigma_{t_0 + \pi}. \end{aligned} \tag{5.18}$$

Hence we have $J^+(\mathcal{O}_q) \cap \Sigma = \Sigma$.

To finish the proof of Proposition 4, we consider the Cauchy surface $\Sigma_{\hat{t}_1}$, defined in Lemma 4.1. Since $\Sigma_{\hat{t}_1}$ intersects $\psi(M)$, it must be contained in $\psi(M)$ and, hence, $T := \inf_{\Sigma_{\hat{t}_1}} t$ is finite. It then follows from Lemma 4.1 that for $t < T$ we have $J^+(\mathcal{O}_p) \cap \Sigma_t \neq \Sigma_t$.

Now let us consider a sequence of points p_i such that $p_i \in J^-(\Sigma_{\hat{t}_1}) \cap \psi(M)$, $p_{i+1} \in I^-(p_i)$, $p_i \in J^+(p)$, and $p_i \to p$, where p is the point we have chosen on $\partial\psi(M)$. Since the points p_i do not have an accumulation point in $\psi(M)$ we have $\lim_{i \to \infty} t(p_i) = -\infty$. This tells us that for any fixed i there is some i_π such that if $j > i_\pi$ then $t(p_j) < t(p_i) - \pi$. It then follows, as shown above, that $J^+(\mathcal{O}_{p_j}) \cap \Sigma_{t_{p_i}} = \Sigma_{t_{p_i}}$. Since $t(p_i) < T$, we have a contradiction, from which it follows that $\psi(M) = \hat{M}$. ∎

Proposition 4 shows that $D^-_{\text{conf}}(\gamma, \pi) \approx D^-(\gamma, \pi)$ if $t \to -\infty$ in D^-_{conf}. To show that $D^-_{\text{conf}}(\gamma, \pi) \approx D^-(\gamma, \pi)$ if $R_0 = 0$, we rely on a more general result, part of which we will need to complete the proof for the expanding direction.

PROPOSITION 5. *Let (M, g) be any globally hyperbolic development of T^2-symmetric initial data. If $R \to 0$ along every past inextendible causal path, then (M, g) covers the past maximal development of (γ, π). If $R \to \infty$ along every future inextendible causal path, then (M, g) covers the future maximal development of (γ, π).*

Proof. If (M, g) is not maximal to the past, then there exists a globally hyperbolic spacetime $(\hat{M}, \hat{g})$, with a smooth proper embedding $\psi(M) \subset \hat{M}$, and with $\partial\psi(M)$ containing a portion $\partial^-\psi(M)$ to the past of $\psi(M)$. Let $p \in \partial^-\psi(M)$. If $R \to 0$ along every past inextendible causal path, then $R(p) = 0$. This disagrees with Proposition 1, so (M, g) must be maximal to the past.

If (M, g) is not maximal to the future, then similarly there is a spacetime $(\check{M}, \check{g})$ with a smooth proper embedding $\psi(M) \subset \check{M}$, and with a portion $\partial^+\psi(M)$ of the boundary to the future of $\psi(M)$. Letting $q \in \partial^+\psi(M)$, we see that if $R \to \infty$ along every future inextendible causal path, then R blows up near q. But since the Killing vector fields in $(\check{M}, \check{g})$ are at least C^1 everywhere, it follows that R must be bounded everywhere. Hence (M, g) must be maximal to the future. ∎

6. ANALYSIS IN THE EXPANDING DIRECTION

The proof of Theorem 1 for the expanding—R increasing—direction is more direct than for the contracting direction, since we work directly with the areal coordinate components of the metric (see Eq. (2.3)) To do this, we need to start with data on an $R = \text{const}$ Cauchy surface. Let R_0 and R_1 be as in Proposition 3; then as shown in Sections 4–5, such surfaces exist for all $R \in (R_0, R_1)$—they lie to the past of the initial surface Σ_0 with data (γ, π) from the hypotheses of Theorem 1. Let us pick one such surface with, say, $R = R_2$. The spacetime $D^-(\gamma, \pi)$ induces initial data for the areal component fields $(U, A, v, \alpha, \underset{1}{G}, \underset{2}{G}, \underset{2}{M}, \underset{1}{M})$ on $\Sigma_{t_2 = R_2}$, and we have local existence for the initial-value problem for these fields. To prove global existence—i.e., to show that we can evolve the fields $(U, A, v, \alpha, \underset{1}{G}, \underset{2}{G}, \underset{2}{M}, \underset{1}{M})$ via the Einstein equations (2.4)–(2.6) to $t \to \infty$—what we need to do (as shown in [14]) is to prove that for any finite interval $[t_2, T)$ on which they exist as a solution to (2.4)–(2.6) these functions are uniformly C^2 bounded. Again, we do this in a series of steps.

Step 1 (Bounds on α, v, and α_t). As in Section 4, we use light cone estimates here to establish the bounds we need for the fields U and A and their derivatives. However, since the wave equations (2.5a) and (2.5b) for U and A involve α and its

derivatives and since the constraint equation (2.4c) for α involves v, we first need to bound these quantities. The first step towards doing this is an energy monotonicity result:

Let us define

$$\mathscr{E}(t) := \int_{S^1} \left[\alpha^{-1/2} U_t^2 + \alpha^{1/2} U_\theta^2 + \frac{e^{4U}}{4t^2} (\alpha^{-1/2} A_t^2 + \alpha^{1/2} A_\theta^2) \right] d\theta. \tag{6.1}$$

Using Eqs. (2.4)–(2.6), and integration by parts, we calculate

$$\begin{aligned} \frac{d}{dt} \mathscr{E} = & \frac{-K^2}{t^3} \int_{S^1} \frac{1}{2\alpha} \left[\alpha^{-1/2} U_t^2 + \alpha^{1/2} U_\theta^2 + \frac{e^{4U}}{4t^2} (\alpha^{-1/2} A_t^2 + \alpha^{1/2} A_\theta^2) \right] e^{2v} \alpha^2 \, d\theta \\ & - \frac{1}{2t^3} \int_{S^1} (e^{4U} \alpha^{1/2} A_\theta^2) \, d\theta - \frac{2}{t} \int_{S^1} (\alpha^{-1/2} U_t^2) \, d\theta < 0. \end{aligned} \tag{6.2}$$

This shows that $\mathscr{E}(t)$ decreases monotonically in t. So in particular, we have

$$\mathscr{E}(t) < \mathscr{E}(t_2) \tag{6.3}$$

for all $t > t_2$.

Now we consider the quantity

$$\tilde{v} := -v - \tfrac{1}{2} \ln \alpha. \tag{6.4}$$

The spatial derivative of $\tilde{v}$, as a consequence of the constraint (2.4b), is given by

$$\tilde{v}_\theta = -2t U_t U_\theta - \frac{e^{4U}}{2t} A_t A_\theta. \tag{6.5}$$

It follows readily from the definition of $\mathscr{E}(t)$—and from the algebraic fact that for any a, b, and $c > 0$ one has $|ab| \leqslant (1/2c)\, a^2 + 2cb^2$—that

$$\int_{S^1} |\tilde{v}_\theta| \, d\theta \leqslant t\mathscr{E}(t). \tag{6.6}$$

Hence, using the monotonicity of $\mathscr{E}(t)$, we find that for all $t \geqslant t_0$,

$$\int_{S^1} |\tilde{v}_\theta| \, d\theta \leqslant t\mathscr{E}(t_2). \tag{6.7}$$

As a consequence of (6.7) and the mean value theorem, we can control the variance of $\tilde{v}$ on a given Cauchy surface at areal time t. That is, for any $\theta_1, \theta_2 \in S^1$ and for any $t \in [t_2, T)$ we have

$$
\begin{aligned}
|\tilde{v}(\theta_2, t) - \tilde{v}(\theta_1, t)| &= \left| \int_{\theta_1}^{\theta_2} \tilde{v}_\theta \, d\theta \right| \\
&\leqslant \int_{\theta_1}^{\theta} |\tilde{v}_\theta| \, d\theta \\
&\leqslant \int_{S^1} |\tilde{v}_\theta| \, d\theta \leqslant t\mathscr{E}(t_2).
\end{aligned} \tag{6.8}
$$

We calculate the time derivative of $\tilde{v}$ from constraints (2.4a) and (2.4c); we get

$$
\tilde{v}_t = -t \left[U_t^2 + \alpha U_\theta^2 + \frac{e^{4U}}{4t^2} (A_t^2 + A_\theta^2) \right] + \lambda \frac{e^{2v}}{4t^3} \alpha K^2. \tag{6.9}
$$

It follows immediately from (6.9) that we have two inequalities for $\tilde{v}_t$:

$$
\tilde{v}_t \geqslant -t \left[U_t^2 + \alpha U_\theta^2 + \frac{e^{4U}}{4t^2} (A_t^2 + \alpha A_\theta^2) \right] \tag{6.10a}
$$

and

$$
\tilde{v}_t \leqslant \lambda \frac{e^{-2\tilde{v}}}{4t^3} K^2. \tag{6.10b}
$$

Using (6.10a), we obtain

$$
\begin{aligned}
\int_{S^1} \tilde{v} \, d\theta &= \int_{t_2}^{t} \frac{d}{dt} \left(\int_{S^1} \tilde{v} \, d\theta \right) d\tau \\
&= \int_{t_2}^{t} \left(\int_{S^1} \tilde{v}_t \, d\theta \right) d\tau \\
&\geqslant -\int_{t_2}^{t} \tau \int_{S^1} \left[U_t^2 + \alpha U_\theta^2 + \frac{e^{4U}}{4\tau^2} (A_t^2 + \alpha A_\theta^2) \right] d\theta \, d\tau \\
&= -\int_{t_2}^{t} \tau \mathscr{E}(\tau) \, d\tau \\
&\geqslant -\int_{t_2}^{t} \tau \mathscr{E}(t_2) \, d\tau \\
&\geqslant -\mathscr{E}(t_2)(t^2 - t_2^2)/2,
\end{aligned} \tag{6.11}
$$

which controls $\int_{S^1} \tilde{v} \, d\theta$ from below. Then if we combine (6.11) and (6.8), we derive a lower bound on $\tilde{v}$ itself. We do this as follows: If we apply (6.11) to the left-hand side of the identity

$$
\int_{S^1} \tilde{v} = \int_{S^1} [\min_{S^1} \tilde{v} + \tilde{v} - \min_{S^1} \tilde{v}] \tag{6.12}
$$

we obtain

$$-\mathscr{E}(t_2)(t^2-t_2^2) \leqslant 2\pi \min_{S^1} \tilde{v} + \int_{S^1} (\tilde{v} - \min_{S^1} \tilde{v}). \tag{6.13}$$

Applying (6.8) to the second term on the right-hand side of (6.13), we get

$$-\mathscr{E}(t_2)(t^2-t_2^2) \leqslant 2\pi \min_{S^1} \tilde{v} + ct \tag{6.14}$$

for some constant c. Rearranging (6.14), we have

$$\min_{S^1} \tilde{v} \geqslant c(t), \tag{6.15}$$

where $c(t)$ is a bounded function of t on $[t_2, T)$.

To obtain an upper bound for $\tilde{v}$, we use (6.10b), together with (6.15). Specifically, we have

$$\begin{aligned} \tilde{v}_t &\leqslant \lambda \frac{e^{-2\tilde{v}}}{4t^3} K^2 \\ &\leqslant \lambda \frac{e^{-2 \min_{S^1} \tilde{v}}}{4t^3} K^2 \\ &\leqslant \lambda \frac{e^{-2c(t)}}{4t^3} K^2 \end{aligned} \tag{6.16}$$

from which it follows that $\tilde{v}_t$ is controlled into the future. Control of $\tilde{v}$ immediately follows; so we have upper and lower bounds for $\tilde{v}$ on $S^1 \times [t_2, T)$.

We now use the bounds just established for $\tilde{v}$ to obtain controls for v and α. We start by noting that the constraint (2.4b), together with the definition of $\tilde{v}$, leads to the expression

$$\partial_t(\ln \alpha) = -\frac{1}{2} \lambda \frac{e^{-2\tilde{v}}}{t^3} K^2. \tag{6.17}$$

With $\tilde{v}$ bounded above and below on $S^1 \times [t_2, T)$, it follows from (6.17) that $\partial_t(\ln \alpha)$—and, hence, $\ln \alpha$ and α—are as well. Since

$$v = -\tilde{v} - \tfrac{1}{2} \ln \alpha, \tag{6.18}$$

we thus obtain bounds for v. Finally, as a consequence of these bounds on v and α, Eq. (2.4c) tells us that α_t is bounded.

This analysis does not lead to bounds for α_θ on $S^1 \times [t_2, T)$. While this might appear, from Eqs. (2.5a) and (2.5b) to be a potential obstacle to using light cone estimates for U and A, we will see in the next step that it is not.

Step 2 (Bounds on U, U_θ, U_t and A, A_θ, A_t). As with the contracting direction, to obtain the light cone estimates we use for bounding U and A and their derivatives in the expanding direction, we find it useful to treat U and A as components of a wave map ϕ. The base geometry and target geometries are different, but the idea is very much the same. We take for the base geometry $(S^1 \times \mathbb{R}, {}^2g)$ with the (nonflat) Lorentz metric

$$ {}^2g = -dt^2 + \frac{1}{\alpha}\, d\theta^2, \tag{6.19}$$

and for the family of target geometries we use $(\mathbb{R}^2, h_{(t)})$, with (t-dependent) Riemannian metrics,

$$ h_{(t)} = dU^2 + \frac{e^{4U}}{4t^2}\, dA^2. \tag{6.20}$$

The maps take the component form

$$\begin{aligned} \phi\colon S^1 \times \mathbb{R}^1 &\to \mathbb{R}^2 \\ (\theta, t) &\mapsto \phi^a(\theta, t) = \begin{pmatrix} U(\theta, t) \\ A(\theta, t) \end{pmatrix}. \end{aligned} \tag{6.21}$$

Since the base geometry is not flat, the metric-compatible[4] covariant derivative D for these maps has nonvanishing Christoffel coefficients

$$\begin{aligned} &\Gamma^t{}_{tt} = 0, \qquad \Gamma^t{}_{t\theta} = 0, \qquad \Gamma^t{}_{\theta\theta} = -\frac{\alpha_t}{2\alpha} \\ &\Gamma^\theta{}_{\theta\theta} = -\frac{\alpha_\theta}{2\alpha}, \qquad \Gamma^\theta{}_{\theta t} = -\frac{\alpha_t}{2\alpha}, \qquad \Gamma^\theta_{tt} = 0, \end{aligned} \tag{6.22}$$

as well as target Christoffel coefficients

$$\begin{aligned} &\Gamma^U{}_{UU} = 0, \qquad \Gamma^U{}_{UA} = 0, \qquad \Gamma^U{}_{AA} = -\frac{e^{4U}}{2t^2} \\ &\Gamma^A{}_{AA} = 0, \qquad \Gamma^A{}_{AU} = 2, \qquad \Gamma^A_{UU} = 0. \end{aligned} \tag{6.23}$$

The wave equation for $\phi = \binom{U}{A}$ now takes the form

$$\Box \phi^a = \psi^a, \tag{6.24a}$$

[4] As in Section 4, D is not fully metric-compatible as a consequence of the t-dependence of $h_{(t)}$ in (6.20).

where

$$\psi^a = \begin{pmatrix} \dfrac{U_t}{t} \\ -\dfrac{A_t}{t} \end{pmatrix}. \tag{6.24b}$$

We have the corresponding energy-momentum tensor, defined as in (4.15), taking the form

$$T_{\mu\nu} = U_\mu U_\nu + \frac{e^{4U}}{4t^2} A_\mu A_\nu + \frac{1}{2} g_{\mu\nu} \left[U_t^2 - \alpha U_\theta^2 + \frac{e^{4U}}{4t^2} (A_t^2 - A_\theta^2) \right].$$

Since the base metric ${}^2g$ is not flat and while we can readily choose a pair of everywhere-independent null vector fields $l = (1/\sqrt{2})(\partial_t + \alpha^{1/2}\partial_\theta)$ and $n = (1/\sqrt{2})(\partial_t - \alpha^{1/2}\partial_\theta)$ for the base, these do not generally define global null coordinates. The light cone argument does not really need such coordinates, however. It is sufficient to work with l, n, and their integral paths, which are well behaved since α is bounded.

Calculating as in Eqs. (4.17)–(4.18), we derive from (6.25) and (6.22)–(6.23) the equations

$$n(T_{ll}) = -\frac{\alpha_t}{\sqrt{2}\,\alpha} T_{ll} - \frac{1}{\sqrt{2}} \left(\frac{U_t^2}{t} + \frac{e^{4U}}{4t^3} \alpha A_\theta^2 + \frac{\alpha^{1/2}}{t} U_\theta U_t + \alpha^{1/2} \frac{e^{4U}}{4t^3} A_\theta A_t \right) \tag{6.25a}$$

and

$$l(T_{nn}) = -\frac{\alpha_t}{\sqrt{2}\,\alpha} T_{nn} - \frac{1}{\sqrt{2}} \left(\frac{U_t^2}{t} + \frac{e^{4U}}{4t^3} \alpha A_\theta^2 - \frac{\alpha^{1/2}}{t} U_\theta U_t - \alpha^{1/2} \frac{e^{4U}}{4t^3} A_\theta A_t \right). \tag{6.25b}$$

It is important here that, while α and α_t appear in Eqs. (6.25), α_θ does not. Thus we find that the right-hand side of (6.27) involves only terms quadratic in U_θ, U_t, $(e^{2U}/2t)\, A_\theta$, and $(e^{2U}/2t)\, A_t$, with the coefficients of these terms all bounded on $S^1 \times [t_2, T)$ as a consequence of the estimates from Step 1. Noting this, and writing $T_{ll} = T_{tt} + \alpha^{1/2} T_{\theta t}$ and $T_{nn} = T_{tt} - \alpha^{1/2} T_{\theta t}$, we see that if we set $E = T_{tt}$, $P = \alpha^{1/2} T_{t\theta}$, and $J =$ (right-hand side of (6.25a)), and $L =$ (right-hand side of (6.25b)), then (6.25) matches (4.6).

Since (6.25) has been determined to have the appropriate form, we see that the rest of the light cone estimate argument goes through more or less as discussed in Step 2 of Section 4, from Eq. (4.19) on. We need to replace null coordinates by null paths generated by the null vector fields n and l, but this does not affect the argument significantly. We thus find that for any point $(\theta, t) \in S^1 \times (t_2, T]$, we have $E(\theta, t)$ bounded by data on the initial hypersurface Σ_{t_2}. It immediately follows that U and A are uniformly C^1 bounded on $S^1 \times (t_2, T)$.

Step 3 (Bounds on $v_t, \alpha_\theta, v_\theta, \alpha_{\theta t}$, and α_{tt}). Since the constraint Eq. (2.4a) expresses v_t in terms of U_t, U_θ, U, A_t, A_θ, v, and t and since we have shown (Steps 1 and 2) that these are all bounded, we immediately obtain from (2.4a) bounds on v_t as well.

The same argument does not work for v_θ, since Eq. (2.4b) involves α_θ, for which we do not yet have bounds. However, we recall from Eq. (5.5) that the expression for $\tilde{v}_\theta$ involves only t, U_t, U_θ, A_t, A_θ, and U, so $\tilde{v}_\theta$ must be bounded. Then, if we write (2.4c) in the form

$$\alpha_t = -\lambda \frac{e^{-2\tilde{v}}}{t^3} \alpha K^2 \tag{6.26}$$

and calculate the θ derivative of both sides (and use local smoothness), we obtain

$$\partial_t \alpha_\theta = \left(-\lambda \frac{e^{-2\tilde{v}}}{t^3} K^2 \right) \alpha_\theta + \left(2\lambda \frac{e^{-2\tilde{v}}}{t^3} K^2 \tilde{v}_\theta \alpha \right). \tag{6.27}$$

Since the quantities in parentheses are controlled, we may integrate this differential equation for α_θ in time and thereby obtain bounds for α_θ.

It then follows from the relation $v_\theta = -\tilde{v}_\theta - \frac{1}{2}(\alpha_\theta/\alpha)$ that v_θ is bounded.

We now have uniform C^1 bounds on all of the primary fields U, A, v, and α. Equation (6.27) tells us that $\alpha_{\theta t}$ is also bounded, and if we calculate the time derivative of Eq. (2.4c), we get

$$\alpha_{tt} = \lambda \frac{e^{2v}}{t^3} \alpha K^2 \left(-2\alpha v_t - 2\alpha_t + \frac{3}{t} \right) \tag{6.28}$$

which implies that α_{tt} is bounded as well.

To go any further, we need to use light cone estimates again.

Step 4 (Bounds on second derivatives). If we take time derivatives of the wave equations (2.5) for U and A, then we get wave equations for U_t and A_t, which we can write in the wave map form:

$$\begin{aligned} \Box U_t = &-\frac{(U_t)_t}{t} + \left[\frac{1}{2}\alpha_{tt} - \frac{\alpha_t^2}{4\alpha} - \frac{e^{4U}}{t^2} \alpha A_\theta^2 + \frac{1}{t^2} \right] U_t \\ &+ \left(\frac{e^{4U}}{t^2} \alpha A_\theta A_t \right) U_\theta - \alpha \frac{e^{4U}}{2t^3} A_\theta^2 \end{aligned} \tag{6.29a}$$

$$\Box A_t = \frac{(A_t)_t}{t} + \left[\frac{1}{2}\alpha_{tt} - \frac{\alpha_t^2}{4\alpha} - 4\alpha U_\theta^2 - \frac{1}{t^2} \right] A_t + 4\alpha U_\theta U_t A_\theta; \tag{6.29b}$$

jointly, we have

$$\Box \phi_t^a = \underset{1}{\psi}^a. \tag{6.30}$$

The two components of $\underset{1}{\psi}^a$ correspondent to the right-hand sides of (6.29).

The important thing to note is that all of the quantities in $\underset{1}{\psi}^a$ except U_{tt} and A_{tt} have been shown in previous steps to be controlled. More importantly, we find that we may infer from (6.29) that the quantities

$$\underset{1}{E} = \frac{1}{2} U_{tt}^2 + \frac{1}{2} \alpha U_{t\theta}^2 + \frac{e^{4U}}{4t} \left(\frac{1}{2} A_{tt}^2 + \frac{1}{2} \alpha A_{t\theta}^2 \right) \tag{6.31a}$$

$$\underset{1}{P} = U_{tt} U_{t\theta} + \frac{e^{4U}}{4t^2} A_{tt} A_{t\theta} \tag{6.31b}$$

satisfy equations of the form

$$n(\underset{1}{E} + \underset{1}{P}) = \underset{1}{J} \tag{6.32a}$$

$$l(\underset{1}{E} - \underset{1}{P}) = \underset{1}{L}, \tag{6.32b}$$

where $\underset{1}{J}$ and $\underset{1}{L}$ involve nothing but controlled quantities, together with terms quadratic in U_{tt}, $U_{t\theta}$, A_{tt}, and $A_{t\theta}$.

Hence, we may repeat the light cone estimate argument as in Step 2 and thereby verify that U_{tt}, $U_{t\theta}$, A_{tt}, and $A_{t\theta}$ are all bounded on $S^1 \times [t_0, T)$. Further, using the wave Eqs. (2.5a)–(2.5b), we get bounds on $U_{\theta\theta}$ and $A_{\theta\theta}$; then, using arguments of the sort discussed in Step 4, we obtain C^2 bounds on v and α as well. Thus we have uniform C^2 bounds on all of the primary fields.

One could repeat this "boot strap"-type argument step-by-step and obtain bounds on higher order derivatives. However, C^2 bounds are sufficient for the theorems we cite [14] to establish global existence, so we have proven existence for $t \to \infty$ of the variables U, A, α, and v.

Step 5 (Extension of the shift functions). It remains to show that the shift functions $\underset{1}{G}$, $\underset{2}{G}$, $\underset{1}{M}$, and $\underset{2}{M}$ extend to $R = t \to \infty$. Since the constraint equations (2.6) for these functions in areal coordinates are essentially identical to the constraint equations (2.10) for them in conformal coordinates, the procedure for proving that they extend is just that found in Step 6 of Section 2.

This completes our proof that the areal coordinate development of the initial data on Σ_{t_2} extends to $R \to \infty$. As shown in Proposition 5, if $R \to \infty$ in the future development of a set of initial data, then that development must be maximal. Hence we have completed the proof of our main result, Theorem 1.

7. CONCLUSIONS

The primary motivation for this work has been to set up a framework—including a geometrically based time foliation—for studying *strong cosmic censorship* and other global issues in a family of spacetimes which is larger and more complicated than the Gowdy spacetimes, but which still can be studied via $(1+1)$-dimensional PDE analysis. Indeed, by relaxing the Gowdy requirement that the twist quantities be nonzero, one obtains field equations which are considerably more intricate than in the Gowdy case.[5] Hence the tools we are developing in working with the T^2-symmetric spacetimes could be more generalizable than those developed in working with Gowdy spacetimes.

Our work here obtains this framework. While the $R=t$ foliation we have obtained here should—as evidenced by its importance in the Gowdy spacetimes—prove to be very useful, there is another geometrically-based foliation of considerable interest for these and other spacetimes: the constant mean curvature (CMC) foliation. Rendall [18] has studied CMC foliations on T^2-symmetric spacetimes—for the Einstein–Vlasov and Einstein–wave map equations as well as for the Einstein vacuum equations—and he has shown that if such a spacetime admits at least one CMC Cauchy surface Σ, then it admits a CMC foliation from a neighbourhood of Σ back to the singularity. We expect that our areal coordinate foliation provides the barriers necessary to guarantee the existence of the needed first CMC Cauchy surface. We also expect that one might be able to use our result to show that the CMC foliation covers the entire maximal domain of dependence of the spacetime.

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