MATHEMATICAL GENERAL RELATIVITY: A SAMPLER

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Abstract. We provide an introduction to selected significant advances in the mathematical understanding of Einstein's theory of gravitation which have taken place in recent years.

Contents

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1. Introduction

Mathematical general relativity is, by now, a well-established vibrant branch of mathematics. It ties fundamental problems of gravitational physics with beautiful questions in mathematics. The object is the study of manifolds equipped with a Lorentzian metric satisfying the Einstein field equations. Some highlights of its history include the discovery by Choquet-Bruhat of a well posed Cauchy problem [156],

subsequently globalized by Choquet-Bruhat and Geroch [80], the singularity theorems of Penrose and Hawking [176, 275], the proof of the positive mass theorem by Schoen and Yau [300], and the proof of stability of Minkowski space-time by Christodoulou and Klainerman [94].

There has recently been spectacular progress in the field on many fronts, including the Cauchy problem, stability, cosmic censorship, construction of initial data, and asymptotic behavior, many of which will be described here. Mutual benefits are drawn, and progress is being made, from the interaction between general relativity and geometric analysis and the theory of elliptic and hyperbolic partial differential equations. The Einstein equation shares issues of convergence, collapse and stability with other important geometric PDEs, such as the Ricci flow and the mean curvature flow. Steadily growing overlap between the relevant scientific communities can be seen. For all these reasons it appeared timely to provide a mathematically oriented reader with an introductory survey of the field. This is the purpose of the current work.

In Section 2 we survey the Lorentzian causality theory, the basic language for describing the structure of space-times. In Section 3 the reader is introduced to black holes, perhaps the most fascinating prediction of Einstein's theory of gravitation, and the source of many deep (solved or unsolved) mathematical problems. In Section 4 the Cauchy problem for the Einstein equations is considered, laying down the foundations for a systematic construction of general space-times. Section 5 examines initial data sets, as needed for the Cauchy problem, and their global properties. In Section 6 we discuss the dynamics of the Einstein equations, including questions of stability and predictability; the latter question known under the baroque name of "strong cosmic censorship". Section 7 deals with *trapped* and *marginally trapped* surfaces, which signal the presence of black holes, and have tantalizing connections with classical minimal surface theory. The paper is sprinkled with open problems, which are collected in Appendix A.

2. A rapid course in Lorentzian geometry and causal theory

2.1. **Lorentzian manifolds.** In general relativity, and related theories, the space of physical events is represented by a *Lorentzian manifold*. A Lorentzian manifold is a smooth (Hausdorff, paracompact) manifold $\mathscr{M} = \mathscr{M}^{n+1}$ of dimension $n+1$, equipped with a Lorentzian metric g. A Lorentzian metric is a smooth assignment to each point $p \in \mathscr{M}$ of a symmetric, nondegenerate bilinear form on the tangent space $T_p\mathscr{M}$ of *signature* $(- + \cdots +)$. Hence, if $\{e_0, e_1, ..., e_n\}$ is an orthonormal basis for $T_p\mathscr{M}$ with respect to g, then, perhaps after reordering the basis, the matrix $[g(e_i, e_j)]$ equals $\operatorname{diag}(-1, +1, ..., +1)$. A vector $v = \sum v^\alpha e_\alpha$ then has 'square norm',

$$g(v, v) = -(v^0)^2 + \sum (v^i)^2 \,, \tag{2.1}$$

which can be positive, negative or zero. This leads to the causal character of vectors, and indeed to the causal theory of Lorentzian manifolds, which we shall discuss in Section 2.3.

On a coordinate neighborhood $(U, x^\alpha) = (U, x^0, x^1, ..., x^n)$ the metric g is completely determined by its metric component functions on U, $g_{\alpha\beta} := g(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta})$, $0 \le \alpha, \beta \le n$: For $v = v^\alpha \frac{\partial}{\partial x^\alpha}, w = w^\beta \frac{\partial}{\partial x^\beta} \in T_p\mathscr{M}$, $p \in U$, $g(v, w) = g_{\alpha\beta} v^\alpha w^\beta$. (Here we have used the Einstein summation convention: If, in a coordinate chart, an index appears repeated, once up and once down, then summation over that

index is implied.) Classically the metric in coordinates is displayed via the "line element", $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$.

The prototype Lorentzian manifold is Minkowski space $\mathbb{R}^{1,n}$, the space-time of special relativity. This is $\mathbb{R}^{n+1}$, equipped with the Minkowski metric, which, with respect to Cartesian coordinates $(x^0, x^1, ..., x^n)$, is given by

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^n)^2 \,.$$

Each tangent space of a Lorentzian manifold is isometric to Minkowski space, and in this way the local accuracy of special relativity is built into general relativity.

Every Lorentzian manifold (or, more generally, pseudo-Riemannian manifold) $(\mathscr{M}^{n+1}, g)$ comes equipped with a *Levi-Civita connection* (or covariant differentiation operator) ∇ that enables one to compute the directional derivative of vector fields. Hence, for smooth vector fields $X, Y \in \mathfrak{X}(\mathscr{M})$, $\nabla_X Y \in \mathfrak{X}(\mathscr{M})$ denotes the covariant derivative of Y in the direction X. The Levi-Civita connection is the unique connection ∇ on $(\mathscr{M}^{n+1}, g)$ that is (i) symmetric (or torsion free), i.e., that satisfies $\nabla_X Y - \nabla_Y X = [X, Y]$ for all $X, Y \in \mathfrak{X}(M)$, and (ii) compatible with the metric, i.e. that obeys the metric product rule, $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$, for all $X, Y, Z \in \mathfrak{X}(M)$.

In a coordinate chart (U, x^α), one has,

$$\nabla_X Y = (X(Y^\mu) + \Gamma^\mu_{\alpha\beta} X^\alpha Y^\beta)\partial_\mu \,, \tag{2.2}$$

where X^α, Y^α are the components of X and Y, respectively, with respect to the coordinate basis $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$, and where the $\Gamma^\mu_{\alpha\beta}$'s are the classical Christoffel symbols, given in terms of the metric components by,

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(\partial_\beta g_{\alpha\nu} + \partial_\alpha g_{\beta\nu} - \partial_\nu g_{\alpha\beta}) \,. \tag{2.3}$$

Note that the coordinate expression (2.2) can also be written as,

$$\nabla_X Y = X^\alpha \nabla_\alpha Y^\mu \partial_\mu \,, \tag{2.4}$$

where $\nabla_\alpha Y^\mu$ (often written classically as ${Y^\mu}_{;\alpha}$) is given by,

$$\nabla_\alpha Y^\mu = \partial_\alpha Y^\mu + \Gamma^\mu_{\alpha\beta} Y^\beta \,. \tag{2.5}$$

We shall feel free to interchange between coordinate and coordinate free notations. The Levi-Civita connection ∇ extends in a natural way to a covariant differentiation operator on all tensor fields.

The Riemann curvature tensor of $(\mathscr{M}^{n+1}, g)$ is the map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(X, Y, Z) \to R(X, Y)Z$, given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \,. \tag{2.6}$$

This expression is linear in $X, Y, Z \in \mathfrak{X}(\mathscr{M})$ with respect to $C^\infty(\mathscr{M})$. This implies that R is indeed tensorial, i.e., that the value of $R(X, Y)Z$ at $p \in M$ depends only on the value of X, Y, Z at p.

Equation (2.6) shows that the Riemann curvature tensor measures the extent to which covariant differentiation fails to commute. This failure to commute may be seen as an obstruction to the existence of parallel vector fields. By Riemann's theorem, a Lorentzian manifold is locally Minkowskian if and only if the Riemann curvature tensor vanishes.

The components ${R^\mu}_{\gamma\alpha\beta}$ of the Riemann curvature tensor R in a coordinate chart (U, x^α) are determined by the equations, $R(\partial_\alpha, \partial_\beta)\partial_\gamma = {R^\mu}_{\gamma\alpha\beta}\partial_\mu$. Equations (2.2)

and (2.6) then yield the following explicit formula for the curvature components in terms of the Christoffel symbols,

$$R^{\mu}{}_{\gamma\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\gamma\beta} - \partial_\beta \Gamma^{\mu}_{\gamma\alpha} + \Gamma^{\nu}_{\gamma\beta}\Gamma^{\mu}_{\nu\alpha} - \Gamma^{\nu}_{\gamma\alpha}\Gamma^{\mu}_{\nu\beta} \,. \tag{2.7}$$

The Ricci tensor, Ric, is a bilinear form obtained by contraction of the Riemann curvature tensor, i.e., its components $R_{\mu\nu} = \mathrm{Ric}(\partial_\mu, \partial_\nu)$ are determined by tracing, $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$. Symmetries of the Riemann curvature tensor imply that the Ricci tensor is symmetric, $R_{\mu\nu} = R_{\nu\mu}$. By tracing the Ricci tensor, we obtain the scalar curvature, $R = g^{\mu\nu}R_{\mu\nu}$, where $g^{\mu\nu}$ denotes the matrix inverse to $g_{\mu\nu}$.

2.2. **Einstein equations.** The Einstein equation (with cosmological constant Λ), the field equation of general relativity, is the tensor equation,

$$\mathrm{Ric} - \frac{1}{2}Rg + \Lambda g = 8\pi \mathcal{T} \,, \tag{2.8}$$

where $\mathcal{T}$ is the energy-momentum tensor. (See, e.g., Section 6.5.2 for an example of an energy-momentum tensor.) When expressed in terms of coordinates, the Einstein equation becomes a system of second order equations for the metric components $g_{\mu\nu}$ and the nongravitational field variables introduced through the energy-momentum tensor. We say that space-time obeys the vacuum Einstein equation if it obeys the Einstein equation with $\mathcal{T} = 0$.

The Riemann curvature tensor has a number of symmetry properties, one of which is the so-called first Bianchi identity:

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0 \,.$$

The curvature tensor also obeys a differential identity known as the second Bianchi identity:

$$\nabla_\sigma R_{\alpha\beta\gamma\delta} + \nabla_\alpha R_{\beta\sigma\gamma\delta} + \nabla_\beta R_{\sigma\alpha\gamma\delta} = 0 \,. \tag{2.9}$$

When twice contracted, (2.9) yields the following *divergence identity*:

$$\nabla_\alpha \left(R^{\alpha\beta} - \frac{R}{2} g^{\alpha\beta} \right) = 0 \,. \tag{2.10}$$

This plays a fundamental role in general relativity, as, in particular, it implies, in conjunction with the Einstein equation, local conservation of energy, $\nabla_\alpha T^{\alpha\beta} = 0$. It also plays an important role in the mathematical analysis of the Einstein equations; see Section 4 for further discussion.

2.3. **Elements of causal theory.** Many concepts and results in general relativity make use of the causal theory of Lorentzian manifolds. The starting point for causal theory is the causal classification of tangent vectors. Let $(\mathscr{M}^{n+1}, g)$ be a Lorentzian manifold. A vector $v \in T_p\mathscr{M}$ is timelike (resp., spacelike, null) provided $g(v,v) < 0$ (resp., $g(v,v) > 0$, $g(v,v) = 0$). The collection of null vectors forms a double cone $\mathcal{V}_p$ in $T_p\mathscr{M}$ (recall (2.1)), called the null cone at p; see Figure 2.1.

The timelike vectors at p point inside the null cone and the spacelike vectors point outside. We say that $v \in T_p\mathscr{M}$ is causal if it is timelike or null. We define the length of causal vectors as $|v| = \sqrt{-g(v,v)}$. Causal vectors $v, w \in T_p\mathscr{M}$ that point into the same half-cone of the null cone $\mathcal{V}_p$ obey the reverse triangle inequality, $|v + w| \geq |v| + |w|$. Geometrically, this is the source of the twin paradox.

These notions of causality extend to curves. Let $\gamma : I \to \mathscr{M}$, $t \to \gamma(t)$, be a smooth curve in $\mathscr{M}$. γ is said to be timelike (resp., spacelike, null, causal)

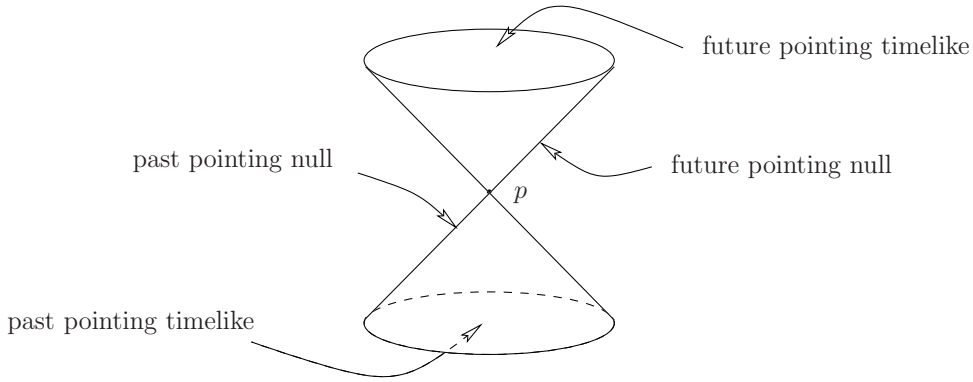

FIGURE 2.1. The light cone at p.

provided each of its velocity vectors $\gamma'(t)$ is timelike (resp., spacelike, null, causal). Heuristically, in accordance with relativity, information flows along causal curves, and so such curves are the focus of attention in causal theory. The notion of a causal curve extends in a natural way to piecewise smooth curves, and we will normally work within this class. As usual, we define a geodesic to be a curve $t \to \gamma(t)$ of zero covariant acceleration, $\nabla_{\gamma'}\gamma' = 0$. Since geodesics γ are constant speed curves ($g(\gamma', \gamma') = const.$), each geodesic in a Lorentzian manifold is either timelike, spacelike or null.

The length of a causal curve $\gamma : [a, b] \to \mathscr{M}$, is defined as

$$L(\gamma) = \text{Length of } \gamma = \int_a^b |\gamma'(t)| dt = \int_a^b \sqrt{-g(\gamma'(t), \gamma'(t))}\, dt\,.$$

If γ is timelike one can introduce an arc length parameter along γ. In general relativity, a timelike curve corresponds to the history of an observer, and arc length parameter, called proper time, corresponds to time kept by the observer. Using the existence and properties of geodesically convex neighborhoods [265] one can show that causal geodesics are locally maximal (i.e., locally longest among causal curves).

Each null cone $\mathcal{V}_p$ consists of two half-cones, one of which may designated as the future cone, and the other as the past cone at p. If the assignment of a past and future cone at each point of $\mathscr{M}$ can be carried out in a continuous manner over $\mathscr{M}$ then $\mathscr{M}$ is said to be time-orientable. There are various ways to make the phrase "continuous assignment" precise, but they all result in the following fact: A Lorentzian manifold $(\mathscr{M}^{n+1}, g)$ is time-orientable if and only if it admits a smooth timelike vector field Z. If $\mathscr{M}$ is time-orientable, the choice of a smooth time-like vector field Z fixes a time orientation on $\mathscr{M}$: For any $p \in \mathscr{M}$, a causal vector $v \in T_p\mathscr{M}$ is future directed (resp. past directed) provided $g(v, Z) < 0$ (resp. $g(v, Z) > 0$). Thus, v is future directed if it points into the same null half cone at p as Z. We note that a Lorentzian manifold that is not time-orientable always admits a double cover that is. By a *space-time* we mean a connected time-oriented Lorentzian manifold (M^{n+1}, g). Henceforth, we restrict attention to space-times.

2.3.1. *Past and futures.* Let $(\mathscr{M}, g)$ be a space-time. A timelike (resp. causal) curve $\gamma : I \to \mathscr{M}$ is said to be *future directed* provided each tangent vector $\gamma'(t)$,

$t \in I$, is future directed. (*Past-directed* timelike and causal curves are defined in a time-dual manner.) $I^+(p)$, the timelike future of $p \in \mathscr{M}$, is the set consisting of all points $q \in \mathscr{M}$ for which there exists a future directed timelike curve from p to q. $J^+(p)$, the causal future of $p \in \mathscr{M}$, is the set consisting of p and all points q for which there exists a future directed causal curve from p to q. In Minkowski space $\mathbb{R}^{1,n}$ these sets have a simple structure: For each $p \in \mathbb{R}^{1,n}$, $\partial I^+(p) = J^+(p) \setminus I^+(p)$ is the future cone at p generated by the future directed null rays emanating from p. $I^+(p)$ consists of the points inside the cone, and $J^+(p)$ consists of the points on and inside the cone. In general, the curvature and topology of space-time can strongly influence the structure of these sets.

Since a timelike curve remains timelike under small smooth perturbations, it is heuristically clear that the sets $I^+(p)$ are in general open; a careful proof makes use of properties of geodesically convex sets. On the other hand the sets $J^+(p)$ need not be closed in general, as can be seen by considering the space-time obtained by removing a point from Minkowski space.

It follows from variational arguments that, for example, if $q \in I^+(p)$ and $r \in J^+(q)$ then $r \in I^+(p)$. This and related claims are in fact a consequence of the following fundamental causality result [265].

Proposition 2.1. *If $q \in J^+(p) \setminus I^+(p)$, i.e., if q is in the causal future of p but not in the timelike future of p then any future directed causal curve from p to q must be a null geodesic.*

Given a subset $S \subset \mathscr{M}$, $I^+(S)$, the timelike future of S, consists of all points $q \in \mathscr{M}$ for which there exists a future directed timelike curve from a point in S to q. $J^+(S)$, the causal future of S consists of the points of S and all points $q \in \mathscr{M}$ for which there exists a future directed causal curve from a point in S to q. Note that $I^+(S) = \bigcup_{p \in S} I^+(p)$. Hence, as a union of open sets, $I^+(S)$ is always open.

The timelike and causal pasts $I^-(p)$, $J^-(p)$, $I^-(S)$, $J^-(S)$ are defined in a time dual manner in terms of past directed timelike and causal curves. It is sometimes convenient to consider pasts and futures within some open subset U of $\mathscr{M}$. For example, $I^+(p, U)$ denotes the set consisting of all points $q \in U$ for which there exists a future directed timelike from p to q contained in U.

Achronal sets play an important role in causal theory. A subset $A \subset \mathscr{M}$ is *achronal* provided no two of its points can be joined by a timelike curve. Of particular importance are *achronal boundaries*. By definition, an achronal boundary is a set of the form $\partial I^+(S)$ (or $\partial I^-(S)$), for some $S \subset \mathscr{M}$.

We consider several structural properties of achronal boundaries. To have a simple example in mind, take S to be a Euclidean disk contained in a time slice of 3-dimensional Minkowski space. Then $\partial I^+(S)$ is the union of the disk and a truncated null cone; see Figure 2.2.

Proposition 2.2. *An achronal boundary $\partial I^+(S)$, if nonempty, is a closed achronal C^0 hypersurface in $\mathscr{M}$.*

The proof of Proposition 2.2 makes use of the notion of the *edge* of an achronal set $S \subset \mathscr{M}$. This is defined as the set of points $p \in \overline{S}$ such that every neighborhood U of p, contains a timelike curve from $I^-(p, U)$ to $I^+(p, U)$ that does *not* meet S. Elementary arguments show that achronal boundaries $\partial I^+(S)$ are achronal and edgeless. Proposition 2.2 then follows from the basic causal theoretic result that if A is achronal, then $A \setminus \operatorname{edge} A$, if nonempty, is a C^0 hypersurface in $\mathscr{M}$ [265].

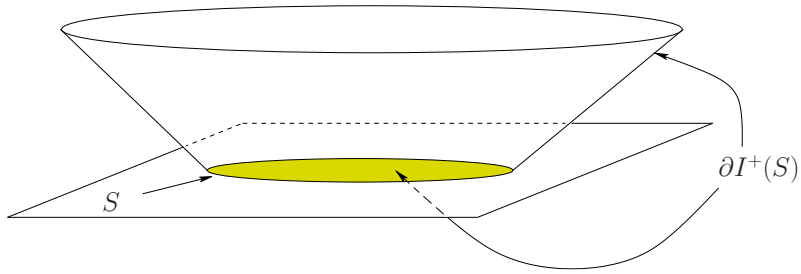

FIGURE 2.2. The boundary of the timelike future of S.

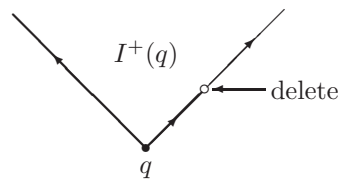

FIGURE 2.3. Two-dimensional Minkowski space-time with a point removed. The thin null geodesic lies on $\partial I^+(q)$, but has no past end point.

The next result shows that, in general, large portions of achronal boundaries are ruled by null geodesics; compare Figure 2.3:

Proposition 2.3. *Let $S \subset \mathscr{M}$ be closed. Then each $p \in \partial I^+(S) \setminus S$ lies on a null geodesic contained in $\partial I^+(S)$, which either has a past end point on S, or else is past inextendible in $\mathscr{M}$.*

The proof uses a limit curve argument, a standard tool in causal theory. The idea is to consider a sequence of past directed timelike curves γ_n from $p_n \in I^+(S)$ to S, such that $p_n \to p$. One can extract a subsequence γ_m that converges to a past directed causal curve γ contained in $\partial I^+(S)$ starting at p. One can then use Proposition 2.1 and the achronality of $\partial I^+(S)$ to show that γ is the desired null geodesic. (That the limit of piecewise smooth curves may not be piecewise smooth is a technicality that can be dealt with; cf. Section 2.3.4).

2.3.2. *Causality conditions.* A number of results in Lorentzian geometry and general relativity require some sort of causality condition. It is perhaps natural on physical grounds to rule out the occurrence of closed timelike curves. Physically, the existence of such a curve signifies the existence of an observer who is able to travel into his/her own past, which leads to variety of paradoxical situations. A space-time $\mathscr{M}$ satisfies the *chronology condition* provided there are no closed timelike curves in $\mathscr{M}$. It can be shown that all compact space-times violate the chronology condition, and for this reason compact space-times have been of limited interest in general relativity.

A somewhat stronger condition than the chronology condition is the *causality condition.* A space-time $\mathscr{M}$ satisfies the causality condition provided there are no closed (nontrivial) causal curves in $\mathscr{M}$. A slight weakness of this condition is

that there are space-times which satisfy the causality condition, but contain causal curves that are "almost closed", see e.g. [179, p. 193].

It is useful to have a condition that rules out "almost closed" causal curves. A space-time $\mathscr{M}$ is said to be *strongly causal* at $p \in \mathscr{M}$ provided there are arbitrarily small neighborhoods U of p such that any causal curve γ which starts in, and leaves, U never returns to U. $\mathscr{M}$ is *strongly causal* if it is strongly causal at each of its points. Thus, heuristically speaking, $\mathscr{M}$ is strongly causal provided there are no closed or "almost closed" causal curves in $\mathscr{M}$. Strong causality is the "standard" causality condition of space-time geometry, and although there are even stronger causality conditions, it is sufficient for most applications. A very useful fact about strongly causal space-times is the following: If $\mathscr{M}$ is strongly causal then any future (or past) inextendible causal curve γ cannot be "imprisoned" or "partially imprisoned" in a compact set. That is to say, if γ starts in a compact set K, it must eventually leave K for good.

We now come to a fundamental condition in space-time geometry, that of *global hyperbolicity*. Mathematically, global hyperbolicity is a basic 'niceness' condition that often plays a role analogous to geodesic completeness in Riemannian geometry. Physically, global hyperbolicity is connected to the notion of strong cosmic censorship, the conjecture that, generically, space-time solutions to the Einstein equations do not admit *naked* (i.e., observable) singularities; see Section 6.2 for further discussion.

A space-time $\mathscr{M}$ is said to be globally hyperbolic provided:

(1) $\mathscr{M}$ is strongly causal.
(2) (Internal Compactness) The sets $J^+(p) \cap J^-(q)$ are compact for all $p, q \in \mathscr{M}$.

Condition (2) says roughly that $\mathscr{M}$ has no holes or gaps. For example Minkowski space $\mathbb{R}^{1,n}$ is globally hyperbolic but the space-time obtained by removing one point from it is not. Leray [224] was the first to introduce the notion of global hyperbolicity (in a somewhat different, but equivalent form) in connection with his study of the Cauchy problem for hyperbolic PDEs.

We mention a couple of basic consequences of global hyperbolicity. Firstly, globally hyperbolic space-times are *causally simple*, by which is meant that the sets $J^{\pm}(A)$ are closed for all compact $A \subset \mathscr{M}$. This fact and internal compactness implies that the sets $J^+(A) \cap J^-(B)$ are compact, for all compact $A, B \subset \mathscr{M}$.

Analogously to the case of Riemannian geometry, one can learn much about the global structure of space-time by studying its causal geodesics. Global hyperbolicity is the standard condition in Lorentzian geometry that guarantees the existence of maximal timelike geodesic segments joining timelike related points. More precisely, one has the following.

Proposition 2.4. *If $\mathscr{M}$ is globally hyperbolic and $q \in I^+(p)$, then there exists a maximal timelike geodesic segment γ from p to q (where by maximal, we mean $L(\gamma) \geq L(\sigma)$ for all future directed causal curves from p to q).*

Contrary to the situation in Riemannian geometry, geodesic completeness does not guarantee the existence of maximal segments, as is well illustrated by anti-de Sitter space, see e.g. [27].

Global hyperbolicity is closely related to the existence of certain 'ideal initial value hypersurfaces', called *Cauchy (hyper)surfaces*. There are slight variations in the literature in the definition of a Cauchy surface. Here we adopt the following

definition: A Cauchy surface for a space-time $\mathscr{M}$ is an achronal subset S of $\mathscr{M}$ which is met by every inextendible causal curve in $\mathscr{M}$. From the definition it is easy to see that if S is a Cauchy surface for $\mathscr{M}$ then $S = \partial I^+(S)$. It follows from Proposition 2.2 that a Cauchy surface S is a closed achronal C^0 hypersurface in $\mathscr{M}$. The following result is fundamental.

Proposition 2.5 (Geroch [169])**.** *$\mathscr{M}$ is globally hyperbolic if and only if $\mathscr{M}$ admits a Cauchy surface. If S is a Cauchy surface for $\mathscr{M}$ then $\mathscr{M}$ is homeomorphic to $\mathbb{R} \times S$.*

With regard to the implication that global hyperbolicity implies the existence of a Cauchy surface, Geroch, in fact, proved something substantially stronger. (We will make some comments about the converse in Section 2.3.3.) A *time function* on $\mathscr{M}$ is a C^0 function t on $\mathscr{M}$ such that t is strictly increasing along every future directed causal curve. Geroch established the existence of a time function t all of whose level sets $t = t_0$, $t_0 \in \mathbb{R}$, are Cauchy surfaces. This result can be strengthened to the smooth category. By a *smooth time function* we mean a smooth function t with everywhere past pointing timelike gradient. This implies that t is strictly increasing along all future directed causal curves, and that its level sets are smooth spacelike[1] hypersurfaces. In [308], a smoothing procedure is introduced to show that a globally hyperbolic space-time admits a smooth time function all of whose levels sets are Cauchy surfaces; see also [41] for a recent alternative treatment. In fact, one obtains a diffeomorphism $\mathscr{M} \approx \mathbb{R} \times S$, where the $\mathbb{R}$-factor corresponds to a smooth time function, such that each slice $S_t = \{t\} \times S$, $t \in \mathbb{R}$, is a Cauchy surface.

Given a Cauchy surface S to begin with, to simply show that $\mathscr{M}$ is homeomorphic to $\mathbb{R} \times S$, consider a timelike vector field Z on $\mathscr{M}$ and observe that each integral curve of Z, when maximally extended, meets S is a unique point. This leads to the desired homeomorphism. (If S is smooth this will be a diffeomorphism.) In a similar vein, one can show that any two Cauchy surfaces are homeomorphic. Thus, the topology of a globally hyperbolic space-time is completely determined by the common topology of its Cauchy surfaces.

The following result is often useful.

Proposition 2.6. *Let $\mathscr{M}$ be a space-time.*

(1) *If S is a compact achronal C^0 hypersurface and $\mathscr{M}$ is globally hyperbolic then S must be a Cauchy surface for $\mathscr{M}$.*
(2) *If t is a smooth time function on $\mathscr{M}$ all of whose level sets are compact, then each level set is a Cauchy surface for $\mathscr{M}$, and hence $\mathscr{M}$ is globally hyperbolic.*

We will comment on the proof shortly, after Proposition 2.8.

2.3.3. *Domains of dependence.* The *future domain of dependence* of S is the set $\mathscr{D}^+(S)$ consisting of all points $p \in \mathscr{M}$ such that every past inextendible causal curve[2] from p meets S.In physical terms, since information travels along causal curves, a point in $\mathscr{D}^+(S)$ only receives information from S. Thus, in principle, $\mathscr{D}^+(S)$ represents the region of space-time to the future of S that is predictable

[1]A hypersurface is called *spacelike* if the induced metric is Riemannian; see Section 2.4.

[2]We note that some authors use past inextendible *timelike* curves to define the future domain of dependence, which results in some small differences in certain results.

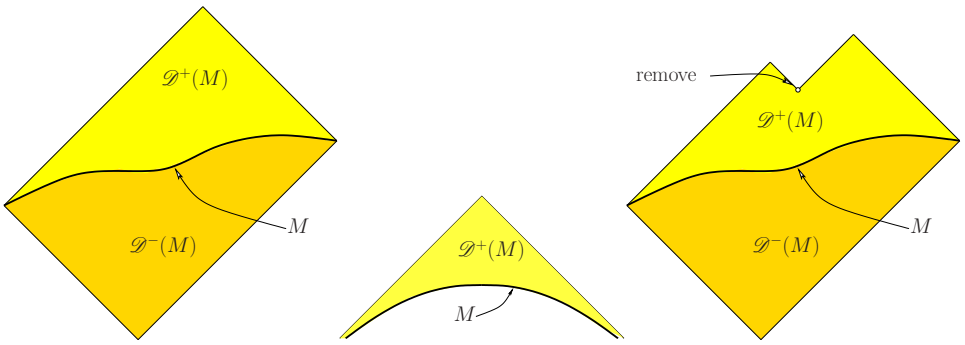

FIGURE 2.4. Examples of domains of dependence and Cauchy horizons.

from S. $\mathscr{H}^+(S)$, the *future Cauchy horizon* of S, is defined to be the future boundary of $\mathscr{D}^+(S)$; in precise terms, $\mathscr{H}^+(S) = \{p \in \overline{\mathscr{D}^+(S)} : I^+(p) \cap \mathscr{D}^+(S) = \emptyset\}$. Physically, $\mathscr{H}^+(S)$ is the future limit of the region of space-time predictable from S. Some examples of domains of dependence, and Cauchy horizons, can be found in Figure 2.4.

It follows almost immediately from the definition that $\mathscr{H}^+(S)$ is achronal. In fact, Cauchy horizons have structural properties similar to achronal boundaries, as indicated in the following.

Proposition 2.7. *Let S be an achronal subset of a space-time $\mathscr{M}$. Then $\mathscr{H}^+(S) \setminus$ edge S, if nonempty, is an achronal C^0 hypersurface of $\mathscr{M}$ ruled by null geodesics, called generators, each of which either is past inextendible in $\mathscr{M}$ or has past end point on* edge S.

The proof of Proposition 2.7 is roughly similar to the proofs of Propositions 2.2 and 2.3.

The past domain of dependence $\mathscr{D}^-(S)$ of S, and the past Cauchy horizon $\mathscr{H}^-(S)$ of S, are defined in a time-dual manner. The total domain of dependence $\mathscr{D}(S)$ and the total Cauchy horizon $\mathscr{H}(S)$, are defined respectively as, $\mathscr{D}(S) = \mathscr{D}^+(S) \cup \mathscr{D}^-(S)$ and $\mathscr{H}(S) = \mathscr{H}^+(S) \cup \mathscr{H}^-(S)$

Domains of dependence may be used to characterize Cauchy surfaces. In fact, it follows easily from the definitions that an achronal subset $S \subset \mathscr{M}$ is a Cauchy surface for $\mathscr{M}$ if and only if $\mathscr{D}(S) = \mathscr{M}$. Using the fact that $\partial\mathscr{D}(S) = \mathscr{H}(S)$, we obtain the following.

Proposition 2.8. *Let S be an achronal subset of a space-time $\mathscr{M}$. Then, S is a Cauchy surface for $\mathscr{M}$ if and only if $\mathscr{D}(S) = \mathscr{M}$ if and only if $\mathscr{H}(S) = \emptyset$.*

Part 1 of Proposition 2.6 can now be easily proved by showing, with the aid of Proposition 2.7, that $\mathscr{H}(S) = \emptyset$. Indeed if $\mathscr{H}^+(S) \neq \emptyset$ then there exists a past inextendible null geodesic $\eta \subset \mathscr{H}^+(S)$ with future end point p imprisoned in the compact set $J^+(S) \cap J^-(p)$ which, as already mentioned, is not possible in strongly causal space-times. Part 2 is proved similarly; compare [59, 164].

The following basic result ties domains of dependence to global hyperbolicity.

Proposition 2.9. *Let $S \subset \mathscr{M}$ be achronal.*

(1) *Strong causality holds at each point of* $\operatorname{int}\mathscr{D}(S)$.
(2) *Internal compactness holds on* $\operatorname{int}\mathscr{D}(S)$*, i.e., for all* $p, q \in \operatorname{int}\mathscr{D}(S)$*,* $J^+(p) \cap J^-(q)$ *is compact.*

Propositions 2.8 and 2.9 immediately imply that if S is a Cauchy surface for a space-time $\mathscr{M}$ then $\mathscr{M}$ is globally hyperbolic, as claimed in Proposition 2.5.

2.3.4. *Lipschitz causal paths.* A significant number of proofs in causality theory involve taking limits of causal curves, but those limits will rarely belong to the class of causal curves as defined so far, which then leads to various technical difficulties. So we close this section on causal theory by describing an approach which overcomes this, as follows: It is convenient to choose once and for all some auxiliary Riemannian metric b on $\mathscr{M}$, such that $(\mathscr{M}, b)$ is complete — such a metric always exists [261]. Let d_b denote the associated distance function. A parameterized path $\gamma : I \to \mathscr{M}$ from an interval $I \subset \mathbb{R}$ to M is called *locally Lipschitz* if for every compact subset K of I there exists a constant $C(K)$ such that

$$\forall\ s_1, s_2 \in K \quad d_b(\gamma(s_1), \gamma(s_2)) \le C(K)|s_1 - s_2|\ .$$

The class of paths so defined is independent of the choice of the background metric b. A path is called *Lipschitz* if the constant $C(K)$ above can be chosen independently of K.

A key theorem of Rademacher [152] shows that Lipschitz maps ϕ are classically differentiable almost everywhere, with "almost everywhere" understood in the sense of the Lebesgue measure in local coordinates on M. Furthermore, the distributional derivatives of ϕ are in $L^\infty_{\rm loc}$ and are equal to the classical ones almost everywhere. Finally, Lipschitz paths are integrals of their distributional derivatives. All these properties imply that locally Lipschitz paths are *as good as differentiable ones* for most purposes.

Let $\dot\gamma$ denote the *classical* derivative of a path γ, wherever defined. A parameterized path γ is then called *causal future directed* if γ is locally Lipschitz, with $\dot\gamma$ causal and future directed almost everywhere. Thus, $\dot\gamma$ is defined almost everywhere; and it is causal future directed almost everywhere on the set on which it is defined. A parameterized path γ will be called *timelike future directed* if γ is locally Lipschitz, with $\dot\gamma$ timelike future directed almost everywhere. *Past directed* parameterized paths are defined by changing "future" to "past" in the definitions above.

The expert reader can check that, with the definitions above, objects such as $I^+(p)$, $J^+(p)$, etc., remain unchanged, while many proofs within causality theory become simpler.

2.4. **Submanifolds.** In addition to curves, one may also speak of the causal character of higher dimensional submanifolds. Let V be a smooth submanifold of a space-time $(\mathscr{M}, g)$. For $p \in V$, we say that the tangent space T_pV is spacelike (resp. timelike, null) provided g restricted to T_pV is positive definite (resp., has Lorentzian signature, is degenerate). Then V is said to be spacelike (resp., timelike, null) provided each of its tangent spaces is spacelike (resp., timelike, null). Hence if V is spacelike (resp., timelike) then, with respect to its *induced metric*, i.e., the metric g restricted to the tangent spaces of V, V is a Riemannian (resp., Lorentzian) manifold.

3. Stationary black holes

Perhaps the first thing which comes to mind when general relativity is mentioned are black holes. These are among the most fascinating objects predicted by Einstein's theory of gravitation. Although they have been studied for years,[3] they still attract tremendous attention in the physics and astrophysics literature. It is seldom realized that, in addition to Einstein's gravity, several other field theories are known to possess solutions which exhibit black hole properties, amongst which:[4]

- The "dumb holes", arising in Euler equations, which are the sonic counterparts of black holes, first discussed by Unruh [319] (compare [98]).
- The "optical" ones – the black-hole-type solutions arising in the theory of moving dielectric media, or in non-linear electrodynamics [223, 262].

The numerical study of black holes has become a science in itself, see [67, 279] and references therein. The evidence for the existence of black holes in our universe is growing [150, 208, 248, 332]. Reviews of, and further references to, the quantum aspects of black holes can be found in [14, 58, 190, 270, 322].

In this section we focus attention on *stationary* black holes that are solutions of the *vacuum* Einstein equations with vanishing cosmological constant, with one exception: the static electro-vacuum Majumdar–Papapetrou solutions, an example of physically significant multiple black holes. By definition, a *stationary* space-time is an asymptotically flat space-time which is invariant under an action of $\mathbb{R}$ by isometries, such that the associated generator — referred to as *Killing vector* — is timelike[5] in the asymptotically flat region. These model steady state solutions. Stationary black holes are the simplest to describe, and most mathematical results on black holes, such as the uniqueness theorems discussed in Section 3.9, concern those. It should, however, be kept in mind that one of the major open problems in mathematical relativity is the understanding of the dynamical behavior of black hole space-times, about which not much is yet known (compare Section 6.6).

3.1. The Schwarzschild metric.

The simplest stationary solutions describing compact isolated objects are the spherically symmetric ones. According to Birkhoff's theorem [44], any $(n+1)$–dimensional, $n \geq 3$, spherically symmetric solution of the vacuum Einstein equations belongs to the family of Schwarzschild metrics, parameterized by a mass parameter m:

$$g = -V^2 dt^2 + V^{-2} dr^2 + r^2 d\Omega^2 \ , \tag{3.1}$$

$$V^2 = 1 - \tfrac{2m}{r^{n-2}} \ , \quad t \in \mathbb{R} \ , \ r \in (2m, \infty) \ . \tag{3.2}$$

Here $d\Omega^2$ denotes the metric of the standard $(n-1)$-sphere. (This is true *without* assuming stationarity.)

From now on we assume $n = 3$, though identical results hold in higher dimension.

[3] The reader is referred to the introduction to [72] for an excellent concise review of the history of the concept of a black hole, and to [71, 206] for a more detailed one.

[4] An even longer list of models and submodels can be found in [16], see also [15, 263].

[5] In fact, in the literature it is always implicitly assumed that the stationary Killing vector K is *uniformly timelike* in the asymptotic region M_{ext}; by this we mean that $g(K, K) < -\epsilon < 0$ for some ϵ and for all r large enough. This uniformity condition excludes the possibility of a timelike vector which asymptotes to a null one. This involves no loss of generality in well behaved space-times: indeed, uniformity always holds for Killing vectors which are timelike for all large distances if the conditions of the positive energy theorem are met [30, 120].

We will assume

$$m > 0 \ ,$$

because $m < 0$ leads to metrics which are called "nakedly singular"; this deserves a comment. For Schwarzschild metrics we have

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48m^2}{r^6} \ , \tag{3.3}$$

in dimension $3+1$, which shows that the geometry becomes singular as $r = 0$ is approached; this remains true in higher dimensions. As we shall see shortly, for $m > 0$ the singularity is "hidden" behind an event horizon, and this is *not* the case for $m < 0$.

One of the first features one notices is that the metric (3.1) is singular as $r = 2m$ is approached. It turns out that this singularity is related to an unfortunate choice of coordinates (one talks about "a coordinate singularity"); the simplest way to see this is to replace t by a new coordinate v defined as

$$v = t + f(r) \ , \quad f' = \frac{1}{V^2} \ , \tag{3.4}$$

leading to

$$v = t + r + 2m \ln(r - 2m) \ .$$

This brings g to the form

$$g = -(1 - \frac{2m}{r})dv^2 + 2dvdr + r^2 d\Omega^2 \ . \tag{3.5}$$

We have $\det g = -r^4 \sin^2\theta$, with all coefficients of g smooth, which shows that g is a well defined Lorentzian metric on the set

$$v \in \mathbb{R} \ , \quad r \in (0, \infty) \ . \tag{3.6}$$

More precisely, (3.5)-(3.6) provides an analytic extension of the original space-time (3.1).

We could have started immediately from the form (3.5) of g, which would have avoided the lengthy discussion of the coordinate transformation (3.4). However, the form (3.1) is the more standard one. Furthermore, it makes the metric g manifestly asymptotically flat (see Section 3.6); this is somewhat less obvious to an untrained eye in (3.5).

It is easily seen that the region $\{r \le 2m\}$ for the metric (3.5) is a *black hole region*, in the sense that

(3.7) observers, or signals, can enter this region, but can never leave it.

In order to see that, recall that observers in general relativity always move on *future directed timelike curves*, that is, curves with timelike future directed tangent vector. For signals, the curves are *causal future directed*. Let, then, $\gamma(s) = (v(s), r(s), \theta(s), \varphi(s))$ be such a timelike curve; for the metric (3.5) the timelikeness condition $g(\dot\gamma, \dot\gamma) < 0$ reads

$$-(1 - \frac{2m}{r})\dot v^2 + 2\dot v \dot r + r^2(\dot\theta^2 + \sin^2\theta \dot\varphi^2) < 0 \ .$$

This implies

$$\dot v\Big(- (1 - \frac{2m}{r})\dot v + 2\dot r\Big) < 0 \ .$$

It follows that $\dot{v}$ does not change sign on a timelike curve. The usual choice of time orientation corresponds to $\dot{v} > 0$ on future directed curves, leading to

$$-(1 - \frac{2m}{r})\dot{v} + 2\dot{r} < 0 \ .$$

For $r \leq 2m$ the first term is non-negative, which enforces $\dot{r} < 0$ on all future directed timelike curves in that region. Thus, r is a strictly decreasing function along such curves, which implies that future directed timelike curves can cross the hypersurface $\{r = 2m\}$ only if coming from the region $\{r > 2m\}$. This motivates the name *black hole event horizon* for $\{r = 2m, v \in \mathbb{R}\}$. The same conclusion (3.7) applies for causal curves: it suffices to approximate a causal curve by a sequence of timelike ones.

The transition from (3.1) to (3.5) is not the end of the story, as further extensions are possible. For the metric (3.1) a maximal analytic extension has been found independently by Kruskal [221], Szekeres [315], and Fronsdal [163]; for some obscure reason Fronsdal is almost never mentioned in this context. This extension is depicted[6] in Figure 3.1. The region I there corresponds to the space-time (3.1), while the extension just constructed corresponds to the regions I and II.

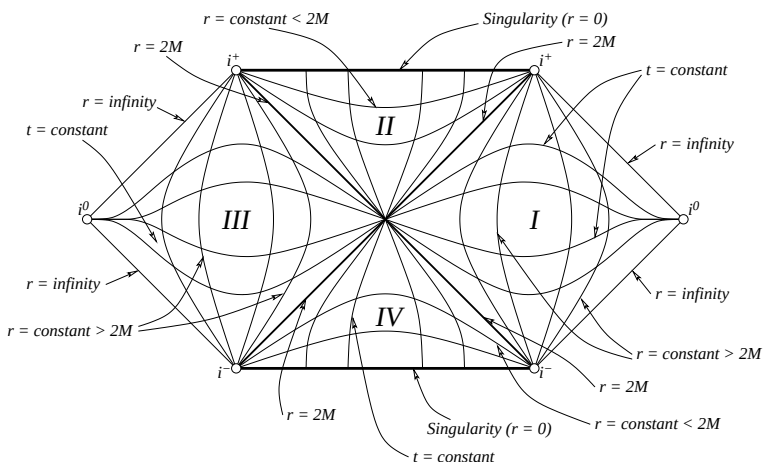

FIGURE 3.1. The Carter-Penrose diagram[6] for the Kruskal-Szekeres space-time with mass M. There are actually two asymptotically flat regions, with corresponding event horizons defined with respect to the second region. Each point in this diagram represents a two-dimensional sphere, and coordinates are chosen so that light-cones have slopes plus minus one.

The Kruskal-Szekeres extension is singled out by being maximal in the class of vacuum, analytic, simply connected space-times, with all maximally extended geodesics γ either complete, or with the curvature scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverging along γ in finite affine time.

[6]We are grateful to J.-P. Nicolas for allowing us to use his figure from [259].

An alternative convenient representation of the Schwarzschild metrics, which makes the space-part of g manifestly conformally flat, is given by

$$g = -\left(\frac{1-m/2|x|^{n-2}}{1+m/2|x|^{n-2}}\right)^2 dt^2 + \left(1+\frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \left(\sum_{1=1}^{n}(dx^i)^2\right) . \tag{3.8}$$

3.2. **Rotating black holes.** Rotating generalizations of the Schwarzschild metrics are given by the family of *Kerr metrics*, parameterized by a mass parameter m and an angular momentum parameter a. One explicit coordinate representation of the Kerr metric is

$$\begin{aligned} \hat{g} &= -\Big(1-\frac{2mr}{\Sigma}\Big)dv^2 + 2drdv + \Sigma d\theta^2 - 2a\sin^2\theta d\phi dr \\ &\quad + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 - \frac{4amr\sin^2\theta}{\Sigma}d\phi dv , \end{aligned} \tag{3.9}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta , \qquad \Delta = r^2 + a^2 - 2mr .$$

Note that (3.9) reduces to the Schwarzschild solution in the representation (3.5) when $a = 0$. The reader is referred to [69, 266] for a thorough analysis. All Kerr metrics satisfying

$$m^2 \geq a^2$$

provide, when appropriately extended, vacuum space-times containing a rotating black hole. Higher dimensional analogues of the Kerr metrics have been constructed by Myers and Perry [256].

A fascinating class of black hole solutions of the $4+1$ dimensional stationary vacuum Einstein equations has been found by Emparan and Reall [148] see also [147, 149]. The solutions, called *black rings*, are asymptotically Minkowskian in spacelike directions, with an event horizon having $S^1 \times S^2$ cross-sections. The "ring" terminology refers to the S^1 factor in $S^1 \times S^2$.

3.3. **Killing horizons.** Before continuing some general notions are in order. By definition, a *Killing field* is a vector field the local flow of which preserves the metric. Killing vectors are solutions of the over-determined system of *Killing equations*

$$\nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 0 . \tag{3.10}$$

One of the features of the metric (3.1) is its *stationarity*, with Killing vector field $X = \partial_t$: As already pointed out, a space-time is called *stationary* if there exists a Killing vector field X which approaches ∂_t in the asymptotically flat region (where r goes to ∞, see Section 3.6 for precise definitions) *and* generates a one parameter group of isometries. A space-time is called *static* if it is stationary and if the distribution of hyperplanes orthogonal to the stationary Killing vector X is integrable.

A space-time is called *axisymmetric* if there exists a Killing vector field Y which generates a one parameter group of isometries and which behaves like a *rotation*: this property is captured by requiring that all orbits are 2π–periodic, and that the set $\{Y = 0\}$, called the *axis of rotation*, is non-empty.

Let X be a Killing vector field on $(\mathscr{M}, g)$, and suppose that $\mathscr{M}$ contains a null hypersurface (see Sections 2.4 and 7.1) $\mathscr{N}_0 = \mathscr{N}_0(X)$ which coincides with a

connected component of the set

$$\mathscr{N}(X) := \{p \in \mathscr{M} \mid g(X_p, X_p) = 0 \ , \ X_p \neq 0\} \ ,$$

with X tangent to $\mathscr{N}_0$. Then $\mathscr{N}_0$ is called a *Killing horizon* associated to the Killing vector X. The simplest example is provided by the "boost Killing vector field"

$$K = z\partial_t + t\partial_z \tag{3.11}$$

in four-dimensional Minkowski space-time $\mathbb{R}^{1,3}$: $\mathscr{N}(K)$ has four connected components

$$\mathscr{N}(K)_{\epsilon\delta} := \{t = \epsilon z \ , \delta t > 0\} \ , \quad \epsilon, \delta \in \{\pm 1\} \ .$$

The closure $\overline{\mathscr{N}(K)}$ of $\mathscr{N}(K)$ is the set $\{|t| = |z|\}$, which is *not* a manifold, because of the crossing of the null hyperplanes $\{t = \pm z\}$ at $t = z = 0$. Horizons of this type are referred to as *bifurcate Killing horizons*.

A very similar behavior is met in the extended Schwarzschild space-time: the set $\{r = 2m\}$ is a null hypersurface $\mathscr{E}$, the Schwarzschild event horizon. The stationary Killing vector $X = \partial_t$ extends to a Killing vector $\hat{X}$ in the extended space-time which becomes tangent to and null on $\mathscr{E}$, except at the "bifurcation sphere" right in the middle of Figure 3.1, where $\hat{X}$ vanishes.

A last noteworthy example in Minkowski space-time $\mathbb{R}^{1,3}$ is provided by the Killing vector

$$X = y\partial_t + t\partial_y + x\partial_y - y\partial_x = y\partial_t + (t+x)\partial_y - y\partial_x \ . \tag{3.12}$$

Thus, X is the sum of a boost $y\partial_t + t\partial_y$ and a rotation $x\partial_y - y\partial_x$. Note that X vanishes if and only if

$$y = t + x = 0 \ ,$$

which is a two-dimensional null submanifold of $\mathbb{R}^{1,3}$. The vanishing set of the Lorentzian length of X,

$$g(X, X) = (t + x)^2 = 0 \ ,$$

is a null hyperplane in $\mathbb{R}^{1,3}$. It follows that, e.g., the set

$$\{t + x = 0 \ , \ y > 0 \ , t > 0\}$$

is a *Killing horizon* with respect to two different Killing vectors, the boost Killing vector $x\partial_t + t\partial_x$, and the Killing vector (3.12).

3.3.1. *Surface gravity.* The *surface gravity* κ of a Killing horizon is defined by the formula

$$d\Big(g(X, X)\Big) = -2\kappa X^\flat \ , \tag{3.13}$$

where $X^\flat$ is the one-form metrically dual to X, i.e. $X^\flat = g_{\mu\nu} X^\nu dx^\mu$. Two comments are in order: First, since $g(X, X) = 0$ on $\mathscr{N}(X)$, the differential of $g(X, X)$ annihilates $T\mathscr{N}(X)$. Now, simple algebra shows that a one–form annihilating a null hypersurface is proportional to $g(\ell, \cdot)$, where ℓ is any null vector tangent to $\mathscr{N}$ (those are defined uniquely up to a proportionality factor, see Section 7.1). We thus obtain that $d(g(X, X))$ is proportional to $X^\flat$; whence (3.13). Next, the name "surface gravity" stems from the following: using the Killing equations (3.10) and (3.13) one has

$$X^\mu \nabla_\mu X^\sigma = -X^\mu \nabla^\sigma X_\mu = \kappa X^\sigma \ . \tag{3.14}$$

Since the left-hand-side of (3.14) is the acceleration of the integral curves of X, the equation shows that, in a certain sense, κ measures the gravitational field at the horizon.

A key property is that the surface gravity κ is constant on bifurcate [209, p. 59] Killing horizons. Furthermore, κ [182, Theorem 7.1] is constant for all Killing horizons, whether bifurcate or not, in space-times satisfying the *dominant energy condition*: this means that

$$T_{\mu\nu}X^\mu Y^\nu \geq 0 \text{ for causal future directed vector fields } X \text{ and } Y. \tag{3.15}$$

As an example, consider the Killing vector K of (3.11). We have

$$d(g(K,K)) = d(-z^2+t^2) = 2(-zdz+tdt)\ ,$$

which equals twice $K^\flat$ on $\mathscr{N}(K)_{\epsilon\delta}$. On the other hand, for the Killing vector X of (3.12) one obtains

$$d(g(X,X)) = 2(t+x)(dt+dx)\ ,$$

which vanishes on each of the Killing horizons $\{t=-x\ , y\neq 0\}$. This shows that the same null surface can have zero or non-zero values of surface gravity, depending upon which Killing vector has been chosen to calculate κ.

The surface gravity of black holes plays an important role in *black hole thermodynamics*; see [58] and references therein.

A Killing horizon $\mathscr{N}_0(X)$ is said to be *degenerate*, or *extreme*, if κ vanishes throughout $\mathscr{N}_0(X)$; it is called *non-degenerate* if κ has no zeros on $\mathscr{N}_0(X)$. Thus, the Killing horizons $\mathscr{N}(K)_{\epsilon\delta}$ are non-degenerate, while both Killing horizons of X given by (3.12) are degenerate. The Schwarzschild black holes have surface gravity

$$\kappa_m = \frac{1}{2m}\ .$$

So there are no degenerate black holes within the Schwarzschild family. Theorem 3.5 below shows that there are no regular, degenerate, static vacuum black holes at all.

In Kerr space-times we have $\kappa=0$ if and only if $m=|a|$. On the other hand, all horizons in the multi-black hole Majumdar-Papapetrou solutions are degenerate.

3.4. **The orbit-space geometry near Killing horizons.** Consider a space-time $(\mathscr{M},g)$ with a Killing vector field X. On any set $\mathscr{U}$ on which X is timelike we can introduce coordinates in which $X=\partial_t$, and the metric may be written as

$$g = -V^2(dt+\theta_i dx^i)^2 + h_{ij}dx^i dx^j\ , \quad \partial_t V = \partial_t\theta_i = \partial_t h_{ij} = 0\ . \tag{3.16}$$

where $h = h_{ij}dx^i dx^j$ has Riemannian signature. The metric h is often referred to as the *orbit-space metric*.

Let M be a spacelike hypersurface in $\mathscr{M}$; then (3.16) defines a Riemannian metric h on $M\cap\mathscr{U}$. Assume that X is timelike on a one-sided neighborhood $\mathscr{U}$ of a Killing horizon $\mathscr{N}_0(X)$, and suppose that $M\cap\mathscr{U}$ has a boundary component S which forms a compact cross-section of $\mathscr{N}_0(X)$, see Figure 3.2. The vanishing, or not, of the surface gravity has a deep impact on the geometry of h near $\mathscr{N}_0(X)$ [104]:

(1) Every differentiable such S, included in a C^2 *degenerate* Killing horizon $\mathscr{N}_0(X)$, corresponds to a *complete* asymptotic end of $(M\cap\mathscr{U},h)$. See Figure 3.3.[7]

[7]We are grateful to C. Williams for providing the figure.

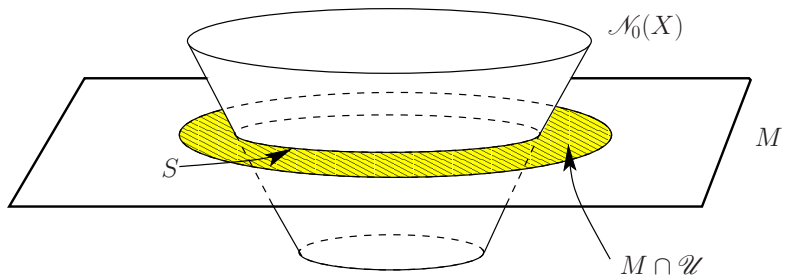

FIGURE 3.2. A space-like hypersurface M intersecting a Killing horizon $\mathscr{N}_0(X)$ in a compact cross-section S.

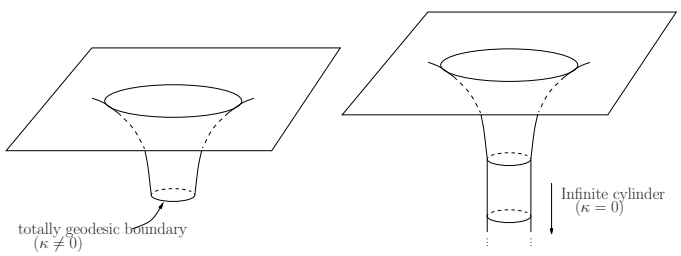

FIGURE 3.3. The general features of the geometry of the orbit-space metric on a spacelike hypersurface intersecting a non-degenerate (left) and degenerate (right) Killing horizon, near the intersection, visualized by a co-dimension one embedding in Euclidean space.

(2) Every such S included in a smooth Killing horizon $\mathscr{N}_0(X)$ on which

$$\kappa > 0 \ ,$$

corresponds to a totally geodesic boundary of $(M \cap \mathscr{U}, h)$, with h being smooth up–to–boundary at S. Moreover
  (a) a doubling of $(\overline{M} \cap \mathscr{U}, h)$ across S leads to a smooth metric on the doubled manifold,
  (b) with $\sqrt{-g(X,X)}$ extending smoothly to $-\sqrt{-g(X,X)}$ across S.

In the Majumdar-Papapetrou solutions of Section 3.9.2, the orbit-space metric h as in (3.16) asymptotes to the usual metric on a round cylinder as the event horizon is approached. One is therefore tempted to think of degenerate event horizons as corresponding to *asymptotically cylindrical ends* of (M, h).

3.5. **Near-horizon geometry.** Following [254], near a smooth null hypersurface one can introduce *Gaussian null coordinates*, in which the metric takes the form

$$g = r\varphi dv^2 + 2dvdr + 2r\chi_a dx^a dv + h_{ab} dx^a dx^b \ . \tag{3.17}$$

The hypersurface is given by the equation $\{r=0\}$. Let S be any smooth compact cross-section of the horizon; then the *average surface gravity* $\langle\kappa\rangle_S$ is defined as

$$\langle\kappa\rangle_S = -\frac{1}{|S|}\int_S \varphi d\mu_h \ , \tag{3.18}$$

where $d\mu_h$ is the measure induced by the metric h on S, and $|S|$ is the volume of S. We emphasize that this is defined regardless of whether or not the stationary Killing vector is tangent to the null generators of the hypersurface; on the other hand, $\langle\kappa\rangle_S$ coincides with κ when κ is constant and the Killing vector equals ∂_v.

On a degenerate Killing horizon the surface gravity vanishes, so that the function φ in (3.17) can itself be written as rA, for some smooth function A. The vacuum Einstein equations imply (see [254, eq. (2.9)] in dimension four and [225, eq. (5.9)] in higher dimensions)

$$\mathring{R}_{ab} = \frac{1}{2}\mathring{\chi}_a\mathring{\chi}_b - \mathring{D}_{(a}\mathring{\chi}_{b)} \ , \tag{3.19}$$

where $\mathring{R}_{ab}$ is the Ricci tensor of $\mathring{h}_{ab} := h_{ab}|_{r=0}$, and $\mathring{D}$ is the covariant derivative thereof, while $\mathring{\chi}_a := h_a|_{r=0}$. The Einstein equations also determine $\mathring{A} := A|_{r=0}$ uniquely in terms of $\mathring{h}_a$ and $\mathring{h}_{ab}$:

$$\mathring{A} = \frac{1}{2}\mathring{h}^{ab}\left(\mathring{\chi}_a\mathring{\chi}_b - \mathring{D}_a\mathring{\chi}_b\right) \tag{3.20}$$

(this equation follows again e.g. from [254, eq. (2.9)] in dimension four, and can be checked by a calculation in all higher dimensions). Equations (3.19) have only been understood under the supplementary assumptions of staticity [123],[8] or axial symmetry in space-time dimension four [225]:

Theorem 3.1 ([123]). *Let the space-time dimension be $n+1$, $n\geq 3$, suppose that a degenerate Killing horizon $\mathscr{N}$ has a compact cross-section, and that $\mathring{\chi}_a = \partial_a\lambda$ for some function λ (which is necessarily the case in vacuum static space-times). Then* (3.19) *implies $\mathring{\chi}_a \equiv 0$, so that $\mathring{h}_{ab}$ is Ricci-flat.*

Theorem 3.2 ([225]). *In space-time dimension four and in vacuum, suppose that a degenerate Killing horizon $\mathscr{N}$ has a spherical cross-section, and that $(\mathscr{M}, g)$ admits a second Killing vector field with periodic orbits. For every connected component $\mathscr{N}_0$ of $\mathscr{N}$ there exists a map ψ from a neighborhood of $\mathscr{N}_0$ into a degenerate Kerr space-time which preserves $\mathring{\chi}_a$, $\mathring{h}_{ab}$ and $\mathring{A}$.*

It would be of interest to classify solutions of (3.19) in a useful manner, in all dimensions, without any restrictive conditions.

In the four-dimensional static case, Theorem 3.1 enforces toroidal topology of cross-sections of $\mathscr{N}$, with a flat $\mathring{h}_{ab}$. On the other hand, in the four-dimensional axi-symmetric case, Theorem 3.2 guarantees that the geometry tends to a Kerr one, up to second order errors, when the horizon is approached. So, in the degenerate case, the vacuum equations impose strong restrictions on the near-horizon geometry. This is not the case any more for non-degenerate horizons, at least in the analytic setting.

[8]Some partial results with a non-zero cosmological constant have also been proved in [123].

3.6. **Asymptotically flat metrics.** In relativity one often needs to consider initial data on non-compact manifolds, with natural restrictions on the asymptotic geometry. The most commonly studied such examples are *asymptotically flat manifolds*, which model isolated gravitational systems. Now, there exist several ways of defining asymptotic flatness, all of them roughly equivalent in vacuum. We will adapt a Cauchy data point of view, as it appears to be the least restrictive; the discussion here will also be relevant for Section 5.

So, a space-time $(\mathscr{M}, g)$ will be said to possess an *asymptotically flat end* if $\mathscr{M}$ contains a spacelike hypersurface M_{ext} diffeomorphic to $\mathbb{R}^n \setminus B(R)$, where $B(R)$ is a coordinate ball of radius R. An end comes thus equipped with a set of Euclidean coordinates $\{x^i, i = 1, \dots, n\}$, and one sets $r = |x| := \left(\sum_{i=1}^n (x^i)^2\right)^{1/2}$. One then assumes that there exists a constant $\alpha > 0$ such that, in local coordinates on M_{ext} obtained from $\mathbb{R}^n \setminus B(R)$, the metric h induced by g on M_{ext}, and the second fundamental form K of M_{ext} (compare (4.15) below), satisfy the fall-off conditions, for some $k > 1$,

$$h_{ij} - \delta_{ij} = O_k(r^{-\alpha}) \ , \qquad K_{ij} = O_{k-1}(r^{-1-\alpha}) \ , \tag{3.21}$$

where we write $f = O_k(r^\beta)$ if f satisfies

$$\partial_{k_1} \dots \partial_{k_\ell} f = O(r^{\beta-\ell}) \ , \quad 0 \le \ell \le k \ . \tag{3.22}$$

In applications one needs (h, K) to lie in certain weighted Hölder or Sobolev space defined on M, with the former better suited for the treatment of the evolution as discussed in Section 6.[9]

3.7. **Asymptotically flat stationary metrics.** For simplicity we assume that the space-time is vacuum, though similar results hold in general under appropriate conditions on matter fields, see [29, 120] and references therein.

Along any spacelike hypersurface M, a Killing vector field X of $(\mathscr{M}, g)$ can be decomposed as

$$X = Nn + Y \ ,$$

where Y is tangent to M, and n is the unit future-directed normal to M. The fields N and Y are called "Killing initial data", or KID for short. The vacuum field equations, together with the Killing equations, imply the following set of equations on M

$$D_i Y_j + D_j Y_i = 2N K_{ij} \ , \tag{3.23}$$

$$R_{ij}(h) + K^k{}_k K_{ij} - 2K_{ik}K^k{}_j - N^{-1}(\mathscr{L}_Y K_{ij} + D_i D_j N) = 0 \ , \tag{3.24}$$

where $R_{ij}(h)$ is the Ricci tensor of h. These equations play an important role in the gluing constructions described in Section 5.3.

Under the boundary conditions (3.21), an analysis of these equations provides detailed information about the asymptotic behavior of (N, Y). In particular one can prove that if the asymptotic region M_{ext} is contained in a hypersurface M satisfying the requirements of the positive energy theorem (see Section 5.2.1), and if X is timelike along M_{ext}, then $(N, Y^i) \to_{r\to\infty} (A^0, A^i)$, where the A^μ's are constants satisfying $(A^0)^2 > \sum_i (A^i)^2$ [30, 120]. Further, in the coordinates of (3.21),

$$\theta_i = O_k(r^{-\alpha}) \ , \quad V - 1 = O_k(r^{-\alpha}) \ . \tag{3.25}$$

[9]The analysis of elliptic operators such as the Laplacian on weighted Sobolev spaces was initiated by Nirenberg and Walker [260] (see also [19, 78, 232–235, 245–247] as well as [77]). A readable treatment of analysis on weighted spaces (not focusing on relativity) can be found in [269].

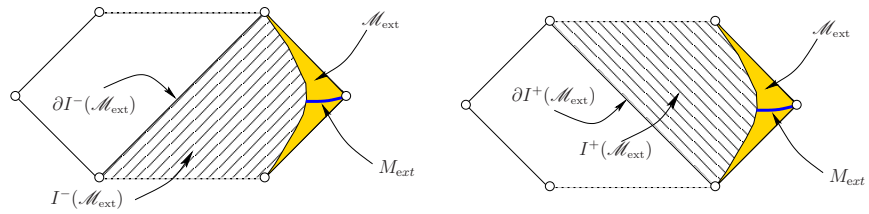

FIGURE 3.4. M_{ext}, $\mathscr{M}_{\rm ext}$, together with the future and the past of $\mathscr{M}_{\rm ext}$. One has $\mathscr{M}_{\rm ext} \subset I^\pm(\mathscr{M}_{\rm ext})$, even though this is not immediately apparent from the figure. The domain of outer communications is the intersection $I^+(\mathscr{M}_{\rm ext}) \cap I^-(\mathscr{M}_{\rm ext})$, compare Figure 3.5.

As discussed in more detail in [31], in h-harmonic coordinates, and in e.g. a maximal (i.e., mean curvature zero) time-slicing, the vacuum equations for g form a quasi-linear elliptic system with diagonal principal part, with principal symbol identical to that of the scalar Laplace operator. It can be shown that, in this "gauge", all metric functions have a full asymptotic expansion in terms of powers of $\ln r$ and inverse powers of r. In the new coordinates we can in fact take

$$\alpha = n - 2 \ . \tag{3.26}$$

By inspection of the equations one can further infer that the leading order corrections in the metric can be written in the Schwarzschild form (3.8).

3.8. **Domains of outer communications, event horizons.** A key notion in the theory of asymptotically flat black holes is that of the *domain of outer communications*, defined for stationary space-times as follows: For $t \in \mathbb{R}$ let $\phi_t[X] : \mathscr{M} \to \mathscr{M}$ denote the one-parameter group of diffeomorphisms generated by X; we will write ϕ_t for $\phi_t[X]$ whenever ambiguities are unlikely to occur. Let M_{ext} be as in Section 3.6, and assume that X is timelike along M_{ext}. The exterior region $\mathscr{M}_{\rm ext}$ and the *domain of outer communications* $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ are then defined as[10]

$$\mathscr{M}_{\rm ext} := \cup_t \phi_t(M_{ext}) \ , \qquad \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle = I^+(\mathscr{M}_{\rm ext}) \cap I^-(\mathscr{M}_{\rm ext}) \ . \tag{3.27}$$

The *black hole region* $\mathscr{B}$ and the *black hole event horizon* $\mathscr{H}^+$ are defined as (see Figures 3.4 and 3.5)

$$\mathscr{B} = \mathscr{M} \setminus I^-(\mathscr{M}_{\rm ext}) \ , \quad \mathscr{H}^+ = \partial \mathscr{B} \ . \tag{3.28}$$

The *white hole region* $\mathscr{W}$ and the *white hole event horizon* $\mathscr{H}^-$ are defined as above after changing time orientation:

$$\mathscr{W} = \mathscr{M} \setminus I^+(\mathscr{M}_{\rm ext}) \ , \quad \mathscr{H}^- = \partial \mathscr{W} \ .$$

It follows that the boundaries of $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ are included in the event horizons. We set

$$\mathscr{E}^\pm = \partial \langle\langle \mathscr{M}_{\rm ext} \rangle\rangle \cap I^\pm(\mathscr{M}_{\rm ext}) \ , \qquad \mathscr{E} = \mathscr{E}^+ \cup \mathscr{E}^- \ . \tag{3.29}$$

Similarly to Proposition 2.3, $\mathscr{E}^+$ is ruled by null geodesics, called *generators*.

[10] See Section 2.3.1 for the definition of $I^\pm(\Omega)$.

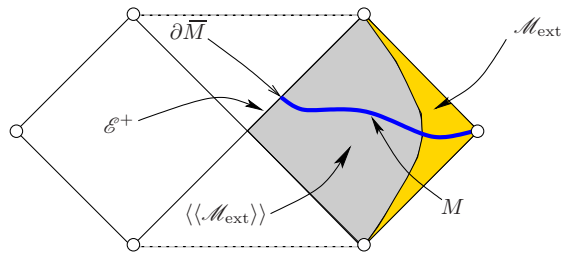

FIGURE 3.5. The hypersurface M from the definition of I^+–regularity. To avoid ambiguities, we note that $\mathscr{M}_{\rm ext}$ is a subset of $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$.

In general, each asymptotically flat end of $\mathscr{M}$ determines a different domain of outer communications. Although there is considerable freedom in choosing the asymptotic region M_{ext} giving rise to a particular end, it can be shown that $I^\pm(\mathscr{M}_{\rm ext})$, and hence $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$, $\mathscr{H}^\pm$ and $\mathscr{E}^\pm$, are independent of the choice of M_{ext}.

3.9. **Uniqueness theorems.** It is widely expected that the Kerr metrics provide the only stationary, regular, vacuum, four-dimensional black holes. In spite of many works on the subject (see, e.g., [70, 109, 182, 192, 193, 257, 292, 325] and references therein), the question is far from being settled.

To describe the current state of affairs, some terminology is needed. A Killing vector X is said to be complete if its orbits are complete, i.e., for every $p \in \mathscr{M}$ the orbit $\phi_t[X](p)$ of X is defined for all $t \in \mathbb{R}$. X is called *stationary* if it is timelike at large distances in the asymptotically flat region.

A key definition for the uniqueness theory is the following:

Definition 3.3. Let $(\mathscr{M}, g)$ be a space-time containing an asymptotically flat end M_{ext}, and let X be a stationary Killing vector field on $\mathscr{M}$. We will say that $(\mathscr{M}, g, X)$ is I^+–regular if X is complete, if the domain of outer communications $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$ is globally hyperbolic, and if $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$ contains a spacelike, connected, acausal hypersurface $M \supset M_{ext}$, the closure $\overline{M}$ of which is a topological manifold with boundary, consisting of the union of a compact set and of a finite number of asymptotically flat ends, such that the boundary $\partial\overline{M} := \overline{M} \setminus M$ satisfies

$$\partial\overline{M} \subset \mathscr{E}^+ \ , \tag{3.30}$$

(see (3.29)) with $\partial\overline{M}$ meeting every generator of $\mathscr{E}^+$ precisely once; see Figure 3.5.

Some comments might be helpful. First one requires completeness of the orbits of the stationary Killing vector because one needs an action of $\mathbb{R}$ on $\mathscr{M}$ by isometries. Next, one requires global hyperbolicity of the domain of outer communications to guarantee its simple connectedness, and to avoid causality violations. Further, the existence of a well-behaved spacelike hypersurface gives reasonable control of the geometry of $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$, and is a prerequisite to any elliptic PDEs analysis, as is extensively needed for the problem at hand. The existence of compact cross-sections of the future event horizon $\mathscr{E}^+$ prevents singularities on the future part of the boundary of the domain of outer communications, and eventually guarantees the smoothness of that boundary.

The proof of the following can be found in [109]:

Theorem 3.4. *Let $(\mathscr{M}, g)$ be an I^+–regular, vacuum, analytic, asymptotically flat, four-dimensional stationary space-time. If $\mathscr{E}^+$ is connected and mean non-degenerate (in the sense that $< \kappa >_{\partial\overline{M}} \neq 0$; (compare (3.18)), then $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ is isometric to the domain of outer communications of a Kerr space-time.*

Theorem 3.4 finds its roots in work by Carter and Robinson [70,292], with further key steps of the proof due to Hawking [177] and Sudarsky and Wald [313]. It should be emphasized that the hypotheses of analyticity and non-degeneracy are highly unsatisfactory, and one believes that they are not needed for the conclusion. One also believes that no solutions with more than one component of $\mathscr{E}^+$ are regular; this has been established so far only for some special cases [227, 326].

Partial results concerning uniqueness of higher dimensional black holes have been obtained by Hollands and Yazadjiev [185], compare [173, 174, 255].

The proof of Theorem 3.4 can be outlined as follows: First, the event horizon in a smooth or analytic space-time is a priori only a Lipschitz surface, so the starting point of the analysis is provided by a result in [114], that event horizons in regular stationary black hole space-times are as differentiable as the differentiability of the metric allows. One then shows [177][11] that *either*

a) the stationary Killing vector is tangent to the generators of the event horizon, *or*
b) there exists a second Killing vector defined near the event horizon. The remaining analysis relies heavily on the fact that the domain of outer communications is simply connected [127] (compare [165]).

In case a) one shows that the domain of outer communications contains a maximal (mean curvature zero) spacelike hypersurface [126]; to be able to use the result from that reference one might need, first, to extend $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ using the construction in [280]. This allows one to establish staticity [313], and one concludes using Theorem 3.5 below.

In case b), analyticity and simple connectedness imply [103] that the isometry group of $(\mathscr{M}, g)$ contains a $U(1)$ factor, with non-empty axis of rotation. A delicate argument, which finds its roots in the work of Carter [70], proves that the area function

$$W := -\det(g(K_a, K_b)) \ , \qquad a, b = 1, 2$$

where K_a are the stationary and the periodic Killing vector, is strictly positive on the domain of outer communications. Classical results on group actions on simply connected manifolds [268, 281] show that the domain of outer communications is diffeomorphic to $\mathbb{R} \times (\mathbb{R}^3 \setminus \overline{B(1)})$, with the action of the isometry group by translations in the first factor, and by rotations around an axis in $\mathbb{R}^3$. The uniformization theorem allows one to establish that $\sqrt{W}$ can be used as the usual polar coordinate ρ on $\mathbb{R}^3$, leading to a coordinate system in which the field equations reduce to a harmonic map with values in two-dimensional hyperbolic space. The map is singular at the rotation axis (compare [108]), with rather delicate singularity structure at points where the event horizon meets the axis. A uniqueness theorem for such maps [292, 325] achieves the proof.

[11] Compare [161]; the result, proved by Hawking in space-dimension $n = 3$ [177, 179], has been generalised to $n \geq 4$ in [184, 202].

3.9.1. *Static case.* Assuming *staticity*, i.e., stationarity and hypersurface-orthogonality of the stationary Killing vector, a more satisfactory result is available in space dimensions less than or equal to seven, and in higher dimensions on manifolds on which the Riemannian rigid positive energy theorem holds: non-connected configurations are excluded, without any *a priori* restrictions on the gradient of the norm of the static Killing vector at event horizons.

More precisely, we shall say that a manifold $\widehat{M}$ is of *positive energy type* if there are no asymptotically flat complete Riemannian metrics on $\widehat{M}$ with positive scalar curvature and vanishing mass except perhaps for a flat one. As made clear in Theorem 5.2, this property has been proved so far for all asymptotically flat n–dimensional manifolds $\widehat{M}$ of dimension $3 \le n \le 7$ [296], or under the hypothesis that $\widehat{M}$ is spin for any $n \ge 3$, and is expected to be true in general.

We have the following result, which finds its roots in the work of Israel [205], with further simplifications by Robinson [293], and with a significant strengthening by Bunting and Masood-ul-Alam [60]; the proof of the version presented here can be found in [104, 109]:

Theorem 3.5. *Let $(\mathscr{M}, g)$ be an I^+–regular, vacuum, static, analytic, $(n+1)$-dimensional space-time, $n \ge 3$. Let $\widehat{M}$ denote the manifold obtained by doubling the hypersurface M of Definition 3.3 across all non-degenerate components of its boundary and smoothly compactifying, in the doubled manifold, all asymptotically flat regions but one to a point. If $\widehat{M}$ is of positive energy type, then $\langle\langle \mathscr{M}_{\rm ext} \rangle\rangle$ is isometric to the domain of outer communications of a Schwarzschild space-time.*

Remark 3.6. As a corollary of Theorem 3.5 one obtains non-existence of static, regular, vacuum black holes with some components of the horizon degenerate. As observed in [123], if the space-time dimension is four, non-existence follows immediately from Theorem 3.1 and from simple connectedness of the domain of outer communications [127], but this does not seem to generalize to higher dimensions in any obvious way.

3.9.2. *Multi-black hole solutions.* In this section we assume that the space-time dimension is four. Space-times containing several black holes seem to be of particular interest, but Theorem 3.5 implies, under the conditions spelled out there, that no such *vacuum* solutions exist in the static class. However, the *Einstein-Maxwell* equations admit static solutions with several black holes: the *Majumdar-Papapetrou* (MP) solutions. The metric g and the electromagnetic potential A take the form [238, 271]

$$g = -u^{-2}dt^2 + u^2(dx^2 + dy^2 + dz^2)\,, \qquad A = \pm u^{-1}dt\,, \tag{3.31}$$

with $\Delta_\delta u = 0$, where Δ_δ is the Laplace operator of the flat metric δ. *Standard MP black holes* are defined by further requiring that

$$u = 1 + \sum_{i=1}^{I} \frac{\mu_i}{|\vec{x} - \vec{a}_i|}\,, \tag{3.32}$$

for some positive constants μ_i, the electric charges (up to the choice of sign for A in (3.31)) carried by the punctures $\vec{x} = \vec{a}_i$. Further, the coordinates x^μ of (3.31) are required to cover the range $\mathbb{R} \times (\mathbb{R}^3 \setminus \{\vec{a}_i\})$ for a finite set of points $\vec{a}_i \in \mathbb{R}^3$, $i = 1, \ldots, I$. It has been shown by Hartle and Hawking [175] that standard MP

space-times can be analytically extended to an electro–vacuum space-time with I black hole regions.

The case $I = 1$ is the special case $m = |q|$ of the so-called *Reissner-Nordström* metrics, which are the charged, spherically symmetric (connected) generalizations of the Schwarzschild black holes with mass m and electric charge q.

The static I^+–regular electro-vacuum black holes are well understood: Indeed, the analysis in [70, 240, 294, 311] (compare [105][12]) leads to:

Theorem 3.7. *Every domain of outer communications in a static, electro-vacuum, I^+–regular, analytic black hole space-time* without degenerate horizons *is isometric to a domain of outer communications of a Reissner-Nordström black hole.*

The relevance of the standard MP black holes follows now from the following result [125]:

Theorem 3.8. *Every domain of outer communications in a static, electro-vacuum, I^+–regular, analytic black hole space-time* containing degenerate horizons *is isometric to a domain of outer communications of a standard MP space-time.*

It thus follows that the MP family provides the only static, electro-vacuum, regular black holes with *non-connected* horizons.

3.10. **Non-zero cosmological constant.** A family of black hole solutions with non-zero cosmological constant has been discovered by Kottler [219] (compare [45, 63]). These metrics are also known as the Schwarzschild-anti de Sitter metrics (when $\Lambda < 0$) or the Schwarzschild-de Sitter metrics (when $\Lambda > 0$), and take the form

$$g = -e^{2\mathring{\lambda}(r)}dt^2 + e^{-2\mathring{\lambda}(r)}dr^2 + r^2\mathring{k} \ , \tag{3.33}$$

where $\mathring{k}$ is an Einstein metric on a compact $(n-1)$–dimensional manifold N, $n \geq 3$. Here

$$e^{2\mathring{\lambda}(r)} = \alpha r^2 + \beta + \frac{2m}{r} \ ,$$

with $\alpha = -2\Lambda/n(n-1)$, $\beta = R(\mathring{k})/(n-1)(n-2)$, where $R(\mathring{k})$ is the scalar curvature of the metric $\mathring{k}$, while $m \in \mathbb{R}$ is a constant, called the *mass* of g. The global structure of (suitably extended) Kottler space-times has been analyzed in [57, 170] in dimension $3 + 1$; the results extend to $n \geq 3$ dimension.

4. The Cauchy problem

The component version of the *vacuum Einstein equations with cosmological constant* Λ (2.8) reads

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0 \ , \tag{4.1}$$

where $G_{\alpha\beta}$ is the Einstein tensor defined as

$$G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} \ , \tag{4.2}$$

[12]See [109] or the arXiv version of [104] for corrections to some of the claims in [105].

while $R_{\alpha\beta}$ is the Ricci tensor and R the scalar curvature. We will refer to those equations as *the vacuum Einstein equations*, regardless of whether or not the cosmological constant vanishes, and in this work we will mostly assume $\Lambda = 0$. Taking the trace of (4.1) one obtains

$$R = \frac{2(n+1)}{n-1}\Lambda \ , \tag{4.3}$$

where, as elsewhere, $n+1$ is the dimension of space-time. This leads to the following equivalent version of (4.1):

$$\mathrm{Ric} = \frac{2\Lambda}{n-1} g \ . \tag{4.4}$$

Thus the Ricci tensor of the metric is proportional to the metric. Pseudo-Riemannian manifolds with metrics satisfying Equation (4.4) are called *Einstein manifolds* in the mathematical literature, see e.g. [42].

Given a manifold $\mathscr{M}$, Equation (4.1) or, equivalently, Equation (4.4) forms a system of second order partial differential equations for the metric, linear in the second derivatives of the metric, with coefficients which are rational functions of the $g_{\alpha\beta}$'s, quadratic in the first derivatives of g, again with coefficients rational in g. Equations linear in the highest order derivatives are called *quasi-linear*, hence the vacuum Einstein equations constitute a second order system of quasi-linear partial differential equations for the metric g.

In the discussion above we assumed that the manifold $\mathscr{M}$ has been given. In the evolutionary point of view, which we adapt in most of this work, all space-times of main interest have topology $\mathbb{R} \times M$, where M is an n–dimensional manifold carrying initial data. Thus, solutions of the Cauchy problem (as defined precisely by Theorem 6.2 below) have topology and differential structure which are determined by the initial data. As will be discussed in more detail in Section 6.2, the space-times obtained by evolution of the data are sometimes extendible; there is then a lot of freedom in the topology of the extended space-time, and we are not aware of conditions which would guarantee *uniqueness* of the extensions. So in the evolutionary approach the manifold is best thought of as being given *a priori* — namely $\mathscr{M} = \mathbb{R} \times M$, but it should be kept in mind that there is no *a priori* known natural time coordinate which can be constructed by evolutionary methods, and which leads to the decomposition $\mathscr{M} = \mathbb{R} \times M$.

Now, there exist standard classes of partial differential equations which are known to have good properties. They are determined by looking at the algebraic properties of those terms in the equations which contain derivatives of highest order, in our case of order two. Inspection of (4.1) shows that this equation does not fall in any of the standard classes, such as hyperbolic, parabolic, or elliptic. In retrospect this is not surprising, because equations in those classes typically lead to unique solutions. On the other hand, given any solution g of the Einstein equations (4.4) and any diffeomorphism Φ, the pull-back metric $\Phi^* g$ is also a solution of (4.4), so whatever uniqueness there might be will hold only *up to diffeomorphisms*. An alternative way of describing this, often found in the physics literature, is the following: suppose that we have a matrix $g_{\mu\nu}(x)$ of functions satisfying (4.1) in some coordinate system x^μ. If we perform a coordinate change $x^\mu \to y^\alpha(x^\mu)$, then

the matrix of functions $\bar{g}_{\alpha\beta}(y)$ defined as

$$g_{\mu\nu}(x) \to \bar{g}_{\alpha\beta}(y) = g_{\mu\nu}(x(y)) \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} \tag{4.5}$$

will also solve (4.1), if the x-derivatives there are replaced by y-derivatives. This property is known under the name of *diffeomorphism invariance*, or *coordinate invariance*, of the Einstein equations. Physicists say that "the diffeomorphism group is the gauge group of Einstein's theory of gravitation".

Somewhat surprisingly, Choquet-Bruhat [156] proved in 1952 that there exists a set of *hyperbolic* equations underlying (4.2). This proceeds by the introduction of so-called "harmonic coordinates", to which we turn our attention in the next section.

4.1. **The local evolution problem.**

4.1.1. *Wave coordinates.* A set of coordinates $\{y^\mu\}$ is called *harmonic* if each of the functions y^μ satisfies

$$\Box_g y^\mu = 0 \ , \tag{4.6}$$

where $\Box_g$ is the d'Alembertian associated with g acting on scalars:

$$\Box_g f := \mathrm{tr}_g \mathrm{Hess}\, f = \frac{1}{\sqrt{|\det g|}} \partial_\mu \left(\sqrt{|\det g|} g^{\mu\nu} \partial_\nu f \right) \ . \tag{4.7}$$

One also refers to these as *"wave coordinates"*. Assuming that (4.6) holds, (4.4) can be written as

$$\begin{aligned} 0 = \hat{E}^{\alpha\beta} \quad &:= \quad \Box_g g^{\alpha\beta} - g^{\epsilon\phi} \Big(2 g^{\gamma\delta} \Gamma^\alpha_{\gamma\epsilon} \Gamma^\beta_{\delta\phi} + (g^{\alpha\gamma} \Gamma^\beta_{\gamma\delta} + g^{\beta\gamma} \Gamma^\alpha_{\gamma\delta}) \Gamma^\delta_{\epsilon\phi} \Big) \\ &\quad - \frac{4\Lambda}{n-1} g^{\alpha\beta} \ . \end{aligned} \tag{4.8}$$

Here the $\Gamma^\alpha_{\beta\gamma}$'s should be calculated in terms of the $g_{\alpha\beta}$'s and their derivatives as in (2.3), and the wave operator $\Box_g$ is as in (4.7). So, *in wave coordinates, the Einstein equation forms a second-order quasi-linear wave-type system of equations* (4.8) for the metric functions $g^{\alpha\beta}$. (This can of course be rewritten as a set of quasi-linear equations for the $g_{\alpha\beta}$'s by algebraic manipulations.)

Standard theory of hyperbolic PDEs [151] gives:[13]

Theorem 4.1. *For any initial data*

$$g^{\alpha\beta}(0, y^i) \in H^{k+1}_{loc} \ , \quad \partial_0 g^{\alpha\beta}(0, y^i) \in H^k_{loc} \ , \qquad k > n/2 \ , \tag{4.9}$$

prescribed on an open subset $\mathscr{O} \subset \{0\} \times \mathbb{R}^n \subset \mathbb{R} \times \mathbb{R}^n$ there exists a unique solution $g^{\alpha\beta}$ of (4.8) *defined on an open neighborhood $\mathscr{U} \subset \mathbb{R} \times \mathbb{R}^n$ of $\mathscr{O}$. The set $\mathscr{U}$ can be chosen so that $(\mathscr{U}, g)$ is globally hyperbolic with Cauchy surface $\mathscr{O}$.*

Remark 4.2. The results in [216–218,312] and references therein allow one to reduce the differentiability threshold above.

[13] If k is an integer, then the Sobolev spaces H^k_{loc} are defined as spaces of functions which are in $L^2(K)$ for any compact set K,with their distributional derivatives up to order k also in $L^2(K)$. In the results presented here one can actually allow non-integer k's, the spaces H^k_{loc} are then defined rather similarly using the Fourier transformation.

Equation (4.8) would establish the *hyperbolic*, evolutionary character of the Einstein equations, if not for the following problem: Given initial data for an equation as in (4.8) there exists a unique solution, at least for some short time. But there is *a priori* no reason to expect that the solution will satisfy (4.6); if it does not, then a solution of (4.8) will *not* solve the Einstein equation. In fact, if we set

$$\lambda^\mu := \Box_g y^\mu \ , \tag{4.10}$$

then

$$R^{\alpha\beta} = \frac{1}{2}(\hat{E}^{\alpha\beta} - \nabla^\alpha\lambda^\beta - \nabla^\beta\lambda^\alpha) + \frac{2\Lambda}{n-1}g^{\alpha\beta} \ , \tag{4.11}$$

so that it is precisely the vanishing – or not – of λ which decides whether or not a solution of (4.8) is a solution of the vacuum Einstein equations.

This problem has been solved by Choquet-Bruhat [156]. The key observation is that (4.11) and the Bianchi identity imply a wave equation for the λ^α's. In order to see that, recall the twice-contracted Bianchi identity (2.10):

$$\nabla_\alpha\Big(R^{\alpha\beta} - \frac{R}{2}g^{\alpha\beta}\Big) = 0 \ .$$

Assuming that (4.8) holds, one finds

$$\begin{aligned} 0 &= -\nabla_\alpha\Big(\nabla^\alpha\lambda^\beta + \nabla^\beta\lambda^\alpha - \nabla_\gamma\lambda^\gamma g^{\alpha\beta}\Big) \\ &= -\Big(\Box_g\lambda^\beta + R^\beta{}_\alpha\lambda^\alpha\Big) \ . \end{aligned}$$

This shows that λ^α necessarily satisfies the second order hyperbolic system of equations

$$\Box_g\lambda^\beta + R^\beta{}_\alpha\lambda^\alpha = 0 \ .$$

Now, it is a standard fact in the theory of hyperbolic equations that we will have

$$\lambda^\alpha \equiv 0$$

on the domain of dependence $\mathscr{D}(\mathscr{O})$, provided that both λ^α and its derivatives vanish at $\mathscr{O}$. To see how these initial conditions on λ^α can be ensured, it is convenient to assume that y^0 is the coordinate along the $\mathbb{R}$ factor of $\mathbb{R}\times\mathbb{R}^n$, so that the initial data surface $\{0\}\times\mathscr{O}$ is given by the equation $y^0 = 0$. We have

$$\begin{aligned} \Box_g y^\alpha &= \frac{1}{\sqrt{|\det g|}}\partial_\beta\Big(\sqrt{|\det g|}g^{\beta\gamma}\partial_\gamma y^\alpha\Big) \\ &= \frac{1}{\sqrt{|\det g|}}\partial_\beta\Big(\sqrt{|\det g|}g^{\beta\alpha}\Big) \ . \end{aligned}$$

Clearly a necessary condition for the vanishing of $\Box_g y^\alpha$ is that it vanishes at $y^0 = 0$, and this allows us to calculate some time derivatives of the metric in terms of space ones:

$$\partial_0\Big(\sqrt{|\det g|}g^{0\alpha}\Big) = -\partial_i\Big(\sqrt{|\det g|}g^{i\alpha}\Big) \ . \tag{4.12}$$

This implies that the initial data (4.9) for the equation (4.8) cannot be chosen arbitrarily if we want both (4.8) and the Einstein equation to be simultaneously satisfied.

Now, there is still freedom left in choosing the wave coordinates. Using this freedom, one can show that there is *no loss of generality* in assuming that on the initial hypersurface $\{y^0 = 0\}$ we have

$$g^{00} = -1 \ , \qquad g^{0i} = 0 \ , \tag{4.13}$$

and this choice simplifies the algebra considerably. Equation (4.12) determines the time derivatives $\partial_0 g^{0\mu}|_{\{y^0=0\}}$ needed in Theorem 4.1, once $g_{ij}|_{\{y^0=0\}}$ and $\partial_0 g_{ij}|_{\{y^0=0\}}$ are given. So, from this point of view, the essential initial data for the evolution problem are the space metric

$$h := g_{ij} dy^i dy^j \ ,$$

together with its time derivatives.

It turns out that further constraints arise from the requirement of the vanishing of the derivatives of λ. Supposing that (4.12) holds at $y^0 = 0$ — equivalently, supposing that λ vanishes on $\{y^0 = 0\}$, we then have

$$\partial_i \lambda^\alpha = 0$$

on $\{y^0 = 0\}$. To obtain the vanishing of all derivatives initially it remains to ensure that some transverse derivative does. A convenient transverse direction is provided by the field n of unit timelike normals to $\{y^0 = 0\}$, and the vanishing of $\nabla_n \lambda^\alpha$ is guaranteed by requiring that

$$\Big(G_{\mu\nu} + \Lambda g_{\mu\nu}\Big) n^\mu = 0 \ . \tag{4.14}$$

This follows by simple algebra from the equation $\hat{E}_{\alpha\beta} = 0$ and (4.11),

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -\Big(\nabla_\mu \lambda_\nu + \nabla_\nu \lambda_\mu - \nabla^\alpha \lambda_\alpha g_{\mu\nu}\Big) \ ,$$

using that $\lambda_\mu|_{y^0=0} = \partial_i \lambda_\mu|_{y^0=0} = 0$.

Equations (4.14) are called the *Einstein constraint equations*, and will be discussed in detail in Section 5.

Summarizing, we have proved:

Theorem 4.3. *Under the hypotheses of Theorem 4.1, suppose that the initial data* (4.9) *satisfy* (4.12), (4.13) *as well as the constraint equations* (4.14). *Then the metric given by Theorem 4.1 satisfies the vacuum Einstein equations.*

4.2. **Cauchy data.** In Theorem 4.1 we consider initial data given in a single coordinate patch $\mathscr{O} \subset \mathbb{R}^n$. This suffices for applications such as the Lindblad-Rodnianski stability theorem discussed in Section 6.5 below, where $\mathscr{O} = \mathbb{R}^n$. But a correct geometric picture is to start with an n–dimensional hypersurface M, and prescribe initial data there; the case where M is $\mathscr{O}$ is thus a special case of this construction. At this stage there are two attitudes one may wish to adopt: the first is that M is a subset of the space-time $\mathscr{M}$ — this is essentially what we assumed in Section 4.1. The alternative is to consider M as a manifold of its own, equipped with an embedding

$$i : M \to \mathscr{M} \ .$$

The most convenient approach is to go back and forth between those points of view, and this is the strategy that we will follow.

A *vacuum initial data set* (M, h, K) is a triple where M is an n–dimensional manifold, h is a Riemannian metric on M, and K is a symmetric two-covariant tensor field on M. Further (h, K) are supposed to satisfy the vacuum constraint

equations that result from (4.14), and which are written explicitly in terms of K and h in Section 5.1. Here the tensor field K will eventually become the second fundamental form of M in the resulting space-time $\mathscr{M}$, obtained by evolving the initial data. Recall that the second fundamental form of a spacelike hypersurface M is defined as

$$\forall X \in TM \qquad K(X,Y) = g(\nabla_X n, Y) \ , \tag{4.15}$$

where n is the future pointing unit normal to M, and K is often referred to as the *extrinsic curvature tensor of* M in the relativity literature. Specifying K is equivalent to prescribing the time-derivatives of the space-part g_{ij} of the resulting space-time metric g; this can be seen as follows: Suppose, indeed, that a space-time $(\mathscr{M}, g)$ has been constructed (not necessarily vacuum) such that K is the extrinsic curvature tensor of M in $(\mathscr{M}, g)$. Consider any domain of coordinates $\mathscr{O} \subset M$, and construct coordinates y^μ in a space-time neighborhood $\mathscr{U}$ such that $M \cap \mathscr{U} = \mathscr{O}$; those coordinates could be wave coordinates, obtained by solving the wave equations (4.6), but this is not necessary at this stage. Since y^0 is constant on M the one-form dy^0 annihilates $TM \subset T\mathscr{M}$, as does the 1–form $g(n,\cdot)$. Since M has codimension one, it follows that dy^0 must be proportional to $g(n,\cdot)$:

$$n_\alpha dy^\alpha = n_0 dy^0$$

on $\mathscr{O}$. The normalization $-1 = g(n,n) = g^{\mu\nu} n_\mu n_\nu = g^{00}(n_0)^2$ gives

$$n_\alpha dy^\alpha = \frac{1}{\sqrt{|g^{00}|}} dy^0 \ .$$

We then have, by (4.15),

$$K_{ij} \quad = \quad -\frac{1}{2} g^{0\sigma} \Big(\partial_j g_{\sigma i} + \partial_i g_{\sigma j} - \partial_\sigma g_{ij} \Big) n_0 \ . \tag{4.16}$$

This shows that the knowledge of $g_{\mu\nu}$ *and* $\partial_0 g_{ij}$ at $\{y^0 = 0\}$ allows one to calculate K_{ij}. Reciprocally, (4.16) can be rewritten as

$$\partial_0 g_{ij} = \frac{2}{g^{00} n_0} K_{ij} + \text{ terms determined by the } g_{\mu\nu}\text{'s and their space–derivatives} \ ,$$

so that the knowledge of the $g_{\mu\nu}$'s and of the K_{ij}'s at $y^0 = 0$ allows one to calculate $\partial_0 g_{ij}$. Thus, K_{ij} is the geometric counterpart of the $\partial_0 g_{ij}$'s.

4.3. **Solutions global in space.** In order to globalize the existence Theorem 4.1 *in space*, the key point is to show that two solutions differing only by the values $g_{0\alpha}|_{\{y^0=0\}}$ are (locally) isometric: so suppose that g and $\tilde{g}$ both solve the vacuum Einstein equations in a globally hyperbolic region $\mathscr{U}$, with the same Cauchy data (h, K) on $\mathscr{O} := \mathscr{U} \cap M$. One can then introduce wave coordinates in a globally hyperbolic neighborhood of $\mathscr{O}$ both for g and $\tilde{g}$, satisfying (4.13), by solving

$$\Box_g y^\mu = 0 \ , \qquad \Box_{\tilde{g}} \tilde{y}^\mu = 0 \ , \tag{4.17}$$

with the same initial data for y^μ and $\tilde{y}^\mu$. Transforming both metrics to their respective wave-coordinates, one obtains two solutions of the reduced equation (4.8) with the same initial data.

The question then arises whether the resulting metrics will be sufficiently differentiable to apply the uniqueness part of Theorem 4.1. Now, the metrics obtained so far are in a space $C^1([0,T], H^s)$, where the Sobolev space H^s involves the space-derivatives of the metric. The initial data for the solutions y^μ or $\tilde{y}^\mu$ of (4.17) may

be chosen to be in $H^{s+1} \times H^s$. However, a rough inspection of (4.17) shows that the resulting solutions will be only in $C^1([0,T], H^s)$, because of the low regularity of the metric. But then (4.5) implies that the transformed metrics will be in $C^1([0,T], H^{s-1})$, and uniqueness can only be invoked *provided that* $s-1 > n/2+1$, which is one degree of differentiability more than what was required for existence. This was the state of affairs for some fifty-five years until the following simple argument of Planchon and Rodnianski [278]: To make it clear that the functions y^μ are considered to be scalars in (4.17), we shall write y for y^μ. Commuting derivatives with $\Box_g$ one finds, for metrics satisfying the vacuum Einstein equations,

$$\Box_g \nabla_\alpha y = \nabla_\mu \nabla^\mu \nabla_\alpha y = [\nabla_\mu \nabla^\mu, \nabla_\alpha] y = \underbrace{R^{\sigma\mu}{}_{\alpha\mu}}_{=R^\sigma{}_\alpha=0} \nabla_\sigma y = 0 \ .$$

Commuting once more one obtains an evolution equation for the field $\psi_{\alpha\beta} := \nabla_\alpha \nabla_\beta y$:

$$\Box_g \psi_{\alpha\beta} + \underbrace{\nabla_\sigma R_\beta{}^\lambda{}_\alpha{}^\sigma}_{=0} \nabla_\lambda y + 2 R_\beta{}^\lambda{}_\alpha{}^\sigma \psi_{\sigma\lambda} = 0 \ ,$$

where the underbraced term vanishes, for vacuum metrics, by a contracted Bianchi identity. So the most offending term in this equation for $\psi_{\alpha\beta}$, involving three derivatives of the metric, disappears when the metric is vacuum. Standard theory of hyperbolic PDEs shows now that the functions $\nabla_\alpha \nabla_\beta y$ are in $C^1([0,T], H^{s-1})$, hence $y \in C^1([0,T], H^{s+1})$, and the transformed metrics are regular enough to invoke uniqueness without having to increase s.

Suppose, now, that an initial data set (M, h, K) as in Theorem 4.1 is given. Covering M by coordinate neighborhoods $\mathscr{O}_p$, $p \in M$, one can use Theorem 4.1 to construct globally hyperbolic developments $(\mathscr{U}_p, g_p)$ of $(\mathscr{U}_p, h, K)$. By the argument just given the metrics so obtained will coincide, after performing a suitable coordinate transformation, wherever simultaneously defined. This allows one to patch the $(\mathscr{U}_p, g_p)$'s together to a globally hyperbolic Lorentzian manifold, with Cauchy surface M. Thus:

Theorem 4.4. *Any vacuum initial data set (M, h, K) of differentiability class $H^{s+1} \times H^s$, $s > n/2$, admits a globally hyperbolic development.*

The solutions are locally unique, in a sense made clear by the proof. The important question of *uniqueness in the large* will be addressed in Section 6.2.

4.4. **Other hyperbolic reductions.** The wave-coordinates approach of Choquet-Bruhat, presented above, is the first hyperbolic reduction discovered for the Einstein equations. It has been given new life by the Lindblad-Rodnianski stability theorem, presented in Section 6.5.2 below. However, one should keep in mind the existence of several other such reductions.

An example is given by the symmetric-hyperbolic first order system of Baumgarte, Shapiro, Shibata and Nakamura [26, 295, 310], known as the BSSN system, widely used in numerical general relativity. Another noteworthy example is the elliptic-hyperbolic system of [8], in which the elliptic character of some of the equations provides increased control of the solution. A notorious problem in numerical simulations is the lack of constraint preservation, see [186, 273] and references therein for attempts to improve the situation. The reader is referred to [158, 162] for a review of many other possibilities.

4.5. **The characteristic Cauchy problem.** Another important systematic construction of solutions of the vacuum Einstein equations proceeds via a *characteristic Cauchy problem*. In this case the initial data are prescribed on Cauchy hypersurfaces which are allowed to be piecewise null. This problem has been considerably less studied than the spacelike one described above. We will not go into any details here; see [46, 61, 62, 93, 143, 282] for further information.

4.6. **Initial-boundary value problems.** Numerical simulations necessarily take place on a finite grid, which leads to the need of considering initial-boundary value problems. In general relativity those are considerably more complicated than the Cauchy problem, and much remains to be understood. In pioneering work, Friedrich and Nagy [160] constructed a system of equations, equivalent to Einstein's, for a set of fields that includes some components of the Weyl tensor, and proved well-posedness of an initial-boundary value problem for those equations. It would seem that the recent work by Kreiss *et al.* [220] might lead to a simpler formulation of the problem at hand.

5. Initial data sets

We now turn our attention to an analysis of the constraint equations, returning to the evolution problem in Section 6.

An essential part of the mathematical analysis of the Einstein field equations of general relativity is the rigorous formulation of the Cauchy problem, which is a means to describe solutions of a dynamical theory via the specification of initial data and the evolution of that data. In this section we will be mainly concerned with the initial data sets for the Cauchy problem. As explained in Section **??**, those initial data sets have to satisfy the relativistic constraint equations (4.14). This leads to the following questions: What are the sets of allowable initial data? Is it possible to parameterize them in a useful way? What global properties of the space-time can be seen in the initial data sets? How does one engineer initial data so that the associated space-time has some specific properties?

5.1. **The constraint equations.** As explained in Section 4.2, an initial data set for a vacuum space-time consists of an n-dimensional manifold M together with a Riemannian metric h and a symmetric tensor K. In the non-vacuum case we also have a collection of non-gravitational fields which we collectively label $\mathcal{F}$ (usually these are sections of a bundle over M). We have already seen the relativistic vacuum constraint equations expressed as the vanishing of the normal components of the Einstein equations (4.14). Now, if h is the metric induced on a spacelike hypersurface in a Lorentzian manifold, it has its own curvature tensor ${R^i}_{jk\ell}$. If we denote by K_{ij} the second fundamental form of M in $\mathscr{M}$, and by ${\mathcal{R}^i}_{jk\ell}$ the space-time curvature tensor, the Gauss-Codazzi equations provide the following relationships:

$$ {R^i}_{jk\ell} = {\mathcal{R}^i}_{jk\ell} + {K^i}_\ell K_{jk} - {K^i}_k K_{j\ell} \ , \tag{5.1} $$

$$ D_i K_{jk} - D_j K_{ik} = \mathcal{R}_{ijk\mu} n^\mu \ . \tag{5.2} $$

Here n is the timelike normal to the hypersurface, and we are using a coordinate system in which the ∂_i's are tangent to the hypersurface M.

Contractions of (5.1)-(5.2) and simple algebra allow one to reexpress (4.14) in the following form, where we have now allowed for the additional presence of non-gravitational fields:

$$\mathrm{div} K - d(\mathrm{tr} K) = J \ , \tag{5.3}$$

$$R(h) - 2\Lambda - |K|_h^2 + (\mathrm{tr} K)^2 = 2\rho \ , \tag{5.4}$$

$$\mathcal{C}(\mathcal{F}, h) = 0 \ , \tag{5.5}$$

where $R(h)$ is the scalar curvature of the metric h, J is the momentum density of the non-gravitational fields, ρ is the energy density,[14] and $\mathcal{C}(\mathcal{F}, h)$ denotes the set of additional constraints that might come from the non-gravitational part of the theory. The first of these equations is known as the *momentum constraint* and is a vector field equation on M. The second, a scalar equation, is referred to as the *scalar*, or *Hamiltonian, constraint*, while the last are collectively labeled the non-gravitational constraints. These are what we shall henceforth call the *Einstein constraint equations*, or simply the *constraint equations* if ambiguities are unlikely to occur.

As an example, for the Einstein-Maxwell theory in 3+1 dimensions, the non-gravitational fields consist of the electric and magnetic vector fields E and B. In this case we have $\rho = \frac{1}{2}(|E|_h^2 + |B|_h^2)$, $J = (E \times B)_h$, and we have the extra (non-gravitational) constraints $\mathrm{div}_h E = 0$ and $\mathrm{div}_h B = 0$.

Equations (5.3)-(5.5) form an underdetermined system of partial differential equations. In the classical vacuum setting of $n = 3$ dimensions, these are locally four equations for the twelve unknowns given by the components of the symmetric tensors h and K. This section will focus primarily on the vacuum case with a zero cosmological constant. However, we will allow arbitrary values of Λ in Section 5.3.5.

The most successful approach so far for studying the existence and uniqueness of solutions to (5.3)-(5.5) is through the conformal method of Lichnerowicz [228], Choquet-Bruhat and York [87]. The idea is to introduce a set of unconstrained "conformal data", which are freely chosen, and find (h, K) by solving a system of determined partial differential equations. In the vacuum case with vanishing cosmological constant [87], the free conformal data consist of a manifold M, a Riemannian metric $\tilde{h}$ on M, a trace-free symmetric tensor $\tilde{\sigma}$, and the *mean curvature function* τ. The initial data (h, K) defined as

$$h = \phi^q \tilde{h} \ , \quad q = \frac{4}{n-2} \ , \tag{5.6}$$

$$K = \phi^{-2}(\tilde{\sigma} + \widetilde{\mathcal{D}} W) + \frac{\tau}{n}\phi^q h \ , \tag{5.7}$$

where ϕ is positive, will then solve (5.3)-(5.4) if and only if the function ϕ and the vector field W solve the equations

$$\mathrm{div}_{\tilde{h}}(\widetilde{\mathcal{D}} W + \tilde{\sigma}) = \frac{n-1}{n}\phi^{q+2}\tilde{D}\tau \ , \tag{5.8}$$

$$\Delta_{\tilde{h}}\,\phi - \frac{1}{q(n-1)}R(\tilde{h})\phi + \frac{1}{q(n-1)}|\tilde{\sigma} + \widetilde{\mathcal{D}} W|_{\tilde{h}}^2 \phi^{-q-3} - \frac{1}{qn}\tau^2\phi^{q+1} = 0 \ . \tag{5.9}$$

[14] If $\mathcal{T}$ is the stress-energy tensor of the non-gravitational fields, and n denotes the unit timelike normal to a hypersurface M embedded in a space-time, with induced data $(M, h, K, \mathcal{F})$, then $J = -\mathcal{T}(n, \cdot)$ and $\rho = \mathcal{T}(n, n)$. However, in terms of the initial data set itself we shall regard (5.3)-(5.4) as the definitions of the quantities J and ρ.

We use the symbol $\tilde{D}$ to denote the covariant derivative of $\tilde{h}$; $\widetilde{\mathcal{D}}$ is the *conformal Killing operator*:

$$\widetilde{\mathcal{D}}W_{ab} = \tilde{D}_a W_b + \tilde{D}_b W_a - \frac{2}{n}\tilde{h}_{ab}\tilde{D}_c W^c \ . \tag{5.10}$$

Vector fields W annihilated by $\mathcal{D}$ are called *conformal Killing vector fields*, and are characterized by the fact that they generate (perhaps local) conformal diffeomorphisms of (M, h). The semi-linear scalar equation (5.9) is often referred to as the *Lichnerowicz equation*.

Equations (5.8)-(5.9) form a determined system of equations for the $(n+1)$ functions (ϕ, W). The operator $\mathrm{div}_{\tilde{h}}(\widetilde{\mathcal{D}}\,\cdot)$ is a linear, formally self-adjoint, elliptic operator on vector fields. What makes the study of the system (5.8)-(5.9) difficult in general is the nonlinear coupling between the two equations.

The explicit choice of (5.6)-(5.7) is motivated by the two identities (for $\tilde{h} = \phi^q h$)

$$R(\tilde{h}) = -\phi^{-q-1}(q(n-1)\Delta_h\phi - R(h)\phi) \ , \tag{5.11}$$

where $q = \frac{4}{n-2}$, which is the unique exponent that does not lead to supplementary $|D\phi|^2$ terms in (5.11), and

$$D^a_{\tilde{h}}(\phi^{-2}B_{ab}) = \phi^{-q-2}D^a_h B_{ab} \tag{5.12}$$

which holds for any trace-free tensor B. Equation (5.11) is the well known identity relating the scalar curvatures of two conformally related metrics.

In the space-time evolution $(\mathscr{M}, g)$ of the initial data set (M, h, K), the function $\tau = \mathrm{tr}_h K$ is the mean curvature of the hypersurface $M \subset \mathscr{M}$. The assumption that the mean curvature function τ is constant on M significantly simplifies the analysis of the vacuum constraint equations because it decouples equations (5.8) and (5.9). One can then attempt to solve (5.8) for W, and then solve the Lichnerowicz equation (5.9).

Existence and uniqueness of solutions of this problem for constant mean curvature ("CMC") data has been studied extensively. For compact manifolds this was exhaustively analysed by Isenberg [194], building upon a large amount of previous work [87,228,264,331]; the proof was simplified by Maxwell in [242]. If we let $\mathcal{Y}([h])$ denote the Yamabe invariant of the conformal class $[h]$ of metrics determined by h (see [222]), the result reads as follows:

Theorem 5.1 ([194]). *Consider a smooth conformal initial data set $(\tilde{h}, \tilde{\sigma}, \tau)$ on a compact manifold M, with constant τ. Then there always exists a solution W of (5.8). Setting $\sigma = \widetilde{\mathcal{D}}W + \tilde{\sigma}$, the existence, or not, of a positive solution ϕ of the Lichnerowicz equation is shown in Table 5.1.*

	$\sigma \equiv 0, \tau = 0$	$\sigma \equiv 0, \tau \neq 0$	$\sigma \not\equiv 0, \tau = 0$	$\sigma \not\equiv 0, \tau \neq 0$
$\mathcal{Y}([h]) < 0$	*No*	*Yes*	*No*	*Yes*
$\mathcal{Y}([h]) = 0$	*Yes*	*No*	*No*	*Yes*
$\mathcal{Y}([h]) > 0$	*No*	*No*	*Yes*	*Yes*

TABLE 5.1. Existence of solutions in the conformal method for CMC data on compact manifolds.

More recently, work has been done on analysing these equations for metrics of low differentiability [75,242]; this was motivated in part by recent work on the evolution problem for "rough initial data" [216–218,312]. Exterior boundary value problems for the constraint equations, with nonlinear boundary conditions motivated by black holes, were considered in [137,243].

The conformal method easily extends to CMC constraint equations for some non-vacuum initial data, e.g. the Einstein-Maxwell system [194] where one obtains results very similar to those of Theorem 5.1. However, other important examples, such as the Einstein-scalar field system [82–84,180], require more effort and are not as fully understood.

Conformal data close to being CMC (e.g. via a smallness assumption on $|\nabla\tau|$) are usually referred to as "near-CMC". Classes of near-CMC conformal data solutions have been constructed [1,81,199,200] and there is at least one example of a non-existence theorem [203] for a class of near-CMC conformal data. However, due to the non-linear coupling in the system (5.8)-(5.9), the question of existence for unrestricted choices of the mean curvature τ appears to be significantly more difficult, and until recently all results assumed strong restrictions on the gradient of τ. The first general result in this context is due to Holst, Nagy, and Tsogtgerel [187,188], who construct solutions with freely specified mean curvature in the presence of matter. In [241], Maxwell provides a sufficient condition, with no restrictions on the mean curvature, for the conformal method to generate solutions to the vacuum constraint equations on compact manifolds. As an application, Maxwell demonstrates the existence of a large class of solutions to the *vacuum* constraint equations with freely specified mean curvature. These results together represent a significant advance in our understanding of how the conformal method may be used to generate solutions of the vacuum constraint equations. However the existence question for generic classes of large conformal data remains wide open.

The analysis of the conformal constraint equations (5.8)-(5.9) discussed above proceeds either via the method of sub- and super-solutions (which is a barrier argument exploiting the maximum principle), or a perturbation or fixed point method. In [180] Hebey, Pacard and Pollack used the mountain pass lemma to analyse Lichnerowicz-type equations arising in certain cases of the Einstein-scalar field system. Such arguments may conceivably prove useful in studying (5.8)-(5.9) for general τ's.

A natural question is whether the set of solutions to the constraint equations forms a manifold. This was first considered by Fisher and Marsden [155], who provided a Fréchet manifold structure; Banach manifold structures have been obtained in [112], and a Hilbert manifold structure (for asymptotically flat initial data sets) in [23].

In [25] the reader will find a presentation of alternative approaches to constructing solutions of the constraints, covering work done up to 2003.

5.1.1. *The constraint equations on asymptotically flat manifolds.* Recall that asymptotically flat initial data sets have been defined in Section 3.6. There is a large number of well-established results concerning the existence of CMC and "near CMC" solutions of the Einstein constraint equations on asymptotically flat manifolds [68, 77, 78, 85, 87, 243, 244]. We mention here only that there is a precise

conformally invariant criterion sufficient to prove the existence of a positive solution to the Lichnerowicz equation relative to a given set of asymptotically flat conformal data [68, 243][15].

5.2. **Mass inequalities.** Among the deepest results in mathematical general relativity are the global mass inequalities for asymptotically flat manifolds. Those have been discussed extensively in the existing literature [51, 52, 55, 222, 296, 298], and therefore will only be given the minimum amount of attention, as needed for the remaining purposes of this work.

5.2.1. *The Positive Mass Theorem.* Using the coordinate system of (3.21), one defines the *Arnowitt-Deser-Misner [11] mass* of (M, h) of an asymptotically flat end as

$$m = \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} \sum_{i,j} \left(\frac{\partial h_{ij}}{\partial x^i} - \frac{\partial h_{ii}}{\partial x^j} \right) dS_i \ . \tag{5.13}$$

Here S_r is the coordinate sphere at radius r and $dS_i = \partial_i \rfloor d\mu$, and $d\mu$ is the Riemannian volume form of h. The factor 16π is a matter of convention and is natural in space-dimension three. The integral converges to a finite, coordinate-independent limit if, for some $\alpha > \frac{n-2}{2}$,

$$|h_{ij} - \delta_{ij}| \le cr^{-\alpha}, \qquad |\partial h| \le cr^{-\alpha-1} \qquad \text{and} \qquad R(h) \in L^1(M_{ext}), \tag{5.14}$$

with those conditions being essentially optimal [19, 99].

For time-symmetric initial data the vacuum constraint equations (5.3)-(5.4) reduce to the condition that the metric h be scalar-flat, i.e. $R(h) = 0$. On the other hand, if one considers time-symmetric data for a non-vacuum space-time, then from (5.4) we see that the scalar curvature is twice the energy density of the matter fields. The non-negativity of R, assuming a vanishing cosmological constant Λ, is then a consequence of the dominant energy condition for initial data (which follows from (3.15)),

$$\rho \ge |J|_h \ , \tag{5.15}$$

where ρ and J are defined in (5.3)-(5.4); see also Section 7.5. One checks that the dominant energy condition (3.15) holds on $\mathscr{M}$ if and only if (5.15) holds relative to each spacelike hypersurface in $\mathscr{M}$.

Now, the ADM mass is thought to represent the total mass of the system as viewed on M, which contains contributions from the matter fields, the gravitational field, as well as their binding energy. The long-standing question of its positivity was resolved by Schoen and Yau [300] in dimension three, and is now known as the Positive Mass Theorem:

Theorem 5.2. *Let (M, h) be an asymptotically flat Riemannian manifold with nonnegative scalar curvature. Suppose that either M is spin, or the dimension $n \le 7$, or that h is conformally flat. Then the total ADM mass m satisfies $m \ge 0$, with equality if and only if (M, h) is isometric to Euclidean space $(\mathbb{R}^n, \delta)$.*

[15] Maxwell [243] uncovered an error in Cantor's definition of the invariant [68] and provided the correct definition.

As remarked in the introduction, this theorem, and its generalizations, stands as one of the cornerstones of mathematical relativity. Accessible introductions to the positive mass theorem may be found in [52, 106, 222, 296]. The restriction on the dimension arises from the use of area minimizing hypersurfaces [296], which are known to sometimes possess singularities in higher dimensions, and it is expected that positivity is true in all dimensions. The Positive Mass Theorem was proven in all dimensions for conformally flat manifolds by Schoen and Yau [305] by a different argument, and in all dimensions for spin manifolds by Witten [329] (see also [19, 74, 272]). The result was generalized in [302, 303] (compare [298]) to asymptotically flat initial data sets $(M, h, K, \mathcal{F})$ satisfying the dominant energy condition (5.15).

5.2.2. *Riemannian Penrose Inequality.* An important generalization of the Positive Mass Theorem is given by the Riemannian Penrose Inequality.

Theorem 5.3. *Let (M, h) be a complete, smooth, asymptotically flat 3-manifold with nonnegative scalar curvature with total mass m and which has an outermost minimal surface Σ_0 of area A_0. Then*

$$m \geq \sqrt{\frac{A_0}{16\pi}}\,, \tag{5.16}$$

with equality if and only if (M, h) is isometric to the Schwarzschild metric $(\mathbb{R}^3 \setminus \{0\}, (1 + \frac{m}{2|x|})^4 \delta)$ outside their respective outermost minimal surfaces.

This theorem was proven in this form by Bray [49] and by Huisken and Ilmanen [191] when we instead take A_0 to be the area of the largest connected component of Σ_0. These results are a beautiful application of geometric flows to a basic problem in relativity and certainly constitute a very significant advance for the field. As such there have already been a number of accessible reviews written on the subject to which we refer the interested reader [50–52, 55]. A generalization of Theorem 5.3 to dimensions $n \leq 7$ has been established in [53].

5.2.3. *Quasi-local mass.* In the context of asymptotically flat space-times, there are well defined global notions of mass and energy, and these are central to the celebrated positive mass theorem discussed in Section 5.2.1. One would, however, like to have a well-defined useful *local* notion of mass or energy, with natural properties – e.g., monotonicity – that one has in other physical theories. Such a definition has been elusive despite a great deal of effort by many people and this remains an important open problem. We refer the reader to the Living Reviews article by Szabados [314] for a survey, and note that there have been interesting recent mathematical developments in the area [54, 309, 323, 324], not described in the currently available version of [314].

5.3. **Applications of gluing techniques.** Over the past 25 years, "gluing techniques" have become a standard tool in geometric analysis. Since the construction by Taubes of self-dual Yang-Mills connections on four-manifolds [317], which played a crucial role in Donaldson's construction of exotic smooth structures in four-dimensions [142], gluing has been applied in important ways across a very wide range of areas. What gluing typically refers to is a construction in which solutions of a nonlinear partial differential equation or system, which correspond to some geometric quantity of interest, e.g. self-dual connections, are fused together to create new solutions. This is done by a mix of geometry and analysis in which

one ultimately studies the linearization of the relevant PDEs, and in most cases one has to overcome analytic degeneracy introduced in the gluing procedure. Thus, from an analytic point of view, gluing should be regarded as a singular perturbation method. Part of the usefulness of the technique lies in the fact that, away from the small set about which one fuses the two solutions, the new solution is very close to the original ones. The fact that the original solutions are *usually* not exactly preserved is a reflection of the fact that the relevant equations satisfy a unique continuation property: any two solutions which agree on an open set must agree everywhere. This is a well known property for, say, a scalar semi-linear elliptic equation.

5.3.1. *The linearized constraint equations and KIDs.* The starting point of gluing constructions for the constraint equations is the linearization of these equations about a given solution (M, h, K). We let $\mathcal{P}^*_{(h,K)}$ denote the L^2 adjoint of the linearization of the constraint equations at this solution. Viewed as an operator acting on a scalar function N and a vector field Y, $\mathcal{P}^*_{(h,K)}$ takes the explicit form [111]

$$\mathcal{P}^*_{(h,K)}(N,Y) = \begin{pmatrix} 2(\nabla_{(i}Y_{j)} - \nabla^l Y_l g_{ij} - K_{ij}N + \mathrm{tr}K\ N g_{ij}) \\ \\ \nabla^l Y_l K_{ij} - 2K^l{}_{(i}\nabla_{j)}Y_l + K^q{}_l \nabla_q Y^l g_{ij} \\ -\Delta N g_{ij} + \nabla_i\nabla_j N + (\nabla^p K_{lp} g_{ij} - \nabla_l K_{ij})Y^l \\ -N\mathrm{Ric}\,(g)_{ij} + 2NK^l{}_i K_{jl} - 2N(\mathrm{tr}\ K)K_{ij} \end{pmatrix} . \tag{5.17}$$

Now this operator does not, on first inspection, appear to be very "user friendly". However, our immediate concern is solely with its kernel, and the pairs (N, Y) which lie in its kernel have a very straightforward geometric and physical characterization. In particular, let Ω be an open subset of M. By definition, the set of "KIDs" on Ω, denoted $\mathscr{K}(\Omega)$, is the set of all solutions of the equation

$$\mathcal{P}^*_{(\gamma,K)|_\Omega}(N,Y) = 0\ . \tag{5.18}$$

Such a solution (N, Y), if nontrivial, generates a space-time Killing vector field in the domain of dependence of $(\Omega, h|_\Omega, K|_\Omega)$ [250]; compare Section 3.6.

From a geometric point of view one expects that solutions with symmetries should be rare. This was made rigorous in [32], where it is shown that the generic behaviour among solutions of the constraint equations is the absence of KIDs on any open set. On the other hand, one should note that essentially every explicit solution has symmetries. In particular, both the flat initial data for Minkowski space, and the initial data representing the constant time slices of Schwarzschild have KIDs.

5.3.2. *Corvino's result.* As we have already pointed out, the Einstein constraint equations form an underdetermined system of equations, and as such, it is unreasonable to expect that they (or their linearizations) should satisfy the unique continuation property. In 2000, Corvino established a gluing result for asymptotically flat metrics with zero scalar curvature which dramatically illustrated this point [129]. In the special case when one considers initial data with vanishing second fundamental form $K \equiv 0$, the momentum constraint equation (5.3) becomes trivial and the Hamiltonian constraint equation (5.3) reduces to simply $R(h) = 0$, i.e. a scalar flat metric. Such initial data sets are referred to as "time-symmetric" because the space-time obtained by evolving them possesses a time-reversing isometry which leaves the initial data surface fixed. Beyond Euclidean space itself, the constant

time slices of the Schwarzschild space-time form the most basic examples of asymptotically flat, scalar flat manifolds. One long-standing open problem [22,306] in the field had been whether there exist scalar flat metrics on $\mathbb{R}^n$ which are not globally spherically symmetric but which are spherically symmetric in a neighborhood of infinity and hence, by Birkhoff's theorem, Schwarzschild there.

Corvino resolved this by showing that he could deform any asymptotically flat, scalar flat metric to one which is exactly Schwarzschild outside of a compact set.

Theorem 5.4 ([129]). *Let (M, h) be a smooth Riemannian manifold with zero scalar curvature containing an asymptotically flat end $M_{ext} = \{|x| > r > 0\}$. Then there is a $R > r$ and a smooth metric $\bar{h}$ on M with zero scalar curvature such that $\bar{h}$ is equal to h in $M \setminus M_{ext}$ and $\bar{h}$ coincides on $\{|x| > R\}$ with the metric induced on a standard time-symmetric slice in the Schwarzschild solution. Moreover the mass of $\bar{h}$ can be made arbitrarily close to that of h by choosing R sufficiently large.*

Underlying this result is a gluing construction where the deformation has compact support. The ability to do this is a reflection of the underdetermined nature of the constraint equations. In this setting, since $K \equiv 0$, the operator takes a much simpler form, as a two-covariant tensor valued operator acting on a scalar function u by

$$\mathcal{P}^* u = -(\Delta_h u)h + \mathrm{Hess}_h u - u\,\mathrm{Ric}(h)\ .$$

An elementary illustration of how an underdetermined system can lead to compactly supported solutions is given by the construction of compactly supported transverse-traceless tensors on $\mathbb{R}^3$ in Appendix B of [130] (see also [28,138]).

An additional challenge in proving Theorem 5.4 is the presence of KIDs on the standard slice of the Schwarzschild solution. If the original metric had ADM mass $m(h)$, a naive guess could be that the best fitting Schwarzschild solution would be the one with precisely the same mass. However the mass, and the coordinates of the center of mass, are in one-to-one correspondence with obstructions arising from KIDs. To compensate for this co-kernel in the linearized problem, Corvino uses these ($n+1$ in dimension n) degrees of freedom as effective parameters in the geometric construction. The final solution can be chosen to have its ADM mass arbitrarily close to the initial one.

The method uncovered in Corvino's thesis has been applied and extended in a number of important ways. The "asymptotic simplicity" model for isolated gravitational systems proposed by Penrose [274] has been very influential. This model assumes existence of smooth conformal completions to study global properties of asymptotically flat space-times. The question of existence of such vacuum space-times was open until Chruściel and Delay [110], and subsequently Corvino [130], used this type of gluing construction to demonstrate the existence of infinite dimensional families of vacuum initial data sets which evolve to asymptotically simple space-times. The extension of the gluing method to non-time-symmetric data was done in [111,131]. This allowed for the construction of space-times which are exactly Kerr outside of a compact set, as well as showing that one can specify other types of useful asymptotic behavior.

5.3.3. *Conformal gluing.* In [196], Isenberg, Mazzeo and Pollack developed a gluing construction for initial data sets satisfying certain natural non-degeneracy assumptions. The perspective taken there was to work within the conformal method, and thereby establish a gluing theorem for solutions of the determined system of PDEs

given by (5.8) and (5.9). This was initially done only within the setting of constant mean curvature initial data sets and in dimension $n = 3$ (the method was extended to all higher dimensions in [195]). The construction of [196] allowed one to combine initial data sets by taking a connected sum of their underlying manifolds, to add wormholes (by performing codimension 3 surgery on the underlying, connected, 3-manifold) to a given initial data set, and to replace arbitrary small neighborhoods of points in an initial data set with asymptotically hyperbolic ends.

In [197] this gluing construction was extended to only require that the mean curvature be constant in a small neighborhood of the point about which one wanted to perform a connected sum. This extension enabled the authors to show that one can replace an arbitrary small neighborhood of a generic point in any initial data set with an asymptotically *flat* end. Since it is easy to see that CMC solutions of the vacuum constraint equations exist on any compact manifold [328], this leads to the following result which asserts that there are no topological obstructions to asymptotically flat solutions of the constraint equations.

Theorem 5.5 ([197]). *Let M be any closed n-dimensional manifold, and $p \in M$. Then $M \setminus \{p\}$ admits an asymptotically flat initial data set satisfying the vacuum constraint equations.*

5.3.4. *Initial data engineering.* The gluing constructions of [196] and [197] are performed using a determined elliptic system provided by the conformal method, which necessarily leads to a global deformation of the initial data set, small away from the gluing site. Now, the ability of the Corvino gluing technique to establish compactly supported deformations invited the question of whether these conformal gluings could be localized. This was answered in the affirmative in [111] for CMC initial data under the additional, generically satisfied [32], assumption that there are no KIDs in a neighborhood of the gluing site.

In [117, 118], this was substantially improved upon by combining the gluing construction of [196] together with the Corvino gluing technique of [110, 129], to obtain a localized gluing construction in which the only assumption is the absence of KIDs near points. For a given n-manifold M (which may or may not be connected) and two points $p_a \in M$, $a = 1, 2$, we let $\tilde{M}$ denote the manifold obtained by replacing small geodesic balls around these points by a neck $S^{n-1} \times I$. When M is connected this corresponds to performing codimension n surgery on the manifold. When the points p_a lie in different connected components of M, this corresponds to taking the connected sum of those components.

Theorem 5.6 ([117,118]). *Let (M, h, K) be a smooth vacuum initial data set, with M not necessarily connected, and consider two open sets $\Omega_a \subset M$, $a = 1, 2$, with compact closure and smooth boundary such that*

$$\text{the set of KIDs, } \mathscr{K}(\Omega_a), \text{ is trivial.}$$

Then for all $p_a \in \Omega_a$, $\epsilon > 0$, and $k \in \mathbb{N}$ there exists a smooth vacuum initial data set $(\tilde{M}, h(\epsilon), K(\epsilon))$ on the glued manifold $\tilde{M}$ such that $(h(\epsilon), K(\epsilon))$ is ϵ-close to (h, K) in a $C^k \times C^k$ topology away from $B(p_1, \epsilon) \cup B(p_2, \epsilon)$. Moreover $(h(\epsilon), K(\epsilon))$ coincides with (h, K) away from $\Omega_1 \cup \Omega_2$.

This result is sharp in the following sense: first note that, by the positive mass theorem, initial data for Minkowski space-time cannot locally be glued to anything which is non-singular and vacuum. This meshes with the fact that for Minkowskian

initial data, we have $\mathscr{K}(\Omega) \neq \{0\}$ for any open set Ω. Next, recall that by the results in [32], the no-KID hypothesis in Theorem 5.6 is generically satisfied. Thus, the result can be interpreted as the statement that for generic vacuum initial data sets the local gluing can be performed around arbitrarily chosen points p_a. In particular the collection of initial data with generic regions Ω_a satisfying the hypotheses of Theorem 5.6 is not empty.

The proof of Theorem 5.6 is a mixture of gluing techniques developed in [195,196] and those of [111,129,131]. In fact, the proof proceeds initially via a generalization of the analysis in [196] to compact manifolds with boundary. In order to have CMC initial data near the gluing points, which the analysis based on [196] requires, one makes use of the work of Bartnik [20] on the plateau problem for prescribed mean curvature spacelike hypersurfaces in a Lorentzian manifold.

An interesting corollary of the above concerns the question of the existence of CMC slices in space-times with compact Cauchy surfaces. In [21], Bartnik showed that there exist maximally extended, globally hyperbolic solutions of the Einstein equations *with dust* which admit no CMC slices. Later, Eardley and Witt (unpublished) proposed a scheme for showing that similar vacuum solutions exist, but their argument was incomplete. It turns out that these ideas can be implemented using Theorem 5.6, which leads to:

Corollary 5.7. *[117, 118] There exist maximal globally hyperbolic vacuum space-times with compact Cauchy surfaces which contain no compact, spacelike hypersurfaces with constant mean curvature.*

Compact Cauchy surfaces with constant mean curvature are useful objects, as the existence of one such surface gives rise to a unique foliation by such surfaces [56], and hence a canonical choice of time function (often referred to as CMC or York time). Foliations by CMC Cauchy surfaces have also been extensively used in numerical analysis to explore the nature of cosmological singularities. Thus the demonstration that there exist space-times with no such surfaces has a negative impact on such studies.

One natural question is the extent to which space-times with no CMC slices are common among solutions to the vacuum Einstein equations with a fixed spatial topology. It is expected that the examples constructed in [117,118] are not isolated. In general, there is a great deal of flexibility (in the way of free parameters) in the local gluing construction. This can be used to produce one parameter families of distinct sets of vacuum initial data which lead to space-times as in Corollary 5.7. What is less obvious is how to prove that all members of these families give rise to *distinct* maximally extended, globally hyperbolic vacuum space-times.

A deeper question is whether a sequence of space-times which admit constant mean curvature Cauchy surfaces may converge, in a strong topology, to one which admits no such Cauchy surface. (See [18,21,168] for general criteria leading to the existence of CMC Cauchy surfaces.)

5.3.5. *Non-zero cosmological constant.* Gluing constructions have also been carried out with a non-zero cosmological constant [113,121,122]. One aim is to construct space-times which coincide, in the asymptotic region, with the corresponding black hole models. In such space-times one has complete control of the geometry in the domain of dependence of the asymptotic region, described there by the Kottler metrics (3.33). For time-symmetric slices of these space-times, the constraint equations

reduce to the equation for constant scalar curvature $R = 2\Lambda$. Gluing constructions have been previously carried out in this context, especially in the case of $\Lambda > 0$, but in [113, 121, 122] the emphasis is on gluing with compact support, in the spirit of Corvino's thesis and its extensions already discussed.

The time-symmetric slices of the $\Lambda > 0$ Kottler space-times provide "Delaunay" metrics (see [122] and references therein), and the main result of [121, 122] is the construction of large families of metrics with exactly Delaunay ends. When $\Lambda < 0$ the focus is on asymptotically hyperbolic metrics with constant negative scalar curvature. With hindsight, within the family of Kottler metrics with $\Lambda \in \mathbb{R}$ (with $\Lambda = 0$ corresponding to the Schwarzschild metric), the gluing in the $\Lambda > 0$ setting is technically easiest, while that with $\Lambda < 0$ is the most difficult. This is due to the fact that for $\Lambda > 0$ one deals with one linearized operator with a one-dimensional kernel; in the case $\Lambda = 0$ the kernel is $(n+1)$–dimensional; while for $\Lambda < 0$ one needs to consider a one-parameter family of operators with $(n+1)$–dimensional kernels.

6. Evolution

In Section 4 we have seen that solutions of the vacuum Einstein equations can be constructed by solving a Cauchy problem. It is then of interest to inquire about the global properties of the resulting space-times. As an illustration of what can happen, we start with a short description of a solution of the vacuum Einstein equations with interesting dynamical properties, the Taub – Newman Unti Tamburino (Taub–NUT) metric.

6.1. **Taub–NUT space-times.** The Taub–NUT metrics provide interesting examples of pathological behavior: incomplete geodesics within compact sets, closed causal curves, inequivalent extensions of maximal globally hyperbolic regions (to be defined shortly) and inequivalent conformal boundary completions at infinity. For all these reasons they are a must in any introductory discussion of general relativity.

The Taub–NUT metrics [258,316] are solutions of the vacuum Einstein equations on space-time manifolds $\mathscr{M}_I$ of the form

$$\mathscr{M}_I := I \times S^3 \ ,$$

where I is an interval. They take the form [249]

$$-U^{-1}dt^2 + (2\ell)^2 U\sigma_1^2 + (t^2+\ell^2)(\sigma_2^2+\sigma_3^2)\,, \tag{6.1}$$

$$U(t) = -1 + \frac{2(mt+\ell^2)}{t^2+\ell^2}\,. \tag{6.2}$$

Here ℓ and m are real numbers with $\ell > 0$. Further, the one-forms σ_1, σ_2 and σ_3 form a basis for the set of left-invariant one-forms on $SU(2) \approx S^3$: If

$$i_{S^3} : S^3 \to \mathbb{R}^4$$

is the standard embedding of S^3 into $\mathbb{R}^4$, then one can take

$$\begin{aligned} \sigma_1 &= 2i^*_{S^3}(x\,dw - w\,dx + y\,dz - z\,dy)\ , \\ \sigma_2 &= 2i^*_{S^3}(z\,dx - x\,dz + y\,dw - w\,dy)\ , \\ \sigma_3 &= 2i^*_{S^3}(x\,dy - y\,dx + z\,dw - w\,dz)\ . \end{aligned} \tag{6.3}$$

The metric (6.1) is invariant under a left $SU(2)$–action, and a further $U(1)$ action consisting of right-rotations of σ_2 and σ_3 amongst themselves, so that the connected component of the identity of the group of isometries of g is $SU(2) \times U(1)$.

The function U always has two zeros,

$$U(t) = \frac{(t_+ - t)(t - t_-)}{t^2 + \ell^2} \ ,$$

where

$$t_\pm := m \pm \sqrt{m^2 + \ell^2} \ .$$

It follows that I has to be chosen so that $t_\pm \notin I$. The space-time $(\mathscr{M}_{(t_-,t_+)}, g)$ will be referred to as *the Taub space-time* [316]. It is not very difficult to show that the Taub space-times exhaust the collection of maximal globally hyperbolic vacuum space-times evolving from $SU(2) \times U(1)$–invariant Cauchy data on S^3.

The significance of these metrics to strong cosmic censorship, which we are about to discuss, stems from the following observation (see [107, 115, 249]):

Theorem 6.1. *Consider the Taub space-time $(\mathscr{M}_{(t_-,t_+)}, g)$ defined above. Then*

(1) *$(\mathscr{M}_{(t_-,t_+)}, g)$ is maximal globally hyperbolic.*
(2) *There exists an uncountable number of analytic, vacuum, simply connected, non-equivalent extensions of $(\mathscr{M}_{(t_-,t_+)}, g)$.*

6.2. **Strong cosmic censorship.** The *strong cosmic censorship (SCC) problem* concerns predictability: Indeed, a fundamental requirement of physically relevant equations is that solutions should be *uniquely determined by initial data*. So it is important to inquire about predictability in general relativity.

In other words, we would like to know whether or not the solutions provided by Theorem 4.4 are unique. Now, it is easy to see that there can be no uniqueness unless some restrictions on the development are imposed: consider for example $(-\infty, 1) \times \mathbb{R}^n$, $\mathbb{R} \times \mathbb{R}^n$ and $(\mathbb{R} \times \mathbb{R}^n) \setminus \{(1, \vec{0})\}$ equipped with the obvious flat metric. All three space-times contain the spacelike surface $(\{0\} \times \mathbb{R}^n, \delta, 0)$, where δ is the Euclidean metric on $\mathbb{R}^n$. The first two are globally hyperbolic developments of the given initial data, but the third is not, as it is not globally hyperbolic. And obviously these are not isometric: e.g,. the second is geodesically complete, while the other two are not. So to guarantee uniqueness some further conditions are needed.

The key existence *and uniqueness* theorem in this context is due to Choquet-Bruhat and Geroch [80] (compare [87, 102, 179]). Some terminology is needed: a space-time $(\mathscr{M}, g)$ is said to be a *development* of an initial data set (M, h, K) if there exists an embedding $i : M \to \mathscr{M}$ such that $i(M)$ is a Cauchy surface for $(\mathscr{M}, g)$, with

$$i^* g = h \ ,$$

and with K being the pull-back to M of the extrinsic curvature tensor (second fundamental form) of $i(M)$. We will say that a development $(\mathscr{M}, g)$ is *maximal globally hyperbolic* if the following implication holds: if $\psi : \mathscr{M} \to \mathscr{M}'$ is an isometric embedding of $\mathscr{M}$ into $(\mathscr{M}', g')$, and if $\psi(\mathscr{M}) \neq \mathscr{M}'$, then $\mathscr{M}'$ is *not* globally hyperbolic.

Note that we are not imposing any field equations on $(\mathscr{M}', g')$. One could similarly define a notion of maximality within the class of *vacuum* space-times, but this would lead to a weaker statement of the Choquet-Bruhat – Geroch theorem, which for simplicity is presented in the smooth case:

Theorem 6.2 (Existence of maximal globally hyperbolic developments [80])**.** *For any smooth vacuum initial data (M, h, K) there exists a unique, up to isometric diffeomorphism, vacuum development $(\mathscr{M}, g)$, which is inextendible in the class of smooth globally hyperbolic Lorentzian manifolds.*

This theorem can be thought of as the equivalent of the usual ODE theorem of existence of maximal solutions. The generalization is, however, highly non-trivial because while the proof for ODEs deals with subsets of $\mathbb{R}$, Theorem 6.2 deals with manifolds which are dynamically obtained by patching together local solutions. The main difficulty is to prove that the patching leads to a Hausdorff topological space. The argument makes use of Lorentzian causality theory, which in turn relies heavily on C^2 differentiability of the metric. To obtain a version of Theorem 6.2 with lower differentiability, as in Theorem 4.4 or in Remark 4.2, one would need to show that the relevant parts of causal theory can be repeated in the wider setting.

While Theorem 6.2 is highly satisfactory, it does not quite prove what one wants, because *uniqueness is claimed in the globally hyperbolic class only.* But we have seen in Theorem 6.1 that there exist vacuum space-times with non-unique extensions of a maximal globally hyperbolic region. In such examples the space-time $(\mathscr{M}, g)$ of Theorem 6.2 is unique in the class of globally hyperbolic space-times, but it can be extended in more than one way to strictly larger vacuum solutions. In such cases the extension always takes places across a *Cauchy horizon*, as defined in Section 2.3.3

So one cannot expect uniqueness in general. However, it has been suggested by Penrose [277] that non-uniqueness happens only in very special circumstances. The following result of Isenberg and Moncrief [198, 202, 254] (compare [184]) indicates that this might indeed be the case:

Theorem 6.3. *Let $(\mathscr{M}, g)$ be an analytic space-time containing an analytic compact Cauchy horizon $\mathscr{H}$. If the null geodesics threading $\mathscr{H}$ are closed, then the Cauchy horizon is a Killing horizon; in particular the isometry group of $(\mathscr{M}, g)$ is at least one-dimensional.*

The hypotheses of analyticity, compactness, and closed generators are of course highly restrictive. In any case it is conceivable that some kind of local isometries need to occur in space-times with Cauchy horizons when those conditions are not imposed; indeed, all known examples have this property. But of course existence of local isometries is a highly non-generic property, even when vacuum equations are imposed [32], so a version of Theorem 6.3 without those undesirable hypotheses would indeed establish SCC.

Whether or not Cauchy horizons require Killing vector fields, a loose mathematical formulation of strong cosmic censorship, as formulated in [101] following Moncrief and Eardley [253] and Penrose [277], is the following:

> *Consider the collection of initial data for, say, vacuum or electro–vacuum space-times, with the initial data surface M being compact, or with asymptotically flat initial data (M, h, K) . For generic such data the maximal globally hyperbolic development is inextendible.*

Because of the difficulty of the strong cosmic censorship problem, a full understanding of the issues which arise in this context seems to be completely out of reach at this stage. There is therefore some interest in trying to understand that question under various restrictive hypotheses, *e.g.*, symmetry. The simplest case, of

spatially homogeneous space-times, has turned out to be surprisingly difficult, because of the intricacies of the dynamics of some of the Bianchi models discussed in Section 6.7, and has been settled in the affirmative in [124] (compare Theorem 6.9 below).

6.2.1. *Gowdy toroidal metrics.* The next simplest case is that of *Gowdy metrics* on $\mathbb{T}^3 := S^1 \times S^1 \times S^1$: by definition,

$$g = e^{(\tau-\lambda)/2}(-e^{-2\tau}d\tau^2 + d\theta^2) + e^{-\tau}[e^P d\sigma^2 + 2e^P Q d\sigma d\delta + (e^P Q^2 + e^{-P})d\delta^2], \tag{6.4}$$

where $\tau \in \mathbb{R}$ and (θ, σ, δ) are coordinates on $\mathbb{T}^3$, with the functions P, Q and λ depending only on τ and θ. The metric of a maximal globally hyperbolic $U(1) \times U(1)$–symmetric vacuum space-time with $\mathbb{T}^3$–Cauchy surfaces can be globally written [100] in the form (6.4) provided that the *twist constants* vanish:

$$c_a := \epsilon_{\alpha\beta\gamma\delta} X_1^\alpha X_2^\beta \nabla^\gamma X_a^\delta = 0 \ , \qquad a = 1, 2 \ , \tag{6.5}$$

where the X_a's are the Killing vectors generating the $U(1) \times U(1)$ action. The condition $c_1 = c_2 = 0$ is equivalent to the requirement that the family of planes $\mathrm{span}\{X_1, X_2\}^\perp$ is integrable.

For metrics of the form (6.4), the Einstein vacuum equations become a set of *wave-map* equations

$$P_{\tau\tau} - e^{-2\tau}P_{\theta\theta} - e^{2P}(Q_\tau^2 - e^{-2\tau}Q_\theta^2) = 0, \tag{6.6}$$

$$Q_{\tau\tau} - e^{-2\tau}Q_{\theta\theta} + 2(P_\tau Q_\tau - e^{-2\tau}P_\theta Q_\theta) = 0, \tag{6.7}$$

which are supplemented by ODE's for the function λ:

$$\lambda_\tau = P_\tau^2 + e^{-2\tau}P_\theta^2 + e^{2P}(Q_\tau^2 + e^{-2\tau}Q_\theta^2), \tag{6.8}$$

$$\lambda_\theta = 2(P_\theta P_\tau + e^{2P}Q_\theta Q_\tau). \tag{6.9}$$

Here we write P_τ for $\partial_\tau P$, etc.

Initial data on $\mathbb{T}^3$ for P and Q have to satisfy an integral constaint,

$$\int_{S^1} (P_\theta P_\tau + e^{2P}Q_\theta Q_\tau) d\theta = 0 \ , \tag{6.10}$$

which is a consequence of (6.9) and of periodicity in θ. The metric function λ is obtained by integrating (6.8)-(6.9). Global existence of solutions to (6.6)-(6.7) was proved in [251] when the initial data are given on a hypersurface $\{\tau = \mathrm{const}\}$, and in [100] for general $U(1) \times U(1)$–symmetric Cauchy surfaces.

The question of SCC in this class of metrics has been settled by Ringström, who proved that the set of smooth initial data for Gowdy models on $\mathbb{T}^3$ that do *not* lead to the formation of Cauchy horizons contains a set which is open and dense within the set of all smooth initial data. More precisely, Ringström's main result (see [290, 291] and references therein) is the following:

Theorem 6.4. *Let $\tau_0 \in \mathbb{R}$ and let $\mathcal{S} = \{(Q(\tau_0), P(\tau_0), Q_\tau(\tau_0), P_\tau(\tau_0))\}$ be the set of smooth initial data for* (6.6)-(6.7) *satisfying (6.10). There is a subset $\mathcal{G}$ of $\mathcal{S}$ which is open with respect to the $C^2 \times C^1$ topology, and dense with respect to the C^∞ topology, such that the space-times of the form (6.4) corresponding to initial data in $\mathcal{G}$ are causally geodesically complete in one time direction, incomplete in the other time direction, and the Kretschmann scalar, $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, becomes unbounded in the incomplete direction of causal geodesics.*

This result does indeed establish SCC in this class of metrics: to see that the resulting space-times are inextendible in the category of C^3 manifolds with C^2 Lorentzian metrics, note that the existence of any such extension would imply existence of geodesics which are incomplete in the original space-time, and along which every curvature scalar is bounded.

Theorem 6.4 is complemented by the results in [73, 116, 252], where infinite dimensional families of (nongeneric) solutions which *are extendible* across a Cauchy horizon are constructed.

The key to the understanding of the global structure of the Gowdy space-times is the analysis of the behavior of the functions P and Q as $\tau \to \pm\infty$. The asymptotic behavior of those functions, established by Ringström, can then be translated into statements about the behavior of the space-time geometry as those limits are approached. A central element of the proof is the existence of a *velocity function*

$$v(\theta) := \lim_{\tau\to\infty} \sqrt{P_\tau^2 + e^{2P} Q_\tau^2}\ .$$

Essential steps in Ringström's analysis are provided by the work on Fuchsian PDEs of Kichenassamy and Rendall [210,286], as well as the study of the action of Geroch transformations by Rendall and Weaver [287] (compare [73]). See also [119] for the related problem of an exhaustive description of Cauchy horizons in those models.

6.2.2. *Other $U(1)\times U(1)$ symmetric models.* The existence of two Killing vectors is also compatible with S^3, $L(p,q)$ ("lens" spaces), and $S^1 \times S^2$ topologies. Thus, to achieve a complete understanding of the set of spatially compact initial data with precisely two Killing vectors one needs to extend Ringström's analysis to those cases. There is an additional difficulty that arises because of the occurrence of axes of symmetry, where the (1+1)–reduced equations have the usual singularity associated with polar coordinates. Nevertheless, in view of the analysis by Christodoulou and Tahvildar-Zadeh [96, 97] (see also [100]), the global geometry of *generic* maximal globally hyperbolic solutions with those topologies is reasonably well understood. This leads one to expect that one should be able to achieve a proof of SCC in those models using simple abstract arguments, but this remains to be seen.

Recall, finally, that general models with two Killing vectors X_1 and X_2 on $\mathbb{T}^3$ have non-vanishing *twist constants* (6.5). The Gowdy metrics are actually "zero measure" in the set of all $U(1)\times U(1)$ symmetric metrics on $\mathbb{T}^3$ because $c_a \equiv 0$ for the Gowdy models. The equations for the resulting metrics are considerably more complicated when the c_a's do not vanish, and only scant rigorous information is available on the global properties of the associated solutions [35, 204, 283]. It seems urgent to study the dynamics of those models, as they are expected to display [36] "oscillatory behavior" as the singularity is approached, in the sense of Section 6.8. Thus, they should provide the simplest model in which to study this behavior.

6.2.3. *Spherical symmetry.* One could think that the simplest possible asymptotically flat model for studying the dynamics of the gravitational field will be obtained by requiring spherical symmetry, since then the equations should reduce to wave equations in only two variables, t and r. Unfortunately, for vacuum space-times this turns out to be useless for this purpose because of Birkhoff's theorem [44], which asserts that spherically symmetric vacuum metrics are static. So, if one wishes to maintain spherical symmetry, supplementary fields are needed. The case of a scalar

field was studied in a series of intricate papers over 13 years by Christodoulou, beginning with [89], and culminating in [91] with the verification of the strong cosmic censorship conjecture within the model. Christodoulou further established "weak cosmic censorship" in this class, an issue to which we return in the next section, and exhibited non-generic examples for which the conclusions of these conjectures fail [90].

The situation changes when electromagnetic fields are introduced. The analysis by Dafermos [132, 133] of the spherically symmetric Einstein-Maxwell-scalar field equations yields a detailed picture of the interior of the black hole for this model, in terms of initial data specified on the event horizon and on an ingoing null hypersurface. When combined with the work by Dafermos and Rodnianski [134] on Price's law, one obtains the following global picture: initial data with a compactly supported scalar field, and containing a trapped surface (see Section 7.2 below), lead to space-times which *either* contain a degenerate (extremal) black hole, *or* develop a Cauchy horizon, with a space-time metric that can be continued past this horizon *in a C^0, but not C^1 manner*. It seems that not much is known about the properties of the degenerate solutions, which are presumably non-generic; it would be of interest to clarify that. In any case, the work shows that strong cosmic censorship holds within the class of nondegenerate solutions with trapped surfaces, at the C^1 level, leaving behind the perplexing possibility of continuous extendability of the metric.

The reader is referred to [2, 101, 285] and references therein for further reading on SCC.

6.3. **Weak cosmic censorship.** The strong cosmic censorship conjecture is an attempt to salvage predictability of Einstein's theory of gravitation. There exists a variant thereof which addresses the fact that we do not seem to observe any of the singularities that are believed to accompany gravitational collapse. The hope is then that, generically, in asymptotically flat space-times, any singular behavior that might form as a result of gravitational collapse, such as causality violations, lack of predictability, or curvature singularities, will be *clothed by an event horizon*. For this, one introduces the notion of *future null infinity*, which is an idealized boundary attached to space-time that represents, loosely speaking, the end points of null geodesics escaping to infinity. (In stationary situations this is closely related to the region $\mathscr{M}_{\rm ext}$ of (3.27).) The *black hole event horizon* is then the boundary of the past of null infinity; compare (3.28) and (3.29). One then wishes the part of the space-time that lies outside the black hole region to be well-behaved and "sufficiently large". This is the content of the *weak cosmic censorship* conjecture, originally due to Penrose [277], as made precise by Christodoulou [92]: *for generic asymptotically flat initial data, the maximal globally hyperbolic development has a complete future null infinity*. Heuristically this means that, disregarding exceptional sets of initial data, no singularities are observed at large distances, even when the observations are continued indefinitely. One should remark that, despite the names, the strong and weak cosmic censorship conjectures are logically independent; neither follows from the other. Note also that some predictability of Einstein's theory would be salvaged if strong cosmic censorship failed with weak cosmic censorship being verified, since then the failure of predictability would be invisible to outside observers.

Both cosmic censorship conjectures are intimately related to the issue of *gravitational collapse*, the dynamical formation of black holes and singularities, first

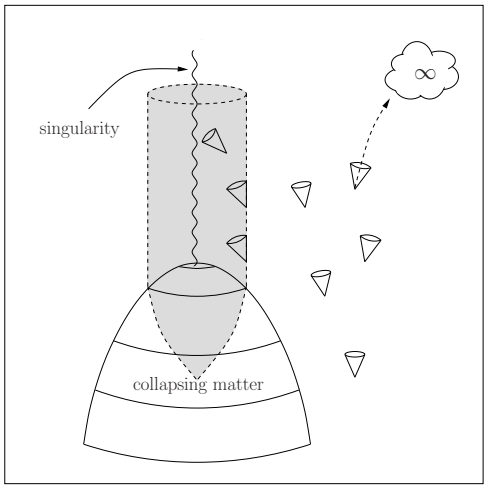

FIGURE 6.1. Light-cones in the Oppenheimer-Snyder collapse.[7]

observed for a homogeneous dust model by Oppenheimer and Snyder in 1939 [267], visualized in Figure 6.1.

So far the only complete analysis of weak cosmic censorship in a field theoretical model is that of the spherically symmetric scalar field model studied by Christodoulou [90, 91], already mentioned in Section 6.2.3.

6.4. **Stability of vacuum cosmological models.** Not being able to understand the dynamics of all solutions, one can ask whether some features of certain particularly important solutions persist under small perturbations of initial data. For example, will geodesic completeness still hold for space-times arising from small perturbations of Minkowskian initial data? Or, will a global, all encompassing, singularity persist under perturbations of Bianchi IX initial data (see Section 6.7). Such questions are the object of stability studies.

6.4.1. $U(1)$ *symmetry.* Our understanding of models with *exactly one* Killing vector is dramatically poorer than that of $U(1) \times U(1)$ symmetric space-times. Here one only has *stability* results, for small perturbations within the $U(1)$ isometry class in the expanding direction ("away from the singularity"): In [76] Choquet-Bruhat considers $U(1)$ symmetric initial data (h, K) for the vacuum Einstein equations on a manifold of the form $M \times S^1$, where M is a compact surface of genus $g > 1$. It is assumed that $\mathrm{tr}_h K$ is constant, and that (h, K) are sufficiently close to (h_0, K_0), where h_0 is a product metric

$$h_0 = \gamma + dx^2 \ ,$$

with γ being a metric of constant Gauss curvature on M, and with K_0 proportional to h_0. The sign of the trace of K_0 determines an expanding time direction and a contracting one. Under those conditions, Choquet-Bruhat proves that the solution exists for an infinite proper time in the expanding direction. The analysis builds upon previous work by Choquet-Bruhat and Moncrief [86], where a supplementary polarization condition has been imposed. Not much is known in the contracting

direction in the $U(1)$-symmetric models (see, however, [201]), where "mixmaster behavior"[16] is expected [34, 38]; compare [39].

6.4.2. *Future stability of hyperbolic models.* The proof of the above result bears some similarity to the *future stability* theorem of Andersson and Moncrief [9], as generalized in [7], for spatially compact hyperbolic models *without any symmetries.* Those authors consider initial data near a negatively curved compact space form, with the extrinsic curvature being close to a multiple of the metric, obtaining future geodesic completeness in the expanding direction. The control of the solution is obtained by studying the Bel-Robinson tensor and its higher-derivatives analogues. A striking ingredient of the proof is an elliptic-hyperbolic system of equations, used to obtain local existence in time [8].

6.5. **Stability of Minkowski space-time.**

6.5.1. *The Christodoulou-Klainerman proof.* One of the flagship results in mathematical general relativity is nonlinear stability of Minkowski space-time, first proved by Christodoulou and Klainerman [94]. One starts with an asymptotically flat vacuum initial data set (h, K) on $\mathbb{R}^3$. Under standard asymptotic flatness conditions, for (h, K) sufficiently close to Minkowskian data, the maximal globally hyperbolic development $(\mathscr{M}, g)$ of the data contains a maximal hypersurface, i.e., a hypersurface satisfying $\mathrm{tr}_h K = 0$; this follows from the results in [18, 24, 95]. So without loss of generality one can, in the small data context, assume that the initial data set is maximal.

The precise notion of smallness needed for the Christodoulou-Klainerman theorem is defined as follows: For $p \in \Sigma \approx \mathbb{R}^3$, $a > 0$, consider the quantity

$$Q(a,p) = a^{-1} \int_\Sigma \{\sum_{\ell=0}^{1} (d_p^2 + a^2)^{\ell+1} |\nabla^\ell \mathrm{Ric}|^2 + \sum_{\ell=1}^{2} (d_p^2 + a^2)^{\ell} |\nabla^\ell K|^2\} d\mu_g \ , \tag{6.11}$$

where d_p is the geodesic distance function from p, Ric is the Ricci tensor of the metric g, $d\mu_g$ is the Riemannian measure of the metric g and ∇ is the Riemannian connection of g. Let

$$Q_* = \inf_{a>0,\, p\in\Sigma} Q(a,p) \ .$$

Christodoulou and Klainerman prove causal geodesic completeness of $(\mathscr{M}, g)$ provided that Q_* is sufficiently small. The proof proceeds via an extremely involved bootstrap argument involving a foliation by maximal hypersurfaces Σ_t together with an analysis of the properties of an *optical function* u. In the context here this is a solution of the eikonal equation

$$g^{\alpha\beta} \partial_a u \partial_\beta u = 0 \ ,$$

the level sets C_u of which intersect Σ_t in spheres which expand as t increases. We have:

Theorem 6.5 (Global Stability of Minkowski space-time)**.** *Assume that* (M, h, K) *is maximal, with*[17]

$$h_{ij} = \delta_{ij} + o_3(r^{-1/2}), \quad K_{ij} = o_2(r^{-3/2}) \ . \tag{6.12}$$

[16] See the discussion after Theorem 6.8, and Section 6.8.

[17] A function f on M is $o_k(r^{-\lambda})$ if $r^{\lambda+i}\nabla^i f \to 0$ as $r \to \infty$ for all $i = 0, \dots k$.

There is an $\epsilon > 0$ such that if $Q_* < \epsilon$, then the maximal globally hyperbolic development $(\mathscr{M}, g)$ of (M, h, K) is geodesically complete.

The above version of Theorem 6.5 is due to Bieri [43]. The original formulation in [94] assumes moreover that

$$h = (1 + 2M/r)\delta + o_4(r^{-3/2}), \quad K = o_3(r^{-5/2}) \ , \tag{6.13}$$

and in the definition (6.11) a term involving K with $\ell = 0$ is added.

By definition, asymptotically flat initial data sets approach the Minkowskian ones as one recedes to infinity. One therefore expects that at sufficiently large distances one should obtain "global existence", in the sense that the maximal globally hyperbolic development contains complete *outgoing* null geodesics. This question has been addressed by Klainerman and Nicolò [213–215]; the reader is referred to those references for precise statements of the hypotheses made:

Theorem 6.6. *Consider an asymptotically flat initial data set (M, h, K), with maximal globally hyperbolic development $(\mathscr{M}, g)$. Let Ω_r denote a conditionally compact domain bounded by a coordinate sphere $S_r \subset \mathscr{M}_{\rm ext}$. There exists $R > 0$ such that for all $r \geq R$ the generators of the boundary $\partial J^+(\Omega_r)$ of the domain of influence $J^+(\Omega_r)$ of Ω_r are future-complete.*

Both in [94] and in [214] one can find detailed information concerning the behavior of null hypersurfaces as well as the rate at which various components of the Riemann curvature tensor approach zero along timelike and null geodesics.

6.5.2. *The Lindblad-Rodnianski proof.* A completely new proof of stability of Minkowski space-time has been given by Lindblad and Rodnianski [230,231]. The method provides less detailed asymptotic information than [94] and [214] on various quantities of interest but is much simpler. The argument is flexible enough to allow the inclusion of a scalar field, or of a Maxwell field [236, 237] (compare [333] for an analysis along the lines of the Christodoulou-Klainerman approach), and generalizes to higher dimensions [79]. Further it allows the following, less restrictive than that in [94, 214], asymptotic behavior of the initial data, for some $\alpha > 0$:

$$h = (1 + 2m/r)\delta + O(r^{-1-\alpha}) \ , \qquad K = O(r^{-2-\alpha}) \ . \tag{6.14}$$

Lindblad and Rodnianski consider the Einstein-Maxwell equations with a neutral scalar field:

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = T_{\mu\nu} + \hat{T}_{\mu\nu} \ , \tag{6.15}$$

with

$$\hat{T}_{\mu\nu} = \partial_\mu \psi \, \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} \big(g^{\alpha\beta} \partial_\alpha \psi \, \partial_\beta \psi \big) \ , \quad T_{\mu\nu} = 2(F_{\mu\lambda} F_\nu{}^\lambda - \frac{1}{4} g_{\mu\nu} F^{\lambda\rho} F_{\lambda\rho}) \ .$$

The initial data are prescribed on $\mathbb{R}^n$, so that the Maxwell field F has a global potential A, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The matter field equations read

$$D_\mu F^{\mu\nu} = 0 \ , \quad \Box_g \psi = 0 \ . \tag{6.16}$$

The initial data, denoted by $(\mathring{h}, \mathring{K}, \mathring{A}, \mathring{E}, \psi_0, \psi_1)$ (where, roughly speaking, $\mathring{A}$ is the initial value for the Maxwell potential and $\mathring{E}$ is the initial value for the electric

field), satisfy the following asymptotic conditions, for $r = |x| \to \infty$, with some $\alpha > 0$:

$$\begin{aligned} &\mathring{h}_{ij} = \begin{cases} (1+\frac{2m}{r})\delta_{ij} + O(r^{-1-\alpha}) \ , \ \text{for } n=3 \ , \\ \delta_{ij} + O(r^{\frac{1-n}{2}-\alpha}) \ , \ \text{for } n \geq 4 \ , \end{cases} \\ &\mathring{A} = O(r^{\frac{1-n}{2}-\alpha}) \ , \quad \mathring{K}_{ij} = O(r^{-\frac{n+1}{2}-\alpha}) \ , \quad \mathring{E} = O(r^{-\frac{n+1}{2}-\alpha}) \ , \\ &\psi_0 := \psi|_{t=0} = O(r^{\frac{1-n}{2}-\alpha}) \ , \quad \psi_1 := \partial_t \psi|_{t=0} = O(r^{-\frac{n+1}{2}-\alpha}) \end{aligned} \tag{6.17}$$

One also supposes that the relevant constraint equations hold initially:

$$\begin{cases} \mathring{R} = \mathring{K}^{ij}\mathring{K}_{ij} - \mathring{K}_i{}^i\mathring{K}_j{}^j + 2\mathring{E}_i\mathring{E}^i + \mathring{F}_{ij}\mathring{F}^{ij} + |\nabla\psi_0|^2 + |\psi_1|^2 \ , \\ \nabla^j\mathring{K}_{ij} - \nabla_i\mathring{K}_j{}^j = \mathring{F}_{0j}\mathring{F}_i{}^j + \nabla_i\psi_0\,\psi_1 \ , \\ \nabla_i\mathring{F}^{0i} = 0 \ , \end{cases} \tag{6.18}$$

where $\mathring{R}$ is the scalar curvature of the metric $\mathring{h}$.

The strategy is to impose *globally* the wave coordinates condition

$$\partial_\mu\left(g^{\mu\nu}\sqrt{|\det g|}\right) = 0 \quad \forall \nu = 0, ..., n, \tag{6.19}$$

as well as the *Lorenz gauge*,

$$\partial_\mu\left(\sqrt{|\det g|}A^\mu\right) = 0 \ . \tag{6.20}$$

Letting $\Box_g = g^{\mu\nu}\partial_\mu\partial_\nu$, the dynamical equations take the form

$$\Box_g \begin{pmatrix} h^1_{\mu\nu} \\ A_\sigma \\ \psi \end{pmatrix} = \begin{pmatrix} S_{\mu\nu} - 2\partial_\mu\psi\partial_\nu\psi \\ S_\sigma \\ 0 \end{pmatrix} - \begin{pmatrix} \Box_g h^0_{\mu\nu} \\ 0 \\ 0 \end{pmatrix} \ , \tag{6.21}$$

where the source terms $S_{\mu\nu}$ and S_σ are bilinear in the derivatives of the fields, with coefficients depending upon the metric. Furthermore,

$$h^1_{\mu\nu} = h_{\mu\nu} - h^0_{\mu\nu} \ , \quad \text{with } h^0_{\mu\nu}(t) = \begin{cases} \chi(r/t)\chi(r)\frac{2m}{r}\delta_{\mu\nu} \quad \text{for } n=3 \ , \\ 0 \quad \text{for } n \geq 4 \ , \end{cases} \tag{6.22}$$

where $\chi \in C^\infty$ is any function such that $\chi(s)$ equals 1 for $s \geq 3/4$ and 0 for $s \leq 1/2$. The proof relies heavily on the structure of the nonlinear terms in wave coordinates.

Recall that there exists an extensive literature on wave equations in $3+1$ dimensions with nonlinearities satisfying the *null condition* [211, 212], but the above nonlinearities *do not* satisfy that condition. The argument works only because different components of h can be treated on a different footing. Indeed, for solutions of the wave equation on Minkowski space-time, the derivatives in directions tangent to the light cones decay faster than the transverse ones. But the wave coordinates condition (6.19) can be used to express the transverse derivatives of some components of $g_{\mu\nu}$ in terms of tangential derivatives of the remaining ones. This provides control of the nonlinearities.

We also note the small data global existence results of [189,226] on $\mathbb{R}^{n+1}$, $n \geq 4$,, and of [88] for odd $n \geq 5$. The structure conditions there are general enough to cover the Einstein equations in wave coordinates, but the assumptions on the fall-off of initial data exclude non-trivial solutions of the vacuum constraint equations[18].

[18]In [189, 226] compactly supported data are considered. In the theorem for general quasilinear systems given in [88] the initial data are in a Sobolev space which requires fall-off at infinity faster than $r^{-n-3/2}$. In both cases the positive energy theorem implies that such initial data lead to Minkowski space-time.

We have:

Theorem 6.7. *Consider smooth initial data* $(\mathring{h}, \mathring{K}, \mathring{A}, \mathring{E}, \psi_0, \psi_1)$ *on* $\mathbb{R}^n$, $n \geq 3$, *satisfying* (6.17) *and* (6.18). *Let* $N \in \mathbb{N}$, *suppose that* $N_n := N + [\frac{n+2}{2}] - 2 \geq 6 + 2[\frac{n+2}{2}]$, *and set*

(6.23)
$$E_{N_n,\gamma}(0) = \sum_{0\leq |I|\leq N_n} \Big(||(1+r)^{1/2+\gamma+|I|}\nabla\nabla^I h_0^1||^2_{L^2} + ||(1+r)^{1/2+\gamma+|I|}\nabla^I \mathring{K}||^2_{L^2} + ||(1+r)^{1/2+\gamma+|I|}\nabla\nabla^I \mathring{A}||^2_{L^2} + ||(1+r)^{1/2+\gamma+|I|}\nabla^I \mathring{E}||^2_{L^2} + \|(1+r)^{1/2+\gamma+|I|}\nabla\,\nabla^I \psi_0\|_{L^2} + \|(1+r)^{1/2+\gamma+|I|}\nabla^I \psi_1\|_{L^2}\Big) \ .$$

Let m *be the ADM mass of* $\mathring{h}$. *For every* $\gamma_0 > 0$ *there exists* $\varepsilon_0 > 0$, *with* $\gamma_0(\varepsilon_0) \to 0$ *as* $\varepsilon_0 \to 0$, *such that if*

$$\sqrt{E_{N_n,\gamma}(0)} + m \leq \varepsilon_0, \tag{6.24}$$

for some $\gamma > \gamma_0$, *then the maximal globally hyperbolic development of the initial data is geodesically complete.*

The proof by Lindblad and Rodnianski is an ingenious and intricate analysis of the coupling between the wave-coordinates gauge and the evolution equations. One makes a clever guess of how the fields decay in space and time, encoded in the following weighted energy functional,

(6.25)
$$\mathscr{E}^{\mathrm{Matter}}_{N_n}(t) = \sup_{0\leq\tau\leq t} \sum_{Z\in\mathscr{Z}, |I|\leq N_n} \int_{\Sigma_\tau} \left(|\partial Z^I h^1|^2 + |\partial Z^I A|^2 + |\partial Z^I \psi|^2\right) w(q)\, d^n x \ ,$$

where

$$w(q) = \begin{cases} 1 + (1+|q|)^{1+2\gamma}, & q > 0 \ , \\ 1 + (1+|q|)^{-2\mu}, & q < 0 \ , \end{cases} \tag{6.26}$$

with $q = r - t$, $\mu > 0$ and $0 < \gamma < 1$. Here $\mathscr{Z}$ denotes the collection of the following generators of the conformal Lorentz group, first used to study the decay of solutions of the Minkowski wave equation by Klainerman [211]:

$$\partial_\alpha, \quad x_\alpha \frac{\partial}{\partial x^\beta} - x_\beta \frac{\partial}{\partial x^\alpha} \ , \quad x^\alpha \frac{\partial}{\partial x^\alpha} \ . \tag{6.27}$$

One argues by continuity: one chooses $0 < \delta < \frac{1}{4}$, and one considers the maximal time T so that the inequality

$$\mathcal{E}^{\mathrm{Matter}}_{N_n}(t) \leq 2C_{N_n}\varepsilon^2(1+t)^{2\delta} \tag{6.28}$$

holds for $0 \leq t \leq T$. A sophisticated method, using the Klainerman-Sobolev inequalities [211], together with a new weighted energy inequality, allows one to show that (6.28) holds for $0 \leq t \leq T$ with a smaller constant on the right-hand-side, contradicting maximality of T, and thus proving global existence.

A long standing question in the study of asymptotically flat space-times is that of the existence of an asymptotic expansion of the metric as one recedes to infinity along outgoing null cones, see [157, 159, 276]. Neither the analysis of [230, 231], nor that in [94, 214], provides sufficient information. It would be of interest to clarify that.

6.6. **Towards stability of Kerr: wave equations on black hole backgrounds.** Since the pioneering work of Christodoulou and Klainerman on stability of Minkowski space-time, many researchers have been looking into ways to address the question of stability of Kerr black holes. The first naive guess would be to study stability of Schwarzschild black holes, but those cannot be stable since a generic small perturbation will introduce angular momentum. The current strategy is to study, as a first step, linear wave equations on black hole backgrounds, with the hope that sufficiently robust linear decay estimates can be bootstrapped to produce a nonlinear stability proof. Due to limited space we will not review those results, referring the reader to recent important papers on the subject [47, 48, 135, 136, 154, 239].

6.7. **Bianchi A metrics.** Another important example of the intricate dynamical behavior of solutions of the Einstein equations is provided by the *"Bianchi A"* vacuum metrics. The key insight provided by these space-times is the supposedly chaotic behavior of large families of metrics in this class when a singularity is approached. This dynamics has been conjectured to be generic; we will return to this issue in Section 6.8. As will be seen shortly, in Bianchi A space-times the Einstein evolution equations reduce to a polynomial dynamical system on an algebraic four-dimensional submanifold of $\mathbb{R}^5$. The spatial parts of the Bianchi geometries provide a realization of six, out of eight, homogeneous geometries in three dimensions which form the basis of Thurston's geometrization program.

For our purposes here we define the Bianchi space-times as *maximal globally hyperbolic vacuum developments of initial data which are invariant under a simply transitive group of isometries*. Here the transitivity of the isometry group is meant at the level of initial data, and *not* for the space-time. The name is a tribute to Bianchi, who gave the classification of three dimensional Lie algebras which underline the geometry here. These metrics split into two classes, Bianchi A and Bianchi B, as follows: Let G be a 3-dimensional Lie group, and let Z_i, $i = 1, 2, 3$ denote a basis of left-invariant vector fields on G. Define the *structure constants* γ_{ij}^k by the formula

$$[Z_i, Z_j] = \gamma_{ij}{}^k Z_k \ .$$

The Lie algebra and Lie group are said to be of class A if $\gamma_{ik}{}^k = 0$; class B are the remaining ones. The classes A and B correspond in mathematical terminology to the *unimodular* and *non-unimodular* Lie algebras. A convenient parameterization of the structure constants is provided by the symmetric matrix n^{ij} defined as

$$n^{ij} = \frac{1}{2}\gamma_{kl}{}^{(i}\epsilon^{j)kl}. \tag{6.29}$$

This implies $\gamma_{ij}{}^k = \epsilon_{ijm}n^{km}$. The Bianchi A metrics are then divided into six classes, according to the eigenvalues of the matrix n^{ij}, as described in Table 6.1. For the Bianchi IX metrics, of particular interest to us here, the group G is $SU(2)$. Thus, the Taub metrics discussed in Section 6.1 are members of the Bianchi IX family, distinguished by the existence of a further $U(1)$ factor in the isometry group.

Let G be any three-dimensional Lie group, the Lie algebra of which belongs to the Bianchi A class. (The G's are closely related to the Thurston geometries, see Table 6.1; compare [2, Table 2]). Denote by $\{\sigma^i\}$ the basis dual to $\{Z_i\}$. It is not too difficult to show that both A and B Bianchi metrics can be globally written as

$$g = -dt^2 + h_{ij}(t)\sigma^i\sigma^j \ , \qquad t \in I \ , \tag{6.30}$$

TABLE 6.1. Lie groups of Bianchi class A.

Bianchi type	n_1	n_2	n_3	Simply connected group
I	0	0	0	Abelian $\mathbb{R}^3$
II	+	0	0	Heisenberg
VI_0	0	+	−	Sol (isometries of the Minkowski plane $\mathbb{R}^{1,1}$)
VII_0	0	+	+	universal cover of Euclid (isometries of $\mathbb{R}^2$)
VIII	−	+	+	universal cover of $SL(2,\mathbb{R})$
IX	+	+	+	$SU(2)$

with a maximal time interval I.

There are various ways to write the Einstein equations for a metric of the form (6.30). We use the formalism introduced by Wainwright and Hsu [320], which has proven to be most useful for analytical purposes [284, 288, 289], and we follow the presentation in [288]. Let

$$\sigma_{ij} = K_{ij} - \frac{1}{3}\mathrm{tr}_h K h_{ij} \ , \qquad \theta := \mathrm{tr}_h K \ ,$$

be the trace-free part of the extrinsic curvature tensor of the level sets of t. Away from the (isolated) points at which θ vanishes, one can introduce

$$\begin{aligned} \Sigma_{ij} &= \sigma_{ij}/\theta \ , \\ N_{ij} &= n_{ij}/\theta \ , \\ B_{ij} &= 2N_i{}^k N_{kj} - N^k_{\ k} N_{ij} \ , \\ S_{ij} &= B_{ij} - \frac{1}{3} B^k_{\ k} \delta_{ij} \ . \end{aligned}$$

Set $\Sigma_+ = \frac{3}{2}(\Sigma_{22} + \Sigma_{33})$ and $\Sigma_- = \sqrt{3}(\Sigma_{22} - \Sigma_{33})/2$. If we let N_i be the eigenvalues of N_{ij}, the vacuum Einstein equations (a detailed derivation of which can be found in [288]) lead to the following *autonomous, polynomial* dynamical system

$$\begin{aligned} N_1' &= (q - 4\Sigma_+) N_1 \ , \\ N_2' &= (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-) N_2 \ , \\ N_3' &= (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-) N_3 \ , \\ \Sigma_+' &= -(2-q)\Sigma_+ - 3S_+ \ , \\ \Sigma_-' &= -(2-q)\Sigma_- - 3S_- \ , \end{aligned} \tag{6.31}$$

where a prime denotes derivation with respect to a new time coordinate τ defined by

$$\frac{dt}{d\tau} = \frac{3}{\theta} \ . \tag{6.32}$$

Further,

$$\begin{aligned} q &= 2(\Sigma_+^2 + \Sigma_-^2) \ , \\ S_+ &= \frac{1}{2}[(N_2 - N_3)^2 - N_1(2N_1 - N_2 - N_3)] \ , \\ S_- &= \frac{\sqrt{3}}{2}(N_3 - N_2)(N_1 - N_2 - N_3) \ . \end{aligned} \tag{6.33}$$

The vacuum constraint equations reduce to one equation,

$$\Sigma_+^2 + \Sigma_-^2 + \frac{3}{4}[N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 + N_3N_1)] = 1 \ . \tag{6.34}$$

The points $(N_1, N_2, N_3, \Sigma_+, \Sigma_-)$ can be classified according to the values of N_1, N_2, N_3 in the same way as the n^i's in Table 6.1. The sets $N_i > 0$, $N_i < 0$ and $N_i = 0$ are invariant under the flow determined by (6.31), and one can therefore classify solutions to (6.31)-(6.34) accordingly. Bianchi IX solutions correspond, up to symmetries of the system, to points with all N_i's positive, while for Bianchi VIII solutions one can assume that two N_i's are positive and the third is negative.

Points with $N_1 = N_2 = N_3 = 0$ correspond to Bianchi I models. The associated vacuum metrics were first derived by Kasner, and take the form

$$ds^2 = -dt^2 + \sum_{i=1}^{3} t^{2p_i} dx^i \otimes dx^i \ , \quad p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \ . \tag{6.35}$$

An important role in the analysis of (6.31) is played by the *Kasner circle*, defined as the set $\{q = 2\}$. These points belong to the configuration space, as determined by (6.34), for Bianchi I models, but the equation $q = 2$ is incompatible with (6.34) for Bianchi IX metrics. Nevertheless, we shall see shortly that the Kasner circle plays an essential role in the analysis of the Bianchi IX dynamics.

The set $\Sigma_- = 0$, $N_2 = N_3$, together with its permutations, is invariant under the flow of (6.31)-(6.34). In the Bianchi IX case these are the Taub solutions. In the Bianchi VIII case the corresponding explicit solutions, known as the NUT metrics, have been found by Newman, Tamburino and Unti [258], and they exhibit properties similar to the Bianchi IX Taub solutions discussed in Section 6.1.

The *ω-limit* of an orbit γ of a dynamical system is defined as the set of accumulation points of that orbit. In [288, 289], Ringström proves the following:

Theorem 6.8. *The ω-limit set of each non-NUT Bianchi VIII orbit contains at least two distinct points on the Kasner circle. Similarly, non-Taub–NUT Bianchi IX orbits have at least three distinct ω-limit points on the Kasner circle.*

The picture which emerges from a numerical analysis of (6.31) (see [37, 128] and references therein) is the following: Every non-Taub–NUT Bianchi IX orbit approaches some point on the Kasner circle; there it performs a "bounce", after which it eventually approaches another point on the Kasner circle, and so on. Theorem 6.8 establishes the validity of this picture. The numerical analysis further suggests that generic orbits will have a dense ω-limit set on the Kasner circle; this is compatible with, but does not follow from, Ringström's analysis. It has been argued that the map which associates to each bounce the nearest point on the Kasner circle possesses chaotic features; this is at the origin of the "mixmaster behavior" terminology, sometimes used in this context. Whether or not this is true, and in which sense, remains to be seen.

The following result of Ringström [288] provides further insight into the geometry of Bianchi IX space-times:

Theorem 6.9. *In all maximal globally hyperbolic developments $(\mathscr{M}, g)$ of non-Taub–NUT Bianchi IX vacuum initial data or of non-NUT Bianchi VIII vacuum initial data the Kretschmann scalar*

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$$

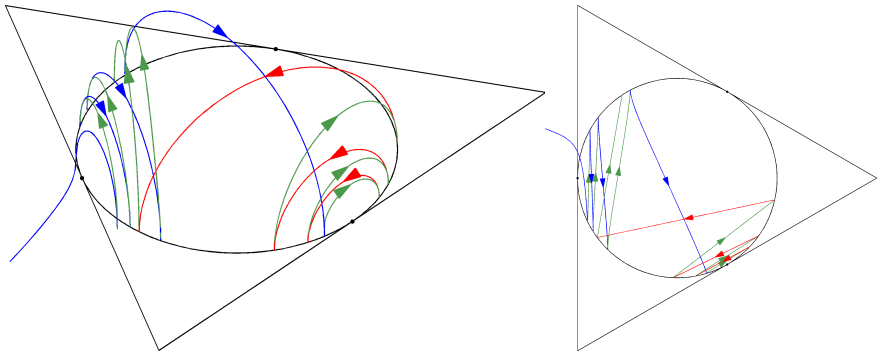

FIGURE 6.2. A few "bounces" in a typical Bianchi IX orbit; figures and numerics by Woei-Chet Lim. The vertical axis represents N_1 (red), N_2 (green), N_3 (blue), with only the biggest of the N_i's plotted. The Kasner circle and the triangle for the Kasner billiard in the (Σ_+, Σ_-)–plane are shown. The projected trajectories can be seen to approach the billiard ones.

is unbounded along inextendible causal geodesics.

Note that the observation of curvature blow-up provides a proof, alternative to that of [124], of the non-existence of Cauchy horizons in generic Bianchi IX models.

We close this section by mentioning that no similar rigorous results are known concerning the global dynamical properties of Bianchi models of class B; compare [183].

6.8. **The mixmaster conjecture.** The *most important question* in the study of the Cauchy problem is that of the *global properties of the resulting space-times.* So far we have seen examples of geodesically complete solutions (e.g., small perturbations of Minkowski space-time), or all-encompassing singularities (e.g., generic Bianchi models), or of Cauchy horizons (e.g., Taub–NUT metrics). The geodesically complete solutions are satisfying but dynamically uninteresting, while the strong cosmic censorship conjecture expresses the hope that Cauchy horizons will almost never occur. So it appears essential to have a good understanding of the remaining cases, presumably corresponding to singularities. Belinski, Khalatnikov and Lifschitz [34] suggested that, near singularities, at each space point the dynamics of the gravitational field resembles that of generic Bianchi metrics, as described in Section 6.7. Whether or not this is true, and in which sense, remains to be seen; in any case the idea, known as *the BKL* conjecture, provided guidance — and still does — to a significant body of research on general relativistic singularities; see [17,139,144] and references therein. This then leads to the mathematical challenge of making sense of the associated slogan, namely that *the singularity in generic gravitational collapse is spacelike, local, and oscillatory.* Here *spacelike* is supposed to mean that strong cosmic censorship holds. The term *local* refers to the idea that, near generic singularities, there should exist coordinate systems in which the metric asymptotes to a solution of equations in which *spatial* derivatives of appropriately chosen fields

have been neglected.[19] Finally, *oscillatory* is supposed to convey the idea that the approximate solutions will actually be provided by the Bianchi IX metrics.

The main rigorous evidence for a relatively large class of vacuum[20] space-times with singularities which are *spacelike and local* in the sense described above is Ringström's Theorem 6.4, describing generic Gowdy metrics, but the resulting singularities are *not* oscillatory. This is not in contradiction with the conjecture, since the Gowdy metrics are certainly not generic, whether in the space of all metrics, or in the space of $U(1)\times U(1)$ symmetric ones: As mentioned in Section 6.4.1, generic $U(1)\times U(1)$ metrics have non-vanishing twist constants c_a as defined by (6.5). The numerical studies of [36] suggest that the switching-on of the twist constants will indeed generically lead to some kind of oscillatory behavior.

In fact, BKL put emphasis on Bianchi IX models, while some other authors seem to favor Bianchi $VI_{-1/9}$, or not-necessarily Bianchi, oscillations [34, 38, 139, 181, 318]. It has moreover been suggested that the oscillatory behavior disappears in space-time dimensions higher than ten [139, 141], and large families of non-oscillatory solutions with singularities have indeed been constructed in [140]. This leads naturally to the following, somewhat loose, conjecture:

Conjecture 6.10 (Mixmaster conjecture)**.** *Let $n+1\le 10$. There exist open sets of vacuum metrics for which some natural geometric variables undergo oscillations of increasing complexity along inextendible geodesics of unbounded curvature.*

The BKL conjecture would thus be a more precise version of the above, claiming moreover genericity of the behavior, and pointing out to the Bianchi dynamics as the right mode. Those properties are so speculative that we decided not to include them in Conjecture 6.10.

The only examples so far of oscillatory singularities *which are not spatially homogeneous* have been constructed by Berger and Moncrief [39]. There, a solution-generating transformation is applied to Bianchi IX metrics, resulting in non-homogeneous solutions governed by the "oscillatory" functions arising from a non-Taub Bianchi IX metric. The resulting metrics have at least one but not more than two Killing vectors. The analysis complements the numerical evidence for oscillatory behavior in $U(1)$ symmetric models presented in [40].

7. Marginally trapped surfaces

There have been some interesting recent developments at the interface of space-time geometry and the theory of black holes associated with the notion of *marginally outer trapped surfaces*. Let Σ be a co-dimension two spacelike submanifold of a space-time $\mathscr{M}$. Under suitable orientation assumptions, there exist two families of future directed null geodesics issuing orthogonally from Σ. If one of the families has vanishing expansion along Σ, then Σ is called a marginally outer trapped surface

[19]The resulting truncated equations should then presumably resemble the equations satisfied by spatially homogeneous metrics. However, different choices of quantities which are expected to be time-independent will lead to different choices of the associated notion of homogeneity; for instance, in [34] the types Bianchi VIII and IX are singled out; the notion of genericity of those types within the Bianchi A class is read from Table 6.1 as follows: "something that can be non-zero is more generic than something that is". On the other hand, the analysis in [181] seems to lead to Bianchi $VI_{-1/9}$ metrics.

[20]See, however, [10, 140] for a class of space-times with sources; [140] also covers vacuum in space dimensions $n\ge 10$.

(or an apparent horizon). The notion of a marginally outer trapped surface was introduced early on in the development of the theory of black holes, as the occurrence of the former signals the presence of the latter. More recently, marginally outer trapped surfaces have played a fundamental role in quasi-local descriptions of black holes, and have been useful in numerical simulations of black hole space-times; see e.g. [13]. Marginally outer trapped surfaces arose in a more purely mathematical context in the work of Schoen and Yau [303] concerning the existence of solutions to the Jang equation, in connection with their proof of the positivity of mass.

Mathematically, marginally outer trapped surfaces may be viewed as space-time analogues of minimal surfaces in Riemannian manifolds. Despite the absence of a variational characterization like that for minimal surfaces,[21] marginally trapped surfaces have recently been shown to satisfy a number of analogous properties, cf., in particular, [3–6, 12, 146, 167]. The aim of this section is to describe some of these mathematical developments.

7.1. **Null hypersurfaces.** Each family of null geodesics issuing orthogonally from Σ, as described above, forms a smooth null hypersurface near Σ. It would be useful at this stage to discuss some general aspects of such hypersurfaces. Null hypersurfaces have an interesting geometry, and play an important role in general relativity. In particular, as we have seen, they represent *horizons* of various sorts, such as the event horizons discussed in Section 3.

Let $(\mathscr{M}^{n+1}, g)$ be a space-time, with $n \geq 2$. A smooth null hypersurface in $\mathscr{M}$ is a smooth co-dimension one submanifold $\mathscr{N} = \mathscr{N}^n$ of $\mathscr{M}$ such that the restriction of g to each tangent space $T_p\mathscr{N}$ of $\mathscr{N}$ is degenerate. This, together with the Lorentz signature, implies that there is a unique direction of degeneracy in each tangent space $T_p\mathscr{N}$. Thus, every null hypersurface $\mathscr{N}$ comes equipped with a smooth future directed *null* vector field K ($g(K, K) = 0$) defined on, and tangent to $\mathscr{N}$, such that the normal space of K at each $p \in \mathscr{N}$ coincides with the tangent space of $\mathscr{N}$ at p, i.e., $K_p^{\perp} = T_p\mathscr{N}$ for all $p \in \mathscr{N}$. Tangent vectors to $\mathscr{N}$, transverse to K, are then necessarily spacelike. The null vector field K associated to $\mathscr{N}$ is unique up to positive pointwise rescaling. However, there is, in general, no canonical way to set the scaling.

Two simple examples arise in Minkowski space $\mathbb{R}^{1,n}$. The past and future cones $\partial I^-(p)$ and $\partial I^+(p)$ are smooth null hypersurfaces away from the vertex p. Each nonzero null vector $v \in T_p\mathbb{R}^{1,n}$ determines a null hyperplane $\Pi = \{q \in \mathbb{R}^{1,n} : \eta(\overrightarrow{pq}, v) = 0\}$, where η is the Minkowski metric, and $\overrightarrow{pq}$ is the tangent vector at p representing the displacement from p to q.

It is a fundamental fact that *the integral curves of K are null geodesics*, though perhaps not affinely parameterized - this will depend on the scaling of K. Thus $\mathscr{N}$ is ruled by null geodesics, called the null generators of $\mathscr{N}$. For example the future cone $\mathscr{N} = \partial I^+(p) \setminus \{p\}$ in Minkowski space is ruled by future directed null rays emanating from p.

The *null expansion scalar* θ of $\mathscr{N}$ with respect to K is a smooth function on $\mathscr{N}$ that gives a measure of the average expansion of the null generators of $\mathscr{N}$ towards the future. In essence, θ is defined as the divergence of the vector field K along $\mathscr{N}$. To be precise, given $p \in \mathscr{N}$, let Π_{n-1} be a co-dimension one subspace of $T_p\mathscr{N}^n$ transverse to K_p. The metric g, restricted to Π_{n-1}, will be positive definite. Let

[21] There seems to be no analogue of the area functional.

$\{e_1, e_2, ..., e_{n-1}\}$ be an orthonormal basis for Π_{n-1} with respect to g. Then θ at p is defined as,

$$\theta(p) = \sum_{i=1}^{n-1} g(\nabla_{e_i} K, e_i)\,. \tag{7.1}$$

Interestingly, due to the fact that K is null, this value is independent of the choice of transverse subspace Π_{n-1}, as well as of the choice of an orthonormal basis for Π_{n-1}, and so the expansion scalar θ is well defined.

While θ depends on the choice of K, it does so in a simple way. As easily follows from Equation (7.1), a positive rescaling of K rescales θ in the same way: If $\tilde{K} = fK$ then $\tilde{\theta} = f\theta$. Thus the *sign* of the null expansion θ does not depend on the scaling of K: $\theta > 0$ means expansion on average of the null generators, and $\theta < 0$ means contraction on average. In Minkowski space, the future null cone $\mathscr{N} = \partial I^+(p) \setminus \{p\}$ has $\theta > 0$, and the past cone, $\mathscr{N} = \partial I^-(p) \setminus \{p\})$ has $\theta < 0$.

It is useful to understand how the null expansion varies as one moves along a null generator of $\mathscr{N}$. Let $s \to \eta(s)$ be a null geodesic generator of $\mathscr{N}$, and assume K is scaled so that η is affinely parameterized. Then it can be shown that the null expansion scalar $\theta = \theta(s)$ along η satisfies the propagation equation,

$$\frac{d\theta}{ds} = -\mathrm{Ric}(\eta', \eta') - \sigma^2 - \frac{1}{n-1}\theta^2\,, \tag{7.2}$$

where $\sigma \geq 0$, the *shear scalar*, measures the deviation from perfect isotropic expansion. Equation (7.2) is known in the relativity community as the Raychaudhuri equation (for a null geodesic congruence) [179], and, together with a timelike version, plays an important role in the proofs of the classical Hawking-Penrose singularity theorems [179]. There are well-known Riemannian counterparts to this equation, going back to work of Calabi [66].

Equation (7.2) shows how the curvature of space-time influences the expansion of the null generators. We consider here a simple application of the Raychaudhuri equation.

Proposition 7.1. *Let $\mathscr{M}$ be a space-time which obeys the null energy condition, $\mathrm{Ric}\,(X, X) = R_{\alpha\beta}X^\alpha X^\beta \geq 0$ for all null vectors X, and let $\mathscr{N}$ be a smooth null hypersurface in $\mathscr{M}$. If the null generators of $\mathscr{N}$ are future geodesically complete then the null generators of $\mathscr{N}$ have nonnegative expansion, $\theta \geq 0$.*

Proof. Suppose $\theta < 0$ at $p \in \mathscr{N}$. Let $\eta : [0, \infty) \to \mathscr{N}$, $s \to \eta(s)$, be the null geodesic generator of $\mathscr{N}$ passing through $p = \eta(0)$; by rescaling K if necessary, we can assume η is affinely parameterized. Let $\theta = \theta(s)$, $s \in [0, \infty)$, be the null expansion of $\mathscr{N}$ along η; hence $\theta(0) < 0$. Raychaudhuri's equation and the null energy condition imply that $\theta = \theta(s)$ obeys the inequality,

$$\frac{d\theta}{ds} \leq -\frac{1}{n-1}\theta^2\,, \tag{7.3}$$

and hence $\theta < 0$ for all $s > 0$. Dividing through by θ^2 then gives,

$$\frac{d}{ds}\left(\frac{1}{\theta}\right) \geq \frac{1}{n-1}\,, \tag{7.4}$$

which implies $1/\theta \to 0$, i.e., $\theta \to -\infty$ in finite affine parameter time, contradicting the smoothness of θ. □

We wish to indicate the connection of Proposition 7.1 with the theory of black holes. In fact, this proposition is the most rudimentary form of Hawking's famous area theorem [179]. Let $\mathscr{M}$ be a standard black hole space-time, as defined for example in [179]. It is not necessary to go into the technical details of the definition. It suffices to say that in $\mathscr{M}$ there exists a region $\mathscr{B}$, the black hole region, from which signals (future directed causal curves) cannot "escape to infinity" (recall the example of the Schwarzschild solution discussed in Section 3.1). The boundary of this region is the event horizon $\mathscr{E}$, which, in general, is a Lipschitz hypersurface ruled by future inextendible null geodesics, called its null generators. If $\mathscr{E}$ is smooth and if its generators are future complete then Proposition 7.1 implies that $\mathscr{E}$ has nonnegative null expansion. This in turn implies that "cross-sections" of $\mathscr{E}$ are nondecreasing in area as one moves towards the future, as asserted by the area theorem. In the context of black hole thermodynamics, the area theorem is referred to as the second law of black mechanics, and provides a link between gravity and quantum physics. As it turns out, the area theorem remains valid without imposing any smoothness assumptions; for a recent study of the area theorem, which focuses on these issues of regularity, see [114].

7.2. **Trapped and marginally trapped surfaces.** We begin with some definitions. Let $\Sigma = \Sigma^{n-1}$, $n \geq 3$, be a spacelike submanifold of co-dimension two in a space-time $(\mathscr{M}^{n+1}, g)$. Regardless of the dimension of space-time, we shall refer to Σ as a surface, which it actually is in the $3+1$ case. We are primarily interested in the case where Σ is compact (without boundary), and so we simply assume this from the outset.

Each normal space of Σ, $[T_p\Sigma]^\perp$, $p \in \Sigma$, is timelike and 2-dimensional, and hence admits two future directed null directions orthogonal to Σ. Thus, if the normal bundle is trivial, Σ admits two smooth nonvanishing future directed null normal vector fields l_+ and l_-, which are unique up to positive pointwise scaling, see Figure 7.1. By convention, we refer to l_+ as outward pointing and l_- as inward pointing.[22] In relativity it is standard to decompose the second fundamental form

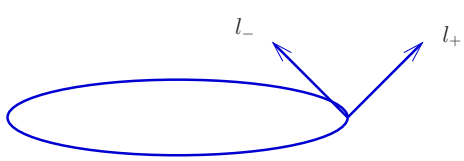

FIGURE 7.1. The null future normals $l_\pm$ to Σ.

of Σ into two scalar valued *null second forms* χ_+ and χ_-, associated to l_+ and l_-, respectively. For each $p \in \Sigma$, $\chi_\pm : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$ is the bilinear form defined by,

$$\chi_\pm(X, Y) = g(\nabla_X l_\pm, Y) \quad \text{for all } X, Y \in T_p\Sigma\,. \tag{7.5}$$

A standard argument shows that $\chi_\pm$ is symmetric. Hence, χ_+ and χ_- can be traced with respect to the induced metric γ on Σ to obtain the null mean curvatures (or null expansion scalars),

$$\theta_\pm = \mathrm{tr}_\gamma\, \chi_\pm = \gamma^{ij}(\chi_\pm)_{ij} = \mathrm{div}_\Sigma l_\pm\,. \tag{7.6}$$

[22]In many situations, there is a natural choice of "inward" and "outward".

$\theta_\pm$ depends on the scaling of $l_\pm$ in a simple way. As follows from Equation (7.5), multiplying $l_\pm$ by a positive function f simply scales $\theta_\pm$ by the same function. Thus, the *sign* of $\theta_\pm$ does not depend on the scaling of $l_\pm$. Physically, θ_+ (resp., θ_-) measures the divergence of the outgoing (resp., ingoing) light rays emanating from Σ.

It is useful to note the connection between the null expansion scalars $\theta_\pm$ and the expansion of the generators of a null hypersurface, as discussed in Section 7.1. Let $\mathscr{N}_+$ be the null hypersurface, defined and smooth near Σ, generated by the null geodesics passing through Σ with initial tangents l_+. Then θ_+ is the null expansion of $\mathscr{N}_+$ *restricted to* Σ; θ_- may be described similarly.

Let M be a spacelike hypersurface in a space-time $(\mathscr{M}, g)$, with induced metric h and second fundamental form K, and suppose Σ is embedded as a 2-sided hypersurface in M. Then the null expansions $\theta_\pm$ can be expressed in terms of the initial data (M, h, K) as follows. Since Σ is 2-sided in M, it admits a smooth unit normal field ν in M, unique up to sign. By convention, we refer to such a choice as outward pointing. Letting u denote the future directed unit normal to M, $l_+ = u + \nu$ (resp., $l_- = u - \nu$) is a future directed outward (resp., future directed inward) pointing null normal vector field along Σ. Let $\theta_\pm$ be the null expansion with respect to the null normal $l_\pm$. Then,

$$\theta_\pm = \mathrm{tr}_\Sigma K \pm H\,, \tag{7.7}$$

where $\mathrm{tr}_\Sigma K$ is the trace of the projection of K into Σ with respect to the induced metric on Σ, and H is the mean curvature of Σ in M.

For round spheres in Euclidean slices of Minkowski space, with the obvious choice of inside and outside, one has $\theta_- < 0$ and $\theta_+ > 0$. In fact, this is the case in general for large "radial" spheres in *asymptotically flat* spacelike hypersurfaces. However, in regions of space-time where the gravitational field is strong, one may have both $\theta_- < 0$ and $\theta_+ < 0$, in which case Σ is called a *trapped surface*. For example the black hole region, $0 < r < 2m$, in $(n+1)$-dimensional Schwarzschild space-time (see Section 3.1) is foliated by spherically symmetric $(n-1)$-spheres, all of which are trapped surfaces. Under appropriate energy and causality conditions, the occurrence of a trapped surface signals the onset of gravitational collapse. This is the implication of the Penrose singularity theorem [275], the first of the famous singularity theorems, which we take a momentary detour to discuss.

Theorem 7.2 (Penrose [275])**.** *Let $\mathscr{M}$ be a globally hyperbolic space-time with noncompact Cauchy surfaces satisfying the null energy condition. If $\mathscr{M}$ contains a trapped surface Σ then $\mathscr{M}$ is future null geodesically incomplete.*

Recall from Section 7.1 that the null energy condition is the curvature requirement, $\mathrm{Ric}\,(X, X) = R_{\alpha\beta}X^\alpha X^\beta \geq 0$ for all null vectors X. If a space-time $\mathscr{M}$ obeys the Einstein equation (2.8), then one can express the null energy condition in terms of the energy momentum tensor: $\mathscr{M}$ obeys the null energy condition if and only if $\mathcal{T}(X, X) = T_{ij}X^iX^j \geq 0$ for all null vectors X.

In studying an *isolated* gravitating system, such as the gravitational collapse of a star, it is customary to model the situation by a space-time which is asymptotically flat. In this context, the assumption of Theorem 7.2 that the space-time admits a noncompact Cauchy surface is natural. The conclusion in the theorem of future null geodesic incompleteness is an indication that space-time "comes to an end" or

develops a singularity somewhere in the causal future. However, the theorem gives no information about the nature of the singularity.

Existence of *vacuum* asymptotically flat initial data sets, with one asymptotic region and containing compact trapped surfaces, has been established by Beig and Ó Murchadha [33]. Very recently, Christodoulou [93] established the dynamical formation of compact trapped surfaces in vacuum, arising solely from the focusing of sufficiently strong incoming gravitational waves.

Remark on the proof of Theorem 7.2. The proof consists of showing that if $\mathscr{M}$ is future null geodesically complete then the achronal boundary $\partial I^+(\Sigma)$ is compact. The first part of Proposition 2.6 then implies that $\partial I^+(\Sigma)$ is a compact Cauchy surface, contrary to assumption. The compactness of $\partial I^+(\Sigma)$ under the completeness assumption may be roughly seen as follows. Each point of $\partial I^+(\Sigma)$ lies on a future directed null geodesic issuing orthogonally from Σ (see Proposition 2.3). By the trapped condition these null geodesics are initially converging. The Raychaudhuri equation, Equation (7.2) and null energy condition imply that these null geodesics continue to converge towards future. Eventually, within a uniform affine parameter value determined by (7.2), each future directed null geodesic η issuing orthogonally from Σ encounters a null caustic (or null focal point) at which $\theta \to -\infty$, after which it necessarily enters $I^+(\Sigma)$, thereby leaving $\partial I^+(\Sigma)$. From this it can be inferred that $\partial I^+(\Sigma)$ is compact. □

To continue our discussion, consider again the general setting of a spacelike surface Σ^{n-1} in a space-time $\mathscr{M}^{n+1}$, with future directed null normal fields $l_\pm$ and associated null expansion scalars $\theta_\pm$. Focusing attention on just the outward null normal l_+, we say that Σ is an *outer trapped surface* (resp., *weakly outer trapped surface*) if $\theta_+ < 0$ (resp., $\theta_+ \leq 0$). If θ_+ vanishes, we say that Σ is a *marginally outer trapped surface*, or MOTS for short. In what follows we will be primarily concerned with properties of MOTSs.

MOTSs arise naturally in a number of situations. As an outgrowth of their work on the positive energy theorem, Schoen and Yau [304] showed that suitable conditions on the energy density and momentum density of an asymptotically flat initial data set insure the presence of a MOTS; see also [330]. Next, as follows from our comments about the area theorem in Section 7.1, cross-sections of the event horizon in black hole space-times have nonnegative expansion $\theta \geq 0$. (By a cross-section, we mean a smooth intersection of the event horizon with a spacelike hypersurface.) In the steady state limit this expansion goes to zero. Thus, it is a basic fact that cross-sections of the event horizon in stationary black hole space-times are MOTSs. For dynamical black hole space-times, MOTSs typically occur in the black hole region, i.e., the region inside the event horizon. While there are heuristic arguments for the existence of MOTSs in this situation, based on looking at the boundary of the 'trapped region' [179, 321] within a given spacelike slice, a result described by Schoen [297], and recently proved by Eichmair [146] and Andersson and Metzger [6], rigorously establishes their existence under physically natural conditions; see Section 7.5.

As noted earlier, MOTSs may be viewed as space-time analogues of minimal surfaces in Riemannian geometry. In fact, as follows from Equation (7.7), in the time-symmetric case ($K = 0$) a MOTS is simply a minimal surface in M. Of importance for certain applications is the fact, first discussed in [3], that MOTS admit a notion of stability analogous to that for minimal surfaces.

7.3. **Stability of MOTSs.** In Riemannian geometry, a minimal surface (surface with vanishing mean curvature) is stable provided, for a suitable class of variations, the second variation of area is nonnegative, $\delta^2 A \geq 0$. Stability of minimal surfaces can also be characterized in terms of the associated stability operator. This latter approach extends to MOTSs, as we now describe.

Let Σ be a MOTS in M with outward unit normal ν. Consider a normal variation of Σ in M, i.e. a map $F : (-\epsilon, \epsilon) \times \Sigma \to M$, such that (i) $F(0, \cdot) = \mathrm{id}_\Sigma$ and (ii) $\frac{\partial F}{\partial t}\big|_{t=0} = \phi\nu$, $\phi \in C^\infty(\Sigma)$. Let $\theta(t)$ denote the null expansion of $\Sigma_t := F(t, \Sigma)$ with respect to $l_t = u + \nu_t$, where u is the future directed timelike unit normal to M and ν_t is the outer unit normal to Σ_t in M. A computation shows,

$$\left.\frac{\partial \theta}{\partial t}\right|_{t=0} = L(\phi)\ , \tag{7.8}$$

where $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ is the operator [4, 167],

$$L(\phi) = -\triangle\phi + 2\langle X, \nabla\phi\rangle + \left(\frac{1}{2}S - (\rho + J(\nu)) - \frac{1}{2}|\chi|^2 + \operatorname{div} X - |X|^2\right)\phi\,. \tag{7.9}$$

In the above, $\triangle$, ∇ and div are the Laplacian, gradient and divergence operator, respectively, on Σ, S is the scalar curvature of Σ, ρ and J are the energy density and momentum density, respectively, as defined in Equations (5.3), (5.4), X is the vector field on Σ defined by taking the tangential part of $\nabla_\nu u$ along Σ, and $\langle\, ,\, \rangle$ denotes the induced metric on Σ.

In the time-symmetric case, θ in (7.8) becomes the mean curvature H, the vector field X vanishes and L reduces to the classical stability operator (linearization of the mean curvature operator) of minimal surface theory. In analogy with the minimal surface case, we refer to L in (7.9) as the stability operator associated with variations in the null expansion θ. Although in general L is not self-adjoint, its principal eigenvalue (eigenvalue with smallest real part) $\lambda_1(L)$ is real. Moreover there exists an associated eigenfunction ϕ which is positive on Σ. Continuing the analogy with the minimal surface case, we say that a MOTS is stable provided $\lambda_1(L) \geq 0$. (In the minimal surface case this is equivalent to the second variation of area being nonnegative.) It follows from basic properties of L that a MOTS Σ is stable if and only if there exists a normal variation of Σ, with $\phi > 0$, such that $\frac{\partial\theta}{\partial t}\big|_{t=0} \geq 0$.

Stable MOTSs arise naturally in physical situations, for example, as *outermost* MOTSs. We say Σ is an *outermost* MOTS in M provided there are no weakly outer trapped ($\theta_+ \leq 0$) surfaces outside of, and homologous to, Σ. We say Σ is a *weakly outermost* MOTS in M provided there are no outer trapped ($\theta_+ < 0$) surfaces outside of, and homologous to, Σ. Clearly, "outermost" implies "weakly outermost". Moreover we have the following.

Proposition 7.3. *Weakly outermost MOTSs are stable.*

To see this consider a variation $\{\Sigma_t\}$ of a weakly outermost MOTS Σ with variation vector field $\mathcal{V} = \phi\nu$, where ϕ is a positive eigenfunction associated to the principal eigenvalue $\lambda_1 = \lambda_1(L)$. If $\lambda_1 < 0$, then Equation (7.8) implies $\frac{\partial\theta}{\partial t}\big|_{t=0} = \lambda_1\phi < 0$. Since $\theta(0) = 0$, this implies that, for small $t > 0$, Σ_t is outer trapped, contrary to Σ being weakly outermost.

A standard fact in the theory of black holes is that, for black hole space-times obeying the null energy condition, there can be no weakly outer trapped surfaces

contained in the domain of outer communications (the region outside of all black holes and white holes). It follows that compact cross-sections of the event horizon in stationary black hole space-times obeying the null energy condition are stable MOTSs. Moreover, results of Andersson and Metzger [5,6] provide natural criteria for the existence of outermost MOTSs in initial data sets containing trapped regions; see Section 7.5.

Stable MOTSs share a number of properties in common with stable minimal surfaces. This sometimes depends on the following fact. Consider the "symmetrized" operator $L_0 : C^\infty(\Sigma) \to C^\infty(\Sigma)$,

$$L_0(\phi) = -\triangle\phi + \left(\frac{1}{2}S - (\rho + J(\nu)) - \frac{1}{2}|\chi|^2\right)\phi\,. \tag{7.10}$$

formally obtained by setting $X = 0$ in (7.9). The key argument in [167] shows the following (see also [4,166]).

Proposition 7.4. $\lambda_1(L_0) \geq \lambda_1(L)$.

We will say that a MOTS is symmetric-stable if $\lambda_1(L_0) \geq 0$; hence "stable" implies "symmetric-stable". In the next subsection we consider an application of stable MOTS to the topology of black holes.

7.4. **On the topology of black holes.** A useful step in the proof of black hole uniqueness (see Section 3.9) is Hawking's theorem on the topology of black holes [179] which asserts that compact cross-sections of the event horizon in $3+1$-dimensional, appropriately regular, asymptotically flat stationary black hole space-times obeying the dominant energy condition are topologically 2-spheres. As shown by Hawking [178], this conclusion also holds for outermost MOTSs in space-times that are not necessarily stationary. The proof in both cases is variational in nature, and relies on the classical Gauss-Bonnet theorem. Developments in physics related to string theory have lead to an increased interest in the study of gravity, and in particular black holes, in higher dimensions; see e.g. [149] for a recent review. The remarkable example of Emparan and Reall of a $4+1$ asymptotically flat stationary vacuum black space-time with horizon topology $S^1 \times S^2$, the so-called "black ring", shows that topological uniqueness of black holes does not hold in higher dimensions. This example naturally led to the question of what are the allowable horizon topologies in higher dimensional black hole space-times. This question was addressed in the papers of [166,167], resulting in a natural generalization of Hawking's topology theorem to higher dimensions, which we now discuss.

Consider a space-time $(\mathscr{M}^{n+1}, g)$, $n \geq 3$, satisfying the Einstein equations (not necessarily vacuum), and for simplicity assume that the cosmological constant vanishes, $\Lambda = 0$. Let M be a spacelike hypersurface in $\mathscr{M}$, which gives rise to the initial data set (M, h, K), as in Section 7.3. Recall from Equation (5.15) that the dominant energy condition holds with respect to this initial data set provided $\rho \geq |J|$ along M, where ρ and J are the energy density and momentum density, respectively, as defined in Section 5.1 (but with Λ set to zero).

The following result, obtained in [167], gives a natural extension of Hawking's black hole topology theorem to higher dimensions. Recall, a Riemannian manifold Σ is of *positive Yamabe type* if it admits a metric of positive scalar curvature.

Theorem 7.5. *Let Σ^{n-1} be a stable MOTS in an initial data set (M^n, h, K), $n \geq 3$.*

(1) *If $\rho > |J|$ along Σ then Σ is of positive Yamabe type.*
(2) *If $\rho \geq |J|$ along Σ then Σ is of positive Yamabe type* unless Σ *is Ricci flat (flat if $n = 2, 3$), $\chi = 0$ and $\rho + J(\nu) = 0$ along Σ.*

In the time-symmetric case, Theorem 7.5 reduces to the classical result of Schoen and Yau [301], critical to their study of manifolds of positive scalar curvature, that a compact stable minimal hypersurface in a Riemannian manifold of positive scalar curvature is of positive Yamabe type.

The key to the proof of Theorem 7.5 is Proposition 7.4, which, since Σ is assumed stable, implies that $\lambda_1(L_0) \geq 0$, where L_0 is the operator given in (7.10). Now, in effect, the proof has been reduced to the Riemannian case. Consider Σ in the conformally related metric, $\tilde{\gamma} = \phi^{\frac{2}{n-2}}\gamma$, where γ is the induced metric on Σ and ϕ is a positive eigenfunction corresponding to $\lambda_1(L_0)$. The scalar curvatures $\tilde{S}$ and S of the metrics $\tilde{\gamma}$ and γ, respectively, are related by (compare (5.11))

$$\begin{aligned}\tilde{S} &= \phi^{-\frac{n}{n-2}}\left(-2\triangle\phi + S\phi + \frac{n-1}{n-2}\frac{|\nabla\phi|^2}{\phi}\right)\\ &= \phi^{-\frac{2}{n-2}}\left(2\lambda_1(L_0) + 2(\rho + J(\nu)) + |\chi|^2 + \frac{n-1}{n-2}\frac{|\nabla\phi|^2}{\phi^2}\right),\end{aligned}$$

where for the second equation we have used (7.10). Since $\rho + J(\nu) \geq \rho - |J| \geq 0$, we have that $\tilde{S} \geq 0$. By further standard metric deformations, the scalar curvature of Σ can be made strictly positive, unless various quantities vanish identically.

According to Theorem 7.5, apart from certain exceptional circumstances, a stable marginally outer trapped surface Σ in a space-time $\mathscr{M}$ obeying the dominant energy condition is of positive Yamabe type. Assume for the following discussion that Σ is orientable. Then, in the standard case: $\dim \mathscr{M} = 3 + 1$ (and hence $\dim \Sigma = 2$), Gauss-Bonnet tells us that if Σ is positive Yamabe then Σ is topologically a two-sphere, and we recover Hawking's theorem. In higher dimensions, much is now known about topological obstructions to the existence of metrics of positive scalar curvature. While the first major result along these lines is the famous theorem of Lichnerowicz [229] concerning the vanishing of the $\hat{A}$ genus, a key advance in our understanding was made in the late 1970s and early 1980s by Schoen and Yau [299, 301], and Gromov and Lawson [171, 172]. Let us focus on the case: $\dim \mathscr{M} = 4 + 1$, and hence $\dim \Sigma = 3$. Then by results of Schoen-Yau and Gromov-Lawson, in light of the resolution of the Poincaré conjecture, Σ must be diffeomorphic to a finite connected sum of spherical spaces (spaces with universal cover the 3-sphere) and $S^2 \times S^1$'s. Indeed, by the prime decomposition theorem, Σ can be expressed as a connected sum of spherical spaces, $S^2 \times S^1$'s, and $K(\pi, 1)$ manifolds (manifolds whose universal covers are contractible). But as Σ admits a metric of positive scalar curvature, it cannot have any $K(\pi, 1)$'s in its prime decomposition. Hence the basic horizon topologies in $\dim \mathscr{M} = 4 + 1$ are S^3 and $S^2 \times S^1$ (in the sense that Σ is "built up" from such spaces), both of which are realized by nontrivial black hole space-times. It remains an interesting open question which topologies of positive Yamabe type can be realized as outermost MOTSs; see [307] for examples involving products of spheres.

A drawback of Theorem 7.5 is that it allows certain possibilities that one would like to rule out: for example, the theorem does not rule out the possibility of a vacuum black hole space-time with toroidal horizon topology. (This borderline case also arises in the proof of Hawking's theorem). In fact, one can construct

examples of stable toroidal MOTSs in space-times obeying the dominant energy condition. Such MOTSs cannot, however, be outermost, as the following theorem asserts.

Theorem 7.6 ([166]). *Let Σ be an outermost MOTS in the spacelike hypersurface M, and assume the dominant energy condition* (3.15) *holds in a space-time neighborhood of Σ.*[23] *Then Σ is of positive Yamabe type.*

As an immediate corollary, we have that *compact cross-sections of event horizons in regular stationary black hole space-times obeying the dominant energy condition are of positive Yamabe type.* In particular, there can be no toroidal horizons.

Theorem 7.6 is an immediate consequence of the following rigidity result.

Theorem 7.7 ([166]). *Let Σ be a weakly outermost MOTS in the spacelike hypersurface M, and assume the dominant energy condition holds in a space-time neighborhood of Σ. If Σ is not of positive Yamabe type then there exists an outer neighborhood $U \approx [0,\epsilon) \times \Sigma$ of Σ in M such that each slice $\Sigma_t = \{t\} \times \Sigma$, $t \in [0,\epsilon)$ is a MOTS.*

Theorem 7.7 is proved in two stages. The first stage, and the main effort, is to establish Theorem 7.7 subject to the additional assumption that M has *nonpositive* mean curvature, $\tau \leq 0$. This is a purely initial data result, and the proof is carried out in two steps. The first step involves an inverse function theorem argument to show that an outer neighborhood of Σ can be foliated by surfaces Σ_t of constant null expansion, $\theta(t) = c_t$. This uses the stability of Σ in a critical way. In the second step it is shown that all of these constants are zero, $c_t = 0$. It is here where the sign of the mean curvature of M needs to be controlled. Once having proved Theorem 7.7, subject to the condition, $\tau \leq 0$, the next stage, which is actually easy, is a deformation argument (specifically, a deformation of M near Σ) that reduces the problem to the case $\tau \leq 0$.

We remark that Riemannian versions of Theorem 7.7 had previously been considered in [64, 65].

7.5. **Existence of MOTSs.** As mentioned earlier, compact cross-sections of the event horizon in regular stationary black hole space-times are necessarily MOTSs. In dynamical black hole space-times, it is typical for trapped or outer trapped surfaces to form in the black hole region. But the occurrence of an outer trapped surface in a spacelike hypersurface that obeys a mild asymptotic flatness condition leads to the existence of a MOTS, as follows from a result described by Schoen [297] and recently proved by Eichmair [146], and also by Andersson and Metzger [6] in somewhat lower dimension.

Let Ω be a relatively compact domain in a spacelike hypersurface M, with smooth boundary $\partial\Omega$. We assume that $\partial\Omega$ decomposes as a disjoint union of components, $\partial\Omega = \Sigma_{\text{inn}} \cup \Sigma_{\text{out}}$, where we think of Σ_{inn} as the "inner" boundary and Σ_{out} as the "outer" boundary of Ω in M. We choose the normal along Σ_{inn} that points into Ω, and the normal along Σ_{out} that points out of Ω, so that both normals point towards the region exterior to Σ_{out}. Thus, Σ_{inn} is outer trapped if $\theta < 0$ with respect to the future directed null normal field along Σ_{inn} that projects into Ω. We say that Σ_{out}

[23]Note that since we are assuming $\Lambda = 0$ here, the dominant energy condition is equivalent to the condition, $G_{\rho\nu}X^\rho Y^\nu \geq 0$ for all future directed causal vectors X, Y, where $G_{\mu\nu}$ is the Einstein tensor.

is *outer untrapped* if $\theta > 0$ with respect to the future directed null normal along $\Sigma_{\rm out}$ that projects out of Ω. Heuristically, if $\Sigma_{\rm out}$ is lying in a region where M is "flattening out" then we expect $\Sigma_{\rm out}$ to be outer untrapped.

With this notation and terminology we have the following existence result for MOTS.

Theorem 7.8 ([146]). *Let M^n be a spacelike hypersurface in a space-time $(\mathscr{M}^{n+1}, g)$, with $n \le 7$. Let Ω be a relatively compact domain in M, with smooth boundary $\partial\Omega = \Sigma_{inn} \cup \Sigma_{out}$, such that the inner boundary Σ_{inn} is outer trapped and the outer boundary Σ_{out} is outer untrapped, as described above. Then there exists a smooth compact MOTS in Ω homologous to Σ_{inn}. Moreover, Σ is symmetric stable.*

In fact, by arguments in [5], the MOTS constructed in Theorem 7.8 will actually be stable.

In the time-symmetric case, Theorem 7.8 reduces to a well-known existence result for stable minimal surfaces. In the time-symmetric case the barrier conditions in Theorem 7.8 simply say that Ω is a mean convex domain. One can then minimize area in the homology class of a surface in Ω parallel to $\Sigma_{\rm inn}$ and apply standard compactness and regularity results of geometric measure theory [153] to obtain a smooth (provided $\dim M \le 7$) stable minimal surface in Ω homologous to $\Sigma_{\rm inn}$. However, since MOTS do not arise as stationary points of some elliptic functional, such a procedure does not work for general initial data. A completely different approach must be taken.

In fact, the proof of Theorem 7.8 is based on Jang's equation [207] which is closely related to the MOTS condition $\theta^+ = 0$. Given an initial data set (M, g, K), consider the Riemannian product manifold, $\hat{M} = \mathbb{R} \times M$, $\hat{g} = dt^2 + g$, and extend K to $\hat{M}$ by taking it to be constant along the t-lines. Given a function f on M, consider its graph, $N_f = \operatorname{graph} f = \{(t, x) \in \hat{M} : t = f(x), x \in M\}$, equipped with the induced metric. Then Jang's equation is the equation,

$$(H + P)(f) = H(f) + P(f) = 0\,, \tag{7.11}$$

where $H(f) =$ the mean curvature of N_f and $P = \operatorname{tr}_{N_f} K$ (compare with Equation (7.7)).

Schoen and Yau [303] established existence and regularity for Jang's equation with respect to asymptotically flat initial sets as part of their approach to proving the positive mass theorem for general, nonmaximal, initial data sets. In the process they discovered an obstruction to global existence: Solutions to Jang's equation tend to blow-up in the presence of MOTS in the initial data (M, g, K). This problematic blow-up behavior that Schoen and Yau had to contend with has now been turned on its head to become a feature of Jang's equation: In order to establish the existence of MOTS, one induces blow-up of the Jang equation. This is the approach taken in [6, 146] in a somewhat different situation.

In order to obtain solutions to Jang's equation one considers the regularized equation,

$$(H + P)(f_t) = t f_t\,. \tag{7.12}$$

In [146] Eichmair uses a Perron method to obtain solutions f_t to (7.12) for t sufficiently small, with values that tend to infinity in a small collared neighborhood of $\Sigma_{\rm out}$, and that tend to minus infinity in a small collared neighborhood of $\Sigma_{\rm inn}$, as $t \to 0$. The construction of Perron sub and super solutions makes use of the

barrier conditions. Using the "almost minimizing" property [146] of the graphs $N_t := \operatorname{graph} f_t$, one is able to pass to a smooth subsequential limit manifold N, bounded away from $\partial\Omega$, each component of which is either a cylinder or a graph that asymptotes to a cylinder. The projection into M of such a cylinder produces the desired MOTS.

Under the barrier conditions of Theorem 7.8, Andersson and Metzger [5] were able to make use of this basic existence result, to establish for 3-dimensional initial data sets, the existence of an *outermost* MOTS, thereby providing a rigorous proof of a long held "folk belief" in the theory of black holes that the boundary of the so-called outer trapped region in a time slice of space-time is a smooth MOTS. The key to proving the existence of an outermost MOTS is a compactness result for stable MOTS which follows from the extrinsic curvature estimates obtained by Andersson and Metzger in [5], together with an area bound obtained in [6]. This area bound follows from an outer injectivity radius estimate for a certain class of MOTS, established by an interesting surgery procedure. The powerful methods developed by Eichmair in [145, 146] can be used to extend this result to n-dimensional initial data sets, $3 \leq n \leq 7$.

Consider, now, a foliation $\{M_t\}$, $a < t < b$ of a region of a space-time $(\mathscr{M}^{n+1}, g)$ by spacelike hypersurfaces M_t. If each slice M_t admits an outermost MOTS Σ_t, then the family of MOTSs $\{\Sigma_t\}$ may form a hypersurface in space-time. A smooth spacelike hypersurface H in a space-time $(\mathscr{M}^{n+1}, g)$ foliated by MOTS is called a *marginally outer trapped tube* (MOTT). In [3, 4], Andersson, Mars and Simon have obtained a rigorous existence result for MOTTs. Consider a spacelike foliation $\{M_t\}$, $a < t < b$, and suppose Σ_{t_0} is a MOTS in M_{t_0}. They prove that if Σ_{t_0} is *strictly* stable, i.e. if the principal eigenvalue of the associated stability operator (7.9) is strictly positive, then there exists a MOTT H such that for t close to t_0, $\Sigma_t := H \cap M_t$ is a MOTS.

Many challenging questions concerning the global existence and behavior of MOTTs remain open. Building on the analysis of Dafermos [133], Williams [327] gave sufficient conditions on spherically symmetric black hole space-times satisfying the dominant energy condition, insuring that a spherically symmetric MOTT exists, is achronal, and is asymptotic to the event horizon. Understanding the generic asymptotic behavior of MOTTs in the nonspherically symmetric case remains an interesting open problem.

A MOTT satisfying certain supplementary conditions gives rise to the notion of a *dynamical horizon*, which provides an alternative, quasi-local description of a black hole. For an extensive review of dynamical horizons and related concepts, including physical applications, see [13].

Appendix A. Open problems

We compile here a list of interesting open problems discussed in the paper. They all appear to be difficult, of varying degrees of difficulty, with some most likely intractable in the foreseeable future.

(1) Classify all vacuum near-horizon geometries with compact cross-sections, i.e., Riemannian metrics on compact manifolds satisfying (3.19), p. 20, in low dimensions.
(2) Remove the hypotheses of analyticity, non-degeneracy, and connectedness in the black-hole uniqueness Theorem 3.4, p. 24.

(3) Show that five dimensional, stationary, I^+–regular vacuum black holes have to have three Killing vectors, or construct a counterexample. More generally, classify such black holes. Compare p. 24.
(4) Prove that the positive energy theorem holds in all dimensions without the spin assumption. Compare pp. 25 and 38.
(5) Find the optimal differentiability conditions for Theorem 4.1. More generally, construct a coherent local well posedness theory for the evolution and constraint equations for metrics with low differentiability; compare Remark 4.2, p. 28.
(6) Describe in a constructive way the set of solutions of the vacuum constraint equations on compact, asymptotically flat, and asymptotically hyperbolic manifolds, with arbitrary smooth initial data. Compare p. 36.
(7) Find a well posed initial boundary value problem for the vacuum Einstein equations which is well suited for numerical treatment, and prove numerical convergence. See Section 4.6, p. 33.
(8) Formulate, and prove, a precise version of Conjecture 6.10, p. 58. More generally, formulate and prove a precise version of the BKL conjecture, or find an open set of metrics developing a singularity which do not exhibit a BKL-type behavior.
(9) Show that generic Bianchi IX orbits have a dense ω–limit set on the Kasner circle; compare p. 56.
(10) Show that uniqueness and existence of maximal globally hyperbolic developments, Theorem 6.2, p. 45, holds in an optimal weak differentiability class. This requires revisiting the whole causality theory for metrics of low differentiability.
(11) Remove the condition of closed generators in Theorem 6.3, p. 45. More generally, show that existence of Cauchy horizons, not necessarily compact or analytic, implies existence of local isometries, or construct a counterexample.
(12) Generalize Ringström's Theorem 6.4, p. 46, to Gowdy models on $S^2 \times S^1$, on S^3, and on lens spaces $L(p, q)$.
(13) Show that some twisting $U(1) \times U(1)$ symmetric vacuum models have mixmaster behavior. More generally, find an open set of such models with mixmaster behavior. Even better, analyze exhaustively the asymptotic behavior of those models; see Section 6.2.2, p. 47.
(14) Show that degenerate asymptotically flat spherically symmetric Einstein-Maxwell-scalar field solutions are non-generic; compare Section 6.2.3, p. 48. Analyze what happens in solutions in which trapped surfaces do not form. More generally, prove weak and strong cosmic censorship within this class of space-times.
(15) Find an open set of $U(1)$ symmetric metrics where the dynamics can be analyzed in the contracting direction. More generally, analyze exhaustively the dynamics of those models, see Section 6.4.1, p. 49.
(16) Show that polyhomogeneous initial data in Theorem 6.7, p. 53, lead to solutions with polyhomogeneous behavior at null infinity.
(17) Show that the Kerr solution is stable against small vacuum perturbations, compare Section 6.6, p. 54.
(18) Determine what topologies of positive Yamabe type can be realized as outermost MOTSs in space-times satisfying the dominant energy condition, compare p. 66.
(19) Prove initial data equivalents of Theorems 7.6 and 7.7, p. 67.
(20) Describe the generic behavior near the event horizon of marginally trapped tubes in black hole space-times, see p. 69.

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