Hyperboloidal Cauchy Data for Vacuum Einstein Equations and Obstructions to Smoothness of Null Infinity

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Various works have suggested that the Bondi-Sachs-Penrose decay conditions on the gravitational field at null infinity are not generally representative of asymptotically flat spacetimes. We have made a detailed analysis of the constraint equations for "asymptotically hyperboloidal" initial data and find that log terms arise generically in asymptotic expansions. These terms are absent in the corresponding Bondi-Sachs-Penrose expansions, and can be related to explicit geometric quantities. We have nevertheless shown that there exists a large class of "nongeneric" solutions of the constraint equations, the evolution of which leads to spacetimes satisfying the Bondi-Sachs-Penrose smoothness conditions.

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One of the milestone predictions of general relativity is that of the existence of gravitational radiation. A framework for analyzing that aspect of the theory was proposed by Bondi, van der Burg, and Metzner [1] and Sachs [2]: In [1,2] it was assumed that the metric admits an expansion in terms of inverse powers of r along lightlike directions, where r is a luminosity distance (cf. also [3] for some significant early analysis). It was subsequently realized by Penrose [4] that the Bondi-Sachs asymptotic conditions are related to (and in fact equivalent to) the possibility of completing the spacetime by adding to it a conformal boundary. In the Penrose approach the hypothesis of the existence of expansions in terms of r^{-i} is replaced by assumptions on the regularity of the conformally rescaled metric near the conformal boundary. Both in the Bondi-Sachs and in the Penrose framework the basic open question was and still is the following: Are the Bondi-Sachs-Penrose asymptotic conditions compatible with the behavior of the gravitational field in a "sufficiently large" class of physical situations? It should be stressed that until 1986, those metrics which were known to satisfy the appropriate asymptotic conditions (cf., e.g., [5] for a review), namely, the boost-rotation symmetric spacetimes and the Robinson-Trautman spacetimes (cf. also [6]), were of a rather special kind.

Significant progress towards the understanding of this question was made by Friedrich, who showed [7] that spacetimes satisfying the Bondi-Sachs-Penrose conditions could be constructed provided that sufficiently well behaved initial data "of hyperboloidal type" could be found. The question of the "largeness" of the set of spacetimes admitting smooth conformal completions was consequently shifted to the question of the largeness of the set of initial data satisfying the conditions spelled out by Friedrich. The possibility of constructing appropriate initial data was demonstrated in [8] under, however, some rather restrictive hypotheses on the extrinsic curvature of the Cauchy surface. In this Letter we present the results of [9] and of [10] concerning the possibility of constructing appropriate initial data using the conformal Choquet-Bruhat-Lichnerowicz-York [11] method. Our main results are the following: For generic "seed fields" the Cauchy data constructed by the conformal method starting from a "smoothly conformally compact" Riemannian manifold do *not* possess the asymptotic regularity compatible with Penrose's smoothness requirements for null infinity ($\mathcal{J}$). [The notion of nongenericity referred to here is made clear by Eqs. (10) and (11) below.] On the other hand, there exists a large class of nongeneric "backgrounds" for which the solutions display the required regularity. The "generic nonsmoothness of $\mathcal{J}$" suggested by our analysis confirms a similar observation of Christodoulou and Klainerman [12]. It is also fully consistent with Winicour's [13] analysis of Bondi-type initial data (cf. also [14]), as well as with various approximation calculations (cf. [15] and references therein).

Let us recall the Lichnerowicz-Choquet-Bruhat-York method for constructing solutions of the constraint equations (1),(2) of general relativity, with asymptotic conditions appropriate to the "hyperboloidal" initial value problem [7]; cf. also [8,9]. Let $\overline{M}$ be a smooth, connected, Hausdorff, compact three-dimensional manifold with smooth boundary, set $M=\mathrm{int}\overline{M}$, $\partial\overline{M}=\partial M\equiv\overline{M}\backslash M$. For $0\le k\le\infty$ let $C^k(\overline{M})$ denote the space of k times differentiable tensor fields on M which, in local coordinates near ∂M, extend continuously to ∂M together with all derivatives of order less than or equal to k. We wish to find initial data $(\tilde g,\tilde K)$ which solve the general relativistic constraint equations

$$R(\tilde g)=\tilde g_{ij}\tilde g_{kl}\tilde K^{ik}\tilde K^{jl}-(\tilde g_{ij}\tilde K^{ij})^2\,, \tag{1}$$

$$\tilde D_i(\tilde K^{ij}-\tilde g_{kl}\tilde K^{kl}\tilde g^{ij})=0\,, \tag{2}$$

where $\tilde D_i$ is the Levi-Civita connection of $\tilde g_{ij}$, and $R(f)$ is the Ricci scalar of a metric f, satisfying the condition

$$\tilde K\equiv\tilde g_{ij}\tilde K^{ij}=3\,. \tag{3}$$

We may construct such data from a set of "seed fields" (g_{ij},A^{ij}), where g_{ij} is any smooth Riemannian metric on M extending smoothly to ∂M, and A^{ij} is any symmetric, traceless tensor field on M extending smoothly to ∂M, via the following procedure: Let x be any defining function for ∂M, i.e., a function satisfying $x \in C^\infty(\overline{M})$, $x \geq 0$, $x(p)=0 \Longleftrightarrow p \in \partial M$, and $dx \neq 0$ at ∂M. Let $A^{ij} \in C^\infty(\overline{M})$ be symmetric traceless and let X be any solution of the equation

$$D_i[x^{-3}(D^iX^j+D^jX^i-\tfrac{2}{3}D_kX^kg^{ij})]=-D_i(x^{-2}A^{ij})\,; \tag{4}$$

define

$$L^{ij}\equiv\frac{\omega^2}{x^3}\left(D^iX^j+D^jX^i-\frac{2}{3}D_kX^kg^{ij}\right)+\frac{\omega^2}{x^2}A^{ij}, \tag{5}$$

where ω is a solution of the equation

$$\omega\Delta_g\omega-\tfrac{3}{2}|D\omega|_g^2+\tfrac{1}{4}\omega^2[R(g)-|L|_g^2]+\tfrac{3}{2}=0\,, \tag{6}$$

satisfying $\omega \geq 0$, with ωx^{-1} uniformly bounded on M from above and uniformly bounded away from zero, and where $|L|_g^2=g_{ij}g_{kl}L^{ik}L^{jl}$. Setting

$$\tilde{g}_{ij}=\omega^{-2}g_{ij}\,,\quad \tilde{K}^{ij}=\omega^3L^{ij}+\tilde{g}^{ij}\,,$$

one obtains a solution of (1),(2) satisfying (3). Moreover, when X is chosen appropriately the fields $(\tilde{g}_{ij},\tilde{K}^{ij})$ satisfy (in a rough way) the asymptotic conditions appropriate to the "asymptotically hyperboloidal" setting (cf. [7,9]). The asymptotic behavior of the initial data as described above is somewhat reminiscent of that which occurs for initial data induced on a standard hyperboloid in Minkowski spacetime, whence the terminology above.

It should be pointed out that Eqs. (4)-(6) constitute a nonlinear system of elliptic equations uniformly degenerating at the boundary ∂M, which is at the origin of various difficulties. Although a vast literature on such problems exists (cf. [16] and references therein), no detailed information of the kind needed, e.g., for Friedrich's evolution theorems [7] can be found.

For simplicity we shall assume throughout this Letter that $\partial M \approx S^2$—the two-dimensional sphere. In [9] the following has been shown: (1) For any (g,A) as above one can find a solution X to (4) such that

$$(x/\omega)^2L^{ij}=U^{ij}+x^2\log x U^{ij}_{\log}\,,\quad U^{ij},U^{ij}_{\log}\in C^\infty(\overline{M})\,.$$

Given any g there exists an open dense set [in the $C^\infty(\overline{M})$ topology] of A's for which $U^{ij}_{\log}|_{\partial M}\neq 0$ (however, there exists an infinite dimensional closed subspace of A's for which $U^{ij}_{\log}|_{\partial M}\equiv 0$). If $U^{ij}_{\log}|_{\partial M}\equiv 0$, then $U^{ij}_{\log}\equiv 0$ and thus $x^2\omega^{-2}L^{ij}\in C^\infty(\overline{M})$. Let us also note that in an orthonormal frame e_i such that $e_A \| \partial M$, $A=2,3$, if we write, in a neighborhood of ∂M,

$$L^{ij}=L_0^{ij}(v)+xL_1^{ij}(v)+\cdots\,,$$

where v denotes coordinates on ∂M, then we have $L_0^{1i}\equiv 0$, while both $L_0^{AB}(v)-\frac{1}{2}L_0^{CD}h_{CD}h^{AB}$ and $L_1^{AB}(v)-\frac{1}{2}L_1^{CD}h_{CD}h^{AB}$ are freely specifiable tensor fields on ∂M. X is unique in an appropriate class of functions; cf. [9] for details. (2) For any (g,A) as above one can find a solution $\omega\in\mathcal{A}_{\rm phg}$ of Eq. (6), where $\mathcal{A}_{\rm phg}$ denotes the space of polyhomogeneous functions on $\overline{M}$: more precisely, there exists a sequence $\{N_j\}_{j=0}^\infty$ with $N_0=N_1=N_2=N_3=0$, $N_4=1$ and functions $\omega_{ij}\in C^\infty(\overline{M})$ such that

$$\omega\sim\sum_{i\geq 0}\sum_{j=0}^{N_i}\omega_{i,j}x^i\log^j x\,, \tag{7}$$

where "$\sim$" means "asymptotic to," in the sense that ω minus an appropriately truncated sum of the form given by the right-hand side of (7) vanishes faster than x^n with any desired n, and that this property is preserved under differentiation in the obvious way. According to standard terminology, functions with these properties are called *polyhomogeneous*; cf., e.g., [17]. For an open dense set of (g,A)'s we have $\omega_{4,1}|_{\partial M}\not\equiv 0$. If $\omega_{4,1}|_{\partial M}\equiv 0$, then $\omega\in C^\infty(\overline{M})$.

Suppose now that one has initial data such that the log terms described above do not vanish. In such a case the metric will immediately pick up log terms when time-evolved with Einstein equations, so that at later times there will be no decomposition of the three-dimensional metric into a smooth up to the boundary background and a conformal factor. This shows that it is natural to consider the above construction under the condition that the seed fields are polyhomogeneous rather than smooth. One can show [9] that for any polyhomogeneous Riemannian metric g on $\overline{M}$ and for any uniformly bounded polyhomogeneous symmetric tensor field A^{ij} on $\overline{M}$ there exist solutions (X,ω) of (4) and (6) such that L^{ij} given by (5) is polyhomogeneous and uniformly bounded on $\overline{M}$, and ω/x is polyhomogeneous, uniformly bounded, and uniformly bounded away from zero on $\overline{M}$.

When $L^{ij}\equiv 0$ and the seed metric is smooth up to the boundary, the obstructions to smoothness of ω have been analyzed in detail in [8]. In that reference it has been shown, in particular, that $\omega_{4,1}|_{\partial M}$ vanishes if the Weyl tensor of the unphysical (conformally rescaled) spacetime metric is bounded near ∂M (cf. [8] for details). In [10] we have extended that analysis to the case $L^{ij}\not\equiv 0$. In order to present our results it is useful to define two tensor fields $\sigma^\pm$ defined on the conformal boundary ∂M of the initial data surface:

$$\sigma^\pm_{AB}\equiv\left(\lambda_{AB}-\frac{h^{CD}\lambda_{CD}}{2}h_{AB}\right)\pm\left(K_{AB}-\frac{h^{CD}K_{CD}}{2}h_{AB}\right)\Bigg|_{\partial M}, \tag{8}$$

which we shall call the *shear tensors of* ∂M. Here h_{AB} is the induced metric on ∂M, λ_{AB} is the extrinsic curvature of ∂M in (M,g), and K_{ij} is the extrinsic curvature of M in the conformally rescaled spacetime metric.

Let us say that a spacetime admits a polyhomogeneous $\mathcal{J}$ if the conformally rescaled metric is polyhomogeneous at the conformal boundary; i.e., in local coordinates the components of the conformally rescaled metric are bounded and polyhomogeneous. In the case of Cauchy data constructed as described above starting from smooth seed fields, the results of [10] linking the geometry of the boundary of the initial data surface with the geometry of the resulting spacetime can be summarized as follows: (1) Suppose that neither σ^+ nor σ^- vanishes. Then *there exists no development of the initial data with a smooth or polyhomogeneous* $\mathcal{J}$. (2) Suppose that $\sigma^+ \equiv 0$ or $\sigma^- \equiv 0$; changing the time orientation if necessary we may without loss of generality assume that $\sigma^+ \equiv 0$. Let $K_{ij}^{\log}$ denote "the logarithmic part" of K_i:

$$K_{ij} = \hat{K}_{ij} + x^2 \log x K_{ij}^{\log} ,$$

$$\hat{K}_{ij} \in C^{\infty}(\overline{M}), \quad K_{ij}^{\log} \in C^0(\overline{M}) \cap \mathcal{A}_{\mathrm{phg}} .$$

The the following holds: (a) If $K_{1A}^{\log}|_{\partial M} \not\equiv 0$, then if there exists a development with a polyhomogeneous $\mathcal{J}$, it is essentially polyhomogeneous, i.e., no development with a smooth $\mathcal{J}$ exists. [The vanishing of $K_{1A}^{\log}|_{\partial M}$ is actually equivalent (under the present assumptions) to the vanishing at the conformal boundary of the Weyl tensor of the conformally rescaled metric.] (b) Suppose instead that $K_{1A}^{\log}|_{\partial M} \equiv 0$ and $K_{11}^{\log}|_{\partial M} \equiv 0$. Then *there exists a development which admits a smooth conformal boundary.*

It should be stressed that the results linking the log terms with the nonvanishing of the Weyl tensor proved in [10] show that the occurrence of shear and of at least some of the log terms in asymptotic expansions of physical fields at $\mathcal{J}$ is not an artifact of a bad choice of a conformal factor, or of a pathological choice of the initial data hypersurface (within the class of uniformly bounded from above and uniformly bounded away from zero, locally C^2, conformal factors and C^1 deformations of the initial data hypersurface which fix ∂M): If $\mathcal{J}$ is *not* shear-free (by which we mean that none of the shear tensors $\sigma^{\pm}$ vanish), then no conformal transformation will make it shear free. Similarly if the Weyl tensor does not vanish at ∂M, then no "gauge transformation" in the above sense will make it vanish (cf. [10] for a more detailed discussion).

The conditions for smoothness up to boundary of an initial data set can be expressed as *local* conditions on the boundary on the seed fields (g_{ij}, A^{ij}). Let (x, v^A) be a Gauss coordinate system near ∂M; the interesting case is the one in which one of the shear tensors of the conformal boundary vanishes, which corresponds to the condition that, changing A_{ij} to $-A_{ij}$ if necessary,

$$(\lambda_{AB} - \tfrac{1}{2} \lambda h_{AB})|_{\partial M} = (A_{AB} - \tfrac{1}{2} h^{CD} A_{CD} h_{AB})|_{\partial M} . \quad (9)$$

It can be shown that without loss of generality in the construction of the initial data one can assume that

$$A_{1j}|_{\partial M} = 0 ,$$

and in what follows we shall assume that this condition holds—the equations below would have been somewhat more complicated without this condition. Similarly, it is useful to choose a "conformal gauge" such that

$$\lambda \equiv h^{AB} \lambda_{AB}|_{\partial M} = 0 .$$

Then the conditions for smoothness up to the boundary of ω and L^{ij} reduce to

$$[\mathcal{D}^A \mathcal{D}^B \lambda_{AB} + R_{AB} \lambda^{AB}]|_{\partial M} = 0 , \quad (10)$$

where $\mathcal{D}$ is the covariant derivative operator of the metric h induced from g on ∂M, R_{ij} is the Ricci tensor of g, and

$$[\partial_x A_{AB} - \tfrac{1}{2} h^{CD} \partial_x A_{CD} h_{AB}]|_{\partial M} = 0 . \quad (11)$$

Failure of (10) or (11) will lead to the occurrence of some log terms in the initial data set.

The overall picture that emerges from the results of [9,12–15] and from the results described here is that the usual hypotheses of smoothness of $\mathcal{J}$ are overly restrictive. These results seem to indicate that a possible self-consistent setup for an analysis of the gravitational radiation is that of *polyhomogeneous* rather than smooth functions on the conformally completed manifold, i.e., functions that have asymptotic expansions in terms of powers of x and $\log x$ rather than of x only. It should, however, be stressed that even though the fact that the physical fields $(\tilde{g}, \tilde{K})$ satisfy the constraint equations guarantees the existence of some vacuum development $(V, \tilde{\gamma})$, it is by no means obvious that in the case when, e.g., $\sigma^+ \equiv 0$, the existence of some kind of compactification of $(M, \tilde{g}, \tilde{K})$ implies the existence of some useful conformal completion of $(V, \tilde{\gamma})$. In the case of *polyhomogeneous hyperboloidal* initial Cauchy data the question of existence of conformal completions of $(V, \tilde{\gamma})$ with three-dimensional boundaries is the most important unsolved mathematical problem of the present theory. Nevertheless we expect that polyhomogeneous initial data of the kind constructed in [9] for which the shear of $\mathcal{J}$ vanishes will lead to spacetimes with metrics which along lightlike directions admit expansions in terms of $r^{-j} \log^i r$ [18], rather than in terms of r^{-j} as postulated in [1,2] (cf. [14] for a more detailed discussion of that question).

The results presented here immediately lead to the following question: how much *physical* generality does one lose by restricting oneself to Cauchy data which satisfy the conditions (10), (11)? These conditions are similar in spirit to those of Bondi, van der Burg, and Metzner [1], who impose conditions on the r^{-2} terms in the "free part of the metric" at $u=0$ to avoid the occurrence of $r^{-j} \log^i r$ terms in the metric at later times. By doing so, or by imposing (10), (11), one gains the luxury of working with smooth conformal completions, avoiding all the complications which arise due to the occurrence of log terms—but, then, does one overlook some *physically* *significant* features of radiating gravitating systems?

To obtain a real understanding of gravitational radiation, it is therefore necessary to establish what asymptotic conditions are appropriate from a physical point of view. The following are some criteria which might be considered as physically desirable: (i) existence of a well defined notion of total energy; (ii) existence of a well defined notion of angular momentum; (iii) existence of a development (V,γ) of the initial data set which admits a $\mathcal{J}$ with a reasonable regularity; and (iv) existence of a development of the data up to i^+. In some situations it might be appropriate to impose only part of the above conditions. On the other hand it might perhaps be appropriate to add to the above the requirement that the function spaces considered include those data sets which arise by evolution from generic initial data which are asymptotically flat at spatial infinity.

We would like to emphasize that *it is not known* what regularity conditions on the conformally compactified metric are *necessary* for any of the above criteria to hold.

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[18] The embedding of the initial data set into an asymptotically flat spacetime constructed in [10] gives a spacetime metric with a Bondi-type expansion which contains in general $r^{-1}\log r$ terms. It could, however, happen that there exist better embeddings for which the first log terms occur with higher powers of r^{-1}.