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# Killing initial data

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**Abstract.** We study spacetime Killing vectors in terms of their 'lapse and shift' relative to some spacelike slice. We give a necessary and sufficient condition in order for these lapse–shift pairs, which we call Killing initial data (KID), to form a Lie algebra under the bracket operation induced by the Lie commutator of vector fields on spacetime. This result is applied to obtain a theorem on the periodicity of orbits for a class of Killing vector fields in asymptotically flat spacetimes.

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*We dedicate this work to Professor Andrzej Trautman on the occasion of his birthday. It is a great pleasure to pay tribute to his lasting contributions to relativity.*

## 1. Introduction

When considering black hole spacetimes with more than one Killing vector field it is customary to assume that one of the Killing vectors has complete periodic orbits. In a recent paper [1] we have shown that this is necessarily the case, under a set of conditions on the spacetimes under consideration. This set of hypotheses includes a 'largeness condition' on the spacetimes, namely that the spacetime contains a 'boost-type domain'. While this hypothesis will be satisfied for many models of matter coupled to gravity, provided the fields under consideration fall off sufficiently fast at spatial infinity [2, 3], there are various cases in which we are not certain *a priori* that this will be the case. For this reason it is useful to have results under hypotheses involving initial data sets only, and that with a minimal set of hypotheses on the matter fields under consideration. It is the aim of this paper to prove the existence of a Killing vector with periodic orbits in a Cauchy data setting, when there are at least two linearly independent Killing vectors, one of which is transverse to the initial data surface (at least in the asymptotic region). The reader should note that the classification of possible isometry groups, or of possible Lie algebras of Killing vectors, follows immediately from this result, as in [1], except for one-dimensional algebras of Killing vectors.

In order to address the issue raised above, it is first necessary to face the following problem. Consider an initial data set with two or more 'candidate Killing vector fields'. Under which conditions do these vector fields lead to Killing vector fields on a corresponding spacetime? We show that this question can be reduced to that of certain properties of an appropriately defined bracket operation on the initial data surface. More precisely, we give a

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necessary and sufficient condition for the bracket operation to form a Lie algebra. We show that, when the bracket operation forms a Lie algebra, the 'candidate Killing vectors' become Killing vectors in the Killing development† associated with any 'transverse candidate Killing vector'.

This paper is organized as follows. In the next section we introduce the notion of a 'Killing initial data' (KID) and discuss some elementary properties thereof. In section 3 we give a sufficient and necessary condition for the set of KIDs to form a Lie algebra. In section 4 we show that Lie algebras of KIDs 'extend' to Lie algebras of Killing vectors of Killing developments. In section 5 we consider asymptotically flat Killing developments of initial data sets with at least two-dimensional Lie algebras of KIDs, and we prove the existence of Killing vectors with periodic orbits in such a case.

## 2. Killing initial data (KIDs)

Let $(M, g_{\mu\nu})$ be a connected spacetime and $X, \bar{X}$ be Killing vector fields, i.e.

$$\mathcal{L}_X g_{\mu\nu} = 0 = \mathcal{L}_{\bar{X}} g_{\mu\nu}. \tag{2.1}$$

Then the commutator $[X, \bar{X}]$ is also a Killing vector field since

$$\mathcal{L}_{[X,\bar{X}]} g_{\mu\nu} = [\mathcal{L}_X, \mathcal{L}_{\bar{X}}] g_{\mu\nu} = 0. \tag{2.2}$$

More generally, let $\boldsymbol{V}$ be the finite-dimensional vector space over $\mathbb{R}$ of Killing vector fields on $(M, g_{\mu\nu})$. Then $\boldsymbol{V}$ is closed under $[\,,\,]$. Let $(\Sigma, g_{ij}, K_{ij})$ be a connected spacelike submanifold of $(M, g_{\mu\nu})$ with induced metric $g_{ij}$ and second fundamental form $K_{ij}$. We can then decompose the Killing vector field $X$ along $\Sigma$ according to

$$X = N n^\mu \partial_\mu + Y^i \partial_i, \qquad N = -X^\mu n_\mu \tag{2.3}$$

where $n^\mu$ is the future unit normal of $\Sigma$. Here we are using a coordinate system $x^\mu$ in which $\Sigma$ is described by the equation $t \equiv x^0 = 0$. In order to translate the Killing equation into a statement in terms of $(N, Y^i)$ and $(g_{ij}, K_{ij})$ it is convenient to choose Gaussian coordinates $x^\mu = (t, x^i)$ on a tubular neighbourhood of $\Sigma$ in $(M, g_{\mu\nu})$. Then

$$g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = -\mathrm{d}t^2 + g_{ij}(t, x^\ell)\, \mathrm{d}x^i\, \mathrm{d}x^j. \tag{2.4}$$

The $(i, j)$-component of

$$\mathcal{L}_X g_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + 2 g_{\rho(\mu} \partial_{\nu)} X^\rho = 0 \tag{2.5}$$

yields

$$2N K_{ij} + 2D_{(i} Y_{j)} = 0, \tag{2.6}$$

where we have used $\partial_t g_{ij} = 2K_{ij}$, valid in Gaussian coordinates. The $(t, t)$-component of (2.5) says that

$$\partial_t N = 0 \tag{2.7}$$

and the $(t, i)$-component that

$$\partial_t Y^i = g^{ij} D_j N. \tag{2.8}$$

† See [4] and section 2 of this paper for the definition of the notion of Killing development. Let us emphasize that, when suitable field equations are imposed, there exists a neighbourhood of the initial data hypersurface in the Killing development which is isometrically diffeomorphic to a neighbourhood of the initial data surface in the spacetime obtained by evolving the initial data using the field equations. Thus statements about the Killing developments are also statements about solutions of the field equations, in this sense.

Another interesting identity results from taking $\partial_t$ of equation (2.6):

$$2N\partial_t K_{ij} + 2D_i D_j N + 4D_{(i}(K_{j)\ell}Y^\ell) - 2Y^\ell(2D_{(i}K_{j)\ell} - D_\ell K_{ij}) = 0 \tag{2.9}$$

where we have used (2.7) and (2.8). We now define

$$G_{\mu\nu}n^\mu n^\nu = \rho, \qquad G_{\mu i}n^\mu = -J_i, \qquad G_{ij} = \tau_{ij}, \tag{2.10}$$

where $G_{\mu\nu}$ is the Einstein tensor of $g_{\mu\nu}$. The quantities $\rho$ and $J_i$ can be expressed in terms of $g_{ij}$ and $K_{ij}$ by the relations

$$2\rho = {}^3R + K^2 - K_{ij}K^{ij} \tag{2.11}$$

$$-J_i = D^j(K_{ij} - Kg_{ij}). \tag{2.12}$$

Using the well known form of $G_{ij}$ in Gaussian coordinates to eliminate $\partial_t K_{ij}$ from (2.9), we obtain

$$\mathcal{L}_Y K_{ij} + D_i D_j N = N({}^3\mathcal{R}_{ij} + KK_{ij} - 2K_{i\ell}K_j{}^\ell) - N\left[\tau_{ij} - \tfrac{1}{2}g_{ij}(\tau - \rho)\right], \tag{2.13}$$

where $\tau := g^{ij}\tau_{ij}$. Clearly, equations (2.6) and (2.11)–(2.13) hold independently of Gaussian coordinates.

In the context of the Cauchy problem it is often convenient to forget about the spacetime and consider only three-dimensional initial data sets $(\Sigma, g_{ij}, K_{ij})$. For the purpose of equation (2.13) we also need to have a tensor field $\tau_{ij}$ defined on $\Sigma$. We shall call a pair $(N, Y^i)$ a Killing initial data (KID), provided (2.6) and (2.13) hold.

It is worth pointing out that, in the context of the Einstein equations, $\tau_{ij}$ can typically be calculated from the initial data for the matter fields present, and is the matter stress tensor. An alternative point of view is the following. Consider a data set $(\Sigma, g_{ij}, K_{ij})$ together with a scalar field $N$ and a vector field $Y^i$ satisfying equation (2.6). We can then use equation (2.11) to define a scalar field $\rho$, and then equation (2.13) to define a tensor field $\tau_{ij}$, at least on the set where $N$ does not vanish. Thus if we have only one solution of equation (2.6), then equation (2.13) is trivial (except perhaps on the boundary of the zero set of $N$, if some regularity of $\tau_{ij}$ is imposed). If, however, more than one pair $(N, Y^i)$ solving equation (2.6) exists, we can use one such solution to define $\tau_{ij}$, and then consider only those solutions of equation (2.6) which satisfy equation (2.13) with that given $\tau_{ij}$.

Given a KID on $(\Sigma, g_{ij}, K_{ij})$, we can ask the converse question: does there exist a spacetime $(M, g_{\mu\nu})$ 'evolving' from $(\Sigma, g_{ij}, K_{ij})$ with a Killing vector $X$ 'evolving' from $(N, Y^i)$? There is an affirmative answer to this question in the following two cases.

*Case (i).* $(N, Y)$ is 'transversal', i.e. by definition, $N \neq 0$. Then we can use the KID $(N, Y^i)$ to define the *Killing development* $(\hat{M}, \hat{g}_{\mu\nu})$ of $(\Sigma, g_{ij}, K_{ij})$ (see [1]) as follows. Let $\hat{M} = \mathbb{R} \times \Sigma$ and define the Lorentz metric

$$\begin{aligned} &\hat{g}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\hat{N}^2\,\mathrm{d}u^2 + \hat{g}_{ij}(\mathrm{d}x^i + \hat{Y}^i\,\mathrm{d}u)(\mathrm{d}x^j + \hat{Y}^j\,\mathrm{d}u), \\ &\hat{N}(u, x^i) = N(x^i), \qquad \hat{g}_{ij}(u, x^\ell) = g_{ij}(x^\ell), \qquad \hat{Y}^i(u, x^j) = Y^i(x^j). \end{aligned} \tag{2.14}$$

Then $\partial_u$ is a Killing vector of $(\hat{M}, \hat{g}_{\mu\nu})$ extending $(N, Y^i)$, i.e. the vector field $X$ defined on $\Sigma$ by the right-hand side of equation (2.3) coincides with the Killing vector field $\partial_u$ there.

*Case (ii).* $\rho = 0$, $J_i = 0$. In this case when $g_{ij}$ and $K_{ij}$ are sufficiently regular, $(\Sigma, g_{ij}, K_{ij})$ has a vacuum Cauchy development $(\bar{M}, \bar{g}_{\mu\nu})$, i.e. $\bar{R}_{\mu\nu} = 0$. If, furthermore, the KID $(N, Y^i)$ is a vacuum KID in the sense that the 'stress tensor' $\tau_{ij}$, defined by equation (2.13) is also zero, it is known (see [5] and references therein; cf also [6]) that the KID extends to a Killing vector on $(\bar{M}, \bar{g}_{\mu\nu})$.

An analogous statement holds when the vacuum equation is modified by the presence of a cosmological constant $\Lambda$, i.e. $\rho = -\Lambda$, $\tau_{ij} = \Lambda g_{ij}$, $J_i = 0$.

Suppose we now have a spacetime $(M, g_{\mu\nu})$ with two Killing vectors $X, \bar{X}$. Their commutator $[X, \bar{X}]$ gives rise, on $(\Sigma, h_{ij}, K_{ij})$, to the bracket

$$\{(N, Y^i), (\bar{N}, \bar{Y}^j)\} := (\mathcal{L}_Y \bar{N} - \mathcal{L}_{\bar{Y}} N, [Y, \bar{Y}]^\ell + N D^\ell \bar{N} - \bar{N} D^\ell N). \tag{2.15}$$

This is the algebra first studied in [7, 8]. Note, however, that, whereas in [7, 8] the above bracket is, loosely speaking, a commutator of vector fields in the infinite-dimensional space of spacelike embeddings of some 3-manifold into spacetime, it arises in our case simply from the commutator of Killing vector fields on spacetime.

We are now ready to ask the following question. Consider an initial-data set $(\Sigma, g_{ij}, K_{ij})$ and two KIDs, i.e. solutions of equation (2.6) and equation (2.13) *for the same* $\tau_{ij}$. Is their bracket, defined by (2.15), also a KID with the same $\tau_{ij}$? An affirmative answer can immediately be given in the vacuum case (case (ii) above): the vacuum development is clearly defined independently of the KIDs, and thus *every* KID extends to a Killing vector field on $(\bar{M}, \bar{g}_{\mu\nu})$. Thus the KIDs, in this case, are closed under $\{\,,\}$. In the non-vacuum case, when one of the KIDs $(N, Y^i)$ has $N \neq 0$, one might consider the Killing development associated with this particular KID. But it is then unclear whether some other KID $(\bar{N}, \bar{Y}^i)$, if present, extends to a Killing vector in the Killing development given by $(N, Y^i)$. In fact, the following example shows that KIDs are, in general, *not* closed under $\{\,,\}$.

*Example.* Let $(\Sigma, h_{ij}, K_{ij}) = (\mathbb{R}^3, \delta_{ij}, 0)$ and take for $\tau_{ij}$

$$\tau_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2. \tag{2.16}$$

Define two KIDs by

$$\begin{aligned} N &= 0, \qquad & Y &= x^2 \partial_{x^3} - x^3 \partial_{x^2} \\ \bar{N} &= \mathrm{e}^{x^3}, \qquad & \bar{Y} &= 0. \end{aligned} \tag{2.17}$$

It is then easy to check that $(N, Y^i)$ and $(\bar{N}, \bar{Y}^i)$ are both KIDs with $\tau_{ij}$ given by (2.16), but their bracket is not.

## 3. The Lie algebra of KIDs

We first show the Jacobi identity for $\{\,,\}$.

*Lemma.* Consider three pairs $(N, Y^i)$, $(\bar{N}, \bar{Y}^i)$, $(\widetilde{N}, \widetilde{Y}^i)$ satisfying equation (2.6). Then

$$\begin{aligned} &\{(\widetilde{N}, \widetilde{Y}^i), \{(N, Y^i), (\bar{N}, \bar{Y}^i)\}\} + \{(\bar{N}, \bar{Y}^i), \{(\widetilde{N}, \widetilde{Y}^i), (N, Y^i)\}\} \\ &\qquad + \{(N, Y^i), \{(\bar{N}, \bar{Y}^i), (\widetilde{N}, \widetilde{Y}^i)\}\} = 0. \end{aligned} \tag{3.1}$$

*Proof.* This is a straightforward computation, based on the Jacobi identity for the commutator of vector fields on $\Sigma$ and relations like

$$\mathcal{L}_Y D^i \bar{N} = D^i \mathcal{L}_Y \bar{N} + 2 N K^{ij} D_j \bar{N}. \tag{3.2}$$

□

We now state the main result of this paper.

*Theorem.* Let $\boldsymbol{W}$ be the vector space over $\mathbb{R}$ of KIDs on $(\Sigma, g_{ij}, K_{ij})$ for some fixed stress tensor $\tau_{ij}$. The linear space $\boldsymbol{W}$ is closed under the bracket $\{\,,\}$, if and only if

$$(N\mathcal{L}_{\bar{Y}} - \bar{N}\mathcal{L}_Y)\tau_{ij} = 2J_{(i}(ND_{j)}\bar{N} - \bar{N}D_{j)}N) \tag{3.3}$$

for all pairs $(N, Y^i)$, $(\bar{N}, \bar{Y}^i)$ of KIDs.

*Proof.* We first have to look at the expression

$$\mathcal{L}_{[Y,\bar{Y}]+ND\bar{N}-\bar{N}DN}\, g_{ij} + 2(\mathcal{L}_Y\bar{N} - \mathcal{L}_{\bar{Y}}N)K_{ij}, \tag{3.4}$$

where $ND\bar{N} - \bar{N}DN$ is a shorthand for the vector $ND^i\bar{N} - \bar{N}D^iN$. Using equation (2.6) for both pairs $(N, Y^i)$ and $(\bar{N}, \bar{Y}^i)$ the expression (3.4) can be written as

$$\mathcal{L}_Y(-2\bar{N}K_{ij}) + 2D_{(i}(ND_{j)}\bar{N}) + 2(\mathcal{L}_Y\bar{N})K_{ij} - ((N, Y) \longleftrightarrow (\bar{N}, \bar{Y})). \tag{3.5}$$

Using equation (2.13) to eliminate $D_iD_jN$ and $D_iD_j\bar{N}$ in (3.5), we find that all terms add up to zero. Here, and in the following, we are repeatedly using the fact that terms which are independent of $Y$ and $\bar{Y}$ and contain $N$ and $\bar{N}$ without derivatives drop out upon antisymmetrization. Thus $\{(N, Y), (\bar{N}, \bar{Y})\}$ also satisfies equation (2.6). We now compute $\mathcal{L}_Y\, {}^3\mathcal{R}_{ij} = \delta\, {}^3\mathcal{R}_{ij}(\mathcal{L}_Y g_{k\ell})$, where $\delta\, {}^3\mathcal{R}_{ij}$ is the linearization of the Ricci tensor at $g_{ij}$. Thus

$$\begin{aligned}\mathcal{L}_Y\, {}^3\mathcal{R}_{ij} &= \Delta(NK_{ij}) + D_iD_j(NK) - 2D_{(i}D^\ell(NK_{j)\ell}) \\ &\quad -2N\, {}^3\mathcal{R}^\ell{}_{(ij)}{}^m K_{\ell m} - 2N\, {}^3\mathcal{R}_{(i}{}^\ell K_{j)\ell}.\end{aligned} \tag{3.6}$$

Consequently,

$$\begin{aligned}(\bar{N}\mathcal{L}_Y - N\mathcal{L}_{\bar{Y}})^3\mathcal{R}_{ij} &= \bar{N}[(\Delta N)K_{ij} + 2(D^\ell N)D_\ell K_{ij} + (D_iD_jN)K + 2(D_{(i}N)D_{j)}K \\ &\quad - 2(D_{(i}D^\ell N)K_{j)\ell} - 2(D^\ell N)D_{(i}K_{j)\ell} \\ &\quad - 2(D_{(i}N)(D_{j)}K - J_{j)})] - (N \longleftrightarrow \bar{N}) \\ &= \bar{N}[(\Delta N)K_{ij} - (\mathcal{L}_YK_{ij})K + 2(\mathcal{L}_YK_{(i}{}^\ell)K_{j)\ell} \\ &\quad + 2D^\ell N(D_\ell K_{ij} - D_{(i}K_{j)\ell}) + 2(D_{(i}N)J_{j)}] \\ &\quad - ((N, Y) \longleftrightarrow (\bar{N}, \bar{Y}))\end{aligned} \tag{3.7}$$

where we have used (2.13) in the last line. Equations (3.6) and (2.13) imply that

$$\mathcal{L}_YK = N({}^3\mathcal{R} + K^2) - \Delta N - N\left(-\tfrac{1}{2}\tau + \tfrac{3}{2}\rho\right). \tag{3.8}$$

Now equations (3.7) and (3.8) and the definition (2.11) of $\rho$ give rise to

$$(N\mathcal{L}_{\bar{Y}} - \bar{N}\mathcal{L}_Y)\rho = 2(ND^i\bar{N} - \bar{N}D^iN)J_i. \tag{3.9}$$

Finally, we compute

$$\mathcal{L}_{[Y,\bar{Y}]+ND\bar{N}-\bar{N}DN}K_{ij} + D_iD_j(\mathcal{L}_Y\bar{N} - \mathcal{L}_{\bar{Y}}N) - (\mathcal{L}_Y\bar{N} - \mathcal{L}_{\bar{Y}}N)M_{ij} \tag{3.10}$$

where $NM_{ij}$ is the right-hand side of (2.13), i.e.

$$M_{ij} := {}^3\mathcal{R}_{ij} + KK_{ij} - 2K_{i\ell}K_j{}^\ell - \tau_{ij} + \tfrac{1}{2}g_{ij}(\tau - \rho). \tag{3.11}$$

Using (2.13), the expression (3.10) turns into

$$\begin{aligned}&-[\mathcal{L}_Y, D_iD_j]\bar{N} + \bar{N}\mathcal{L}_YM_{ij} + N(D^\ell\bar{N})D_\ell K_{ij} \\ &\qquad +2K_{\ell(i}D_{j)}(ND^\ell\bar{N}) - ((N, Y) \longleftrightarrow (\bar{N}, \bar{Y})) \\ &\quad = \bar{N}\mathcal{L}_YM_{ij} - 2\bar{N}(D^\ell N)(D_\ell K_{ij} - D_{(i}K_{j)\ell}) \\ &\qquad +2\bar{N}K_{\ell(i}(\mathcal{L}_YK_{j)}{}^\ell) - ((N, Y) \longleftrightarrow (\bar{N}, \bar{Y})).\end{aligned} \tag{3.12}$$

We now insert equations (3.7) and (3.8) into $(\bar{N}\mathcal{L}_Y - N\mathcal{L}_{\bar{Y}})M_{ij}$ and substitute this in the third line of (3.12). Remarkably, all terms not involving $\tau_{ij}$, $J_i$, $\rho$ drop out. In order for $\{(N, Y), (\bar{N}, \bar{Y})\}$ to again satisfy equation (2.13), we are then left with the condition

$$(N\mathcal{L}_{\bar{Y}} - \bar{N}\mathcal{L}_Y)\tau_{ij} - \tfrac{1}{2}g_{ij}(N\mathcal{L}_{\bar{Y}} - \bar{N}\mathcal{L}_Y)(\tau - \rho) = 2J_{(i}(ND_{j)}\bar{N} - \bar{N}D_{j)}N). \tag{3.13}$$

It is easily seen from (3.9) that (3.13) is equivalent to (3.3). Thus we are left with (3.3) as the necessary and sufficient condition for $\boldsymbol{W}$ to form a Lie algebra under $\{\,,\}$, and the proof is complete. □

We also record, for later use, the identity

$$(N\mathcal{L}_{\bar{Y}} - \bar{N}\mathcal{L}_Y)J_i = (ND^j\bar{N} - \bar{N}D^jN)\tau_{ij} + (ND_i\bar{N} - \bar{N}D_iN)\rho. \tag{3.14}$$

Equation (3.14) follows from the definition (2.12) and equations (2.6) and (2.13), independently of the condition (3.3), in much the same way as (3.9) follows from (2.11).

There are situations, in addition to the vacuum case, where the condition (3.3) is 'automatically satisfied'. Let $\rho$ be everywhere positive and suppose that

$$\tau_{ij} = \frac{1}{\rho}J_iJ_j. \tag{3.15}$$

Then equation (3.3) follows from (3.9) and (3.14). If there is a transversal KID $(N, Y^i)$ and if, in addition to (3.15), there holds

$$\rho = \sqrt{J_iJ^i}, \tag{3.16}$$

the Killing development associated with $(N, Y^i)$ is a null dust spacetime, i.e.

$$\hat{G}_{\mu\nu} = \hat{\rho}\xi_\mu\xi_\nu, \qquad \hat{g}^{\mu\nu}\xi_\mu\xi_\nu = 0 \tag{3.17}$$

with $\hat{\rho}(u, x^i) = \rho(x^i)$, $\hat{J}_\ell(u, x^i) = J_\ell(x^i)$, and

$$\xi_\mu\,\mathrm{d}x^\mu = \hat{N}\,\mathrm{d}u - \frac{1}{\hat{\rho}}\hat{J}_i(\mathrm{d}x^i + \hat{Y}^i\,\mathrm{d}u). \tag{3.18}$$

Another possibility would be to have $\rho \geqslant 0$ (and not necessarily identically vanishing), $\tau_{ij} = 0$, $J_i = 0$. Then any Killing development is a (standard, i.e. non-null) dust spacetime.

Finally, there is the situation where $(\rho, J_i, \tau_{ij})$ are built from some other ('good matter') fields, i.e. fields with the property that the combined Einstein-matter system allows a properly posed initial-value problem. For example, $(\rho, J_i, \tau_{ij})$ could be built from the $(E_i, B_i)$-fields derived from a Maxwell field $F_{\mu\nu}$. Then, when there is a spacetime Killing field $X$ satisfying in addition that $\mathcal{L}_XF_{\mu\nu} = 0$, the KID associated with $X$ would satisfy some further equations involving $(E_i, B_i)$. This is discussed in more detail in section 5. Conversely (see [9]), any KID satisfying these latter equations extends to a Killing vector on the Einstein–Maxwell spacetime evolving from $(\Sigma, g_{ij}, K_{ij}; E_i, B_i)$. Thus the condition (3.3) is again automatically satisfied in this case, when $E_i$ and $B_i$ are invariant in an appropriate sense, cf equation (5.2) below.

## 4. Killing developments

Now suppose that condition (3.3) is valid and we have a (non-trivial) Lie algebra of KIDs. Suppose, further, that $(N, Y^i)$, one of these KIDs, has $N \neq 0$, so that we can consider the Killing development defined by $(N, Y^i)$. Then we have

*Proposition.* Consider an initial data set $(\Sigma, g_{ij}, K_{ij})$ and suppose that the set of KIDs forms a Lie algebra $\boldsymbol{W}$. Assume further that there exists a KID $(N, Y^i)$ in $\boldsymbol{W}$ such that $N > 0$, and denote by $(M, g_{\mu\nu})$ the Killing development of $(\Sigma, g_{ij}, K_{ij})$ based on $(N, Y^i)$. Then there is a one-to-one correspondence between the Killing vectors of $(M, g_{\mu\nu})$ and KIDs, which preserves the Lie algebra structure of $\boldsymbol{W}$.

*Proof.* In the Killing development of $(N, Y^i)$, the extension $\hat{X}$ of $(N, Y^i)$ is given by

$$\hat{X} = \hat{N} n^\mu \partial_\mu + \hat{Y}^i \partial_i = \partial_u \tag{4.1}$$

when $u_\mu$ is the unit future normal to $u =$ constant. When $(N_\alpha, Y^i_\alpha)$ is any other KID we have by assumption that

$$\{(N, Y), (N_\alpha, Y_\alpha)\} = c_\alpha (N, Y) + c_\alpha{}^\beta (N_\beta, Y_\beta) \tag{4.2}$$

for some constants $c_\alpha, c_\alpha{}^\beta$. We now define extensions $\hat{X}_\alpha$ of these KIDs by the system of linear homogeneous ODEs

$$\partial_u \hat{N}_\alpha = c_\alpha \hat{N} + c_\alpha{}^\beta \hat{N}_\beta \qquad \partial_u \hat{Y}_\alpha{}^i = c_\alpha \hat{Y}^i + c_\alpha{}^\beta \hat{Y}_\beta{}^i \tag{4.3}$$

with $\hat{N}_\alpha(0, x^i) = N_\alpha(x^i)$, $\hat{Y}_\alpha{}^i(0, x^j) = Y_\alpha{}^i(x^j)$ and

$$\hat{X}_\alpha = \hat{N}_\alpha n^\mu \partial_\mu + \hat{Y}_\alpha{}^i \partial_i \tag{4.4}$$

$$= \frac{1}{\hat{N}} \hat{N}_\alpha \partial_u + \left( \hat{Y}_\alpha{}^i - \frac{\hat{N}_\alpha}{\hat{N}} \hat{Y}^i \right) \partial_i . \tag{4.5}$$

We now compute $\mathcal{L}_{\hat{X}_\alpha} \hat{g}^{\mu\nu}$ for $u = 0$ with $\hat{g}_{\mu\nu}$ given by equation (2.14), i.e.

$$\hat{g}^{\mu\nu} \partial_\mu \partial_\nu = -\frac{1}{\hat{N}^2} (\partial_u - \hat{Y}^i \partial_i)(\partial_u - \hat{Y}^j \partial_j) + \hat{g}^{ij} \partial_i \partial_j . \tag{4.6}$$

We find that the $(uu)$-component of $\mathcal{L}_{\hat{X}_\alpha} \hat{g}^{\mu\nu}$ vanishes by virtue of

$$\partial_u \hat{N}_\alpha = \mathcal{L}_Y N_\alpha - \mathcal{L}_{\vec{Y}_\alpha} N, \tag{4.7}$$

which follows from (2.15) and (4.2), (4.3). Furthermore, the $(ui)$-components vanish by virtue of

$$\partial_u \hat{Y}_\alpha{}^i \big|_{u=0} = [Y, Y_\alpha]^i + N D^i N_\alpha - N_\alpha D^i N . \tag{4.8}$$

Finally, the $(ij)$-component of $\mathcal{L}_{\hat{X}_\alpha} \hat{g}^{\mu\nu}$ is zero for $u = 0$, by virtue of $(N, Y)$, $(N_\alpha, Y_\alpha)$ all obeying equation (2.6) (equation (2.6) actually coincides with the $(ij)$-component of $\mathcal{L}_{\hat{X}_\alpha} \hat{g}_{\mu\nu} = 0$). Furthermore, we see from (4.5) and $\hat{X} = \partial_u$ that

$$[\hat{X}, \hat{X}_\alpha] = c_\alpha \hat{X} + c_\alpha{}^\beta \hat{X}_\beta \tag{4.9}$$

for all $u \in \mathbb{R}$. Thus

$$\frac{\partial}{\partial u} \left( \mathcal{L}_{\hat{X}_\alpha} \hat{g}_{\mu\nu} \right) = c_\alpha{}^\beta \mathcal{L}_{\hat{X}_\beta} \hat{g}_{\mu\nu}, \tag{4.10}$$

which, combined with $(\mathcal{L}_{\hat{X}_\alpha} \hat{g}^{\mu\nu})\big|_{u=0} = 0$, gives the result that $\hat{X}_\alpha$ is a Killing vector of $\hat{g}_{\mu\nu}$, as required. □

We can now interpret the meaning of the condition (3.3) in terms of Killing developments. Suppose $X$ is a Killing vector of $(M, g_{\mu\nu})$ with complete orbits, intersecting exactly once an everywhere transversal spacelike submanifold with induced metric $g_{ij}$. It follows that there exist coordinates $(u, x^i)$, $-\infty < u < \infty$, such that

$$g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = -N^2\, \mathrm{d}u^2 + g_{ij} (\mathrm{d}x^i + Y^i\, \mathrm{d}u)(\mathrm{d}x^j + Y^j\, \mathrm{d}u)$$

with $N$, $Y^i$ and $g_{ij}$ all independent of $u$ and $X = Nn^\mu\partial_\mu + Y^i\partial_i = \partial_u$. Suppose there exists another Killing vector

$$\bar{X} = \bar{N}n^\mu\partial_\mu + \bar{Y}^i\partial_i = \frac{\bar{N}}{N}(\partial_u - Y^i\partial_i) + \bar{Y}^i\partial_i. \tag{4.11}$$

By the $(uu)$- and $(ui)$-components of $\mathcal{L}_{\bar{X}}g^{\mu\nu} = 0$ we find that

$$\begin{aligned} \partial_u\bar{N} &= \mathcal{L}_Y\bar{N} - \mathcal{L}_{\bar{Y}}N \\ \partial_u\bar{Y}^i &= [Y, \bar{Y}]^i + ND^i\bar{N} - \bar{N}D^iN. \end{aligned} \tag{4.12}$$

We also know that

$$\mathcal{L}_{\bar{X}}G_{\mu\nu} = 0, \tag{4.13}$$

where $G_{\mu\nu}$ is the Einstein tensor of $g_{\mu\nu}$. Writing this out, using (4.11), (4.12) and

$$G_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = N^2\rho\,\mathrm{d}u^2 + [\tau_{ij}(\mathrm{d}x^i + Y^i\,\mathrm{d}u) - 2NJ_j\,\mathrm{d}u](\mathrm{d}x^j + Y^j\,\mathrm{d}u), \tag{4.14}$$

we find after straightforward manipulations that (4.13) is equivalent to equations (3.3), (3.9) and (3.14). Since (3.9) and (3.14) are just identities, we have thus found that (3.3) is merely the condition for $\bar{X}$ defined by (4.11) to be a vector field in the Killing development $(M, g_{\mu\nu})$ of $(N, Y)$ which Lie derives $G_{\mu\nu}$. We can now ask whether (4.13) is already sufficient for $\bar{X}$ to be a Killing vector of $g_{\mu\nu}$. In other words, suppose we have a transversal KID $(N, Y)$. Define $\tau_{ij}$ by (2.13). Suppose further we have another KID $(\bar{N}, \bar{Y})$ compatible with $(N, Y)$ in that it satisfies (2.13) *with the same* $\tau_{ij}$, and $\tau_{ij}$ satisfies (3.3). Then, is $\bar{X}$, defined by

$$\bar{X}^\mu(u, x)\partial_\mu = \frac{\bar{N}(u, x)}{N(x)}(\partial_u - Y^i(x)\partial_i) + \bar{Y}^i(u, x)\partial_i \tag{4.15}$$

with $\bar{N}(u, x)$, $\bar{Y}^i(u, x)$ obeying equations (4.12), a Killing vector of $(M, g_{\mu\nu})$? We believe the answer in general will be no, for the following reason. Suppose $(\Sigma, g_{ij}, K_{ij})$, $(N, Y)$, $(\bar{N}, \bar{Y})$ are all analytic for $u = 0$. Then, by the Cauchy–Kowalewskaja theorem, the evolution equations (4.12) can be solved for $(\bar{N}, \bar{Y})$, whence four components of $\mathcal{L}_{\bar{X}}g^{\mu\nu} = 0$ are already satisfied. Equation (2.6), however, is *a priori* only valid for $u = 0$. The condition for the $u$-derivative of this equation to vanish is precisely equation (3.3). This, in turn, is again only valid for $u = 0$, and there is no guarantee that equation (3.3) will propagate. The previous proposition circumvents this problem by assuming that (3.3) is satisfied for all pairs of KIDs with the same tensor field $\tau_{ij}$. In the case where $(\rho, J_i, \tau_{ij})$ is built from 'good matter fields', the Killing development will, in the domain of dependence of $\Sigma$, be the same as the solution to the coupled system, in which case the propagation of equation (3.3) is automatically taken care of.

## 5. An application: periodicity of Killing orbits

A prerequisite for the classification of stationary black holes is an understanding of possible isometry groups of asymptotically flat spacetimes. A classification of the latter has been recently established in [1], under a 'largeness condition' on the spacetimes under consideration. As an application of our results in the preceding sections, we show below that the results of [1] can be recovered without any spacetime 'largeness' conditions, when two or more Killing vector fields are present, one of them being transverse to the initial data hypersurface $\Sigma$.

Consider, thus, as in the preceding section, an initial data set $(\Sigma, g_{ij}, K_{ij})$ with a KID $(N_0, Y_0^i)$, with $N_0 > 0$. If one imposes well behaved evolution equations on the metric, one expects that in the resulting spacetime $(\hat{M}, \hat{g}_{\mu\nu})$ there will exist a neighbourhood $\mathcal{O} \subset \hat{M}$ of $\Sigma$ and an appropriate coordinate system on $\mathcal{O}$ such that the metric will take the form (2.14) (with $\hat{N}$ replaced by $\hat{N}_0$, etc), with $u \in (u^-(p), u^+(p))$, $p \in \Sigma$, $u^+ \in \mathbb{R}^+ \cup \{\infty\}$, $u^- \in \mathbb{R}^- \cup \{-\infty\}$. (This will be the case if, for example, the vacuum Einstein equations are imposed.) This provides us with an isometric diffeomorphism $\Psi$ between $\mathcal{O}$ and the subset $\mathcal{U} = \{p \in \Sigma, u^-(p) < u < u^+(p)\} \subset M$, where $(M, g_{\mu\nu})$ denotes the Killing development of $(\Sigma, g_{ij}, K_{ij})$ based on $(N_0, Y_0^i)$†. One can thus gain insight into the structure of the Killing orbits in $\hat{M}$ by studying that of the Killing orbits of $M$: indeed, if the orbit of a Killing vector field through a point $q \in \mathcal{U}$ *always* remains in $\mathcal{U}$, then one will obtain a complete description of the corresponding orbit of the corresponding Killing vector field on $\hat{M}$. We wish to point out the following result, which is a straightforward consequence of the results of section 4 and of [1, 4].

*Theorem.* Let $(\Sigma, g_{ij}, K_{ij})$ be an asymptotically flat end in the sense of [4], i.e. $\Sigma \equiv \Sigma_R \equiv \mathbb{R}^3 \backslash B(R)$, $R > 0$ with $(g_{ij}, K_{ij})$ satisfying‡

$$g_{ij} - \delta_{ij} = O_k(r^{-\alpha}), \qquad K_{ij} = O_{k-1}(r^{-1-\alpha}), \tag{5.1}$$

with $k \geqslant 3$ and $\alpha > \frac{1}{2}$. Let $|\rho| + |J^i| = \mathrm{O}(r^{-3-\varepsilon})$, $\varepsilon > 0$, and let the ADM 4-momentum of $\Sigma$ be timelike. Consider a tensor field $\tau_{ij}$ on $\Sigma$ satisfying $|\tau_{ij}| = \mathrm{O}(r^{-3-\varepsilon})$, and let $\boldsymbol{W}$ denote the set of solutions $(N, Y^i)$ of the equations (2.6) and (2.13), suppose that $\boldsymbol{W}$ is closed under the bracket (2.15). Assume finally that there exists $(N_0, Y_0^i) \in \boldsymbol{W}$ such that $N_0 > 0$, and let $(M, g_{\mu\nu})$ be the Killing development of $(\Sigma, g_{ij}, K_{ij})$ based on $(N_0, Y_0^i)$. Then for every $(N, Y^i) \in \boldsymbol{W}$ there exists a constant $a \in \mathbb{R}$ such that the KID $(\hat{N}, \hat{Y}^i)$ defined as $(N, Y) + a(N_0, Y_0^i)$ gives rise to a Killing vector on $(M, g_{\mu\nu})$ which has *complete periodic orbits*, through those points $p$ in the asymptotically flat region for which $r(p)$ is large enough. Moreover, the set $\{\hat{N} = \hat{Y}^i = 0\}$ is not empty.

*Remarks.* (i) The theorem proved in section 3 gives a necessary and sufficient condition for $\boldsymbol{W}$ to be closed under the bracket $\{\cdot, \cdot\}$. This condition is trivially satisfied in vacuum $(\rho = J_i = \tau_{ij} = 0)$. As mentioned at the end of section 3, it is also satisfied in electrovacuum when the KIDs correspond moreover to 'a symmetry' of the initial data of the Maxwell field. More precisely, let $E_i$, $B_i$ be vector fields on $(\Sigma, g_{ij}, K_{ij})$ satisfying

$$\begin{aligned}
&\rho = \tfrac{1}{2}(E_i E^i + B_i B^i) \\
&J_i = \varepsilon_i{}^{jk} E_j B_k \equiv (E \times B)_i \\
&\tau_{ij} = \tfrac{1}{2} g_{ij} \rho - (E_i E_j + B_i B_j) \\
&\mathcal{L}_Y E_i = N K E_i - 2N K_i{}^j E_j - N(D \times B)_i - (DN \times B)_i \\
&\mathcal{L}_Y B_i = N K B_i - 2N K_i{}^j B_j + N(D \times E)_i + (DN \times E)_i .
\end{aligned} \tag{5.2}$$

These conditions arise as follows. Let $F_{\mu\nu} = F_{[\mu\nu]}$ be a 2-form on spacetime $(M, g_{\mu\nu})$ with $(\Sigma, g_{ij}, K_{ij})$ a spacelike submanifold and Killing vector $X = N n^\mu \delta_\mu + Y^i \delta_i$. Write

$$F_{\mu\nu} = 2E_{[\mu} n_{\nu]} + \epsilon_{\mu\nu\rho\sigma} B^\rho n^\sigma , \tag{5.3}$$

with $E_\mu n^\mu = 0 = B_\mu n^\mu$ and $\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = -24$, $\epsilon_{0123} > 0$. Then let

$$G_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho - 1/4\, g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} . \tag{5.4}$$

† Some results concerning the question, under which conditions $\mathcal{U} = \hat{M}$, $\mathcal{O} = M$, can be found in [10–12].

‡ We write $f = O_k(r^{-\alpha})$ if there exists a constant $C$ such that $|r^{-\alpha} f| + \cdots + |r^{-\alpha-k} \partial_{i_1 \ldots i_k} f| \leqslant C$.

Equation (5.4) implies the first three conditions of (5.2). Now impose $\mathcal{L}_X F_{\mu\nu} = 0$ and

$$\nabla^\mu F_{\mu\nu} = 0 \qquad \nabla_{[\mu} F_{\nu\rho]} = 0. \tag{5.5}$$

Then the first of equations (5.5) implies the fourth of (5.2) and the second of equations (5.5) implies the fifth of (5.2).

We claim that equations (5.2) imply equation (3.3). In checking that one can use the following identity:

$$A_i(B \times C)_j + C_i(A \times B)_j + B_i(C \times A)_j = g_{ij} A^k (B \times C)_k, \tag{5.6}$$

where $A_i, B_i, C_i$ are vector fields on $\Sigma$.

(ii) The condition that $\{\hat{N} = \hat{Y}^i = 0\} \neq 0$ is the usual condition of axi-symmetry. This condition is needed, for example, to be able to perform the reduction of the axi-symmetric stationary electrovacuum equations to the well known harmonic map equation.

*Proof.* By the proposition in section 3 the Lie algebra of Killing vector fields of $(M, g_{\mu\nu})$ is isomorphic to the Lie algebra of KIDs. The result is obtained by a repetition of the arguments of [1]; no details will be given. Let us simply point out that the hypothesis of completeness of Killing orbits made in theorem 1.2 of [1] was done purely for the sake of simplicity of the presentation of the results proved. □

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