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On Space-Times with $U(1) \times U(1)$ Symmetric Compact Cauchy Surfaces

PIOTR T. CHRUŚCIEL*

Yale University, Physics Department, New Haven, Connecticut 06511

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General space-times evolving from $U(1) \times U(1)$ symmetric Cauchy data prescribed on compact Cauchy surfaces are studied. Existence and properties of solutions of the constraint equations are analyzed. Some "canonical" forms of the metric are derived. When the spatial topology is S^3 or $S^2 \times S^1$ or $L(p, q)$ we show that no singularities form before "the spacelike boundary of Gowdy's square" is reached. © 1990 Academic Press, Inc.

1. INTRODUCTION

One of the outstanding problems in classical general relativity is the understanding of the long time behavior of solutions of Einstein equations. The nonlinearity and the diffeomorphism invariance of these equations makes this problem extremely difficult to handle. As a first step towards development of methods which would eventually lead to an understanding of the general picture one might envisage trying to attack this problem under some restrictive conditions. In the cosmological context it seems natural to consider space-times which develop from Cauchy data prescribed on a compact, connected, orientable Cauchy surface.[1] To further simplify things one may wish to assume that the Cauchy data are invariant under some symmetry group. With the topological restrictions as noted above several choices of the symmetry group are possible (cf. [7] for an exhaustive list), the smallest of which being the group $U(1)$. Although remarkable simplifications of the problem may be achieved by this choice of the symmetry group [15], the understanding of the long time behavior of such vacuum metrics seems still to be a remote goal. The next simplest possible isometry group is $U(1) \times U(1)$, and it is the object of this paper to discuss the most general metrics on globally hyperbolic space-times which evolve from Cauchy data invariant under an effective action of $G = U(1) \times U(1)$ on a compact three dimensional (connected, orientable) manifold ${}^3\Sigma$. It seems that the first fairly exhaustive (and elegant) treatment of metrics with two commuting Killing vectors is due to Geroch [9] (under the assumption,

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[1] In what follows all manifolds are Hausdorff, orientable, connected, paracompact, and smooth.

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however, that one of the Killing vectors is timelike). In the cosmological context $U(1)\times U(1)$ symmetric metrics with spacelike group orbits have been probably first discussed by Gowdy [10], who implicitly or explicitly imposes some supplementary restrictions on the metrics. In this paper we derive some properties of the most general $U(1)\times U(1)$ symmetric metrics (in particular we give some properties of a large class of $U(1)\times U(1)$ symmetric metrics on T^3 which were not covered by Gowdy's paper [10]). In Section 2 we present a "Gowdy-type" way of reducing Einstein equations. In Section 3 an "Ernst-type" parametrization of the metric is presented, and a generalization of the so-called "Ernst equations" is discussed. In Section 4 we discuss existence and some properties of solutions of constraint equations on T^3. In the special case in which some of the constants appearing in the problem are set to zero we derive a "canonical form" of the metric for this spatial topology. In Section 5 we discuss the constraint equations under the hypothesis ${}^3\Sigma \approx S^2\times S^1$ or S^3. In this case we isolate a subset of the set of all $U(1)\times U(1)$ invariant Cauchy data which we show to be generic, and we show that generic Cauchy data can be realized by embedding the Cauchy surface in a "canonical" space-time. In Section 6 we show that "canonical" space-times are "as large as the area function R allows," where R is the area (up to a multiplicative constant) of the group orbits. This result can be thought of as a long-time existence theorem for the problem at hand, since it consists in proving no blow-up of the metric before R goes to zero. The proof is a simple consequence of a difficult result due to Christodoulou [4].

Throughout this paper we shall assume the metric to be smooth: this condition can be considerably relaxed, and because of the essentially two dimensional character of the problem one can even go beyond the weak differentiability conditions of [12]. Let us however note that some of the constructions presented here lead to a loss of one or even two degrees of differentiability of the metric, and it is thus for smooth metrics that the theory is the most elegant.

Before closing this section, let us recall some well-known topological constraints which follow from the hypothesis of a $U(1)\times U(1)$ effective[2] action on a compact three manifold ${}^3\Sigma$ [18]: if one moreover assumes that ${}^3\Sigma$ is connected and orientable it follows that ${}^3\Sigma \simeq T^3$ $(=S^1\times S^1\times S^1$; S^n stands for the n-dimensional sphere) or $S^2\times S^1$ or S^3 or $L(p,q)$ (= the lens space), moreover the action is unique up to equivalence (i.e., up to an automorphism of $U(1)\times U(1)$ and a diffeomorphism of ${}^3\Sigma$ [19]; cf. also [7] for a dicussion of such results in a general relativistic context). One can check that the arguments presented in this paper in the S^3 case remain valid in the $L(p,q)$ case, and in what follows when speaking of S^3 we shall implicitly have in mind $L(p,q)$ as well.[3] For $a=1,2$, let $\bar{x}^a$ be standard coordinates on $U(1)\times U(1)$, i.e., $U(1)\times U(1)=\{(e^{i\bar{x}^1}, e^{i\bar{x}^2}),\ \bar{x}^a\in[0,2\pi]\,|_{\mathrm{mod}\,2\pi}\}$; the

[2] Recall that an action of G is effective if the implication $(\forall p\in M,\ gp=p)\Rightarrow(g=e)$ holds.

[3] At the origin of the nonappearance of any supplementary phenomena in the $L(p,q)$ case, as compared to S^3, is the fact that the covering $S^3\to L(p,q)$ is the projection map of the quotient of S^3 by a subgroup of the isometry group of the metric.

parameters $\bar{x}^a$ are defined uniquely up to a $SL(2, \mathbb{Z})$ transformation. Choosing on each orbit of G on ${}^3\Sigma$ a reference point p_0 one sets $x^a(p) = (\bar{x}^1, \bar{x}^2)$, where $g = (e^{i\bar{x}^1}, e^{i\bar{x}^2})$ is such that $gp = p_0$. From the theory of [18, 19] it follows that

(1) for ${}^3\Sigma \approx T^3$ the orbit space ${}^3\Sigma/G$ is a circle, the action of G on ${}^3\Sigma$ is free and thus ${}^3\Sigma$ can be parametrized by x^a and $\theta \in {}^3\Sigma/G \approx [0, 2\pi]|_{\mathrm{mod}\, 2\pi} \approx S^1$; the coordinates x^a are defined uniquely up to a $SL(2, \mathbb{Z})$ transformation and a "translation" on each orbit,

(2) for ${}^3\Sigma \approx S^3$ the orbit space ${}^3\Sigma/G$ is a closed interval, say $[0, \pi]$. There exist two (linked) circles S^1_1 and S^2_1 such that $S^3 \backslash \{S^1_1 \cup S^1_2\}$ is diffeomorphic to $(0, \pi) \times G$, and $S^3 \backslash \{S^2_1 \cup S^1_2\}$ can thus be parametrized by x^a and $\theta \in (0, \pi)$. One can require that on S^1_1 the, say, second factor $U(1)$ of G acts trivially, and that on S^1_2 the first factor $U(1)$ of G acts trivially and these requirements remove the freedom of performing $SL(2, \mathbb{Z})$ transformations of the x^a's, which thus become defined uniquely up to a "translation" on each orbit,

(3) ${}^3\Sigma \approx S^2 \times S^1$; the orbit space ${}^3\Sigma/G$ is again an interval, say $[0, \pi]$. If one removes the, say, north and south poles of S^2 one obtains a set diffeomorphic to $(0, \pi) \times G$. The requirement that the second $U(1)$ factor of G acts as a rotation of S^2 around the, say, z-axis, removes the $SL(2, \mathbb{Z})$ freedom and the x^a's are again defined modulo translations on each orbit.

The conventions and notations used in this paper are listed in Appendix F.

2. The Field Equations—A "Gowdy-Type" Parametrization of the Metric

Let ${}^3\Sigma$ denote one of the manifolds discussed in Section 1, ${}^3\Sigma \approx S^3(L(p, q))$ or $S^2 \times S^1$ or T^3, let[4] (g_{ij}, P_{ij}) be $U(1) \times U(1)$ invariant Cauchy data on ${}^3\Sigma$; by Cauchy data we shall always mean smooth fields (g_{ij}, P_{ij}) satisfying the vacuum constraint equations. According to the standard theory (cf., e.g., [3]) there exists an $\varepsilon > 0$ and a four dimensional Lorentzian manifold $({}^4\mathcal{M}, g_{\mu\nu})$, ${}^4\mathcal{M} \approx (-\varepsilon, \varepsilon) \times {}^3\Sigma$, called a development of $(g_{ij}, P_{ij}, {}^3\Sigma)$, on which vacuum Einstein equations are satisfied, with $g_{\mu\nu}|_{\{0\} \times {}^3\Sigma}$, $\partial_\sigma g_{\mu\nu}|_{\{0\} \times {}^3\Sigma}$ related to g_{ij} and P_{ij} by well-known equations [3]. From the uniqueness of $g_{\mu\nu}$ in harmonic coordinates it follows that if t is harmonic, then the action of G on ${}^4\mathcal{M}$ defined by $g(t, p) = (t, gp)$, $t \in (-\varepsilon, \varepsilon)$, $p \in {}^3\Sigma$, is an action by isometries; in particular if ε is chosen small enough the orbits of G are spacelike in ${}^4\mathcal{M}$. Let ${}^4\tilde{\mathcal{M}} = \{p \in {}^4\mathcal{M}$: the orbit of G through p is two-dimensional$\}$. From what has been said it follows that

(1) if ${}^3\Sigma = T^3$, then ${}^4\tilde{\mathcal{M}} = {}^4\mathcal{M}$;

(2) if ${}^3\Sigma = S^2 \times S^1$ or ${}^3\Sigma = S^3$, then ${}^4\tilde{\mathcal{M}}$ is ${}^4\mathcal{M}$ with two (smooth, embedded) submanifolds each diffeomorphic to $(-\varepsilon, \varepsilon) \times S^1$ removed.

[4] K_{ij} denotes the extrinsic curvature of ${}^3\Sigma$, $P_{ij} = g^{kl}K_{kl}g_{ij} - K_{ij}$.

${}^4\tilde{\mathscr{M}}/G$ is a two dimensional manifold and one can use the methods developed by Geroch [9] to obtain a "reduced form" of the evolution part of Einstein equations. Let X_a^μ, $a=1, 2$, be two linearly independent Killing vector fields on ${}^4\tilde{\mathscr{M}}$. In most situations a convenient choice is $X_1=\partial/\partial x^1$, $X_2=\partial/\partial x^2$; unless explicitly stated otherwise this choice will be made; in some cases, however, as discussed in Section 3, it is useful not to make this hypothesis. Let us define

$$\lambda_{ab}=g_{\mu\nu}X_a^\mu X_b^\nu. \tag{2.1}$$

Since the X_a's are linearly independent and spacelike it follows that on ${}^4\tilde{\mathscr{M}}$,

$$R^2\equiv\det\lambda_{ab}>0, \tag{2.2}$$

and therefore the matrix

$$\lambda^{ab}=(\lambda_{ab})^{-1}$$

exists. In what follows the indices μ, ν, ... are raised and lowered with $g^{\mu\nu}$ and $g_{\mu\nu}$, while the indices a, b, etc., are raised and lowered with λ^{ab} and λ_{ab} (e.g., $X^{a\mu}=\lambda^{ab}X_b^\mu$, etc.). As has been discussed by Geroch [9], the tensor field

$$h_{\mu\nu}=g_{\mu\nu}-\lambda^{ab}X_{a\mu}X_{b\nu} \tag{2.3}$$

defines naturally a metric on ${}^4\tilde{\mathscr{M}}/G$. Let

$$c_a=\varepsilon_{\mu\nu\rho\sigma}X_1^\mu X_2^\nu\nabla^\rho X_a^\sigma.$$

As has been shown by Geroch [9] (cf. also [14])

$$\partial_\mu c_a=0. \tag{2.4}$$

It has also been shown by Geroch that it follows from Einstein equations that the functions λ_{ab} satisfy the equations[5] on ${}^4\tilde{\mathscr{M}}/G$,

$$\mathscr{D}^A\mathscr{D}_A\lambda_{ab}=\lambda^{cd}(\mathscr{D}^A\lambda_{ac}\mathscr{D}_A\lambda_{bd}-\mathscr{D}^A\lambda_{ab}\mathscr{D}_A\lambda_{cd})-c_ac_b/R^2, \tag{2.5}$$

where $\mathscr{D}_A$ denotes the covariant Riemannian derivative of the metric h_{AB}; and the Lorentzian metric h_{AB} satisfies the evolution equation

$$R(h)=-\frac{3}{2}\frac{\lambda^{ab}c_ac_b}{R^2}+\frac{1}{4}\varepsilon^{ad}\varepsilon^{bc}\mathscr{D}^A\lambda_{ac}\mathscr{D}_A\lambda_{bd}, \tag{2.6}$$

where $R(h)$ is the scalar curvature of h. To discuss the constraint equations it will be useful to introduce on ${}^4\tilde{\mathscr{M}}$ coordinates compatible with the action of the group:

[5] Equations (2.5)–(2.6) can be obtained from Geroch's [9] Eqs. (A18)–(A21) by substituting $\tau=\sqrt{2}iR$; the "i" ($i^2=-1$) accounting for the spacelike rather than timelike character of the group orbits considered here: the reader who feels uneasy with such formal complex substitutions may wish to derive (2.5)–(2.6) directly using the methods of Ref. [9]—the results of Appendix B of this paper can be used to make easier some of the calculations outlined in [9].

so let x^a be the coordinates on the group orbits as discussed in Section 1, let t, $x^3=\theta$ be coordinates on ${}^4\tilde{\mathscr{M}}/G$. The metric can be written in the form

$$g_{\mu\nu}dx^\mu\,dx^\nu = h_{AB}\,dx^A\,dx^B + \lambda_{ab}(dx^a + M^a\,dt + g^a\,d\theta) \times (dx^b + M^b\,dt + g^d\,d\theta) \qquad (x^A = t, \theta), \tag{2.7}$$

where

$$g_a \equiv g_{\theta a}, \qquad M_a \equiv g_{ta}, \qquad g^a \equiv \lambda^{ab} g_b, \qquad M^a \equiv \lambda^{ab} M_b,$$
$$h_{AB}\,dx^A dx^B \equiv g_{AB} dx^A\,dx^B - \lambda_{ab}(M^a\,dt + g^a\,d\theta) \times (M^b\,dt + g^b\,d\theta) \tag{2.8}$$

and all the functions depend on t and θ only. We shall use the following notation

$$n_{ab} \equiv \lambda_{ab}/R,$$
$$h^{AB} = (h_{AB})^{-1}, \qquad h_{\theta\theta} = e^{2B}, \tag{2.9}$$
$$N = (-g^{00})^{-1/2}\ (= (-h^{00})^{-1/2}), \qquad N^\theta = N^2 g^{0\theta}.$$

As discussed in Appendix A the constraints $X_{aj} D_i P^{ij} = 0$ are equivalent to (2.4) with $\mu = \theta$, while the remaining constraints can be written in the form (cf. Eqs. (A.36) and (A.41) of Appendix A).

$$\partial_\theta \frac{\partial R}{\partial n} = \frac{\partial_\theta R(\partial R/\partial n)}{2R} + \left(\frac{\partial B}{\partial n} - \partial_\theta N^\theta\right)\partial_\theta R - \frac{R}{4} n^{ac} n^{bd} \frac{\partial n_{ab}}{\partial n}\frac{\partial n_{cd}}{\partial \theta}, \tag{2.10}$$

$$\partial_\theta^2 R = \frac{(\partial_\theta R)^2 + e^{2B}(\partial R/\partial n)^2}{4R} + e^{2B}\left(\frac{\partial B}{\partial n} - \partial_\theta N^\theta\right)\frac{\partial R}{\partial n}$$
$$+ \partial_\theta B\,\partial_\theta R - \frac{e^{2B} n^{ab} c_a c_b}{4R^2} - \frac{R}{8} n^{ab} n^{cd}\left\{\partial_\theta n_{ac}\,\partial_\theta n_{bd} + e^{2B}\frac{\partial n_{ac}}{\partial n}\frac{\partial n_{bd}}{\partial n}\right\}, \tag{2.11}$$

where

$$\frac{\partial f}{\partial n} \equiv \frac{1}{N}\frac{\partial f}{\partial t} - \frac{N^\theta}{\theta}\frac{\partial f}{\partial \theta}.$$

It is useful to introduce the "null derivatives"

$$f_+ \equiv \partial_\theta f + e^B \frac{\partial f}{\partial n},$$
$$f_- \equiv \partial_\theta f - e^B \frac{\partial f}{\partial n}, \tag{2.12}$$

in terms of which (2.10)–(2.11) can be rewritten in the form

$$\partial_\theta R_\pm = \frac{R_\pm^2}{4R} + (B_\pm - \partial_\theta N^\theta) R_\pm - \frac{e^{2B} n^{ab} c_a c_b}{4R^2} - \frac{R}{8} n^{ac} n^{bd} (n_{ab})_\pm (n_{cd})_\pm . \tag{2.13}$$

The four equations (2.4) with $\mu = t, \theta$ together with the three equations (2.5), the one equation (2.6) and the two equations (2.13) are equivalent to the vacuum Einstein equations $R_{\mu\nu} = 0$ for the metric (2.7).

We shall show that Eqs. (2.5) can be written as a coupled set of one scalar wave equation and a harmonic-type set of equations for a map from ${}^4\tilde{\mathscr{M}}/G$ to the two dimensional hyperbolic space $\mathscr{H}^2$ (= two dimensional, complete, simply connected Riemannian manifold with constant negative scalar curvature). Let us start by recalling the form of the harmonic map equations with target manifold a submanifold S of $\mathbf{R}^{n,m}$ given by the equation $\eta_{KL} y^K y^L = \varepsilon$, $\varepsilon^2 = 1$, where K, $L = 1, ..., n+m$, and where η_{KL} is a constant coefficient symmetric matrix of signature $m-n$. The metric f induced on S by η_{KL} can be identified with the following tensor field on $\mathbf{R}^{n,m}$:

$$f_{KL} = \eta_{KL} - \varepsilon y_K y_L . \tag{2.14}$$

Let (N, h) be a Riemannian or Lorentzian manifold, let $\phi: N \to S$ be a smooth map from N to S. In local coordinates $\phi^K(x) = y^K(x^A)$ the harmonic map equation takes the form

$$D^A(f, h)\, Y_A \equiv h^{AB} D_B(f, h)\, Y_A = 0, \tag{2.15}$$

where Y_B is the vector field tangent to the image of N in S defined by

$$Y_B = \frac{\partial y^K}{\partial x^B} \frac{\partial}{\partial y^K}, \tag{2.16}$$

$D_A(f, h)$ is defined by

$$\begin{aligned} D_A(f, h)\, Y_B &= D_A(f)\, Y_B - \Gamma^C_{AB}(h)\, Y_C, \\ D_A(f) &= \frac{\partial y^K}{\partial x^A} D_K(f), \end{aligned} \tag{2.17}$$

$D_K(f)$ is the Riemannian covariant derivative of the metric f on S, and $\Gamma(h)$ are the Christoffel symbols of the metric h on N. For the metric (2.14) we have

$$\begin{aligned} D_K(f)\, Z^L &= (\delta^M_K - \varepsilon y^M y_K)(\delta^L_N - \varepsilon y^L y_N)\, \partial_M Z^N, \\ y_L &\equiv \eta_{LM} y^M, \end{aligned} \tag{2.18}$$

so that

$$D_A(f)\, Y^K_B = (\delta^K_N - \varepsilon y^K y_N) \frac{\partial^2 y^N}{\partial x^A\, \partial x^B}$$

$$= \frac{\partial^2 y^K}{\partial x^A\, \partial x^B} + \varepsilon \frac{\partial y^N}{\partial x^A} \frac{\partial y_N}{\partial x^B}\, y^K$$

$$= \frac{\partial^2 y^K}{\partial x^A\, \partial x^B} + \varepsilon f_{MN} \frac{\partial y^M}{\partial x^A} \frac{\partial y^N}{\partial x^B}\, y^K, \tag{2.19}$$

therefore the harmonic map equations (2.17) read

$$\Box(h)\, y^K + \varepsilon h^{AB} f_{MN} \frac{\partial y^M}{\partial x^A} \frac{\partial y^N}{\partial x^B}\, y^K = 0, \tag{2.20}$$

where $\Box(h)$ is the Laplace (d'Alembert) operator of the metric h acting on scalars.

Returning to Eq. (2.5), a contraction of (2.5) with λ^{ab} yields

$$\Box(h) R = -\frac{n^{ab} c_a c_b}{2R^2}, \tag{2.21}$$

from which one can obtain

$$\frac{1}{R} \mathscr{D}_A(R \mathscr{D}^A n_{ab}) + (\mathscr{D}^A n_{00} \mathscr{D}_A n_{11} - \mathscr{D}_A n_{01} \mathscr{D}^A n_{01})\, n_{ab}$$

$$= -\frac{c_a c_b}{R^3} + \frac{(n^{cd} c_c c_d)\, n_{ab}}{2R^3}. \tag{2.22}$$

In the space of invertible symmetric 2×2 matrices λ_{ab} we can introduce the Lorentzian scalar product

$$\langle \lambda^1_{ab}, \lambda^2_{ab} \rangle = -\tfrac{1}{2}(\lambda^1_{00} \lambda^2_{11} + \lambda^2_{00} \lambda^1_{11} - 2\lambda^1_{01} \lambda^2_{01})$$

$$= -\tfrac{1}{2}(\det \lambda^1)\, \lambda^{1ab} \lambda^2_{ab} \qquad (\lambda^{1ab} = (\lambda^1_{ab})^{-1}). \tag{2.23}$$

With this scalar product the set $\mathscr{H}^2$ of matrices of unit determinant is given by an equation of the type discussed at the beginning of this section, $\mathscr{H}^2 = \{\lambda_{ab} : \langle \lambda_{ab}, \lambda_{ab} \rangle = -1\}$. If we denote by y^K the collection of the n_{ab}'s, then one recognizes (2.22) as being of the form

$$\frac{1}{R} D^A(f, h)(R Y_A) = Z, \tag{2.24}$$

where $D^A(f, h)$ is the harmonic map derivative operator for maps from ${}^4\tilde{M}/G$ with Lorentzian metric h_{AB} to the unit hyperboloid in $\mathbf{R}^{1,2}$, and Z is the vector field

$$Z = \left[\frac{n^{cd} c_c c_d}{2R^3} n_{ab} - \frac{c_a c_b}{2R^3} \right] \frac{\partial}{\partial n_{ab}} \tag{2.25}$$

(which one can check as being tangent to $\mathscr{H}^2$, as of course it should be in order for Eq. (2.24) to make sense). If one adopts the Gowdy parametrization of $\mathscr{H}^2$ [10],

$$\begin{aligned} n_{11} &= \operatorname{ch} w + \cos\phi \operatorname{sh} w \\ n_{12} &= \operatorname{sh} w \sin\phi \\ (\Rightarrow n_{22} &= \operatorname{ch} w - \cos\phi \operatorname{sh} w), \end{aligned} \tag{2.26}$$

the metric induced from (2.23) on $\mathscr{H}^2$ becomes, after some algebra,

$$f_{KL}\, dy^K\, dy^L = dw^2 + \operatorname{sh}^2 w\, d\phi^2, \tag{2.27}$$

and some more algebra gives

$$\begin{aligned} Z &= Z^w \partial_w + Z^\phi \partial_\phi \\ &= \frac{1}{2R^2}\Big\{[c_1^2(\operatorname{sh} w - \cos\phi \operatorname{ch} w) + c_2^2(\operatorname{sh} w + \operatorname{ch} w \cos\phi) \\ &\quad - 2c_1 c_2 \operatorname{ch} w \sin\phi]\frac{\partial}{\partial w} \\ &\quad + \frac{1}{\operatorname{sh} w}[(c_1^2 - c_2^2)\sin\phi - 2c_1 c_2 \cos\phi]\frac{\partial}{\partial\phi}\Big\}. \end{aligned} \tag{2.28}$$

Equation (2.24) can be written as

$$\frac{1}{R}\mathscr{D}_A(R\mathscr{D}^A y^K) + h^{AB}\Gamma^K_{LM}\frac{\partial y^M}{\partial x^A}\frac{\partial y^N}{\partial x^B} = Z^K, \tag{2.29}$$

and by calculating the Christoffel symbols of (2.27) one finds

$$\begin{aligned} &\frac{1}{R}\mathscr{D}_A(R\mathscr{D}^A\phi) + 2\frac{\operatorname{ch} w}{\operatorname{sh} w}\mathscr{D}^A\phi\mathscr{D}_A w = Z^\phi, \\ &\frac{1}{R}\mathscr{D}_A(R\mathscr{D}^A w) - \operatorname{sh} w \operatorname{ch} w\, \mathscr{D}^A\phi\mathscr{D}_A\phi = Z^w, \end{aligned} \tag{2.30}$$

Z^ϕ, Z^w given by (2.28). In terms of the parametrization (2.26) the $n^{ab}c_a c_b$ term appearing in (2.21) and in (2.6) reads

$$\begin{aligned} n^{ab}c_a c_b &= (\operatorname{ch} w - \operatorname{sh} w \cos\phi)\, c_1^2 + (\operatorname{ch} w + \operatorname{sh} w \cos\phi)\, c_2^2 \\ &\quad - 2 \operatorname{sh} w \sin\phi\, c_1 c_2. \end{aligned} \tag{2.31}$$

To rewrite the constraint equations in terms of these variables let us note that since $\det n_{ab} = 1$ we have

$$\begin{aligned} n^{ac}n^{bd}n_{ab+}n_{cd+} &= (n^{ac}n^{bd} - n^{ab}n^{cd})\, n_{ab+}n_{cd+} \\ &= -\tilde{\varepsilon}^{ad}\tilde{\varepsilon}^{bc}n_{ab+}n_{cd+} \\ &= 2(-n_{00+}n_{11+} + n_{01+}^2) \\ &= 2f_{KL}\, y_+^K y_+^L \end{aligned} \tag{2.32}$$

(where $\tilde{\varepsilon}^{(ab)} = 0$, $\tilde{\varepsilon}^{12} = 1$), which is twice the norm of the vector $Y_+ = \partial_+ y^K(\partial/\partial y^K)$ in the metric on $\mathscr{H}^2$. Therefore, (2.13) reads

$$\begin{aligned} \partial_\theta R_\pm = \frac{R_\pm^2}{4R} + (B_\pm - \partial_\theta N^\theta)\, R_\pm - \frac{e^{2B}n^{ab}c_a c_b}{4R^2} \\ - \frac{R}{4}(w_\pm^2 + \mathrm{sh}^2\, w\, \phi_\pm^2). \end{aligned} \tag{2.33}$$

Let us for completeness write the equation for $R(h)$ in terms of this parametrization of λ_{ab}; from (2.32) one obtains:

$$R(h) = \frac{\mathscr{D}_A R \mathscr{D}^A R}{2R^2} - \frac{1}{2}(\mathscr{D}_A w \mathscr{D}^A w + \mathrm{sh}^2\, w \mathscr{D}_A \phi \mathscr{D}^A \phi) - \frac{3}{2R^3} n^{ab}c_a c_b, \tag{2.34}$$

($n^{ab}c_a c_b$ given by (2.31)). Let us remark that the parametrization (2.26) is not very useful either on $S^2 \times S^1$ or on S^3 because on the axes of symmetry $\lambda_{11}\lambda_{22} = 0$, $R = 0$, and (2.26) becomes singular there. One can try to remedy this by a somewhat more flexible parametrization of the metric as presented in Gowdy [10], the resulting equations seem however to lose their simple geometric interpretation and become quite unpleasant to handle. This problem is (at least in part) solved by an "Ernst-type" parametrization of the metric which we discuss in the next section.

3. The Field Equations—An Ernst-Type Parametrization

In the Ernst-type parametrization of Einstein equations, the symmetry between the Killing fields is lost: one arbitrarily chooses a $U(1)$ subgroup of $U(1) \times U(1)$, say the first factor, which we shall for definiteness call G_1. Let X_1 be the corresponding Killing vector field on ${}^4\mathscr{M}$, let ${}^4\tilde{\mathscr{M}}_1 = \{p \in {}^4\mathscr{M}$: the orbit of p under G_1 is one-dimensional$\}$. ${}^4\tilde{\mathscr{M}}_1$ is a circle fibre bundle over the manifold ${}^4\tilde{\mathscr{M}}_1/G_1$ so that one can use the methods of [8] (cf. also [15]) to obtain the field equations on ${}^4\tilde{\mathscr{M}}_1/G_1$, in this way one obtains well-posed geometric equations on a three dimensional manifold with a smooth metric even if X_2 "has an axis." (For completeness we discuss shortly this reduction method in Appendix E.) In this section rather than starting from the twist potentials to obtain the equations for n_{ab}, as done in

Appendix E, we shall derive the equations for the twist potentials by a reparametrization of the equations of the previous section. Let us define

$$e^{2u} \equiv \lambda_{11}, \qquad A \equiv e^{-2u}\lambda_{12}$$
$$(\Rightarrow \lambda_{22} = R^2 e^{-2u} + e^{2u}A^2),$$
$$\chi_{AB} = e^{2u} h_{AB}. \tag{3.1}$$

(Let us stress that the functions A, u, and χ_{AB} are regular whenever $\lambda_{11} > 0$, while the regularity of $n_{ab} = \lambda_{ab} R^{-1}$ requires $R > 0$.) Let $\bar{\mathcal{D}}_A$ denote the Riemannian covariant derivative of the metric χ_{AB}. With some work (2.5) leads to

$$\bar{\mathcal{D}}_A \bar{\mathcal{D}}^A R = -\frac{e^{-4u}c_1^2}{2R} - \frac{(c_2 - Ac_1)^2}{2R^3}, \tag{3.2a}$$

$$\frac{1}{R}\bar{\mathcal{D}}_A(R\bar{\mathcal{D}}^A u) = \frac{e^{4u}\bar{\mathcal{D}}_B A \bar{\mathcal{D}}^B A}{2R^2} - \frac{e^{-4u}c_1^2}{2R^2}, \tag{3.2b}$$

$$R\bar{\mathcal{D}}_B\left(\frac{e^{4u}\bar{\mathcal{D}}^B A}{R}\right) = -\frac{c_1(c_2 - Ac_1)}{R^2}. \tag{3.2c}$$

Equation (2.6) reads,

$$R(\chi) = 2\bar{\mathcal{D}}_A u \bar{\mathcal{D}}^A u - \frac{e^{4u}\bar{\mathcal{D}}_B A \bar{\mathcal{D}}^B A}{2R^2} - \frac{1}{2}e^{-4u}c_1^2 - \frac{3}{2R^2}(c_2 - Ac_1)^2, \tag{3.3}$$

where $R(\chi)$ is the scalar curvature of χ, and the constraint equations (2.33) take the simple form

$$\partial_\theta R_\pm = (\beta_\pm - \partial_\theta N^\theta) R_\pm - R\left(u_\pm^2 + \frac{e^{4u}A_\pm^2}{4R^2}\right)$$
$$-\frac{e^{2(\beta+u)}}{4R^3}(c_2 - Ac_1)^2 - \frac{e^{2(\beta-u)}c_1^2}{4R}, \tag{3.4}$$
$$\beta \equiv B + u.$$

The form (3.4) seems to be the simplest one, and therefore the most useful, for discussing the constraint equations. Let us now present a reformulation of Eqs. (3.2) in terms of Ernst potentials. Recall that for any Killing vector field X the twist vector

$$\omega_\mu = \varepsilon_{\mu\nu\rho\sigma} X^\nu \nabla^\rho X^\sigma \tag{3.5}$$

satisfies [8]

$$\omega_{[\mu,\nu]}=0. \tag{3.6}$$

Let $X=X_1$; from (3.1) and (B.10) one finds

$$\begin{aligned}\omega_A &= -\frac{1}{R}\varepsilon_{AB}\,e^{4u}\mathscr{D}^B A - c_1 X_A^2\\ &= -\frac{1}{R}\bar{\varepsilon}_{AB}\,e^{4u}\bar{\mathscr{D}}^B A - c_1 X_A^2,\end{aligned} \tag{3.7}$$

where $\bar{\varepsilon}_{AB}$ is the "alternating symbol" of the metric χ_{AB}: $\bar{\varepsilon}_{(AB)}=0$, $\bar{\varepsilon}_{03}=\sqrt{-\det\chi_{AB}}$ (one can check that (3.6) holds using (3.2c) and (E.9b)). Inverting the relationship (3.7) one finds

$$\begin{aligned}\mathscr{D}_A A = \bar{\mathscr{D}}_A A &= Re^{-4u}\varepsilon_{AB}h^{BC}(\omega_C + c_1 X_C^2)\\ &= Re^{-4u}\bar{\varepsilon}_{AB}\chi^{BC}(\omega_C + c_1 X_C^2),\end{aligned} \tag{3.8}$$

and the equation $\bar{\varepsilon}^{AB}\bar{\mathscr{D}}_B\bar{\mathscr{D}}_A A=0$ gives

$$\bar{\mathscr{D}}^A[Re^{-4u}(\omega_A + c_1 X_A^2)]=0. \tag{3.9}$$

It follows from the discussion of Section 1 that we may without loss of generality suppose that the first deRham cohomology group $H^1({}^4\tilde{\mathscr{M}}/G)=\mathbf{R}$ or $\{0\}$. Let $\mathring{\omega}_A\,dx^A$ (possibly equal to zero) be a fixed generator of $H^1({}^4\tilde{\mathscr{M}}/G)$, thus there exists a real constant $c(\omega_A)$ such that

$$\omega_A=\omega_{,A}+\bar{\omega}_A, \qquad \bar{\omega}_A\equiv c(\omega_A)\,\mathring{\omega}_A,$$

for some scalar function ω (defined up to a constant); (3.9) can be written in the form

$$\begin{aligned}&\bar{\mathscr{D}}_A(Re^{-4u}\bar{\mathscr{D}}^A\omega)=\psi^\omega,\\ &\psi^\omega\equiv -\bar{\mathscr{D}}^A[Re^{-4u}(\bar{\omega}_A + c_1 X_A^2)],\end{aligned} \tag{3.10}$$

and Eqs. (3.2b) and (3.10) can be recognized as equations of a harmonic type for a map from ${}^4\tilde{\mathscr{M}}_1/G$ to the two dimensional space $(y^K)=(u,\omega)\in\mathbf{R}\times\mathbf{R}$ with the metric

$$f_{KL}\,dy^K\,dy^L = du^2 + \frac{e^{-4u}}{4}\,d\omega^2, \tag{3.11}$$

and with a source term,

$$\begin{aligned}&D^A(f,\chi)(RY_A)=\psi,\\ &\psi=\psi^\omega\frac{\partial}{\partial\omega}+\psi^u\frac{\partial}{\partial u}\\ &\quad= -\bar{\mathscr{D}}^A[Re^{-4u}(\bar{\omega}_A + c_1 X_A^2)]\frac{\partial}{\partial\omega} - \frac{e^{-4u}}{2R^2}c_1^2\frac{\partial}{\partial u}.\end{aligned} \tag{3.12}$$

It is well known that the metric (3.11) is again the constant curvature metric on the hyperbolic two-space.[6] Due to the explicit presence of undifferentiated factors of A at the right-hand sides of (3.2), (3.4), and (3.13), it seems at first sight that there is no genuine twist potential reformulation of the equations of the previous section unless $c_1 = 0$. Now the constants c_a must vanish on $S^2 \times S^1$ by a theorem due to Papapetrou (cf. [14]), thus problems can arise only on T^3, let us discuss this case in some more detail. If the variables x^a and $x^a + 2\pi$ were not identified we could simply rotate the axes appropriately and obtain a coordinate system in which $c_1 = 0$: indeed under rotations of the Killing vectors

$$\begin{aligned} X_{\hat{1}} &= \cos\kappa\, X_1 + \sin\kappa\, X_2 \\ X_{\hat{2}} &= -\sin\kappa\, X_1 + \cos\kappa\, X_2 \end{aligned} \tag{3.13}$$

the constants c_a rotate as a vector. On the torus we cannot perform such coordinate transformations: if we insist on $X_a = \partial/\partial x^a$, we only have the freedom of performing $SL(2, \mathbb{Z})$ rather than $SL(2, \mathbf{R})$ transformations. We can however set

$$\begin{gathered} \lambda_{\hat{a}\hat{b}} = g_{\mu\nu} X^{\mu}_{\hat{1}} X^{\nu}_{\hat{2}}, \\ e^{2\hat{u}} \equiv \lambda_{\hat{1}\hat{1}}, \qquad \hat{A} = e^{-2\hat{u}} \lambda_{\hat{2}\hat{1}}, \\ \hat{\omega}_{\mu} = \eta_{\mu\nu\rho\sigma} X^{\nu}_{\hat{1}} \nabla^{\rho} X^{\sigma}_{\hat{1}}, \end{gathered} \tag{3.14}$$

and from the covariance of the formalism of Section 2 under rotations (3.13) it follows that the equations for $\hat{A}$, $\hat{\omega}$, $\hat{u}$, etc., can be obtained by "hatting" the equations derived in this section. Choosing κ appropriately in (3.13) so that $c_{\hat{1}} = 0$, $c_{\hat{2}} = c > 0$, one obtains:

$$\begin{aligned} &\hat{\omega}_A = \hat{\omega}_{,A} + \hat{\bar{\omega}}_A, \qquad \hat{\bar{\omega}}_{[A,B]} = 0, \\ &\bar{\mathscr{D}}_A \bar{\mathscr{D}}^A R = -\frac{c^2}{2R^3}, \\ &\frac{1}{R} \bar{\mathscr{D}}_A (R \bar{\mathscr{D}}^A \hat{u}) = -\frac{e^{-4\hat{u}}}{2} \chi^{AB} \hat{\omega}_A \hat{\omega}_B, \\ &\bar{\mathscr{D}}^A [R e^{-4\hat{u}} (\hat{\omega}_{,A} + \hat{\bar{\omega}}_A)] = 0, \\ &R(\chi) = 2 \bar{\mathscr{D}}_A \hat{u} \bar{\mathscr{D}}^A \hat{u} + \frac{e^{-4\hat{u}}}{2} \chi^{AB} \hat{\omega}_A \hat{\omega}_B - \frac{3c^2}{2R^2}, \\ &\partial_\theta R_{\pm} = (\beta_{\pm} - \partial_\theta N^\theta) R_{\pm} - R \left(\hat{u}^2_{\pm} + \frac{e^{-4\hat{u}}}{4} \hat{\omega}^2_{\pm} \right) - \frac{e^{2(\beta + \hat{u})}}{4R^3} c^2. \end{aligned} \tag{3.15}$$

[6] The coordinate transformation $\operatorname{sh} w \cos\phi = \omega e^{-2u}$, $2 \operatorname{sh} \omega \sin\phi = e^{-2u}(e^{4u} + \omega^2 - 1)$ brings (3.11) to the form (2.27). The author is grateful to P. Mansfield for providing the explicit form of this transformation.

Given a set of functions $\hat{\omega}_A$, $\hat{u}$, R, χ_{AB} satisfying (3.15) and a constant κ one can obtain $\hat{A}$ from

$$\bar{\mathscr{D}}_A \hat{A} = Re^{-4\hat{u}} \bar{\varepsilon}_{AB} \chi^{BC} \hat{\omega}_C, \tag{3.16}$$

then (3.14) determines $\lambda_{\hat{a}\hat{b}}$; letting

$$\begin{gathered} X^{\mu}_{\hat{a}} = \kappa^{a}_{\hat{a}} X^{\mu}_{a}, \\ \left(\kappa^{a}_{\hat{a}} = \begin{pmatrix} \cos\kappa & \sin\kappa \\ -\sin\kappa & \cos\kappa \end{pmatrix}, \ \kappa^{\hat{a}}_{a} \equiv (\kappa^{a}_{\hat{a}})^{-1} \right) \end{gathered} \tag{3.17}$$

one has

$$\lambda_{ab} = \kappa^{\hat{a}}_{a} \kappa^{\hat{b}}_{b} \lambda_{\hat{a}\hat{b}}. \tag{3.18}$$

The metric h_{AB} is determined from $h_{AB} = e^{-2u} \chi_{AB}$, and given any two functions g^a, M^a satisfying (E.14),

$$\partial_t g^a - \partial_\theta M^a = -\frac{\sqrt{-\det h_{AB}}\, \lambda^{ab} c_b}{R} \tag{3.19}$$

one obtains a metric (2.7) satisfying vacuum Einstein equations (g^a can be made θ-independent by an appropriate coordinate transformation, cf. next section).

For further reference let us note that if $\chi_{AB}\, dx^A\, dx^B = e^{2\beta}(-dt^2 + d\theta^2)$, then (3.3) can be written as the dynamical equation for β,

$$\begin{aligned} \eta^{AB} \partial_A \partial_B \beta = \eta^{AB} \left(-\partial_A u \partial_B u + \frac{e^{4u} \partial_A A \partial_B A}{4R^2} \right) \\ + \frac{e^{-4u-2\beta} c_1^2}{4R^2} + \frac{3e^{-2\beta}(c_2 - Ac_1)^2}{4R^4}, \end{aligned} \tag{3.20}$$

where $\eta_{AB} = \mathrm{diag}(-1, 1)$.

4. The T^3 Case

When ${}^3\Sigma = T^3$ all group orbits are two-dimensional and for any globally hyperbolic development ${}^4\mathscr{M}$ there exists a neighbourhood $\mathscr{O}$ of ${}^3\Sigma$ such that the set $({}^4\mathscr{M} \cap \mathscr{O})/G$ is a manifold with topology $(-\varepsilon, \varepsilon) \times S^1$. All the three parametrizations of the metric—(λ_{ab}), (u, A, R) and $(\hat{u}, \hat{\omega}, R)$—are equivalent in the sense that all three are smooth if any of them is. As noted in the previous section the (u, A, R) parametrization gives the simplest framework to discuss the constraint equations. Let us rewrite (3.4) in the form

$$\partial_\theta R_\pm = f_\pm R_\pm - h_\pm,$$

$$f_\pm \equiv \beta_\pm - \partial_\theta N^\theta,$$

$$h_\pm \equiv R\left(u_\pm^2 + \frac{e^{4u}A_\pm^2}{4R^2}\right) + \frac{e^{2B}}{4R^3}(c_2 - Ac_1)^2 + \frac{e^{2B-4u}c_1^2}{4R} \geqslant 0. \tag{4.1}$$

LEMMA 4.1. *Let $R_+ \in C^1([a,b])$ satisfy* (4.1).

(1) *If $R_+(a)=0$ then $R_+(b) \leqslant 0$, and*

$$R_+(b)=0 \Rightarrow R_+|_{[a,b]} = h_+|_{[a,b]} = 0.$$

(2) *If $R_+(b)=0$ then $R_+(a) \geqslant 0$, and*

$$R_+(a)=0 \Rightarrow R_+|_{[a,b]} = h_+|_{[a,b]} = 0.$$

The same result holds with (R_+, h_+) replaced by (R_-, h_-).

Proof. Suppose that $R_+(x_0)=0$, solving (4.1) for R_+ we have

$$R_+(x) = -\int_{x_0}^{x} h_+(y)\, e^{\int_y^x f_+(z)\,dz}\, dy,$$

and therefore in case (1) we get

$$R_+(b) = -\int_a^b h_+(y)\, e^{\int_y^b f_+(z)\,dz}\, dy \leqslant 0,$$

with $R_+(b)=0 \Rightarrow h_+|_{[a,b]}=0 \Rightarrow R_+|_{[a,b]}=0$. Case (2) is established in a similar way. ∎

COROLLARY 4.1. *Let $R_+ \in C^1(S^1)$ satisfy* (4.1). *Then either*

(1) $R_+ \equiv h_+ \equiv 0$, *or*

(2) $R_+ > 0$, *or*

(3) $R_+ < 0$.

The same holds with (R_+, h_+) replaced by (R_-, h_-).

Proof. A function on S^1 can be identified with a periodic function on $\mathbf{R}$; if there exists $a \in \mathbf{R}$ such that $R_+(a) = R_+(a+2\pi) = 0$, then by Lemma 4.1 with $b = a + 2\pi$ we have $R_+ \equiv 0$, so that a solution of (4.1) on S^1 must either vanish identically or must have a definite sign. ∎

THEOREM 4.1. *Let* (g_{ij}, P^{ij}) *be* $U(1)\times U(1)$ *symmetric* C^k *Cauchy data on* T^3, $k\geqslant 2$. *Then either*

(1) $c_a=\partial R/\partial x^\mu=\partial\lambda_{ab}/\partial x^\mu=0$ *and* h_{AB} *is flat, or,*

(2) ∇R *is timelike* $(g^{\mu\nu}R_{,\mu}R_{,\nu}<0)$, *where* R *is (up to a multiplicative constant) the area of the orbits of the symmetry group.*

Proof. From Lemma 4.1 it follows that either $R_+R_->0$, or $R_+R_-<0$, or $R_+R_-\equiv 0$. At a maximum θ_0 of R we have $R_\theta(\theta_0)=0$ and (2.12) gives

$$(R_+R_-)(\theta_0)=-\frac{e^{2B}}{N^2}R_t^2(\theta_0)\leqslant 0,$$

thus $R_+R_->0$ is impossible. From (2.12) and from $R_{,\mu}X^\mu_a=0$ one finds

$$\begin{aligned} e^{-2B}R_+R_- &= h^{AB}\partial_A R\,\partial_B R=h^{\mu\nu}\partial_\mu R\,\partial_\nu R\\ &= g^{\mu\nu}\partial_\mu R\,\partial_\nu R, \end{aligned}\tag{4.2}$$

so that (2) holds unless $R_+R_-\equiv 0$. In this last case changing the time orientation if necessary we may suppose that $R_-\equiv 0$. This implies $\partial_\theta R=e^B\,\partial R/\partial n$, therefore $R_+=2\partial_\theta R$, and (4.1) reads

$$\partial_\theta^2 R=f_+\partial_\theta R-\tfrac{1}{2}h_+.\tag{4.3}$$

Lemma 4.1 applied to $\partial_\theta R$ gives either $\partial_\theta R>0$, or $\partial_\theta R<0$, or $\partial_\theta R\equiv 0$; neither of the first two cases can occur because every function on S^1 has an extremum, therefore R is constant, $R_+=R_-\equiv 0$, $R_t\equiv 0$, and also $h_+=h_-\equiv 0$, thus $\partial_\mu u=\partial_\mu A=c_a=0$, and (2.6) implies that the metric h_{AB} is flat. ∎

Whenever $g^{\mu\nu}R_{,\mu}R_{,\nu}<0$ a time orientation of ${}^4\mathscr{M}$ can be chosen so that $\nabla^\mu R$ is past pointing, and we shall implicitly assume in the remainder of this paper that this orientation has been chosen. In this case compactness of ${}^3\Sigma$ implies the existence of $\varepsilon>0$ such that

$$R_+|_{{}^3\Sigma}\geqslant\varepsilon;\qquad R_-|_{{}^3\Sigma}\leqslant\varepsilon.\tag{4.4}$$

Equations (3.4) and (4.4) allow us to algebraically determine $\partial\beta/\partial n$ and β_θ in terms of R, u, A, N, N^θ, R_μ, u_μ, A_μ, and β (one of the equations so obtained becomes thus a differential equation for β). We shall not discuss what the most general conditions on the remaining functions are, which allow us to determine β from the equation so obtained, and we shall simply show that the set of solutions of the constraint equations is not empty. The Cauchy data $R|_{{}^3\Sigma}=R_0$, $(\partial R/\partial t)|_{{}^3\Sigma}=\lambda$, R_0, λ—positive constants, are compatible with Corollary 4.1. Set

$$N=e^BN_0,\qquad \frac{\partial f}{\partial n_0}\equiv\frac{1}{N_0}\left(\frac{\partial f}{\partial t}-N^\theta\frac{\partial f}{\partial\theta}\right),$$

then (3.4) can be written as

$$\partial_\theta \beta = \partial_\theta N^\theta + \frac{2R_0}{\lambda}\left(\partial_\theta u \frac{\partial u}{\partial n_0} + \frac{e^{4u}}{4R_0^2}\partial_\theta A \frac{\partial A}{\partial n_0}\right), \tag{4.5}$$

$$\lambda \frac{\partial \beta}{\partial n} = R_0(u_+^2 + u_-^2) + \frac{e^{4u}}{4R_0}(A_+^2 + A_-^2) + \frac{e^{2(\beta+u)}}{4R_0^3}(c_2 - Ac_1)^2 + \frac{e^{2(\beta-u)}c_1^2}{4R_0}. \tag{4.6}$$

Given $(u, A, N_0, u_\mu, A_\mu, N^\theta)|_{{}^3\Sigma}$ one can solve (4.5) for β by direct integration provided

$$\oint_{S^1}\left(\partial_\theta u \frac{\partial u}{\partial n_0} + \frac{e^{4u}}{4R_0^2}\partial_\theta A \frac{\partial A}{\partial n_0}\right) = 0, \tag{4.7}$$

and then (4.6) can be used to determine $\partial\beta/\partial t$.

Let us discuss in some more detail those space-times for which $c_a = 0$. We shall show that every $U(1)\times U(1)$ symmetric Cauchy data set (g_{ij}, P_{ij}) on ${}^3\Sigma \approx T^3$ with this property can be obtained by embedding ${}^3\Sigma$ as a hypersurface in a "canonical" space-time ${}^4\mathscr{M} \approx (0, \infty)\times T^3$ in which $R = \lambda t$, for an appropriately chosen constant λ. We begin with the following, well-known result (which holds irrespective of whether the constant c_a vanish or not):

LEMMA 4.2. *Let h be a C^k metric on ${}^2\mathscr{M} = (-\varepsilon, \varepsilon)\times S^1$, $k \geqslant 2$, let the slices $\{c\}\times S^1$, $c \in (-\varepsilon, \varepsilon)$, be spacelike. There exists $\varepsilon' > 0$ and an open neighborhood $\mathscr{O}$ of $\{0\}\times S^1$ in ${}^2\mathscr{M}$ together with a C^{k-1} coordinate system $(T, \psi): \mathscr{O} \to (-\varepsilon', \varepsilon')\times S^1$ such that*

$$h_{AB}\, dx^A\, dx^B = e^{2B}(-dT^2 + d\psi^2),$$

$$T|_{\{0\}\times S^1} = 0,$$

with some C^{k-2} function B.

Proof. Let $(t, \theta) \in (-\varepsilon, \varepsilon)\times S^1$ be coordinates on ${}^2\mathscr{M}$, let n_T denote the field of future directed unit normals to the slices $\{t\}\times S^1$, let n_θ be the field of unit tangents to $\{t\}\times S^1$, set $n_\pm = n_\theta \pm n_T$. Let $\Gamma_\pm(p)$ denote an affinely parametrized null geodesic through p with tangent $n_\pm(p)$ at p. It follows from spacelikeness of $\{t\}\times S^1$ that for δ small enough and for all $p \in (-\delta, \delta)\times S^1$ all geodesics $\Gamma_\pm(p)$ meet $\{0\}\times S^1$ once and only once. For $p \in (-\delta, \delta)\times S^1$ let $u(p)$, respectively $v(p)$, be the value of θ for which Γ_-, respectively Γ_+, meet $\{0\}\times S^1$. u and v are C^{k-1} maps from $(-\delta, \delta)\times S^1$ to S^1. Since u and v are constant along null geodesics it follows that

$$h^{uu} = h(du, du) = 0, \qquad h^{vv} = h(dv, dv) = 0. \tag{4.8}$$

A straightforward calculation shows that at $t=0$ we have

$$\left|\det \frac{\partial(u, v)}{\partial(T, \theta)}\right| = \frac{2}{\sqrt{-\det h_{MN}}},$$

so decreasing δ if necessary the map $(T, \theta) \to (u, v)$ is a local diffeomorphism. Let

$$\psi = u + v \in S^1, \qquad T = u - v.$$

Still decreasing δ if necessary one shows that T is a well-defined real number, and from what has been said it follows that $(T, \psi) \in (-\varepsilon', \varepsilon') \times S^1$ form a coordinate system in a neighborhood of $\{0\} \times S^1$, with $T|_{\{0\} \times S^1} = 0$. From (4.8) one has

$$h_{AB}\, dx^A\, dx^B = 4e^{2B}\, du\, dv = e^{2B}(-dT^2 + d\psi^2),$$

for some C^{k-2} function B, and the result follows. ∎

In coordinates of Lemma 4.2 a metric on ${}^4M = (0, \infty) \times T^3$ for which $R = \lambda t$ takes the form

$$ds^2 = e^{2B}(-dt^2 + d\theta^2) + \lambda t n_{ab}(dx^a + M^a\, dt + g^a\, d\theta)(dx^b + M^b\, dt + g^b\, d\theta),$$
$$\theta, x^a \in [0, 2\pi]|_{\text{mod } 2\pi}, \quad \det n_{ab} = 1, \ t \in (0, \infty), \tag{4.9}$$

the constraint equations are given by (4.5)–(4.6) with $c_a = 0$, the dynamical equations are the harmonic map type equations

$$D^A(tY_A) = 0$$
$$\left(Y_A = \frac{\partial y^K}{\partial x^A}\frac{\partial}{\partial y^K};\ (y^K) = (n_{ab})\right), \tag{4.10}$$

and D^A is the harmonic map operator discussed in Section 3. A theorem proved by Moncrief [16] shows that for any given (appropriately differentiable, say—smooth) $n_{ab}(t_0, \theta)$, $(\partial n_{ab}/\partial t)(t_0, \theta)$, $t_0 > 0$, a solution of (4.10) exists for all $t \in (0, \infty)$. Choosing arbitrary $B(t_0, \theta)$, $(\partial B/\partial t)(t_0, \theta)$, $M^a(t, \theta)$, and $g^a(t_0, \theta)$ one obtains a vacuum solution of Einstein equations by integrating (4.5), (4.6), and (E.14) (with $c_a = 0$), provided (4.7) holds. The arbitrariness of $M^a(t, \theta)$ reflects some arbitrariness left in our coordinate system: the coordinates x^a are not rigidly fixed yet since any transformation $x^a \to x^a + f^a(t, \theta)$ preserves the form (4.9) of the metric. One could hope to get rid of the g^a's and M^a's altogether, this is however not possible on $(0, \infty) \times T^3$ in general. Let us set

$$\bar{x}^a = x^a + f^a,$$
$$\frac{\partial f^a}{\partial \theta} = -g^a + \bar{g}^a, \tag{4.11a}$$

$$\bar{g}^a \equiv \frac{1}{2\pi}\int_0^{2\pi} g^a(t, \theta)\, d\theta. \tag{4.11b}$$

Equation (E.14) shows that $\bar{g}^a$ is time-independent. From (4.11) one has $f^a(t, \theta+2\pi) = f^a(t, \theta)$ so that the map $(t, \theta, x^a) \to (t, \theta, \bar{x}^a)$ is well defined from $(0, \infty) \times T^3$ to itself. (More generally, let n^a be the largest integer smaller than or equal to $\bar{g}^a$: set $\partial f^a/\partial\theta = -g^a + \bar{\bar{g}}^a$, $\bar{\bar{g}}^a = \bar{g}^a - n^a$; one again obtains a well-defined map with the "leftover" $\bar{\bar{g}}^a$ satisfying $0 \leqslant \bar{\bar{g}}^a < 1$.) Dropping bars, in the new coordinates x^a we have

$$\frac{\partial g^a}{\partial\theta} = 0 \Rightarrow g^a = g^a(t),$$

and (E.11) gives $M^a = M^a(t)$. Choosing the constant of integration in (4.11a) suitably one can obtain

$$M^a = 0, \qquad \partial_\mu g^a = 0. \tag{4.12}$$

Let us note that the coordinate system we have obtained is defined uniquely up to

(1) a rigid (space-time point independent) rotation of the θ's, $\theta \to \theta + \alpha$, $\partial_\mu \alpha = 0$;

(2) an $SL(2, \mathbb{Z})$ transformation of the coordinates x^a;

(3) rigid (space-time point independent) rotations of the x^a's.

THEOREM 4.2. *Let (g_{ij}, P_{ij}) be smooth $U(1) \times U(1)$ symmetric Cauchy data on ${}^3\Sigma \approx T^3$, $c_a = 0$. Let*

$$\lambda = \left| \frac{1}{(2\pi)^3} \int_{{}^3\Sigma} (g^{ij} - \lambda^{ab} X^i_a X^j_b)\, P_{ij}\, d\mu(g) \right|.$$

There exists a smooth embedding i from ${}^3\Sigma$ to a four dimensional vacuum manifold $({}^4\mathcal{M}, ds^2)$, such that (g_{ij}, P_{ij}) are Cauchy data on $i({}^3\Sigma)$ for ${}^4\mathcal{M}$, where

(1) *if $\lambda = 0$ then ${}^4\mathcal{M} = \mathbf{R} \times T^3$ with a time-independent flat metric*

$$\begin{aligned} ds^2 &= -dt^2 + g_{ij}\, dx^i\, dx^j, \\ \partial_\mu g_{ij} &= 0, \end{aligned} \tag{4.13}$$

(2) *if $\lambda \neq 0$, then ${}^4\mathcal{M} \approx (0, \infty) \times T^3$, with a metric of the form* (4.9), (4.12).

Remark. It should be stressed that $i({}^3\Sigma)$ is in general *not* given by an equation $t = t_0$ in the coordinates in which the metric is of the form (4.13) or (4.9).

Proof. (1) Let $\lambda = 0$, from (A.32) we have

$$\lambda = \frac{1}{2\pi} \left| \int_0^{2\pi} \left(e^B \frac{\partial R}{\partial n} \right) (T_0, \theta)\, d\theta \right|,$$

and it follows that point (1) of Theorem 4.1 holds, thus $\partial_\mu \lambda_{ab} = 0$ and h_{AB} is flat. Let us use the coordinates of Lemma 4.2, $h_{AB}\, dx^A\, dx^B = e^{2B}(-dT^2 + d\psi^2)$, in which flatness of h_{AB} gives

$$(\partial_T^2 - \partial_\psi^2) B = 0,$$

therefore there exist two 2π-periodic functions $f, g \in C^\infty(\mathbf{R})$ such that $B(T, \psi) = f(T+\psi) + g(T-\psi)$. Let

$$\mu^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2g(x)}\, dx, \qquad \nu^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2f(x)}\, dx.$$

Set

$$\theta = \frac{1}{2\nu^2}\int_0^{T+\psi} e^{2f(x)}\, dx + \frac{1}{2\mu^2}\int_0^{T-\psi} e^{2g(x)}\, dx,$$

$$t = \frac{1}{2\mu\nu^3}\int_0^{T+\psi} e^{2f(x)}\, dx - \frac{1}{2\mu^3\nu}\int_0^{T-\psi} e^{2g(x)}\, dx.$$

(The origin of the above formulae becomes transparent in the null coordinates systems $U = \psi + T$, $V = \psi - T$, $u = \theta + \mu\lambda t$, $v = \theta - \mu\lambda t$; the above expressions are however more suited to topological interpretation than the corresponding expressions in terms of u, v, U, V.) From the definition of μ and ν we have

$$\theta(T, \psi + 2\pi) = \theta(T, \psi) + 2\pi,$$

$$t(T, \psi + 2\pi) = t(T, \psi).$$

Thus the map $(T, \psi) \to (t, \theta)$ is well defined from $(-\varepsilon, \varepsilon) \times S_1$ to $(-\infty, \infty) \times S_1$; one easily finds that it is a diffeomorphism between $(-\varepsilon, \varepsilon) \times S^1$ and its image, and

$$h_{AB}\, dx^A\, dx^B = -dt^2 + \mu^{-2}\nu^{-2}\, d\theta^2.$$

The argument leading to (4.12) applies and point (1) follows.

(2) Let ${}^4\mathscr{M}$ be any development of the Cauchy data, let (T, ψ) be the coordinates of Lemma 4.2 in a neighborhood $(-\varepsilon, \varepsilon) \times S^1 \times G$ of ${}^3\Sigma$ in ${}^4\mathscr{M}$. Changing the time orientation if necessary we may assume $\partial_T R|_{{}^3\Sigma} > 0$. We have

$$\lambda = \frac{1}{2\pi}\int_0^{2\pi} \partial_T R(T, \theta)\, d\theta$$

(one easily checks that the integral above is T-independent). Let

$$\theta_{,T} = \lambda^{-1}\partial_\psi R, \qquad \theta_{,\psi} = \lambda^{-1}\partial_T R, \tag{4.14a}$$

$$t = \lambda^{-1} R. \tag{4.14b}$$

Note that

$$\theta(T, \psi + 2\pi) = \theta(T, \psi) + 2\pi,$$

so that (4.14a) defines a map from $(-\varepsilon, \varepsilon) \times S^1$ to S^1, moreover

$$\frac{\partial(\theta, t)}{\partial(\psi, T)} = -\lambda^2 R_+ R_-,$$

so that the map $(\psi, T) \to (\theta, t)$ is a diffeomorphism from an appropriately small neighborhood of $(-\varepsilon, \varepsilon) \times S^1$ to its image, and therefore the map

$$(\psi, x^a) \xrightarrow{i} (t = \lambda^{-1} R(0, \psi), \theta(0, \psi), x^a)$$

defines an embedding of ${}^3\Sigma$ in a spacetime ${}^4\mathscr{M}$ for which $R = \lambda t$ ((4.14) preserves the manifestly conformal form of the metric). A straightforward argument along the lines of the proof of Theorem 6.3, using either Proposition 6.1 or the methods of [16], shows that the metric which, so far, is defined only in a neighborhood of ${}^3\Sigma$, can be extended as a vacuum metric to all $t \in (0, \infty)$. ∎

Let us stress that the construction of Lemma 4.2 gives a loss of two degrees of differentiability of the metric, and the embedding of Theorem 4.2 gives a loss of one more degree of differentiability for the components of the metric tensor. This is of course irrelevant if one considers smooth Cauchy data only, but seems to be quite unpleasant if finite differentiability conditions—e.g., some Sobolev conditions—are assumed to be satisfied by the data. One way out of this problem is to declare the Cauchy data for a $U(1) \times U(1)$ symmetric metric to be of a given differentiability class, say W, if the Cauchy data are obtained by a coordinate transformation from "canonical data," as described in Theorem 4.2, of differentiability class W.

It should be noted that while all metrics of point (1) of Theorem 4.2 are flat, it is not true that all flat $U(1) \times U(1)$ symmetric metrics take the form (4.13)—the flat Kasner metrics have a "canonical form" (4.9). This is related to an interesting corollary of the results of this section, that the space of $U(1) \times U(1)$ symmetric Cauchy data on T^3 has at least three connected components, in a C_k or H_k (Sobolev) topology, $k \geqslant 1$.[7] It should be of some interest to find out whether or not this phenomenon is purely an artefact of the high symmetry assumed.

Let us briefly comment on possible generalizations of the $c_a = 0$ results to the $c_a \neq 0$ case. It follows from Theorem 1 that in a neighborhood of the Cauchy surface in any development of (g_{ij}, P_{ij}) one can choose a time coordinate T such that $R = \lambda T$ (though of course a general ${}^3\Sigma$ will not be given by the equation $T = T_0$), where λ is a nonzero constant. In the case of nonvanishing c_a's the equation $R = \lambda T$ is incompatible with a manifestly conformal flat form of

[7] One of the components consists of data for the flat metrics (4.13), and two for the metrics (4.9), with R_+ either positive or negative.

$h_{AB}dx^A\,dx^B$, but N^θ can be set to zero, so that one has, at least in a small neighborhood of ${}^3\Sigma$,

$$\begin{aligned} ds^2 = e^{2B}(-N_0^2\,dT^2 + d\psi^2) + \lambda T n_{ab}(dx^a + g^a\,d\psi + M^a\,dT) \\ \times (dx^b + g^b\,d\psi + M^b\,dT), \\ \det n_{ab} = 1. \end{aligned} \tag{4.15}$$

By an appropriate transformation of x^a's one can obtain $g^a = g^a(T)$, and (E.14) gives

$$\begin{aligned} \frac{\partial g^a}{\partial T} &= -\frac{1}{2\pi\lambda T}\oint (N_0 e^{2B} c^a)\,d\psi, \\ \frac{\partial M^a}{\partial \psi} &= \frac{N_0 e^{2B} c^a}{\lambda T} - \frac{1}{2\pi\lambda T}\oint (N_0 e^{2B} c^a)\,d\psi. \end{aligned} \tag{4.16}$$

The constraint equations give two ODEs for β, Eqs. (4.5)–(4.6), and the evolution equation (2.21) for R becomes an ODE for $N_0 = e^{-2B}N$:

$$\frac{\partial}{\partial T}\left(\frac{1}{N_0^2}\right) = -\frac{e^{2B} n^{ab} c_a c_b}{\lambda^3 T^2}. \tag{4.17}$$

Unlike the $c_a = 0$ case, all the equations are coupled which makes them more difficult to handle, in particular no long time existence theorem of the type proved by Moncrief [16] is available yet.

5. The $S^2 \times S^1$ and S^3 Case—Constraints

Both in the $S^2 \times S^1$ and in the S^3 case an important simplification occurs at the outset, due to the occurrence of symmetry axes: on those one of the Killing vectors vanishes, therefore the defining expression for the c_a's evaluated there gives zero: since the c_a's are constant [9] it follows $c_a = 0$ (cf. also [14]).

In the case ${}^3\Sigma = S^2 \times S^1$ let G_1 be the subgroup of G which rotates the S^1 factor of ${}^3\Sigma$, let X_1 be the corresponding Killing vector field. The quotient ${}^3\Sigma/G_1 \approx S^2$ is a manifold and Eq. (3.1) defines smooth functions on ${}^3\Sigma/G_1$. According to the discussion of Section 1 the second factor of $U(1) \times U(1)$ is, up to diffeomorphism, a rotation of S^2 around the z-axis so that we can identify θ with the angle away from the z-axis and x^2 with the rotation angle ϕ around this axis, when S^2 is considered as embedded in $\mathbf{R}^3$. Let us recall the following well-known result which we prove for completeness in Appendix C:

Lemma 5.1. *Let A_{ij} be a smooth tensor field on the open ball $B(\varepsilon) \subset \mathbf{R}^3$, $\varepsilon > 0$, A_{ij} invariant under rotations around the z-axis. There exist smooth functions*

$\alpha, \beta, \gamma, \delta, \varepsilon, \phi$, invariant under rotations around the z-axis, such that A_{ij} takes the form (r, ϕ, z-cylindrical coordinates)

$$A_{ij}\, dx^i\, dx^j = \alpha\, dz^2 + r^2\beta\, dz\, d\phi + \gamma r\, dz\, dr + \delta r^2\, dr^2 + \varepsilon(dr^2 + r^2\, d\phi^2) + \phi r^3\, dr\, d\phi.$$

Since this result is purely local, the same holds when z and $z + 2\pi$ are identified. In neighborhoods of the poles of S^2, θ (or $\pi - \theta$) plays the role of r, x^2 plays the role of ϕ, so that smoothness of the metric g_{ij}

$$g_{ij}\, dx^i\, dx^j = (h_{\theta\theta} + \lambda_{ab}\, g^a g^b)\, d\theta^2 + 2g_1\, d\theta\, dx^1 + 2g_2\, d\theta\, dx^2 + \lambda_{11}(dx^1)^2 + 2\lambda_{12}\, dx^1\, dx^2 + \lambda_{22}(dx^2)^2$$

implies, for small $\sin\theta$,

$$\begin{gathered} g_1 = O(\sin\theta),\ g_2 = O(\sin^3\theta),\ \lambda_{12} = O(\sin^2\theta),\ R = O(\sin\theta), \\ \lambda^{11} = O(1),\ \lambda^{12} = O(1),\ \lambda^{22} = O(\sin^{-2}\theta),\ \lambda_{ab}\, g^a g^b = O(\sin^2\theta), \\ \lambda_{22} = h_{\theta\theta}(\sin\theta = 0)\sin^2\theta + O(\sin^4\theta), \\ R_{,\theta} = (\lambda_{11} h_{\theta\theta})^{1/2}(\sin\theta = 0)\cos\theta + O(\sin^2\theta), \end{gathered} \tag{5.1}$$

in particular

$$R_{,\theta}(0) > 0, \qquad R_{,\theta}(\pi) < 0. \tag{5.2}$$

Since P_{ij} is G-invariant so is $K_{ij} = Pg_{ij}/2 - P_{ij}$, and we have

$$K_{12} = O(\sin^2\theta), \qquad K_{22} = O(\sin^2\theta). \tag{5.3}$$

From (A.32) we have $K_{ab} = -\frac{1}{2}\,\partial\lambda_{ab}/\partial n$ and with the parametrization

$$\lambda_{11} = e^{2u}, \qquad \lambda_{21} = Ae^{2u}, \qquad \lambda_{22} = R^2 e^{-2u} + A^2 e^{4u} \tag{5.4}$$

one finds (recall that $\partial/\partial n = (1/N)\,\partial/\partial t - (N^\theta/N)\,\partial/\partial\theta$)

$$u = O(1), \qquad A = O(\sin^2\theta), \qquad \frac{\partial u}{\partial n} = O(1)$$

$$\Rightarrow \frac{\partial A}{\partial n} = O(\sin^2\theta), \qquad \frac{\partial R}{\partial n} = O(\sin\theta),$$

in particular

$$\frac{\partial R}{\partial n}(\sin\theta = 0) = 0. \tag{5.5}$$

In the S^3 case the situation is slightly different since one of the Killing vectors, say X_1, vanishes at $\theta = \pi$ and is nonzero at $\theta = 0$, while $X_2(0) = 0$, $X_2(\pi) \neq 0$. In this

case (5.4) defines smooth functions on ${}^3\tilde{\Sigma}_1/G_1$, where ${}^3\tilde{\Sigma}_1 = \{p \in {}^3\Sigma$: the orbit of G_1 through p is one dimensional$\} \approx B(\pi)$, the open ball of radius π in $\mathbf{R}^3$, and the parametrization

$$\lambda_{22} = e^{2\tilde{u}}, \qquad \lambda_{21} = \tilde{A}e^{2\tilde{u}}, \qquad \lambda_{11} = R^2 e^{-2\tilde{u}} + \tilde{A}^2 e^{4\tilde{u}} \tag{5.6}$$

defines smooth functions on ${}^3\tilde{\Sigma}_2/G_2$, where ${}^3\tilde{\Sigma}_2 = \{p \in {}^3\Sigma$: the orbit of G_2 is one dimensional$\}$. An argument similar to the previous one again leads to (5.2) and (5.5) in the S^3 case. We thus have

LEMMA 5.2. *Let (g_{ij}, P^{ij}) be smooth $U(1) \times U(1)$ symmetric Cauchy data on ${}^3\Sigma \approx S^3 \times S^1$ or S^3. On the axes of symmetry the vector ∇R does not vanish and is spacelike.*

The following is a straightforward corollary of Lemmas 4.1 and 5.2.

COROLLARY 5.1. *Under the hypotheses of Lemma 5.2 let*

$$I_+ = \{\theta : R_+(\theta) = 0\},$$
$$I_- = \{\theta : R_-(\theta) = 0\}.$$

(1) $I_+ \neq \emptyset$, $I_- \neq \emptyset$, $\{0, \pi\} \cap I_\pm = \emptyset$.

(2) *I_+ and I_- are connected.*

(Thus each of I_- and I_+ either consists of a single point, or is a closed interval interior to $(0, \pi)$).

Proof. We have $R_+(0) = R_-(0) > 0$, $R_+(\pi) = R_-(\pi) < 0$, therefore by continuity $I_\pm \neq \emptyset$. By continuity also there exists $\varepsilon > 0$ such that $I_+ \cup I \subset [\varepsilon, \pi - \varepsilon]$, thus it is sufficient to establish (2) in $[\varepsilon, \pi - \varepsilon]$. In this range of θ's the functions A, R, and u are smooth both for S^3 or for $S^2 \times S^1$, and Lemma 4.1 shows that if $\theta^1_\pm \leqslant \theta^2_\pm \in I_\pm$, then $[\theta^1_\pm, \theta^2_\pm] \subset I_\pm$, which establishes point (2). ∎

Let us note a result concerning "asymptotically flat" cylindrically symmetric vacuum Cauchy data which seems worth mentioning here, though it lies somehow out of the scope of this paper. A metric of the form (2.7) will be said to be "cylindrically asymptotically flat" if $x^1 \in \mathbf{R}$, $x^2 \in [0, 2\pi]|_{\mathrm{mod}\, 2\pi}$, $\theta \in [0, \infty)$, the set $\theta = 0$ is a rotation axis and $\lim_{\theta \to \infty} R_+ - R_- = 0$, $\lim_{\theta \to \infty} R_\theta = \mathrm{const} > 0$.[8]

COROLLARY 5.2. *Smooth cylindrically symmetric "asymptotically flat" Cauchy data do not contain trapped surfaces which are either compact or invariant under the isometry group.*

[8] For the validity of Corollary 5.2 it would actually be sufficient to assume $R_+ > c_0$, $R_- > c_0$ for all $\theta > \theta_0$, for some $\theta_0, c_0 \in \mathbf{R}^+$.

Proof. The inequalities $R_{\pm}(0)>0$ together with $\lim_{\theta\to\infty} R_{\pm}(\theta)>0$ and Lemma 4.1 imply that $R_{\pm}>0$. A surface ${}^2S\subset{}^3\Sigma$ invariant under the isometry group $(=\mathbf{R}\times U(1))$ is given by the equation $\theta=\theta_0$; the trapped surface condition

$$D_i n^i + K^{ij} n_i n_j - K^{ij} g_{ij} \leqslant 0,$$

where n^i is the unit exterior normal to ${}^2S$ in ${}^3\Sigma$, by (A.32) takes the form[9]

$$R_+(\theta_0)\leqslant 0$$

($K^{\theta}_a=0$ because of the symmetry axes) and therefore no trapped surfaces invariant under the symmetry group exist. Now if ${}^2S$ is a compact trapped surface, let p be any of the points on ${}^2S$ such that $\theta(p)=\max_{q\in{}^2S}\theta(q)$. It is easily seen that the surface $\theta=\theta(p)$ is trapped, and the result follows from the previous one. ∎

The regularity conditions (5.1) impose an integral constraint on the Cauchy data which we derive now; without loss of generality we may suppose $N^{\theta}=0$. From (4.1) we have

$$\beta_{\theta}=\frac{1}{2}\left\{\frac{\partial_{\theta}R_+ + h_+}{R_+}+\frac{\partial_{\theta}R_- + h_-}{R_-}\right\}, \tag{5.7a}$$

$$\frac{\partial\beta}{\partial n}=\frac{1}{2}\left\{\frac{\partial_{\theta}R_+ + h_+}{R_+}-\frac{\partial_{\theta}R_- + h_-}{R_-}\right\}, \tag{5.7b}$$

and smoothness of $\beta_{\pm}$ implies that the ratios $(\partial_{\theta}R_{\pm}+h_{\pm})/R_{\pm}$ are smooth. (Let us note that this imposes nontrivial constraints on the Cauchy data on $I_{\pm}$, $I_{\pm}$ as in Corollary 5.1, and in neighborhoods of $I_{\pm}$; we, e.g., have $(\partial_{\theta}R_{\pm}+h_{\pm})|_{I_{\pm}}=0$.) Integrating (5.7) gives

$$\beta(\pi)-\beta(0)=\frac{1}{2}\int_0^{\pi}\left(\frac{\partial_{\theta}R_+ + h_+}{R_+}+\frac{\partial_{\theta}R_- + h_-}{R_-}\right)d\theta. \tag{5.8}$$

From (5.1) and (5.4) on $S^2\times S^1$ one has

$$R^2_{,\theta}(0)=e^{2\beta(0)}, \qquad R^2_{,\theta}(\pi)=e^{2\beta(\pi)}. \tag{5.9}$$

Thus (5.8) gives

$$\ln|R_{\theta}(\pi)|-\ln|R_+(0)|=\frac{1}{2}\int_0^{\pi}\left(\frac{\partial_{\theta}R_+ + h_+}{R_+}+\frac{\partial_{\theta}R_- + h_-}{R_-}\right)d\theta. \tag{5.10}$$

When ${}^3\Sigma\approx S^3$ (5.9) is replaced by

$$R^2_{,\theta}(0)=e^{2\beta(0)}, \qquad R^2_{,\theta}(\pi)=e^{2\bar{\beta}(\pi)}. \tag{5.11}$$

[9] The author is grateful to D. Christodoulou for pointing out this fact.

In order to avoid the singularity of the (u, A, β) parametrization of the metric at $\theta=\pi$ it is convenient to parametrize $g_{\mu\nu}$ by (u, A, β) for θ in, say, $[0, \pi/2]$, and to use $(\tilde{u}, \tilde{A}, \tilde{\beta})$ for θ in $[\pi/2, \pi]$. Equation (5.11) and the "$\sim$" equivalent of (5.7),

$$\begin{aligned}
\partial_\theta \tilde{\beta} &= \frac{1}{2}\left\{\frac{\partial_\theta R_+ + \tilde{h}_+}{R_+} + \frac{\partial_\theta R_- + \tilde{h}_-}{R_-}\right\}, \\
\frac{\partial \tilde{\beta}}{\partial n} &= \frac{1}{2}\left\{\frac{\partial_\theta R_+ + \tilde{h}_+}{R_+} - \frac{\partial_\theta R_- + \tilde{h}_-}{R_-}\right\}, \\
\tilde{h}_\pm &= R\left(\tilde{u}_\pm^2 + \frac{e^{-4\tilde{u}}\tilde{A}_\pm^2}{4R^+}\right),
\end{aligned} \tag{5.12}$$

give

$$\begin{aligned}
&\ln|R_\theta(\pi)| - \ln|R_\theta(0)| \\
&\quad = \frac{1}{2}\int_0^{\pi/2}\left(\frac{\partial_\theta R_+ + Rh_+}{R_+} + \frac{\partial_\theta R_- + Rh_-}{R_-}\right) d\theta \\
&\qquad + \frac{1}{2}\int_{\pi/2}^{\pi}\left(\frac{\partial_\theta R_+ + R\tilde{h}_+}{R_+} + \frac{\partial_\theta R_- + R\tilde{h}_-}{R_-}\right) d\theta + u\left(\frac{\pi}{2}\right) - \tilde{u}\left(\frac{\pi}{2}\right),
\end{aligned} \tag{5.13}$$

where (β, u, A, R) and $(\tilde{\beta}, \tilde{u}, \tilde{A}, R)$ are smoothly related to each other on $(0, \pi)$ via

$$\begin{aligned}
e^{2\tilde{u}} &= R^2 e^{-2u} + A^2 e^{2u} \qquad (e^{2u} = R^2 e^{-2\tilde{u}} + \tilde{A}^2 e^{2\tilde{u}}), \\
\tilde{A} e^{2\tilde{u}} &= A e^{2u}, \qquad \tilde{\beta} + \tilde{u} = \beta + u.
\end{aligned} \tag{5.14}$$

We have the following equivalent of Lemma 4.2:

LEMMA 5.3. *Let $g_{\mu\nu}$ be a $U(1)\times U(1)$ invariant smooth metric on ${}^4\mathcal{M} = (-\varepsilon, \varepsilon)\times {}^3\Sigma$, suppose that the sets $X^\mu_a X_{a\mu} = 0$ (no summation over a) are smooth embedded Lorentzian submanifolds of ${}^4\mathcal{M}$, ${}^3\Sigma = S^3$, or $S^2\times S^1$, with $\{0\}\times {}^3\Sigma$ spacelike in ${}^4\mathcal{M}$. There exists an open neighborhood O of $\{0\}\times {}^3\Sigma$ and functions $(T, \psi): O \to \{T\in(-\varepsilon', \varepsilon'), \psi\in[\psi_-(T), \psi_+(T)]\}$, with some $\varepsilon'>0$ and some smooth functions $\psi_\pm(T)$, such that $\{t\}\times\{\psi_-(t), \psi_+(t)\}$ are axes of rotation, and*

$$h_{AB}\, dx^A\, dx^B = e^{2B}(-dT^2 + d\psi^2) \tag{5.15}$$

for some function B. The curves $t\to(t, \psi_-(t))$ and $t\to(t, \psi_+(t))$ are timelike for the metric (5.15).

Proof. The result is obtained by essentially the same geodesic construction as in the proof of Lemma 4.2: in ${}^4\mathcal{M}$ one considers null geodesics lying in the (t, θ) plane, these fall into three classes:

(1) geodesics which meet $\{0\}\times {}^3\Sigma$ without meeting first the symmetry axes;

(2) geodesics which before meeting $\{0\} \times {}^3\Sigma$ meet the left symmetry axis;

(3) geodesics which before meeting $\{0\} \times {}^3\Sigma$ meet the right symmetry axis.

For geodesics of point (1) above, u and v are defined as in the proof of Lemma 2. Let t be any smooth $U(1) \times U(1)$ invariant function defined on the symmetry axes, with $\nabla^\mu t$-timelike (with respect to the geometry induced on the axes from ${}^4\mathscr{M}$). For geodesics of points (2) and (3) set $u = t$ or $v = -t$, where t is the value of t at the intersection point of the geodesic with the axis, the details are left to the reader. ∎

Let us recall a well known result[10]:

LEMMA 5.4. *Let $U(1)$ act smoothly and effectively on ${}^3\Sigma$, let (g_{ij}, P_{ij}) be $U(1)$ invariant data on a compact manifold ${}^3\Sigma$, suppose that the set*

$$S = \{ p \in {}^3\Sigma : \forall g \in U(1),\ gp = p \}$$

is a smooth submanifold of ${}^3\Sigma$, let ${}^4\mathscr{M}$ be a vacuum development of ${}^3\Sigma$. There exists a neighborhood ${}^4O$ of ${}^3\Sigma$ in ${}^4\mathscr{M}$ on which $U(1)$ acts smoothly (and effectively), and the set

$$\tilde{S} = \{ p \in {}^4O : \forall g \in U(1),\ gp = p \}$$

is a smooth Lorentzian submanifold of ${}^4O$.

Proof. Let t be a harmonic time coordinate, let ${}^4O \approx (-\varepsilon, \varepsilon) \times {}^3\Sigma$, ε-small enough. We have $\tilde{S} = (-\varepsilon, \varepsilon) \times S$ (cf. the discussion in the first paragraph of Section 2). ∎

Lemma 5.4 implies

COROLLARY 5.3. *Let $U(1) \times U(1)$ act smoothly and effectively on ${}^3\Sigma$, ${}^3\Sigma \approx S^2 \times S^1$ or S^3. Let (g_{ij}, P_{ij}) be $U(1) \times U(1)$ invariant data on ${}^3\Sigma$. In any vacuum development ${}^4\mathscr{M}$ of (g_{ij}, P_{ij}) there exists a neighborhood of ${}^3\Sigma$ in which the sets $X^\mu_a X_{a\mu} = 0$ (no summation over a) are smooth Lorentzian submanifolds.*

It follows from Corollary 5.3 that the hypotheses of Lemma 5.3 hold for ${}^3\Sigma = S^3$ or $S^2 \times S^1$ and we can thus construct the "explicitly conformally flat" coordinate system of this Lemma, at least in a neighborhood of ${}^3\Sigma$ in ${}^4\mathscr{M}$. In these coordinates $N^\theta = 0$, $N = e^B$, $\partial_\pm = \partial_\theta \pm \partial_t$, thus the right-hand sides of (5.10) and (5.13) do not involve B. Given therefore (R, A, u, R_t, A_t, u_t) which satisfy (5.10) or (5.13) and for which $(\partial_\theta R_\pm + h_\pm)/R_\pm$ are smooth one can solve (5.7) for a smooth β which will satisfy (5.9) or (5.11). It is shown in Appendix C that if (R, A, u, R_t, A_t, u_t) satisfy appropriate conditions on the axes then the resulting metric will be smooth on ${}^3\Sigma$. To establish existence of solutions of the constraint equations one thus has to find (R, A, u, R_t, A_t, u_t) for which $(\partial_\theta R_\pm + h_\pm)/R_\pm$ are smooth and which satisfy the integral constraints. The Cauchy data set will be called *generic* if $I_\pm = \{\theta_\pm\}$, $I_\pm$ as

[10] Lemma 5.4 holds of course for any group, not necessarily $U(1)$.

in Corollary 5.1, and $\partial_\theta R_+(\theta_+)<0$, $\partial_\theta R_-(\theta_-)<0$ (such functions satisfy the (necessary) conditions of Corollary 5.1). This terminology is motivated by the following result, which turns out to be quite useful in studying issues such as cosmic censorship (cf. [5]):

PROPOSITION 5.1. *The set of generic $U(1)\times U(1)$ symmetric Cauchy data on ${}^3\Sigma$, ${}^3\Sigma \approx S^2\times S^1$ or S^3, is open and dense in a C^∞ topology on the set of all $U(1)\times U(1)$ symmetric Cauchy data.*

Proof. Openness is straightforward, let us show density. For notational simplicity, let us assume $N^\theta=0$. Let $\Phi\equiv\{R, A, u, \beta, R_t, A_t, u_t, \beta_t\}$ be smooth functions satisfying the constraint equations, suppose that Φ is not generic and thus $\exists\theta_+ : \partial_\theta R_+(\theta_+)=R_+(\theta_+)=0$, or $\exists\theta_-:\ \partial_\theta R_-(\theta_-)=R_-(\theta_-)=0$, or both. The only (slight) difficulty is to find a sequence $\Phi_n\equiv\{R_n, A_n, u_n, \beta_n, (R_t)_n, (A_t)_n, (u_t)_n, (\beta_t)_n\}$ for which the ratios $(\partial_\theta(R_\pm)_n+(h_\pm)_n)/(R_\pm)_n$ are smooth, the rest is relatively straightforward. Suppose that I_+ is an interval, then $R_+=0$ on this interval and thus $\partial_\theta R_+|_{I_+}=0=-h_+|_{I_+}$. If $I_+=\{\theta_+\}$ then again $h_+(\theta_+)=-\partial_\theta R_+(\theta_+)=0$, and thus in either case

$$h_\pm|_{I_\pm}=0\Rightarrow u_\pm|_{I_\pm}=0,\qquad A_\pm|_{I_\pm}=0. \tag{5.16}$$

Let $\theta_\pm$ be any two points of $I_\pm$, let ε be small enough so that $I_+\cup I_-\subset(\varepsilon,\pi-\varepsilon)$, note that there exists $\delta>0$ such that $R|_{(\varepsilon,\pi-\varepsilon)}\geqslant\delta$. Let ϕ_n, ψ_n be any sequence of nonnegative functions such that $\operatorname{supp}\phi_n$, $\operatorname{supp}\psi_n\subset(\varepsilon,\pi-\varepsilon)$, $\psi_n\searrow 0$, $\phi_n\searrow 0$ in C_∞ topology, and

$$\phi_n|_{I_+}>0,\qquad \psi_n|_{I_-}>0. \tag{5.17}$$

For $\theta\in(\varepsilon,\pi-\varepsilon)$ set

$$(R_+)_n(\theta)=-\int_{\theta_+}^{\theta} R\left[(u_++\phi_n)^2+\frac{e^{4u}A_+^2}{R^2}\right]e^{-\int_\rho^\theta\beta_+(\psi)\,d\psi}\,d\rho, \tag{5.18a}$$

$$(R_-)_n(\theta)=-\int_{\theta_-}^{\theta} R\left[(u_-+\psi_n)^2+\frac{e^{4u}A_-^2}{R^2}\right]e^{-\int_\rho^\theta\beta_-(\psi)\,d\psi}\,d\rho, \tag{5.18b}$$

(cf. the proof of Lemma 4.1 for the origin of these integral expressions),

$$R_n(\theta)=R(\theta_+)+\int_{\theta_+}^{\theta}\frac{(R_+)_n(\psi)+(R_-)_n(\psi)}{2}\,d\psi, \tag{5.18c}$$

$$(u_+)_n=\sqrt{\frac{R}{R_n}}\,(u_++\phi_n), \tag{5.18d}$$

$$(u_-)_n=\sqrt{\frac{R}{R_n}}\,(u_-+\psi_n), \tag{5.18e}$$

$$u_n(\theta)=u(\theta_+)+\int_{\theta_+}^{\theta}\frac{(u_+)_n(\psi)+(u_-)_n(\psi)}{2}\,d\psi, \tag{5.18f}$$

$$(A_\pm)_n = \sqrt{\frac{R_n}{R}}\, e^{2(-u_n+u)} A_\pm, \tag{5.18g}$$

$$(\beta_\pm)_n = \beta_\pm, \tag{5.18h}$$

$$A_n(\theta) = A(\theta_+) + \int_{\theta_+}^{\theta} \frac{(A_+)_n(\psi) + (A_-)_n(\psi)}{2}\, d\psi, \tag{5.18i}$$

with $(R_t)_n$, $(u_t)_n$, etc., defined on $(\varepsilon, \pi-\varepsilon)$ via

$$(R_t)_n = \frac{(R_+)_n - (R_-)_n}{2}, \text{ etc.} \tag{5.19}$$

It follows from (5.17)–(5.19) that for $\theta \in (\varepsilon, \pi-\varepsilon)$ and for n large enough:

(1) the collection Φ_n consists of smooth functions satisfying the constraint equations,

(2) Φ_n fulfill the genericity condition,

(3) $\Phi_n \to \Phi$ in C_∞ topology,

(4) in sufficiently small neighborhoods of ε and $\pi-\varepsilon$, R_n and u_n differ from R and u by some n-dependent constants.

We can smoothly extend ϕ_n in the intervals $[\varepsilon/2, \varepsilon]$ and $[\pi-\varepsilon, \pi-\varepsilon/2]$ so that $(R_n, A_n, u_n, (R_t)_n, (A_t)_n, (u_t)_n)|_{(0,\varepsilon/2)\cup(\pi-\varepsilon/2,\pi)} = (R, A, u, R_t, A_t, u_t)|_{(0,\varepsilon/2)\cup(\pi-\varepsilon/2,\pi)}$. In the S^3 case let Δ_n denote the difference of the left- and right-hand sides of (5.13) (with $(R_+)_n$ for R_+, etc.), similarly for $S^2 \times S^1$ let Δ_n denote the difference of the left- and right-hand sides of (5.10). If Δ_n is positive, an appropriate modification of $(u_t)_n$ on $(\varepsilon/4, \varepsilon/2)$ can make Δ_n vanish; similarly if Δ_n is negative a modification of $(u_t)_n$ on $(\pi-\varepsilon/2, \pi-\varepsilon/4)$ can achieve the same result. Then β_n can be defined by integrating (5.7a) (with R_n, etc. in the r.h.s. of (5.7a)), $(\beta_t)_n$ is obtained from (5.7b). It is easy to see that $(\beta_\pm)_n \equiv \partial_\theta \beta_n \pm (1/N)(\partial_t \beta)_n$ coincides with $\beta_\pm$ on $(\varepsilon, \pi-\varepsilon)$ and is thus smooth. Since Φ_n coincides with Φ on $[0, \varepsilon/4) \cup (\pi-\varepsilon/4, \pi]$, Φ_n satisfies the regularity condition on the axes. ∎

For generic R, R_t, etc., the discussion of existence of solutions of the constraint equations is rather simple: if $R_\pm$ are generic, then $\partial_\theta R_\pm(\theta_\pm) < 0$ and thus, as is well known, there exist smooth functions $\rho_\pm(\theta)$ such that

$$R_\pm = -\rho_\pm(\theta)(\theta - \theta_\pm), \qquad \rho_\pm > 0. \tag{5.20}$$

Let (u, A, u_t, A_t) be such that

$$\left\{R\left(u_\pm^2 + \frac{e^{4u}A_\pm^2}{4R^2}\right)\right\}(\theta_\pm) = \rho_\pm(\theta_\pm) \qquad (= -\partial_\theta R_\pm(\theta_\pm)). \tag{5.21}$$

Therefore there exist smooth functions $\chi_\pm$ such that

$$\left\{\partial_\theta R_\pm + R\left[u_\pm^2 + \frac{e^{4u}A_\pm^2}{4R}\right]\right\}(\theta) = \chi_\pm(\theta)(\theta - \theta_\pm). \tag{5.22}$$

Thus

$$\frac{\partial_\theta R_\pm + h_\pm}{R_\pm} = -\frac{\chi_\pm}{\rho_\pm}, \tag{5.23}$$

which is a smooth function on $(0, \pi)$ since $\rho_\pm > 0$. The regularity conditions on the axes are easily taken care of by the arguments of Appendix C, and the arguments of the last part of the proof of Proposition 5.1 show that the set of solutions of the constraint equations is nonempty for generic $R_\pm$. One can similarly show existence of smooth solutions of the constraint equations when $I_\pm = \{\theta_\pm\}$ and the functions $h_\pm$, $h_\pm$ as in (4.1), have a zero of any given finite order (the appropriate condition on u and A being then that the functions $h_\pm$ have a zero of an appropriate order, with appropriately chosen coefficients determined by $R_\pm$); the case in which I_+ or I_- or both consist of intervals is discussed in Appendix D.

6. $S^2 \times S^1$ AND S^3—MAXIMAL DEVELOPMENTS

The aim of this section is twofold: (1) to show that all $U(1) \times U(1)$ symmetric generic Cauchy data on ${}^3\Sigma$, ${}^3\Sigma \approx S^2 \times S^1$ or S^3, can be realized by embedding ${}^3\Sigma$ in a space-time in which

$$R = \lambda \sin t \sin \theta \tag{6.1}$$

for an appropriate constant $\lambda > 0$, and (2) to show that the maximal globally hyperbolic development of such Cauchy data contains the set $(t, \theta) \in (0, \pi) \times [0, \pi]$, in coordinates (t, θ) in which (6.1) holds. Let us start with a result analogous to Theorem 4.2.

THEOREM 6.1. *Let ${}^3\Sigma = S^2 \times S^1$ or S^3, let (g_{ij}, P_{ij}) be generic $U(1) \times U(1)$ invariant smooth Cauchy data on ${}^3\Sigma$, i.e.,*

$$\{\theta : R_\pm(\theta) = 0\} = \{\theta_\pm\}, \qquad \partial_\theta R_\pm(\theta_\pm) < 0.$$

Reversing the time orientation if necessary we may assume $\theta \leqslant \theta_+$. There exists a smooth embedding of ${}^3\Sigma$ in a space-time ${}^4\mathcal{M} = {}^2\mathcal{M} \times U(1) \times U(1)$, ${}^2\mathcal{M} = \{(T, \psi) : \psi \in [0, \pi],\ T \in (T_-(\psi), T_+(\psi))\}$, for some smooth functions $0 \leqslant T_-(\psi) < T_+(\psi) \leqslant \pi$, such that ${}^4\mathcal{M}$ is a development of (g_{ij}, P_{ij}), and the metric on ${}^4\mathcal{M}$ is of the form

$$\begin{aligned} ds^2 &= e^{2B}(-dT^2 + d\psi^2) + \lambda_{ab}\, dx^a\, dx^b, \\ \sqrt{\det \lambda_{ab}} &= \lambda \sin T \sin \psi, \end{aligned} \tag{6.2}$$

where λ is a positive constant given by

$$\lambda = \frac{1}{2}(R(\theta_+) + R(\theta_-)) - \frac{1}{8\pi^2}\int_{V(R_\mu)} \pi^{ij}(g_{ij} - \lambda_{ab} X_i^a X_j^b)\, d^3x,$$

$$V(R_\mu) = \{p \in {}^3\Sigma : g^{\mu\nu} R_\mu R_\nu \leqslant 0\}, \qquad \pi^{ij} = \sqrt{\det g_{kl}}\, P^{ij}. \tag{6.3}$$

Remarks. (1) Let us note that the genericity requirement is both necessary and sufficient for the existence of the embedding i described above, thus Gowdy's claim [10] that this result holds under more general conditions is incorrect.

(2) It will be shown in Theorem 6.3 that one can assume $T_-(\psi) = 0$, $T_+(\psi) = \pi$.

Proof. Let ${}^4\mathscr{M}_0$ be any development of ${}^3\Sigma$, (decreasing ${}^4\mathscr{M}_0$ if necessary) by Corollary 5.3 and Lemma 5.3 there exist coordinates (t, θ) on ${}^4\mathscr{M}_0$ such that $h_{AB}\, dx^A\, dx^B = e^{2B_0}(-dt^2 + d\theta^2)$. Let

$$f = \frac{1}{2}\int_{\theta_+}^{\theta} R_+(0, x)\, dx + \lambda, \tag{6.4a}$$

$$g = -\frac{1}{2}\int_{\theta_-}^{\theta} R_-(0, x)\, dx, \tag{6.4b}$$

$$\lambda = \frac{1}{2}[R(0, \theta_+) + R(0, \theta_-)] + \frac{1}{2}\int_{\theta_-}^{\theta_+} R_t(0, x)\, dx. \tag{6.4c}$$

(For $\theta \in (\theta_-, \theta_+)$ we have $R_+ > 0$, $R_- < 0$, and thus $R_t > 0$, in particular the integrand in (6.4c) is positive. One also finds that $V(R_\mu) = [\theta_-, \theta_+] \times U(1) \times U(1)$ and the equality of (6.4c) and (6.3) follows from (A.32).) From (6.4) we have $f' - g' = \partial_\theta R(0, \theta)$ and $f(\theta_-) - g(\theta_-) = R(0, \theta_-)$, thus

$$R(0, \theta) = f(\theta) - g(\theta). \tag{6.5}$$

In the coordinates (t, θ), R satisfies the flat space-time wave-equation,

$$(\partial_t^2 - \partial_\theta^2) R = 0, \tag{6.6}$$

it follows that for (t, θ) such that $0 \leqslant u := t + \theta \leqslant \pi$, $0 \leqslant v := \theta - t \leqslant \pi$, one has

$$R(t, \theta) = f(t + \theta) - g(\theta - t). \tag{6.7}$$

We claim that f and g are positive: indeed we have

$$f'(\theta) = 0 \Rightarrow \theta = \theta_+, \qquad g'(\theta) = 0 \Rightarrow \theta = \theta_-. \tag{6.8}$$

Thus θ_+ and θ_- are the only interior extrema of f and g. Now for $\theta > \theta_-$, respectively $\theta < \theta_-$, we have $R_- < 0$, respectively $R_- > 0$; thus g is positive. To show that f is positive it is sufficient to prove that $f(0) > 0$, $f(\pi) > 0$. From (6.4) we have

$$\begin{aligned} f(0) &= \frac{1}{2}\int_0^{\theta_-} R_- > 0 \qquad \text{because } R_- > 0 \text{ on } (0, \theta_-), \\ f(\pi) &= -\frac{1}{2}\int_{\theta_-}^{\pi} R_- > 0 \qquad \text{because } R_- < 0 \text{ on } (\theta_-, \pi); \end{aligned} \tag{6.9}$$

thus

$$0 < f \leqslant \lambda. \tag{6.10}$$

Let us also note that

$$\begin{aligned} g(0) &= \lambda - \frac{1}{2}\int_0^{\theta_+} R_+ < \lambda \qquad \text{because } R_+ > 0 \text{ on } (0, \theta_+), \\ g(\pi) &= \lambda + \frac{1}{2}\int_{\theta_+}^{\pi} R_+ < \lambda \qquad \text{because } R_- < 0 \text{ on } (\theta_+, \pi), \end{aligned} \tag{6.11}$$

so that we also have

$$0 \leqslant g < \lambda. \tag{6.12}$$

f and g can be written in the form

$$\begin{aligned} &f(\theta) = \lambda(1 - f_1(\theta)(\theta - \theta_+)^2), \\ &\left(f_1(\theta) = -\frac{1}{\lambda}\int_0^1 dt \int_0^1 ds\, t f''(\theta_+ + ts(\theta - \theta_+))\right), \end{aligned} \tag{6.13}$$

$$\begin{aligned} &g(\theta) = \lambda g_1(\theta)(\theta - \theta_-)^2, \\ &\left(g_1(\theta) = \frac{1}{\lambda}\int_0^1 dt \int_0^1 ds\, t g''(\theta_- + ts(\theta - \theta_-))\right). \end{aligned} \tag{6.14}$$

Now $f''(\theta_+) = \frac{1}{2}\partial_\theta R_+(0, \theta) < 0$ thus $f_1(\theta_+) > 0$, and $f \in C^2$ implies that f_1 is positive in a neighborhood of θ_+. Equation (6.10) shows that for all $\theta \in [0, \pi]$

$$0 < f_1. \tag{6.15}$$

Similarly $g_1(\theta_-) > 0$ and by (6.12) for all $\theta \in [0, \pi]$,

$$0 < g_1. \tag{6.16}$$

Define

$$\begin{aligned} \psi(t, \theta) &= \arcsin[\sqrt{g_1(\theta - t)}(\theta - t - \theta_-)] \\ &\quad - \arccos[\sqrt{f_1(\theta + t)}\,(t + \theta - \theta_+)], \\ T(t, \theta) &= -\arcsin[\sqrt{g_1(\theta - t)}(\theta - t - \theta_-)] \\ &\quad - \arccos[\sqrt{f_1(\theta + t)}(t + \theta - \theta_+)], \end{aligned} \tag{6.17}$$

where the arccos and arcsin branches are chosen so that

$$\arcsin(0)=0, \qquad \arccos(0)=\pi/2.$$

By (6.17) and by what has been said it follows that ψ and T are smooth functions of t and θ. The origin of (6.17) is easier to understand if one rewrites (6.17) in the form

$$\lambda \sin^2 \frac{V}{2} = g(v), \qquad \lambda \sin^2 \frac{U}{2} = f(u), \tag{6.18}$$

with $U=T+\psi$, $V=\psi-T$. Equations (6.7) and (6.18) imply

$$R(T,\psi)=\lambda \sin T \sin \psi. \tag{6.19}$$

From (6.17) and (6.18) for $v \neq \theta_-$ and $u \neq \theta_+$ it follows immediately that $dU/du>0$, $dV/dv>0$, but one also has

$$\lim_{v\to\theta_-} \frac{dV}{dv} = 2\sqrt{g_1(\theta_-)}>0,$$

$$\lim_{u\to\theta_+} \frac{dU}{du} = 2\sqrt{f_1(\theta_+)}>0,$$

and therefore $(t,\theta)\to(T,\psi)$ is a diffeomorphism in a neighborhood ${}^4\mathcal{O}$ of ${}^3\Sigma$ in ${}^4\mathcal{M}_0$ (smoothness at the axes of symmetry can be checked using methods of Appendix C). ${}^4\mathcal{M}$ can be defined as the diffeomorphic image of ${}^4\mathcal{O}$ by this diffeomorphism. Equation (6.18) implies that the manifestly conformally flat form of the metric preserved. To establish the claimed range of the coordinates (T,ψ) let us note that (6.10) and (6.12) imply

$$0<T<\pi. \tag{6.20}$$

Since on the axes $R=0$, (6.19) and (6.20) show that in the (T,ψ) coordinates the axes are given by $\psi=0$, π. At this stage we have established all our claims but the fact, that M^a and g^a can be set to zero. Now Eq. (E.14) shows that $\partial_t g^a=\partial_\theta M^a$ since $c_a=0$; thus the one forms $\alpha^a=g^a\,d\theta+M^a\,dt$ are closed. A straightforward computation using (5.1) and the results of Appendix C shows that the functions g^a and M^a are smooth and bounded up to the axes $\theta=0$, π;[11] by simple connectedness of ${}^2\mathcal{M}$ it follows that there exist f^a such that $\alpha^a=df^a$. Using the methods of Appendix C one can check that the transformation

$$(t,\theta,x^a)\to(t,\theta,\bar{x}^a=x^a+f^a)$$

is a smooth diffeomorphism of ${}^4\mathcal{M}$ to itself, and in the new coordinates M^a and g^a are equal to zero. ∎

[11] Note that g^a, M^a are defined as $\lambda^{ab}g_{0b}$, $\lambda^{ab}M_b$ and λ^{ab} is singular at the axes.

Let us comment somewhat on formula (6.2). As we have seen the functions $n_{ab}=\lambda_{ab}R^{-1}$ are *not* smooth on ${}^4\mathscr{M}$ because R vanishes on the axes, but the λ_{ab}'s *are* smooth everywhere. In the $S^2\times S^1$ case smoothness of λ_{ab} is equivalent to smoothness (and appropriate boundary behavior for $\theta=0$, π, cf. Appendix C) of R, A, and u (cf. (3.1)). In the S^3 case smoothness of λ_{ab} is equivalent to smoothness (and boundary behavior) of R, A, u on ${}^2\mathscr{M}\setminus\{\theta=\pi\}$ and smoothness of R, $\tilde{A}$, $\tilde{u}$ (cf. (5.6)) on ${}^2\mathscr{M}\setminus\{\theta=0\}$.

To prove the main result of this section, Theorem 6.3, we shall need the following difficult result due to Christodoulou [4]:

THEOREM 6.2 (D. Christodoulou). *Let $a>0$, let $B(a)$ be an open ball in $\mathbf{R}^2$, let*

$$\mathring{x}\in C^\infty(B(a),\mathscr{H}^2),\qquad \mathring{X}\in C^\infty(B(a),T\mathscr{H}^2),$$

$\mathring{X}(p)\in T_{\mathring{x}(p)}\mathscr{H}^2$, where $\mathscr{H}^2$ is the two dimensional hyperbolic space. Let $C(a)$ be the domain of dependence of $B(a)$ in $\mathbf{R}^{1,2}$, $C(a)=\{(t,x)\in\mathbf{R}^3,\ t\in[0,a), r<a-t\}$, suppose that $\mathring{x}$ and $\mathring{X}$ are invariant under rotations of $B(a)$. There exists a unique smooth map $x\in C^\infty(C(a),\mathscr{H}^2)$ satisfying the harmonic map equation,

$$D^A(\eta,h)\,X_a=0,$$

$$x(0,p)=\mathring{x}(p),\qquad \frac{\partial x}{\partial t}(0,p)=\mathring{X}(p),$$

where η is the Minkowski metric on $\mathbf{R}^{1,2}$ ($\eta=\mathrm{diag}(-1,1,1)$) and h is the standard (constant curvature) metric on $\mathscr{H}^2$.

PROPOSITION 6.1. *For $t_0<\tau_0\in\mathbf{R}$, $a<b\in\mathbf{R}$, $\tau\leqslant t_0+b-a$, let $C_\tau=\{(t,\theta)\in\mathbf{R}^2$: $t\in[t_0,\tau)$, $a+t-t_0<\theta<b-t+t_0\}$, let $\mathring{x}\in C^\infty((a,b),\mathscr{H}^2)$, $\mathring{X}\in C^\infty((a,b),T\mathscr{H}^2)$, $\mathring{X}(\theta)\in T_{\mathring{x}(\theta)}\mathscr{H}^2$, let $R\in C^\infty(C_{\tau_0})$, suppose that for every $\tau<\tau_0$ there exists $\varepsilon(\tau)>0$ such that $R|_{C_\tau}\geqslant\varepsilon(\tau)$. There exists a unique map $x\in C^\infty(C_{\tau_0},\mathscr{H}^2)$ satisfying the harmonic map-type equation*

$$D^a(\eta,h)(RX_A)=0,$$

$$x(0,\theta)=\mathring{x}(\theta),\qquad \frac{\partial x}{\partial t}(0,\theta)=\mathring{X}(\theta).$$

Outline of Proof. For $\tau<\tau_0$ the existence of $x\in C^\infty(C_\tau,\mathscr{H}^2)$ can be established either by using Nirenberg–Gagliardo inequalities and appropriate energy functionals (cf., e.g., [1]; rather than using integrals over the whole space as in that reference one should use energy integrals on slices $t=\text{const}$ intersected with compact globally hyperbolic subsets of C_τ, cf., e.g., [11, Chap. 7]); an alternative simple proof may be given using a generalization of C_1 estimates derived for this kind of equations by Moncrief [17], as in [6]. A simple argument shows that one can pass to the limit $\tau\to\tau_0$ and obtain a solution defined on C_{τ_0} (note, however,

that if $\inf_{C_{\tau_0}} R = 0$ the solution may blow up in H_k^{loc} seminorms, $k \geqslant 2$, as one approaches τ_0). ∎

THEOREM 6.3. *Let (g_{ij}, P_{ij}) be generic $U(1) \times U(1)$ symmetric Cauchy data on ${}^3\Sigma$, ${}^3\Sigma \approx S^2 \times S^1$ or S^3, let ${}^4\mathscr{M}$ be the maximal globally hyperbolic development*[12] *of (g_{ij}, P_{ij}). There exists an open subset ${}^4\mathscr{O}$ of ${}^4\mathscr{M}$ (possibly equal to ${}^4\mathscr{M}$) and a coordinate system on ${}^4\mathscr{O}$ such that*

$$\begin{aligned} ds^2 &= e^{2B}(-dT^2 + d\psi^2) + \lambda_{ab}\, dx^a\, dx^b, \\ \det \lambda_{ab} &= \lambda \sin T \sin \psi, \qquad \lambda_{,\mu} = 0, \\ x^a &\in [0, 2\pi]|_{\mathrm{mod}\, 2\pi},\ T \in (0, \pi),\ \psi \in [0, \pi]. \end{aligned} \tag{6.21}$$

Remark. Theorem 6.3 shows that for generic Cauchy data no singularities develop in ${}^4\mathscr{M}$ before the boundaries $T = 0$, π are reached. One expects these boundaries "to contain" either curvature singularities, or Cauchy horizons, or both (cf., e.g., [13, 5]).

To prove Theorem 6.3 we still need three auxiliary lemmas:

LEMMA 6.1. *Let* $I = \{(T, \psi) \in (0, \pi) \times [0, \pi] : \psi < T < \pi - \psi\}$, $II = \{(T, \psi) \in (0, \pi) \times [0, \pi] : \pi - \psi < T < \psi\}$ (cf. Fig. 1), *let* $R = \sin T \sin \psi$. *There exist smooth coordinates t, r: $I \leftrightarrow \mathscr{O} \subset \mathbf{R}^2$, $\bar{t}\, \bar{r}$: $II \leftrightarrow \mathscr{O} \subset \mathbf{R}^2$, such that*

$$R|_I = r, \qquad R|_{II} = \bar{r}, \tag{6.22}$$

and

$$(-dT^2 + d\psi^2)|_I = \Omega^2(-dt^2 + dr^2) \tag{6.23a}$$

$$(-dT^2 + d\psi^2)|_{II} = \bar{\Omega}^2(-d\bar{t}^2 + d\bar{r}^2) \tag{6.23b}$$

for some positive functions Ω, $\bar{\Omega} \in C^\infty(\mathscr{O})$.

Remark. In the interiors of the remaining two triangles in Fig. 1 one can, essentially repeating the proof below, introduce coordinates such that $R = t$.

Proof. In I set

$$r = R, \qquad t_{,T} = R_{,\psi}, \qquad t_{,\psi} = R_{,T},$$

and from $\det \partial(r, t)/\partial(R, T) = R_{,\theta}^2 - R_{,T}^2 > 0$ in I it follows that the map so defined is a diffeomorphism. The coordinates $(\bar{r}, \bar{t})$ are obtained from (r, t) by replacing ψ by $\pi - \psi$. ∎

We shall say that a closed subset Σ of $(0, \pi) \times [0, \pi]$ is a spacelike Cauchy surface for a subset $\mathscr{A}$ of $(0, \pi) \times [0, \pi]$ if it is the disjoint union of $\mathscr{A}_+$, Σ, and $\mathscr{A}_-$,

[12] Cf., e.g., [2] for a definition of the notion of maximal globally hyperbolic development.

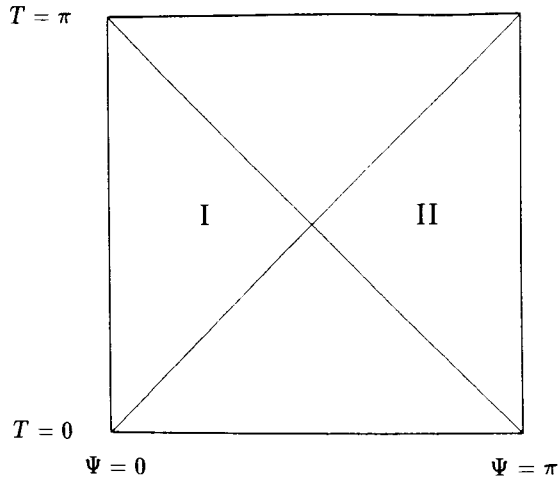

FIGURE 1

and $\mathscr{A}_+ (\mathscr{A}_-)$ have the property that for every $p \in \mathscr{A}_+$ ($p \in \mathscr{A}_-$) every past (future) directed past (future) inextendible causal curve through p meets either one of the axes or meets Σ; and if the curve meets Σ, it does so only once. $\mathscr{A}$ will be said to be globally hyperbolic if $\mathscr{A}$ contains a (spacelike) Cauchy surface. It is easily seen that this definition of global hyperbolicity coincides with the standard notion of global hyperbolicity on $\mathscr{A} \times U(1) \times U(1)$ with a metric of the form (6.21). Domains of dependence in $(0, \pi) \times [0, \pi]$ are defined in an analogous way.

LEMMA 6.2. *Let $\mathscr{A}$ be a globally hyperbolic open subset of* $(0, \pi) \times [0, \pi]$, *let $g_{\mathscr{A}}$ be a smooth vacuum metric on $\mathscr{A} \times U(1) \times U(1)$ of the form* (6.21). *Let* (t, r), $(\bar{t}\,\bar{r})$ *be*

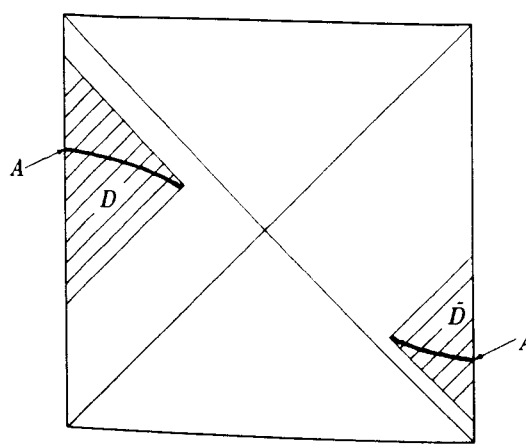

FIGURE 2

coordinates of Lemma 6.1., *let* $A = \{t_0\} \times [0, r_0) \subset \mathscr{A} \cap I$, $\bar{A} = \{\bar{t}_0\} \times [0, \bar{r}_0) \subset \mathscr{A} \cap II$, *let* $\mathscr{D}$, $\bar{\mathscr{D}}$ *be the domains of dependence of* A, $\bar{A}$ (*cf. Fig.* 2). *The metric* $g_{\mathscr{A}}$ *can be uniquely extended to a smooth vacuum metric* g *on* $\mathscr{A} \cup \mathscr{D} \cup \bar{\mathscr{D}}$, *such that*

$$g|_{\mathscr{A}} = g_{\mathscr{A}}.$$

Proof. In the coordinates (t, r), $(\bar{t}\,\bar{r})$ on $\mathscr{D}$ and $\bar{\mathscr{D}}$ the σ-model equation (3.12) reads

$$D^A(rY_A) = 0, \qquad D^A(\bar{r}Y_A) = 0 \tag{6.24}$$

(by simple connectedness of $\mathscr{D}$ and $\bar{\mathscr{D}}$ there is no $\bar{\omega}$-piece). (On $S^2 \times S^1$ one should use the Ernst formulation of the field equations to obtain the "reduced" form of Einstein equations with respect to the same nonvanishing Killing vector in $\mathscr{D}$ or $\bar{\mathscr{D}}$; on S^3 one should use different (appropriate) Killing vectors on $\mathscr{D}$ and on $\bar{\mathscr{D}}$.) Equation (6.24) is the harmonic map equation for a rotation invariant map from $\mathbf{R}^{1,2}$ to $\mathscr{H}^2$, the metric $g_{\mathscr{A}}$ induces rotation invariant smooth Cauchy data for this equation on $B(r_0)$, $B(\bar{r}_0) \subset \mathbf{R}^2$—existence and uniqueness of solutions in $\mathscr{D}$, $\bar{\mathscr{D}}$ follows from Christodoulou's Theorem 6.2. Since $R_+ R_- > 0$ on $\mathscr{D} \cup \bar{\mathscr{D}}$, B can be obtained on $\mathscr{D} \cup \bar{\mathscr{D}}$ by integrating (5.7). ∎

LEMMA 6.3. *Let* $\mathscr{A}$ *be as in Lemma* 6.2, *let* $\{T_0\} \times [\psi_1, \psi_2] \subset \mathscr{A}$, $0 < \psi_1 < \psi_2 < \pi$, *let* $\mathscr{D}$ *be the domain of dependence of* $\{T_0\} \times [\psi_1, \psi_2]$ *in* $(0, \pi) \times [0, \pi]$, $\mathscr{D} = \{(T, \psi) \in (0, \pi) \times [0, \pi] : \psi_1 - \psi < T - T_0 < \psi < \psi_1$ *and* $-\psi_2 + \psi < T - T_0 < \psi_2 - \psi\}$ (*cf. Fig.* 3). *The metric* $g_{\mathscr{A}}$ *can be uniquely extended to a smooth vacuum metric* g *on* $\mathscr{A} \cup \mathscr{D}$, *such that*

$$g|_{\mathscr{A}} = g_{\mathscr{A}}.$$

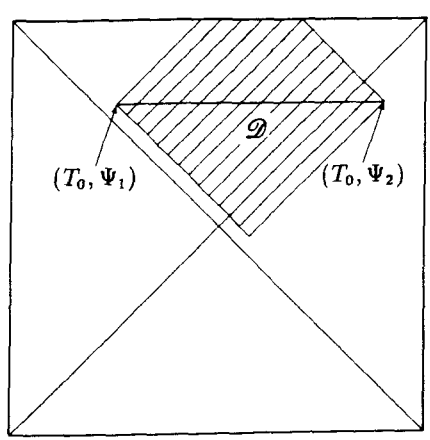

FIGURE 3

Proof. The metric $g_{\mathscr{A}}$ induces smooth Cauchy data for the harmonic-type map equation (2.24) with $c^a = 0$, which by Proposition 6.1 has a unique solution on $\mathscr{D}$. One can solve the linear wave equation (3.20) for β in $\mathscr{D}$ to obtain a smooth metric (6.21) there. The equality $g|_{\mathscr{A}} = g_{\mathscr{A}}$ follows from well-known uniqueness results on globally hyperbolic sets.

Proof of Theorem 6.3.[13] By Theorem 6.1 there exists a globally hyperbolic subset $\mathscr{A}$ of $(0, \pi) \times [0, \pi]$ with a metric of the form (6.21) defined on $\mathscr{A}$. If $\mathscr{A} \neq (0, \pi) \times [0, \pi]$ one can find spacelike submanifolds in $\mathscr{A}$ satisfying either the hypotheses of Lemma 6.2 or of Lemma 6.3 and extend the metric to a larger, still globally hyperbolic, subset of $(0, \pi) \times [0, \pi]$. Let $\mathfrak{A}$ denote the set of all open subsets of $(0, \pi) \times [0, \pi]$ for which ${}^3\Sigma$ is a Cauchy surface and on which a smooth vacuum metric, development of the Cauchy data, of the form (6.21) is defined. Let

$$ {}^2\mathfrak{M} = \bigcup_{A \in \mathfrak{A}} \mathscr{A}. $$

Let $p \in {}^2\mathfrak{M}$, by definition there exists an $A \in \mathfrak{A}$ and an open neighborhood $\mathscr{O}$ of p on which a metric $g_{\mathscr{A}}$ is defined. By uniqueness theorems it follows that $g_{\mathscr{A}}$ is $\mathscr{A}$-independent, thus the equation

$$ g|_{\mathscr{O}} = g_{\mathscr{A}}|_{\mathscr{O}} $$

defines a smooth metric g on ${}^2\mathfrak{M}$. ${}^2\mathfrak{M}$ is an open globally hyperbolic subset of $(0, \pi) \times [0, \pi]$; suppose that ${}^2\mathfrak{M} \neq (0, \pi) \times [0, \pi]$, therefore there exists $(\tilde{T}_0, \tilde{\psi}_0) \in \partial^2\mathfrak{M} = \overline{{}^2\mathfrak{M}} \setminus {}^2\mathfrak{M}$. Changing the time orientation if necessary we may suppose that $(\tilde{T}_0, \tilde{\psi}_0)$ is to the future of Σ. We claim that there also exists $(T_0, \psi_0) \in \partial^2\mathfrak{M}$ such that the causal past $J^-((T_0, \psi_0))$ intersected with $\partial^2\mathfrak{M}$ is empty[14]: let $\Gamma_\pm$ be past-directed null geodesics in the (T, ψ) plane through $(\tilde{T}_0, \tilde{\psi}_0)$. If $(\Gamma_+ \cup \Gamma_-) \cap \partial^2\mathfrak{M} = \emptyset$ set $(T_0, \psi_0) = (\tilde{T}_0, \tilde{\psi}_0)$, and global hyperbolicity of ${}^2\mathfrak{M}$ implies $J^-((T_0, \psi_0)) \cap \partial^2\mathfrak{M} = \emptyset$. If $\Gamma_\pm \cap \partial^2\mathfrak{M} \neq \phi$, say $\Gamma_+ \cap \partial^2\mathfrak{M} \neq \emptyset$ it follows by closedness of $\partial^2\mathfrak{M}$ that there exists a point (T_0, ψ_0) on Γ_+ such that $J^-((T_0, \psi_0)) \cap \partial^2\mathfrak{M} = \emptyset$. Suppose that $\psi_0 \neq 0$ or π, let ε be small enough so that $(T_0 - \varepsilon, \psi_0) \in {}^2\mathfrak{M}$; since $\partial^2\mathfrak{M}$ is closed and $J^-((T_0, \psi_0)) \cap \partial^2\mathfrak{M} = \emptyset$ it follows that if ε is small enough there exists $\delta > 0$ such that $S = \{T_0 - \varepsilon\} \times [\psi_0 - \varepsilon - \delta, \psi_0 + \varepsilon + \delta] \subset {}^2\mathfrak{M}$; by Lemma 6.3 the metric g can smoothly be extended to a vacuum metric in the domain of dependence of S which contains a neighborhood of p, contradicting $p \in \partial^2\mathfrak{M}$. If $\psi_0 = 0$ or π, a repetition of this argument using Lemma 6.2 shows that the metric cannot blow up also on the axes before reaching $T = 0$ or $T = \pi$, which establishes the theorem. ∎

[13] The idea of the proof is essentially the same as in [2].

[14] To simplify notations we assume that $p \notin J^+(p)$.

7. Conclusions

The object of this paper was to develop a useful framework for discussing the evolution problem for general space-times with $U(1)\times U(1)$ symmetric compact Cauchy surfaces; the author believes to have achieved this goal. In the $c_a=0$ case we have corrected some erroneous, essentially minor, claims by Gowdy [10] and have filled in several technical details which were missing in Gowdy's arguments. We have shown that the maximal globally hyperbolic development of $U(1)\times U(1)$ symmetric Cauchy data on S^3 or $S^2\times S^1$ contains the "Gowdy square" $R>0$. In the case, hitherto undiscussed in the literature, $c_a\neq 0$ ($\Rightarrow {}^3\Sigma\approx T^3$) we have presented two possible sets of reduced variables, derived the equations of motions satisfied by them, and derived the restrictions imposed on the free Cauchy data (say λ_{ab}, $\partial\lambda_{ab}/\partial n$) which follow from the constraint equations. The results of this paper in the $c_a\neq 0$ case seem to be a good starting point for an attempt to prove that the maximal globally hyperbolic development of $U(1)\times U(1)$ symmetric data on T^3 contains the set $R>0$, and coincides with this set. It is tempting to raise the question, does strong cosmic censorship hold in the space of metrics considered in this paper? Encouraging results towards an affirmative answer to this question have been presented in [5] in a restricted setting, the general problem seems still out of reach at this stage.

APPENDIX A: Constraint Equations

The formalism developed by Geroch in [9] to analyze Einstein equations on ${}^4\mathscr{M}/G_2$ can be used to obtain a "reduced form" of the constraint equations on ${}^3\tilde{\Sigma}/G$, $G=U(1)\times U(1)$, where ${}^3\tilde{\Sigma}=\{p\in {}^3\Sigma$: the orbit of G through p is two-dimensional$\}$. *Let* $X_a=X^i_a\,\partial/\partial x^i$ be the Killing vectors on ${}^3\tilde{\Sigma}$, let

$$\lambda_{ab}\equiv g_{ij}X^i_aX^j_b,\qquad \lambda^{ab}\equiv(\lambda_{ab})^{-1},$$

$$R\equiv(\det\lambda_{ab})^{1/2},$$

$$\hbar_{ij}=g_{ij}-\lambda^{ab}X_{ai}X_{bj},$$

and we use the convention that indices $i, j, \ldots$, etc., are raised and lowered with $g^{ij}\equiv(g_{ij})^{-1}$ and g_{ij}, while $a, b, \ldots$, etc., are raised and lowered with λ^{ab} and λ_{ab}. As discussed by Geroch [9], tensors on ${}^3\tilde{\Sigma}/G$ can be identified with tensors $t^{j_1\cdots j_l}_{i_1\cdots i_k}$ on ${}^3\tilde{\Sigma}$ satisfying

$$t^{j_1\cdots j_l}_{i_1\cdots i_s\cdots i_k}X^{i_s}_a=t^{j_1\cdots j_s\cdots j_l}_{i_1\cdots i_k}X_{aj_s}=0, \tag{A.1}$$

$$\mathscr{L}_{X_a}t^{j_1\cdots j_l}_{i_1\cdots i_k}=0, \tag{A.2}$$

where $\mathscr{L}_{X_a}$ denotes the Lie derivative with respect to X_a. Under such identifications $\hbar_{ij}$ defines a natural metric on ${}^3\tilde{\Sigma}/G$. For $X\in T({}^3\tilde{\Sigma}/G)$, the derivative operator $\bar{D}_i$ defined by

$$\bar{D}_iX^j=\hbar^{i_1}_i\hbar^j_{j_1}D_{i_1}X^{j_1}, \tag{A.3}$$

where D_i is the Riemannian covariant derivative of the metric g_{ij}, is the Riemannian covariant derivative of the metric $\hbar_{ij}$. Since the Killing vectors commute we have

$$X^j_b D_j X^i_a = X^j_a D_j X^i_b, \qquad D_i X_{a_j} + D_j X_{a_i} = 0 \tag{A.4}$$

$$\Rightarrow D_j \lambda_{ab} X^j_c = 0, \qquad X^i_b D_j X_{ai} = X^i_a D_j X_{bi} = \tfrac{1}{2} D_j \lambda_{ab}. \tag{A.5}$$

Let K_{ij} be the extrinsic curvature of ${}^3\Sigma \subset {}^4\mathscr{M}$, let $P_{ij} = Kg_{ij} - K_{ij}$, $K = g^{kl}K_{kl}$, let

$$\begin{aligned} K_{ab} &= K_{ij} X^i_a X^j_b, & P_{ab} &= P_{ij} X^i_a X^j_b, \\ \bar{K}_{aj} &= \hbar^i_j X^k_a K_{ik}, & \bar{P}_{aj} &= \hbar^i_j X^k_a P_{ik}, \\ \bar{K} &= \hbar^{ij} K_{ij}, & \bar{P} &= \hbar^{ij} P_{ij}, \end{aligned} \tag{A.6}$$

and we have

$$\begin{aligned} \bar{K}_{ij} &= \bar{K}\hbar_{ij}, \qquad \bar{P}_{ij} = \bar{P}\hbar_{ij}, \\ K_{ij} &= \bar{K}\hbar_{ij} + \bar{K}_{ai} X^a_j + \bar{K}_{aj} X^a_i + K_{ab} X^a_i X^b_j, \\ P_{ij} &= \bar{P}\hbar_{ij} + \bar{P}_{ai} X^a_j + \bar{P}_{aj} X^a_i + P_{ab} X^a_i X^b_j. \end{aligned} \tag{A.7}$$

Let us introduce the "twist vectors,"

$$\omega^i_a = \tfrac{1}{2} \varepsilon^i{}_{jk} D^j X^k_a, \tag{A.8}$$

$$\bar{\omega}^i_a = \hbar^i_j \omega^j_a, \qquad \omega_{ab} = X^i_a \omega_{bi}. \tag{A.9}$$

Equations (A.8)–(A.9) imply

$$D_i X_{aj} = \varepsilon_{ijk} (\bar{\omega}^k_a + \omega^b{}_a X^k_b). \tag{A.10}$$

From (A.8)–(A.9) one also has

$$\begin{aligned} \bar{\omega}^i_a &= \hbar^i_{i_1} \varepsilon^{i_1 jk} D_j X_{ak} \\ &= \hbar^i_{i_1} (\hbar^j_{j_1} + X^j_c X^c_{j_1}) \varepsilon^{i_1 j_1 k} D_j X_{ak} \\ &= -\varepsilon^{i_1 j_1 k} \hbar^i_{i_1} X^c_{j_1} D_k \lambda_{ac} \\ &= -\varepsilon^{i_1 j_1 k_1} \hbar^i_{i_1} \hbar^k_{k_1} X^c_{j_1} D_k \lambda_{ac} \\ &= 0, \end{aligned} \tag{A.11}$$

and we have used (A.5) and $\hbar^{[i}_j \hbar^{k]}_l = 0$ since $\hbar_{ij}$ has rank 1. Equation (A.10) gives

$$D_i X_{aj} = \varepsilon_{ijk} X^k_c \omega^c_a. \tag{A.12}$$

Contracting (A.12) with $\varepsilon^{ilm} X^d_l X^j_d X_{bm}$ and making use of (A.5) one obtains

$$\omega_{ab} = \varepsilon^{ijk} X^c_j X_{ak} D_i \lambda_{bc}. \tag{A.13}$$

It is simple now to obtain the reduced version of the vector constraint

$$D_i P^{ij} = 0. \tag{A.14}$$

Equations (A.14) and (A.4) give

$$D_i(P^i_j X^j_a) = 0, \tag{A.15}$$

and the invariance of P_{ij} under the action of G yields

$$\mathscr{L}_{X_a} P_{ij} = 0 \Rightarrow X^k_a D_k P^{ij} = P^{ik} D_k X^j_a + P^{kj} D_k X^i_a. \tag{A.16}$$

From (A.3), (A.7), and (A.16) we have

$$\begin{aligned}
\bar{D}_i \bar{P}^{ij} &= \hbar^k_i \hbar^j_{j_1} D_k (P^{ij_1} - \bar{P}^{aj_1} X^i_a - \bar{P}^{ai} X^{j_1}_a - P^{ab} X^i_a X^{j_1}_b) \\
&= -X^a_i \hbar^j_{j_1} X^a_a D_k P^{ij_1} \\
&= -P^{ak} \hbar^{jj_1} (\hbar^{k_1}_k + X^b_k X^{k_1}_b) D_{k_1} X_{aj_1} - P^{ij} D_j R/R \\
&= \tfrac{1}{2} P^{ab} \bar{D}^j \lambda_{ab} - \bar{P}^{ij} \bar{D}_j R/R
\end{aligned} \tag{A.17}$$

(several terms vanish because $D_i X_{aj}$ is antisymmetric and $\hbar^{[i}_j \hbar^{k]}_l = 0$), in a similar way one derives

$$\bar{D}_i \bar{P}^i_a = -\bar{P}^i_a \bar{D}_i R/R. \tag{A.18}$$

Equations (A.17) and (A.18) can be finally written in the form

$$\begin{aligned}
\bar{D}_i (R \bar{P}^i_a) &= 0, \\
\bar{D}_i (R \bar{P}) &= \frac{R}{2} P^{ab} \bar{D}_i \lambda_{ab}.
\end{aligned} \tag{A.19}$$

Let us now derive the reduced version of the scalar constraint,

$$R(g) = K^{ij} K_{ij} - (g^{ij} K_{ij})^2. \tag{A.20}$$

In order to achieve this we shall calculate the (vanishing) curvature scalar of $\hbar_{ij}$ in terms of the curvature of g_{ij}. Let X^i be an arbitrary vector field on $T({}^3\tilde{\Sigma}/G)$, thus X^i can be represented by a vector field on ${}^3\Sigma$ satisfying

$$X^i_a X_i = 0, \qquad \mathscr{L}_{X_a} X^i = 0.$$

Let us temporarily "forget" that[15] $\hbar^k_{[l} \hbar^i_{j]} = 0$; from

$$\bar{D}_i \bar{D}_j X^i - \bar{D}_j \bar{D}_i X^i = 2\hbar^k_{[i} \hbar^l_{j]} D_k (\hbar^m_l \hbar^i_n D_m X^n)$$

[15] The idea is to derive a formula which holds regardless of the dimension of ${}^3\Sigma/G$, and to use $R(\hbar) = 0$ at the end of the calculation only.

one obtains

$$R_{jl}(\hbar)\, X^l = \{\hbar^k_n \hbar^m_j R^n_{lkm} - 2\lambda^{ab}\hbar^k_n \hbar^l_j D_k X_{al} D_l X^n_b - 2\lambda^{ab}\hbar^k_{[i}\hbar^m_{j]} D_k X_{bi} D_m X_{al}\}\, X^l, \tag{A.21}$$

which in turn leads to

$$\begin{aligned} \hbar^{jl} R_{jl}(\hbar) = R(\hbar) &= 3\lambda^{ab}\hbar^{ij}\hbar^{kl} D_j X_{al} D_i X_{bk} + \hbar^{ij}\hbar^{kl} R_{iljk}(g) \\ &= 6\omega^{ab}\omega_{ab} + \tfrac{3}{2}\bar{D}^i\lambda^{ab}\bar{D}_i\lambda_{ab} + R(g) \\ &\quad - 2X^{ai}X^j_a R_{ij}(g) + X^{ai}X^j_a X^{bk}X^l_b R_{iljk}(g). \end{aligned} \tag{A.22}$$

The well-known formula for the second derivatives of a Killing vector,

$$D_i D_j X_{ak} = R_{lijk}(g)\, X^l_a,$$

gives

$$\begin{aligned} X^i_a X^j_b X^k_c X^l_d R_{iljk} &= \frac{1}{4}(\bar{D}_i\lambda_{ac}\bar{D}^i\lambda_{bd} - \bar{D}_i\lambda_{cd}\bar{D}^i\lambda_{ab}), \\ X^i_a X^j_b R_{ij} &= -\frac{\bar{D}^i\bar{D}_i\lambda_{ab}}{2} + 2\omega_{ca}\omega^c_b, \end{aligned} \tag{A.23}$$

which inserted in (A.22) yields

$$\begin{aligned} 0 = R(\hbar) &= 2\omega_{ab}\omega^{ab} + K^{ij}K_{ij} - K^2 + \frac{2\bar{D}^i\bar{D}_i R}{R} \\ &\quad - \frac{3}{2}\frac{\bar{D}^i R \bar{D}_i R}{R^2} + \frac{1}{4}\bar{D}^i n^{ab}\bar{D}_i n_{ab}, \end{aligned} \tag{A.24}$$

where we have introduced

$$n_{ab} \equiv \lambda_{ab} R^{-1}. \tag{A.25}$$

It is useful to reexpress (A.19) and (A.24) in terms of the tangential and normal derivatives of the components of the metric. Let x^a be the coordinates along the orbits of G, let θ parametrize the orbit space ${}^3\tilde{\Sigma}/G$, as discussed in Section 1. From the definition of $\hbar_{ij}$ one has

$$\begin{aligned} g_{ij}\, dx^i\, dx^j &= \hbar_{\theta\theta}\, d\theta^2 + \lambda_{ab}(dx^a + g^a\, d\theta)(dx^b + g^b\, d\theta), \\ g^a &\equiv \lambda^{ab} g_{\theta b}, \end{aligned} \tag{A.26}$$

which can be also written in the matrix form

$$(g_{ij}) = \begin{bmatrix} 1 & (g^a)^t \\ 0 & (1^a_b) \end{bmatrix} \begin{bmatrix} \hbar & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (g^a) & (1^a_b) \end{bmatrix}. \tag{A.27}$$

Equation (A.27) implies in particular

$$\sqrt{\det g} = \sqrt{\det \hbar} R. \tag{A.28}$$

Let us set

$$\hbar_{\theta\theta} = e^{2B}. \tag{A.29}$$

Equation (A.27) gives

$$[g_{ij}]^{-1} = \begin{bmatrix} 1 & 0 \\ -(g^a) & 1 \end{bmatrix} \begin{bmatrix} \hbar^{-1} & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} 1 & -(g^a)^t \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow g^{ij}\,\partial_i\,\partial_j = e^{-2B}\,\partial_\theta^2 - 2e^{-2B} g^a\,\partial_a\,\partial_\theta + (\lambda^{ab} + e^{-2B} g^a g^b)\,\partial_a\,\partial_b. \tag{A.30}$$

The definition of K_{ij}

$$K_{ij} = -\frac{1}{2}\frac{\partial g_{ij}}{\partial n} + \frac{1}{2N}(g_{ik} N^k_{\cdot j} + g_{jk} N^k_{\cdot i}), \tag{A.31}$$

where we have set

$$\frac{\partial f}{\partial n} \equiv \frac{1}{N}\frac{\partial f}{\partial t} - \frac{N^\theta}{N}\frac{\partial f}{\partial \theta},$$

$$N_i = g_{0i}, \qquad N = (-g^{00})^{-1/2}, \qquad N^i = g^{ij} N_j,$$

(note that $N_a = M_a$, M_a as in (2.8), but $N^a \neq M^a \equiv \lambda^{ab} M_b$) together with (A.28) gives

$$K_{ab} = -\frac{1}{2}\frac{\partial \lambda_{ab}}{\partial n},$$

$$K = -\frac{1}{R}\frac{\partial R}{\partial n} - \frac{\partial B}{\partial n} + \frac{\partial_\theta N^\theta}{N},$$

$$K^a_a = \lambda^{ab} K_{ab} = -\frac{1}{R}\frac{\partial R}{\partial n},$$

$$\bar{K} = K - K^a_a = -\frac{\partial B}{\partial n} + \frac{\partial_\theta N^\theta}{N}, \tag{A.32}$$

$$\bar{K}^\theta_a = K^\theta_a = -\frac{e^{-2B}}{2}\left(\lambda_{ab}\frac{\partial g^b}{\partial n} - \frac{N^i_{\cdot\theta}}{N} g_{ia}\right) \tag{A.33}$$

(note that (A.33) is a little messy to derive by brute force: a simple derivation may be obtained by combining (A.40) and (E.12)). From (A.7) one has

$$K^{ij} K_{ij} - K^2 = 2\bar{K}^{ia}\bar{K}_{ia} + K^{ab} K_{ab} + (K^a_a)^2 - 2K K^a_a. \tag{A.34}$$

In the adapted coordinate system (x^a, θ), Eq. (A.13) reads

$$\omega_{ab} = e^{-B}\varepsilon_a{}^c \lambda_{bc,\theta}$$
$$\Rightarrow \omega_{ab}\omega^{ab} = \tfrac{1}{4}\lambda^{ab}\lambda^{cd}\bar{D}^\theta\lambda_{ac}\bar{D}_\theta\lambda_{bd},$$

which together with (A.32) and (A.34), inserted into (A.24), gives

$$\bar{D}^\theta\bar{D}_\theta R = \frac{\bar{D}^\theta R\bar{D}_\theta R + (\partial R/\partial n)^2}{4R} + \left(\frac{\partial B}{\partial n} - \partial_\theta N^\theta\right)\frac{\partial R}{\partial n}$$
$$- R\bar{K}^{ia}\bar{K}_{ia} - \frac{R}{8}n^{ab}n^{cd}\left\{\frac{\partial n_{ac}}{\partial n}\frac{\partial n_{bd}}{\partial n} + \bar{D}^\theta n_{ac}\bar{D}_\theta n_{bd}\right\}. \tag{A.35}$$

The last of Eqs. (A.19) can be written as

$$\partial_\theta\frac{\partial R}{\partial n} = \frac{\partial_\theta R(\partial R/\partial n)}{2R} + \left(\frac{\partial B}{\partial n} - \partial_\theta N^\theta\right)\partial_\theta R - \frac{R}{4}n^{ac}n^{bd}\frac{\partial n_{ab}}{\partial n}\frac{\partial n_{cd}}{\partial\theta}, \tag{A.36}$$

while the first two Eqs. (A.19) read

$$\partial_\theta(Re^B K_a^\theta) = 0, \tag{A.37}$$

which shows that $Re^B K_a^\theta$ is θ-independent. This quantity is in fact related to the "twist charges,"

$$c_a = \varepsilon_{\mu\nu\rho\sigma}X_1^\mu X_2^\nu \nabla^\rho X_a^\sigma \tag{A.38}$$

(where ∇_μ is the Riemannian covariant derivative of the metric $g_{\lambda\nu}$), which are known to be both space *and time* independent (cf. [9, 14]). Let us in the standard way identify vectors X^i tangent to ${}^3\tilde{\Sigma}$ with vectors X^μ on ${}^4\mathscr{M}$ satisfying $X^\mu n_\mu = 0$, where n_μ is the unit future pointing normal to ${}^3\tilde{\Sigma}$. One has the well-known formula

$$D_\alpha X^\beta = (\delta_\alpha^{\alpha_1} + n_\alpha n^{\alpha_1})(\delta_{\beta_1}^\beta + n^\beta n_{\beta_1})\nabla_{\alpha_1}X^{\beta_1}$$
$$= \nabla_\alpha X^\beta - n_\alpha n^{\alpha_1}\nabla_{\alpha_1}X^\beta - X^{\beta_1}n^\beta\nabla_\alpha n_{\beta_1}$$
$$+ n_\alpha n^\beta n_{\beta_1}n^{\alpha_1}\nabla_{\alpha_1}X^{\beta_1}, \tag{A.39}$$

and we have

$$K_{\mu\nu} = -(\delta_\mu^{\mu_1} + n_\mu n^{\mu_1})(\delta_\nu^{\nu_1} + n_\nu n^{\nu_1})\nabla_{\nu_1}n_{\mu_1}.$$

Equation (A.39) can be used to express $\nabla^\rho X^\sigma$ in terms of $D^\rho X^\sigma$ and the remaining terms; inserted in (A.38) the $D^\rho X^\sigma$ term gives no contributions since it carries only "indices tangent" to ${}^3\tilde{\Sigma}$, thus giving zero when contracted with $\varepsilon_{\mu\nu\rho\sigma}$; using then antisymmetry of $\nabla_\alpha X_{a\mu}$ one obtains

$$
\begin{aligned}
c_a &= \varepsilon_{\mu\nu\rho\sigma} X_1^\mu X_2^\nu (n^\sigma X_a^\beta \nabla^\rho n_\beta + n^\rho n^\alpha \nabla^\sigma X_{a\alpha}) \\
&= 2\varepsilon_{\mu\nu\rho\sigma} X_1^\mu X_2^\nu n^\sigma (g^{\rho\rho_1} + n^\rho n^{\rho_1})(\delta_\beta^{\beta_1} + n_\beta n^{\beta_1}) X_a^\beta \nabla_{\rho_1} n_{\beta_1} \\
&= 2\varepsilon_{ijk} X_1^i X_2^j K_l^k X_a^l = 2\varepsilon_{ijk} X_1^i X_2^j \bar{K}_a^k \\
&= 2Re^B K_a^\theta .
\end{aligned}
$$

Therefore

$$
K_a^\theta = \frac{e^{-B} c_a}{2R}, \qquad c_{a,\mu} = 0. \tag{A.40}
$$

Equation (A.40) and the definition of $\bar{D}_i \bar{D}^i$ allow one to rewrite (A.35) in the form

$$
\begin{aligned}
\partial_\theta^2 R = {} & \frac{(\partial_\theta R)^2 + e^{2B}(\partial R/\partial n)^2}{4R} + e^{2B}\left(\frac{\partial B}{\partial n} - \partial_\theta N^\theta\right)\frac{\partial R}{\partial n} + \partial_\theta B\, \partial_\theta R \\
& - e^{2B}\frac{n^{ab} c_a c_b}{4R^2} - \frac{R}{8} n^{ad} n^{cd}\left\{e^{2B}\frac{\partial n_{ac}}{\partial n}\frac{\partial n_{bd}}{\partial n} + \partial_\theta n_{ac}\, \partial_\theta n_{bd}\right\}.
\end{aligned} \tag{A.41}
$$

Let us note that (A.33) and (A.40) give an evolution equation for g^a:

$$
\frac{\partial g^a}{\partial n} = -\frac{e^B c^a}{R} + \frac{\partial_\theta N^a}{N} + \frac{g^a\, \partial_\theta N^\theta}{N}. \tag{A.42}
$$

APPENDIX B: TWIST POTENTIALS ON $^4\tilde{\mathscr{M}}$

In this appendix we shall derive some formulae for the derivatives of the Killing vectors which are needed in Section 3. Let λ_{ab} and $h_{\mu\nu}$ be defined by (2.1) and (2.3). We have

$$
\nabla_\mu X_{a\nu} + \nabla_\nu X_{a\mu} = 0, \qquad X_a^\mu \nabla_\mu X_b^\nu = X_b^\mu \nabla_\mu X_a^\nu, \tag{B.1}
$$

because the X_a's are commuting Killing vectors, which implies

$$
\begin{gathered}
X_a^\alpha \nabla_\mu X_{b\alpha} = X_b^\alpha \nabla_\mu X_{a\alpha} = \tfrac{1}{2}\nabla_\mu \lambda_{ab}, \\
X_c^\mu \nabla_\mu \lambda_{ab} = 0.
\end{gathered} \tag{B.2}
$$

Let us define

$$
\omega_{\mu ab} = \varepsilon_{\mu\alpha\beta\gamma} X_a^\alpha \nabla^\beta X_b^\gamma, \tag{B.3}
$$

$$
v_{ab}^\mu = h_\nu^\mu \omega_{ab}^\nu, \tag{B.4}
$$

$$
C_{abc} = \omega_{ab\mu} X_c^\mu. \tag{B.5}
$$

From the antisymmetry of C_{abc} in a and c one has

$$
C_{abc} = -\frac{c_b}{R}\varepsilon_{ac}, \qquad c_b = \varepsilon_{\alpha\beta\gamma\delta} X_1^\alpha X_2^\beta \nabla^\gamma X_b^\delta. \tag{B.6}
$$

Contracting (B.3) with $\varepsilon^{\rho\sigma\mu\lambda}X_{c\lambda}$ one finds

$$\varepsilon_{\rho\sigma\mu\lambda}X^{\lambda}_{c}\omega^{\mu}_{ab}=2\lambda_{ac}\nabla_{\sigma}X_{b\rho}+2X_{a[\sigma}\nabla_{\rho]}\lambda_{bc}, \tag{B.7}$$

and a contraction with λ^{ac} yields

$$\nabla_{\alpha}X_{a\beta}=\tfrac{1}{4}\,\varepsilon_{\alpha\beta\mu\nu}X^{b\mu}\omega^{\nu}_{ba}+\tfrac{1}{2}X^{b}_{[\beta}\nabla_{\alpha]}\lambda_{ab}. \tag{B.8}$$

Contracting (B.7) with $\varepsilon^{\rho\alpha\beta\gamma}X^{\sigma}_{d}X^{d}_{\alpha}X^{c}_{\beta}$ leads to

$$\nu^{\alpha}_{ab}=\varepsilon^{\alpha\beta\gamma\delta}X^{c}_{\beta}X_{a\gamma}\nabla_{\delta}\lambda_{bc}. \tag{B.9}$$

Equations (B.4)–(B.6) and (B.9) give

$$\begin{aligned}\omega^{\alpha}_{ab}&=\varepsilon^{\alpha\beta\gamma\delta}X^{c}_{\beta}X_{a\gamma}\nabla_{\delta}\lambda_{bc}-\frac{c_b}{R}\,\varepsilon_{a}{}^{c}X^{\alpha}_{c}\\&=\varepsilon^{\alpha\beta\gamma\delta}X^{c}_{\beta}X_{a\gamma}\mathscr{D}_{\delta}\lambda_{bc}-\frac{c_b}{R}\,\varepsilon_{a}{}^{c}X^{\alpha}_{c},\end{aligned} \tag{B.10}$$

so that (B.8) finally implies

$$\nabla_{\alpha}X_{a\beta}=X^{b}_{[\beta}\nabla_{\alpha]}\lambda_{ba}-\frac{c_a}{2R^2}\,\varepsilon_{\alpha\beta\mu\nu}X^{\mu}_{1}X^{\nu}_{2}. \tag{B.11}$$

Let us mention that (B.10)–(B.11) are quite useful when deriving the reduced Einstein equations (2.5)–(2.6) using Geroch's method.

APPENDIX C: Functions and Tensors Invariant Under the Rotation Group

In this appendix, we shall derive some results about smooth functions invariant under the rotation group, which will be needed in Section 5. As is usual, a rotation invariant function on $\mathbf{R}^n$ will be identified with a function $f(r)$, $r\in[0,\infty)$.

Proposition C1. *Let $f\in C^{2k}(\mathbf{R}^n)$ be invariant under rotations, $0\leqslant k\leqslant\infty$. There exists $\tilde{f}\in C^{k}(\mathbf{R})$ such that*

$$f(x^i)=f(r)=\tilde{f}(r^2),\qquad r^2\equiv\sum(x^i)^2. \tag{C.1}$$

Remark. Equation (C.1) is obviously sufficient for smoothness of f, we show that it is also necessary.

The proof proceeds via a series of lemmas:

Lemma C.1. *For all $0\leqslant l\leqslant k-1$*

$$\left[\left(\frac{d}{dr}\right)^{2l+1}f\right](0)=0. \tag{C.2}$$

Proof. Let $\tilde{f}(x) = f(x, 0, ..., 0)$. By rotation invariance $\tilde{f}(x) = \tilde{f}(-x)$, thus

$$0 = \left[\left(\frac{d}{dx}\right)^{2l+1} \tilde{f}\right](0) = \left[\left(\frac{d}{dr}\right)^{2l+1} f\right](0). \quad \blacksquare$$

By a straightforward induction argument one shows

LEMMA C.2. *Let $\tilde{f}(s) = f(\sqrt{s})$. There exist real numbers $a_{i,n}$ such that*

$$\frac{d^{(n)}\tilde{f}}{ds^n}(s) = s^{-n/2} \sum_{i=0}^{n-1} a_{n,i} f^{(n-i)}(\sqrt{s})\, s^{-i/2}.$$

LEMMA C.3. *Let $g(r) \in C^{2n}([0, \infty))$, suppose that*

$$g(r) = o(r^{2n}), \qquad g'(r) = o(r^{2n-1}), ..., \; g^{(2n)}(r) = o(1)$$

(we write $f = o(r^\alpha)$ if $\lim_{r \to 0} r^{-\alpha} f = 0$). Let $\bar{g}(s) = g(\sqrt{s})$. Then $\bar{g} \in C^n([0, \infty))$ and

$$\lim_{s \to 0} \bar{g}'(s) = \lim_{s \to 0} \bar{g}''(s) = \cdots = \lim_{s \to 0} \bar{g}^{(n)}(s) = 0.$$

Proof. By Lemma C.2,

$$\lim_{s \to 0} \bar{g}^{(j)}(s) = \lim_{s \to 0} \sum_{i=0}^{j-1} a_{j,i}\, g^{(j-i)}(\sqrt{s})\, s^{-(i+j)/2} = 0.$$

Proof of Proposition C.1. Lemma C.2 and Taylor's theorem imply

$$f(r) = \sum_{i=0}^{k} \frac{f^{(2i)}(0)\, r^{2i}}{(2i)!} + R_k,$$

R_k satisfying the hypotheses of Lemma C.3. From the definition (C.1) of $\tilde{f}$ we have

$$\tilde{f}(s) = \sum_{i=0}^{k} \frac{f^{(2i)}(0)\, s^i}{(2i)!} + \bar{R}_k(s), \qquad \bar{R}_k(s) = R_k(\sqrt{s}).$$

For any finite k the first term can be C^∞ extended to negative s in an obvious way, while $\bar{R}_k$ can, by Lemma C.3, be C^k extended to negative s by setting $\bar{R}_k(s) = 0$ for $s < 0$. If f is smooth, a smooth extension of $\tilde{f}$ can be constructed using standard asymptotic expansion methods. $\blacksquare$

Let us outline a proof of Lemma 5.1: first we claim that any smooth rotation invariant one form ω on the plane is of the form $\omega = \alpha(r^2) r\, dr + \beta(r^2) r^2\, d\phi$, for some $\alpha, \beta \in C^\infty(\mathbf{R})$, which can be proved as follows: any one form can be written in the form $\omega = a(x^i)\, dr + b(x^i)\, d\phi$ (a, b possibly singular at the origin). Now $f = \omega_i x^i$ is a smooth rotation invariant function on $\mathbf{R}^2$, with $f(0) = 0$, thus $f = r^2 \alpha(r^2)$, and we have $r a(x^i) = f(x^i) = r^2 \alpha(r^2)$, $a = r\alpha(r^2)$. Since $r\alpha(r^2)\, dr = \alpha(r^2)\, x^i\, dx^i$ is obviously smooth, so must be $b(x^i)\, d\phi$. Let $*$ be the Hodge dual

operator of the metric dx^2+dy^2, then $*\,b(x^i)\,d\phi$ is a smooth rotation invariant form, and $*\,b(x^i)\,d\phi = -(b(x^i)/r)\,dr$ so from the previous argument $b(x^i)=r^2\beta(r^2)$, and the claim follows. Let now $A_{ij}\,dx^i\,dx^j$ be a symmetric rotation invariant tensor on $\mathbf{R}^2$, thus $A_{ij}\,dx^i\,dx^j = a(x^i)\,dr^2 + 2b(x^i)\,dr\,d\phi + c(x^i)\,d\phi^2$ with some (possibly singular at the origin) functions a, b, c. From rotation invariance of $A_{ij}x^ix^j$ one finds $a(x^i)=\alpha(r^2)$. Consider the rotation-invariant one form $\omega = A_{ij}Y^i\,dx^j$, where $Y^i\partial_i = x\,\partial/\partial y - y\,\partial/\partial x$: one has $\omega = b(x^i)\,dr + c(x^i)\,d\phi$, therefore $b(x^i)=r\beta(r^2)$, $c(x^i)=r^2\gamma(r^2)$, and $A_{ij}\,dx^i\,dx^j = (\alpha(r^2)-\gamma(r^2))\,dr^2 + 2r\beta\,dr\,d\phi + \gamma(r^2)(dr^2+r^2\,d\phi^2)$. The last term is obviously smooth and rotation invariant, so must be therefore the sum of the first two, say $\bar{A}_{ij}$. Now $\bar{A}_{ij}\,\delta^{ij}$ is a smooth rotation invariant function, and we have $\bar{A}_{ii} = (\alpha(r^2)-\gamma(r^2))/r^2$, thus $\alpha(r^2)-\gamma(r^2)=\delta(r^2)\,r^2$, and it follows that $B_{ij}\,dx^i\,dx^j = 2r\beta\,dr\,d\phi$ is a smooth rotation invariant tensor. Now $B_{ij} = (\varepsilon_{jk}x^kx^i + \varepsilon_{ik}x^kx^j)(\beta(r^2)/r^2)$, which will have an angle independent limit at the origin only if $\beta(0)=0$, therefore $\beta = r^2\phi(r^2)$ and Lemma 5.1 follows.

Proposition C.1 and Lemma 5.1 show that a necessary condition for smoothness of $g_{ij}\,dx^i\,dx^j$, when an axis of symmetry is present, is the existence of functions $\bar{A}$, $\bar{u}$, $\bar{R}$, $\bar{\beta}\in C^\infty(\mathbf{R})$ such that, say for small θ,

$$A(\theta)=\theta^2\bar{A}(\theta^2), \qquad u=\bar{u}(\theta^2), \qquad R(\theta)=\theta\bar{R}(\theta^2), \tag{C.3}$$

$$\beta=\bar{\beta}(\theta^2), \tag{C.4}$$

while smoothness and rotational invariance of K_{ij} implies existence of functions $\overline{\partial u/\partial n}$, $\overline{\partial R/\partial n}$, $\overline{\partial A/\partial n}$, $\overline{\partial\beta/\partial n}\in C^\infty(\mathbf{R})$ such that

$$\frac{\partial u}{\partial n}=\overline{\frac{\partial u}{\partial n}}(\theta^2), \qquad \frac{\partial A}{\partial n}=\theta^2\,\overline{\frac{\partial A}{\partial n}}(\theta^2), \qquad \frac{\partial R}{\partial n}=\theta\,\overline{\frac{\partial R}{\partial n}}(\theta^2), \tag{C.5}$$

$$\frac{\partial\beta}{\partial n}=\overline{\frac{\partial\beta}{\partial n}}(\theta^2). \tag{C.6}$$

If (C.3), (C.5) hold, a careful examination of the right-hand sides of (5.7a) and (5.7b) leads to

$$\beta_\theta=\theta\psi(\theta^2)\Rightarrow\beta=\bar{\beta}(\theta^2),$$

$$\frac{\partial\beta}{\partial n}=\chi(\theta^2),$$

for some ψ, $\chi\in C^\infty(\mathbf{R})$, which shows that if (C.3), (C.5) hold then β, $\partial\beta/\partial n$ obtained as solutions of (5.7) will satisfy (C.4) and (C.6), leading thus to a smooth Cauchy data set (g_{ij}, P_{ij}).

APPENDIX D: Existence of Nongeneric Solutions of Constraint Equations

Corollary 5.1 establishes several properties of the sets $I_\pm$, and it is of some interest to find out, whether given any sets $I_\pm$ allowed by that corollary, there exist

Cauchy data satisfying the constraint equations for which $I_\pm$ are the sets of zeros of $R_\pm$. This is indeed the case, in this appendix we shall briefly outline how appropriate smooth Cauchy data can be constructed. The idea is similar to the proof of Proposition 5.1. Let $\varepsilon > 0$ be such that $I_+ \cup I_- \subset (\varepsilon, \pi - \varepsilon)$, let $\beta_\pm$, $\phi_\pm$, $\psi_\pm \in C^\infty([\varepsilon, \pi - \varepsilon])$ be arbitrary functions such that

$$\begin{aligned} \{\theta : \phi_+^2(\theta) + \psi_+^2(\theta) = 0\} &= I_+, \\ \{\theta : \phi_-^2(\theta) + \psi_-^2(\theta) = 0\} &= I_-. \end{aligned} \tag{D.1}$$

Let $\theta_\pm \in I_\pm$. For $\theta \in [\varepsilon, \pi - \varepsilon]$ we define

$$\begin{aligned} R_\pm(\theta) &= -\int_{\theta_+}^{\theta} (\phi_\pm^2 + \psi_\pm^2)(\rho)\, e^{-\int_\rho^\theta \beta_\pm(\psi)\, d\psi}\, d\rho, \\ R(\theta) &= R_0 + \frac{1}{2}\int_{\theta_+}^{\theta} (R_+ + R_-)(\psi)\, d\psi, \end{aligned} \tag{D.2}$$

where R_0 is large enough so that $R > 0$ on $[\varepsilon, \pi - \varepsilon]$. Equations (D.1) and (D.2) imply

$$\{\theta \in [\varepsilon, \pi - \varepsilon] : R_\pm(\theta) = 0\} = I_\pm. \tag{D.3}$$

Let u_0 be an arbitrary constant. For $\theta \in [\varepsilon, \pi - \varepsilon]$ set

$$\begin{aligned} u_\pm = \frac{\phi_\pm}{\sqrt{R}}, \qquad u(\theta) &= u_0 + \frac{1}{2}\int_{\theta_+}^{\theta} (u_+ + u_-)(\psi)\, d\psi, \\ A_\pm &= \sqrt{R}\, e^{-2u}\psi_\pm. \end{aligned} \tag{D.4}$$

At this stage one can proceed as in the proof of Proposition 5.1: extend u, A, R, $\partial u/\partial n$, $\partial A/\partial n$, and $\partial R/\partial n$ to $[0, \pi]$ in a way consistent with the regularity conditions on the axes, cf. Section 5 and Appendix C, with $R > 0$, $R_\pm$ positive to the left of $I_\pm$ and negative to the right of $I_\pm$, if necessary modify $\partial u/\partial n$ on $(\varepsilon/4, \varepsilon/2)$ or $(\pi - \varepsilon/2, \pi - \varepsilon/4)$ to satisfy the integral constraint (5.10) or (5.13), and solve (5.7) for β, $\partial\beta/\partial n$—the details are left to the reader.

APPENDIX E: Ernst Equations

In this appendix the Ernst equations of Section 3 will be obtained by a $3+1$ reduction of Einstein equations, following [15]. The methods of that reference lead explicitly to geometric equations on a three-dimensional manifold even if axes of symmetry occur—this fact lies at the heart of the method of the proof of Lemma 6.3.

Let X be a spacelike Killing vector field on ${}^4\mathcal{M}$, let ${}^4\tilde{\mathcal{M}}_1 = \{p \in \mathcal{M} : X(p) \neq 0\}$, let

$$e^{2u} \equiv g_{\mu\nu} X^\mu X^\nu. \tag{E.1}$$

The tensor field [15]

$$b_{\alpha\beta} = e^{2u}(g_{\alpha\beta} - e^{-2u}X_\alpha X_\beta) \tag{E.2}$$

defines a smooth Riemannian metric on ${}^4\tilde{\mathscr{M}}_1/G_1$, where G_1 is the group acting by translations on the orbits of X, which we shall for definiteness assume to be closed. We can choose coordinates $x^{\tilde{\alpha}} = (t, x^2, x^3 = \theta)$ on ${}^4\tilde{\mathscr{M}}_1/G_1$ and x^1 along the group orbits ($X = \partial/\partial x^1$); $g_{\mu\nu}dx^\mu dx^\nu$ takes the form ([15])

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = e^{-2u} b_{\tilde{\alpha}\tilde{\beta}}\, dx^{\tilde{\alpha}}\, dx^{\tilde{\beta}} + e^{2u}(dx^1 + f^1_{\tilde{\alpha}}\, dx^{\tilde{\alpha}})^2. \tag{E.3}$$

Let $X^\mu = X^\mu_1$ be the first Killing vector field of the $U(1)\times U(1)$ symmetric metric (2.7), define

$$\begin{aligned} A &= \lambda_{21} e^{-2u} \quad (\Rightarrow \lambda_{22} = R^2 e^{-2u} + A^2 e^{2u}), \\ e^a &= X^a_A\, dx^A = M^a\, dt + g^a\, d\theta, \qquad \chi_{AB} = e^{2u} h_{AB}. \end{aligned} \tag{E.4}$$

Comparing (E.3) with (2.7) one finds

$$\begin{aligned} g_{\mu\nu}\, dx^\mu\, dx^\nu &= h_{AB}\, dx^A\, dx^B + R^2 e^{-2u}(dx^2 + e^2)^2 + e^{2u}(dx^1 + A\, dx^2 + e^1 + Ae^2)^2 \\ &\Rightarrow b_{\tilde{\alpha}\tilde{\beta}}\, dx^{\tilde{\alpha}}\, dx^{\tilde{\beta}} = \chi_{AB}\, dx^A\, dx^B + R^2(dx^2 + M^2\, dt + g^2\, d\theta)^2. \end{aligned} \tag{E.5}$$

Two of Einstein equations read [8, 15]

$$\tilde{\mathscr{D}}_{\tilde{\alpha}}\tilde{\mathscr{D}}^{\tilde{\alpha}} u = -\tfrac{1}{2} e^{-4u} b^{\tilde{\alpha}\tilde{\beta}} \omega_{\tilde{\alpha}} \omega_{\tilde{\beta}}, \tag{E.6}$$

$$\tilde{\mathscr{D}}_{\tilde{\alpha}}(e^{-4u}\omega^{\tilde{\alpha}}) = 0, \tag{E.7}$$

where $\tilde{\mathscr{D}}$ is the covariant Riemannian derivative of the metric $b_{\tilde{\alpha}\tilde{\beta}}$, and $\omega_{\tilde{\alpha}} \equiv \omega_{\tilde{\alpha}11}$ is the twist vector (B.3) of the Killing vector X_1. Equations (E.6)–(E.7) are harmonic map equations for a map from $({}^4\tilde{\mathscr{M}}_1/G_1,\ b_{\tilde{\alpha}\tilde{\beta}})$ to $\mathscr{H}^2$ [15]. The arguments leading to Eqs. (A.28) and (A.30) show that

$$\begin{aligned} &\sqrt{-\det b_{\tilde{\alpha}\tilde{\beta}}} = R\sqrt{-\det h_{AB}}, \\ &b^{\tilde{\alpha}\tilde{\beta}}\, \partial_{\tilde{\alpha}}\, \partial_{\tilde{\beta}} = \chi^{AB}\partial_A\, \partial_B - 2\chi^{AB} X^2_B\, \partial_A\, \partial_2 + (\chi^{AB} X^2_A X^2_B + R^{-2})\, \partial_2^2, \\ &\chi^{AB} \equiv (\chi_{AB})^{-1}. \end{aligned} \tag{E.8}$$

From (B.10) one finds

$$\begin{aligned} \omega_A &= -\frac{\varepsilon_{AB}}{R} e^{4u} \mathscr{D}^B A - c_1 X^2_A \\ &= -\frac{\tilde{\varepsilon}_{AB}}{R} e^{4u} \tilde{\mathscr{D}}^B A - c_1 X^2_A, \end{aligned} \tag{E.9a}$$

$$\omega_2 = -c_1, \tag{E.9b}$$

where $\bar{\varepsilon}_{AB}$ is the "alternating symbol" of the metric χ_{AB}, $\bar{\varepsilon}_{(AB)}=0$, $\bar{\varepsilon}_{03}=\sqrt{-\det \chi_{AB}}$. Equations (E.8) and (E.9) lead to the two dimensional form of (E.6),

$$\frac{1}{R}\bar{\mathscr{D}}_A(R\bar{\mathscr{D}}^A u)=\frac{e^{4u}\bar{\mathscr{D}}_A A\bar{\mathscr{D}}^A A}{2R^2}-\frac{e^{-4u}c_1^2}{2R^2}, \tag{E.10}$$

which is precisely Eq. (3.2b). It is instructive to recover (3.2c) from $\omega_{[A,B]}=0$ [8]. Contracting (E.9a) with $\bar{\varepsilon}^{AC}\bar{\mathscr{D}}_C$ one obtains

$$\bar{\mathscr{D}}_B\left(\frac{e^{4u}\bar{\mathscr{D}}^B A}{R}\right)=c_1\bar{\varepsilon}^{AC}\bar{\mathscr{D}}_C X_A^2. \tag{E.11}$$

From $\partial_a X_\mu^b=0$, $\partial_A X_c^b=\partial_A\delta_c^b=0$ it follows

$$\begin{aligned} c^a &= \varepsilon^{\alpha\beta\gamma\delta}X_{1\alpha}X_{2\beta}\,\partial_\gamma X_\delta^a \\ &= \varepsilon^{\alpha\beta A\delta}X_{1\alpha}X_{2\beta}\,\partial_A X_\delta^a \\ &= \varepsilon^{\alpha\beta AB}X_{1\alpha}X_{2\beta}\,\partial_A X_B^a \\ &= \varepsilon^{cd}\varepsilon^{AB}X_{1c}X_{2d}\,\partial_A X_B^a \\ &= R\varepsilon^{AB}\,\partial_A X_B^a = Re^{2u}\bar{\varepsilon}^{AB}\,\partial_A X_B^a, \end{aligned} \tag{E.12}$$

thus

$$R\bar{\mathscr{D}}_B\left(\frac{e^{4u}\bar{\mathscr{D}}^B A}{R}\right)=-e^{-2u}c_1c^2=-\frac{c_1(c_2-Ac_1)}{R^2}, \tag{E.13}$$

which is precisely (3.2c). Equation (E.12) also gives

$$\partial_t g^a-\partial_\theta M^a=-\frac{\sqrt{-\det h_{AB}}\,c^a}{R}, \tag{E.14}$$

which is equivalent to (A.42). (Equation (E.12) is actually the simplest derivation of (A.42).) Let us finally rewrite (E.7) in two dimensional form

$$\bar{\mathscr{D}}_A\{e^{-4u}[\bar{\mathscr{D}}^A\omega+\chi^{AB}(\bar{\omega}_B+c_1X_B^2)]\}=0, \tag{E.15}$$

where $\omega_A=\omega_{.A}+\bar{\omega}_A$, which is Eq. (3.10).

APPENDIX F: NOTATION

The signature is $-+++$, the indices have the following ranges: i, j, k—1, 2, 3; greek indices—0, ..., 3; a, b, ...—1, 2; A, B, ...—0, 3 ($x^A=t,\theta$). The "ε" symbols all have the appropriate weighting factor included: $\varepsilon_{0123}=\sqrt{-\det g_{\mu\nu}}$, $\varepsilon_{123}=\sqrt{\det g_{ij}}$, $\varepsilon_{12}=\sqrt{\det \lambda_{ab}}$, $\varepsilon_{03}=\sqrt{-\det h_{AB}}$; in Appendix E we also use $\bar{\varepsilon}_{03}=\sqrt{-\det \chi_{AB}}$. The

covariant derivatives of various metrics are denoted as follows: $g_{\mu\nu} - \nabla_\mu$; $g_{ij} - D_i$; $h_{AB} - \mathscr{D}_A$; $\bar{h}_{ij} - \bar{D}_i$; $\chi_{AB} - \tilde{\mathscr{D}}_A$; and in Appendix E we also have $b_{\bar{\alpha}\bar{\beta}} - \tilde{\mathscr{D}}_{\bar{\alpha}}$. All coordinate systems used in this paper are assumed to satisfy $g^{00} < 0$, $g_{00} < 0$ $(t \equiv x^0)$. $B(a)$ denotes an open ball of radius a; $S(a)$ denotes a sphere of radius a.

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