On the structure of spatial infinity. I. The Geroch structure

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Theorems on uniqueness and "quasi-uniqueness" of the differentiable structure of pointwise singular conformal structures are derived. This allows the classification of all ambiguities in the differentiable structure of conformally completed asymptotic three-dimensional ends.

I. INTRODUCTION

A considerable amount of work on the behavior of the gravitational field at spatial infinity has been done by several authors[1-5] in order to understand how to overcome the apparent lack of definitional uniqueness of four-momentum in general relativity. An interesting framework has been developed by Geroch,[2,6] who adapted the conformal completion technique used in the description of null infinity to the spatial setting, the main idea underlying the construction being to replace "coordinate conditions" by "local differential geometry." In Geroch's framework, the problem of a possibly ill-defined energy momentum reappears in the possibility of existence of inequivalent conformal completions of spatial infinity. In this paper we show that there exists only the well-known three-parameter family of inequivalent—in a sense to be made precise—completions of a three-dimensional asymptotically flat end. The proof we give is almost a filling-in of necessary technicalities in the proof outlined in the Appendix of Ref. 7, the (mild) difficulties arising from the fact that standard results of the PDE theory do not apply here in such a simple manner as in Ref. 7, as a result of the essentially singular character of the conformally rescaled metric at i_0. We shall use two sets of conditions to be satisfied by the conformal completions, the first under which uniqueness of the structure holds, the second in which the three-parameter (n parameter in n dimensions) family of inequivalent completions appears. The conditions we shall use are in spirit those of Geroch,[2,6] though our conditions are (slightly) stronger than those originally introduced. It would be interesting to find out whether the results obtained here can be reproduced under the conditions of Ref. 2. It should be stressed that here we establish uniqueness or quasi-uniqueness of pointwise singular conformal structures rather than of conformal compactifications; a simpler proof of the latter problem, under slightly different conditions, can be obtained using the results of Refs. 4 or 5 (cf., e.g., Ref. 8). To be specific only the three-dimensional problem is considered; similar results hold in any dimension.

II. UNIQUE POINTWISE SINGULAR CONFORMAL STRUCTURES

Following Geroch we shall consider a one-point compactification $\overline{\Sigma} = \Sigma \cup \{i_0\}$ of the initial data (three-dimensional Riemannian) slice Σ, giving $\overline{\Sigma}$ the standard topology. Specifically, such a compactification will be obtained, e.g., by performing the inversion $x^i \to y^i = x^i/r(x)^2$ in some asymptotically flat coordinate system and adding the origin $y^i = 0$ to the set so obtained. This allows one to use the coordinates y^i to define the differentiable structure of $\overline{\Sigma}$ in a neighborhood of i_0. If one applies this procedure to a metric of the form

$$g_{ij} = \delta_{ij} + h_{ij}, \quad h_{ij} = O(r^{-1}), \tag{2.1}$$

the resulting conformally rescaled metric in coordinates y^i will only be Lipschitz continuous at i_0 rather than differentiable, therefore a natural procedure is to include in the atlas those coordinate transformations, the derivatives of which are merely Lipschitz continuous at i_0. (If one does not enlarge the, say C_k, atlas fixed by the coordinates y to the $A_{k,\alpha}$ atlas, as described below, one will end up with infinitely many conformally inequivalent completions of any asymptotically flat space-time, which seems to be quite a luxury. Another way of looking at this issue is that none of the $A_{k,\alpha}$ coordinate systems is singled out by physics, so that an $A_{k,\alpha}$ rather than a C_k structure on $\Sigma \cup \{i_0\}$ is physically appropriate.) To make things precise, let $B(R)$ be a ball of radius R and let us define the set $A_{k,\alpha}(B(R))$, $k \geqslant 1$, $\alpha \in (0,1]$, of functions on $B(R)$, such that $f \in C_1(B(R))$, f is C_k on $B(R) \setminus \{0\}$ and f satisfies

$$|\partial_i f(y) - \partial_i f(0)| \leqslant Cr(y)^{\alpha}, \ldots, |\partial_{i_1} \cdots \partial_{i_k} f| \leqslant C r^{\alpha - k + 1}, \tag{2.2}$$

for some constant C. An $A_{k,\alpha}$ differentiable structure on $\overline{\Sigma}$ will be defined by a maximal atlas on $\overline{\Sigma}$ such that (a) $\overline{\Sigma} \setminus \{i_0\}$ is a C_k manifold and (b) in local coordinates, in a neighborhood of $i_0 = 0$, the transition functions are in $A_{k,\alpha}(B(R))$.

The $A_{k,\alpha}(\overline{\Sigma}, i_0)$ functions on $\overline{\Sigma}$ are defined in an obvious manner. We shall write $A_{k,\alpha}$ rather than $A_{k,\alpha}(\overline{\Sigma}, i_0)$, since confusion is unlikely to occur. A tensor field will be called of class $B_{l,\alpha}(\overline{\Sigma}, i_0)$ ($B_{l,\alpha}$), $l \geqslant 0$, if its components t_I in a map belonging to the $A_{k,\alpha}$ atlas, $k \geqslant l + 1$, where I is a set of covariant and contravariant indices, are C_l on Σ and satisfy

$$\forall y \in \mathcal{O} \setminus \{0\} \quad |t_I(y) - t_I(0)| \leqslant C' r(y)^{\alpha}, \ldots, |\partial_{i_1} \cdots \partial_{i_l} t_I| \leqslant C' r^{\alpha - l} \tag{2.3}$$

for some constant C', where $\mathcal{O}$ is a coordinate neighborhood

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of $0 = i_0$. (It is simple to check that if such a constant exists for one map, there will be some constant for any map from the $A_{k,\alpha}$ atlas.) We shall say that a Riemannian manifold (Σ, g) is asymptotically flat if there exists an $A_{k,\alpha}$, $k \geqslant 3$, Riemannian manifold $\overline{\Sigma} = \Sigma \cup \{i_0\}$ with $B_{k-1,\alpha}$ metric $\bar{g}$ and a function Ω: $\overline{\Sigma} \to \mathbb{R}^+ \cup \{0\}$, such that (1) on Σ we have $g_{ij} = \Omega^{-2}\bar{g}_{ij}$ and (2) $\Omega(i_0) = 0$, $\nabla\Omega(i_0) = 0$, $\forall i\, \partial_i \Omega \in A_{1,\alpha}$, and $\partial_i\, \partial_j \Omega(i_0) = 2\bar{g}_{ij}(i_0)$.

To prove our uniqueness results we shall need two auxiliary lemmas.

Lemma 2.1: Let $\alpha \in (0,1]$, let g be a metric in $B(r_0)$ (= ball of radius r_0), satisfying

$$|g_{ij}(x) - g_{ij}(0)| \leqslant Cr^{\alpha}, \quad |\partial_k g_{ij}| \leqslant Cr^{\alpha-1}, \ldots, \quad |\partial_{i_1} \cdots \partial_{i_l} g_{ij}| \leqslant Cr^{\alpha-l}, \tag{2.4}$$

$l \geqslant 1$, and let c: $B(r_0) \setminus \{0\} \to \mathbb{R}$ satisfy

$$|c| \leqslant Cr^{\alpha-2}, \quad |\partial_i c| \leqslant Cr^{\alpha-3}, \ldots, |\partial_{i_1} \cdots \partial_{i_k} c| \leqslant Cr^{\alpha-2-k}, \tag{2.5}$$

$k \geqslant 0$. There exists $0 < r_1 \leqslant r_0$ and a function f: $B(r_1) \setminus \{0\} \to \mathbb{R}$, a weak solution of

$$(\Delta_g + c) f = 0, \tag{2.6}$$

in $B(r_1) \setminus \{0\}$, satisfying $\frac{1}{4} \leqslant f \leqslant 4$.

Proof: Equation (2.6) for f is equivalent to the equation

$$(\Delta_{\bar{g}} + \bar{c})\, \bar{f} = 0, \tag{2.7}$$

where $\bar{g}_{ij} = u^{4/(n-2)} g_{ij}$, $\bar{f} = f/u$, and $\bar{c} = cu^{-4/(n-2)} + u^{-(n+2)/(n-2)} \Delta_g u$ (in dimension n). Let $u = 1 - ar^{\alpha}$, $a \in \mathbb{R}^+$. For $r < r_1$, small enough, one finds

$$\Delta_g u \leqslant -a\alpha(n - 2 + \alpha) r^{-2+\alpha}/2,$$

so that increasing a and decreasing r_1 if necessary one has (for $r \leqslant r_1$) $\bar{c} \leqslant 0$ and $u \geqslant \frac{1}{2}$, and the metric $\bar{g}$ satisfies inequalities of the form (2.4). The same calculation shows that for positive C_1 and C_2 large enough, again decreasing r_1 if necessary, the functions $u^+ = 2 - C_1 r^{\alpha}$ and $u^- = \frac{1}{2} + C_2 r^{\alpha}$ are supersolutions and subsolutions, respectively, for Eq. (2.7), and we have $u^+_{|S(r_1)} \geqslant 1$, $u^-_{|S(r_1)} \leqslant 1$, where $S(r_1)$ denotes a sphere of radius r_1. Let $\bar{f}_k$ be the sequence of solutions of (2.7) in $B(r_1) \setminus B(r_1/k)$ satisfying $\bar{f}_{k|S(r_1)} = \bar{f}_{k|S(r_1/k)} = 1$. By the comparison principle we have $u^- \leqslant \bar{f}_k \leqslant u^+$, wherever defined. By $C_{1,\epsilon}$ estimates, as given, e.g., in Ref. 9 and a standard diagonalization procedure one can extract a sequence f_{i_k} converging to a function $\bar{f} = \bar{f}_{\infty}$, a weak solution of (2.7) in $B(r_1) \setminus \{0\}$, satisfying $1/2 \leqslant \bar{f} \leqslant 2$. f is defined by $f \equiv u\bar{f}$.

Lemma 2.2: Under the hypotheses of Lemma 2.1, let f: $B(r_1) \setminus \{0\} \to \mathbb{R}$ be a weak solution of (2.6) in $B(r_1) \setminus \{0\}$ satisfying $c_0 \leqslant f \leqslant c_1$, $c_0, c_1 \in \mathbb{R}$. Then f satisfies (2.6) in $B(r_1)$ in a weak sense, and there exist constants $f(0)$ and C_2, such that

(a) $|f(x) - f(0)| \leqslant C_2 r^{\alpha}$,

$$|\partial f| \leqslant C_2 r^{\alpha-1}, \ldots, |\partial_{i_1} \cdots \partial_{i_m} f| C_2 r^{\alpha-m}, \quad \text{if } 0 < \alpha < 1,$$

(b) $|f(x) - f(0)| \leqslant C_2 r \ln r$,

$$|\partial f| \leqslant C_2 \ln r, \ldots, |\partial_{i_1} \cdots \partial_{i_m} f| C_2 r^{\alpha-m} \ln r, \quad \text{if } \alpha = 1,$$

where $m = \min(l, k+1)$.

Proof: Because f is bounded from above and below the removable singularity theorem of Serrin[10] (Theorem 1) implies that f is a weak solution in $B(r_1)$ so that by Theorem 8 of Ref. 11, f is Hoelder continuous at 0 with some exponent ϵ. Performing an inversion, $r \to \rho = 1/r$, $f \to \bar{f} = rf$, one finds that $\bar{f}$ satisfies the following equation (It is not too difficult, using, e.g., the methods of Ref. 13, to prove the desired estimates directly without performing an inversion. We use the inversion argument for simplicity, to be able to use the classical results of Ref. 12):

$$\Delta_{\bar{g}} \bar{f} = O(\rho^{-3-\epsilon}), \quad \text{in } \mathbb{R}^n \setminus B(1/r_1),$$

where $\bar{g}$ is the conformally rescaled metric in $\mathbb{R}^n \setminus B(1/r_1)$, $\bar{g}_{ij} = \rho^4 g_{ij}$. An iterative application of the estimates of Meyers (Ref. 12, Lemma 5) (cf. also, e.g., Ref. 13) implies $\bar{f} = O(\rho^{-1-\alpha})$ for $0 < \alpha < 1$ or $\bar{f} = O(\rho^{-2} \ln \rho)$ for $\alpha = 1$, a standard scaling argument (cf., e.g., Ref. 12) gives the derivatives estimates, inverting back to the original variables our claims follow.

The idea of the proof of Proposition 2.3, which is the key result to prove our theorems, is due to Geroch (cf. the Appendix in Ref. 7).

Proposition 2.3: Let $0 < \alpha < 1$, and let g^1 and g^2 be two metrics in neighborhoods Ω_1 and Ω_2 of the origin conformally related to each other in $\Omega_i \setminus \{0\}$:

$$\text{for } x \neq 0, \ g^1_{ij}(x) = \phi^2(x) g^2_{kl}(y(x)) \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j}, \quad \phi(x) > 0;$$

$y(x)$ continuous in Ω_1, $y(0) = 0$, and $y(x) C_1$ for x different from 0, and let the metrics g^a satisfy the inequalities (2.4) with $l \geqslant 2$. The function ϕ can be extended to a continuous strictly positive function on Ω_1, satisfying

$$|\partial\phi| \leqslant Cr^{\alpha-1},$$

for some constant C.

Proof: Suppose first that $l \geqslant 3$. Lemma 2.1 implies the existence of functions ϕ_1 and ϕ_2, $0 < \frac{1}{4} < \phi_{\alpha} < 4$, such that

$$(\Delta_{g^a} - R_a/8)\phi_a = 0 \tag{2.8}$$

in suitable neighborhoods of the origins, where R_a denotes the Ricci scalar of g^a. (2.8) implies $\bar{R}_a = 0$, where $\bar{R}_a$ is the Ricci scalar of the conformally rescaled metric $\bar{g}^a_{ij} = \phi_a^4 g^a_{ij}$, and the estimates of Lemma 2.2 show that the rescaled metrics $\bar{g}^a_{ij}$ also satisfy inequalities of the form (2.4) with $l = 2$. Let $\psi = \phi_2^{-1} \phi^{1/2} \phi_1$. $\bar{R}_a = 0$ and $\bar{g}^1_{ij} = \psi^4 \bar{g}^2_{ij}$ imply

$$\forall x \in \Omega_1 \setminus \{0\} (\Delta_{\bar{g}^1} - \bar{R}_1/8)\psi = \Delta_{\bar{g}^1}\psi = 0,$$
$$\forall y \in \Omega_2 \setminus \{0\} (\Delta_{\bar{g}^2} - \bar{R}_2/8)(\psi^{-1}) = \Delta_{\bar{g}^2}(\psi^{-1}) = 0. \tag{2.9}$$

Since $\psi \geqslant 0$, a theorem by Serrin (Theorem 2, Ref. 10) implies that either ψ behaves as $1/r$ or ψ is bounded and satisfies (2.9) throughout Ω_1—whichever case occurs the maximum

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principle applies and ψ is bounded in $\Omega_1\setminus\{0\}$ from below by the minimum of its values on $\partial\Omega_1$, which is strictly positive by construction of ϕ_a and by the hypothesis on ϕ. Similarly, ψ^{-1} either blows up as $1/r$ or is bounded from above and is bounded from below by a strictly positive constant. If ψ behaves like $1/r$ then ψ^{-1} must vanish at the origin, which leads to a contradiction; therefore both ψ and ψ^{-1} are bounded and Lemma 2.2 gives derivative estimates for ψ. Since g^1 is conformally equivalent to g^2 with a bounded conformal factor, a standard argument (cf., e.g., the proof of Lemma 1 in Ref. 5) implies $c^{-1}r(x)\leqslant r(y(x))\leqslant cr(x)$ for some constant c, so that from $|\nabla\phi_1|_{g_1}\leqslant Cr(x)^{\alpha-1}$, $|\nabla\phi_2|_{g_2}\leqslant Cr(y)^{\alpha-1}$ and $\phi=\psi^2\phi_1^{-2}\phi_2^2$ one obtains the derivatives estimate for ϕ. The case $l=2$ is obtained by an approximation argument.

Theorem 2.4 (Uniqueness of conformal $A_{k,\alpha}$ structure): Let (Σ_1,i_1,g_1), (Σ_2,i_2,g_2) be $A_{k,\alpha}$ manifolds with $B_{k-1,\alpha}$ metrics, $\alpha\in(0,1)$, $k\geqslant 3$; let Ψ be a continuous conformal mapping from Σ_1 to Σ_2, $\Psi(i_1)=i_2$, Ψ differentiable in $\Sigma_1\setminus\{i_1\}$. Then Ψ is $A_{k,\alpha}$.

Proof: In local coordinates in neighborhoods of i_1 and i_2 the hypotheses of Proposition 2.3 are satisfied, therefore the metric $\bar g^1_{ij}=\phi^{-2}g^1_{ij}$, ϕ as in Proposition 2.3, is of class $B_{1,\alpha}$. Replacing g^1 by $\bar g^1$ and $\lim_{r\to\infty}$ by $\lim_{r\to 0}$ in the proof of Lemma 1 of Ref. 5 shows that Ψ is $A_{2,\alpha}$. The formula for the transformation of the Ricci tensor under conformal changes of the metric can be written in the form

$$\phi^{-1}_{;ij}=\frac{1}{(n-2)}\{\phi^{-1}(R^{\,1}_{ij}-R^{\,2}_{ij})+2(n-2)\phi\phi_{,i}^{-1}\phi_{,j}^{-1}-(\Delta_{g_1}\phi^{-1}+(n-3)\phi|\nabla\phi^{-1}|^2_{g_1})g^1_{ij}\},\tag{2.10}$$

and we also have

$$\Delta_{g_1}\phi^{-1}=\{-\phi^{-3}R_2+\phi^{-1}R_1-(n-1)(n-4)\phi|\nabla\phi^{-1}|^2_{g_1}\}/2\{(n-1)\}.\tag{2.11}$$

In local coordinates x^i in a neighborhood of i_1, the right-hand side of (2.11) is bounded by $Cr^{\alpha-2}$, so that (2.10) implies that ϕ is twice differentiable outside the origin and

$$|\partial_i\,\partial_j\phi|\leqslant Cr^{\alpha-2}.\tag{2.12}$$

The right-hand side of the equation

$$\frac{\partial^2 y^i}{\partial x^k\,\partial x^l}=\bar\Gamma^{1m}_{kl}\frac{\partial y^i}{\partial x^m}-\Gamma^{2i}_{mn}\frac{\partial y^m}{\partial x^k}\frac{\partial y^n}{\partial x^l},\tag{2.13}$$

where $\bar\Gamma^{1m}_{kl}$ and Γ^{2i}_{mn} are the Christoffel symbols of the metrics $\phi^{-2}g^1_{ij}$ and g^2_{ij}, respectively, is in view of (2.12) differentiable with derivatives bounded by $Cr^{\alpha-2}$ for some constant C, which shows that ψ is $A_{3,\alpha}$. The equations obtained by differentiation of (2.10), (2.11), and (2.13) imply $\psi\in A_{k,\alpha}$ by induction.

III. THE LOGARITHMIC AMBIGUITIES

In Sec. II we have established uniqueness of $A_{k,\alpha}$ conformal structures, $\alpha\in(0,1)$. In Geroch's analysis one needs some more structure, which we introduce below, it should, however, be noted that the well-known condition $\alpha>\frac{1}{2}$ (Ref. 4, 5, and 14) is sufficient for a meaningful definition of energy in Geroch's framework (cf., e.g., Ref. 8). We shall say that a function f is of class $\overline{A}_k(\overline{\Sigma},i_0)$ ($\overline{A}_k$) if f is C_1 on $\overline{\Sigma}$ and if the following limits

$$\lim_{r\to 0}\partial_i f(r\mathbf{n}),\ldots,\lim_{r\to 0}r^{k-2}\partial_{i_1}\cdots\partial_{i_k}f(r\mathbf{n})$$

exist, where $\mathbf{n}$ is any unit vector. As in Sec. II we can define an $\overline{A}_k$ differentiable structure and $\overline{B}_l$, $l\leqslant k-1$ tensors. The following theorem shows that there exists only a three-parameter family (parametrized by the vector $\overline{C}^i$) of inequivalent $\overline{A}_k$ structures compatible with a given conformal geometry.

Theorem 3.1 (Quasi-uniqueness of conformal $\overline{A}_k$ structures): Let (Σ_1,i_1,g_1), (Σ_2,i_2,g_2) be $\overline{A}_k$ manifolds, $k\geqslant 3$, with $\overline{B}_{k-1}$ metrics, let Ψ be a continuous conformal mapping from Σ_1 to Σ_2, $\Psi(i_1)=i_2$, Ψ of class C_1 in $\Sigma_1\setminus\{i_1\}$. Then there exist $\overline{A}_k$ charts $\{\bar x^i\}$, $\{\bar y^i\}$ in neighborhoods of i_1 and i_2 and a (constant) vector $\overline{C}^k$, such that, in local coordinates, Ψ takes the form

$$\bar y^i=\bar x^i+(\overline{C}^i\bar r^2-2\overline{C}^k\bar x^k\bar x^i)\ln\bar r,\quad \bar r=r(\bar x).$$

In particular, ψ is $\overline{A}_k$ if and only if $\overline{C}^i=0$.

Proof: For any $\alpha<1$, ψ is of class $A_{k,\alpha}$ by Theorem 2.4. There exist $\overline{A}_k$ coordinates $\{x^i\}$, $\{y^i\}$ in neighborhoods of i_1 and i_2, such that $x^i(i_1)=y^i(i_2)=0$, $g^1_{ij}(0)=g^2_{ij}(0)=\delta_{ij}$, $\phi(0)=1$. The estimates of Lemma 2.2, part (b) and Eqs. (2.10) and (2.11) yield an equation of the form

$$\Delta_{\text{flat}}\phi=\Delta_{\text{flat}}\phi-\Delta_{g_2}\phi+\Delta_{g_2}\phi=c(\theta,\varphi)/r+o(1/r),\tag{3.1}$$

for some function $c(\theta,\varphi)$, so that, e.g., an inversion argument, as in the proof of Lemma 2.2 together with the estimates of Ref. 12, lead to

$$\phi=1+C^ix^i\ln r+rg(\theta,\varphi)+o(r),\tag{3.2}$$

for some constants C^i and a function $g(\theta,\varphi)$, which implies

$$\bar\Gamma^{1m}_{kl}=(\delta^m_kC^l+\delta^m_lC^k-\delta^k_lC^m)\ln r+A^{\,m}_{kl}(\theta,\varphi)+o(1),\tag{3.3}$$

for some functions $A^{\,m}_{kl}(\theta,\varphi)$. (3.3) inserted in (2.13) gives

$$\frac{\partial}{\partial r}\frac{\partial y^k}{\partial x^i}=n^l\frac{\partial^2y^k}{\partial x^i\,\partial x^l}=(n^kC^i+\delta^k_iC^jn^j-n^iC^k)\ln r+A^{\,k}_i(\theta,\varphi)+o(1),\tag{3.4}$$

$n^i=x^i/r$, with some functions $A^{\,k}_i(\theta,\varphi)$. Twice integrating (3.4) along rays, rigidly rotating the coordinates x if necessary, $x^i\to\bar x^i=\omega^i_jx^j$, $\omega^i_j\in O(3)$, one obtains

$$y^k=\bar x^k+(2\overline{C}^j\bar x^j\bar x^k-\bar r^{\,2}\overline{C}^k)\ln\bar r+\bar r^{\,2}A^{\,k}(\theta,\varphi)+o(\bar r^{\,2}),\tag{3.5}$$

for some functions $A^{\,k}(\theta,\varphi)$. Let

$$\bar y^k=\bar x^k+(2\overline{C}^j\bar x^j\bar x^k-\bar r^{\,2}\overline{C}^k)\ln\bar r.$$

Then (3.5) yields

$$y^k=\bar y^k+r(\bar y)^2A^{\,k}(\theta,\varphi)+o(\bar r^{\,2}),\tag{3.6}$$

which is an $\overline{A}_k$ transformation and the theorem follows.

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IV. CONCLUSIONS

Our results show that Geroch's description of spatial infinity is a fairly natural one, since the conformal $\overline{A}_k$ completions are "almost uniquely" defined by the asymptotic behavior of the metric: the only ambiguities consist of a three-parameter family of "logarithmic transformations." It is well known (cf., e.g., Ref. 15) that these transformations do not affect the numerical value of four-momentum. It must be stressed that the $A_{k,\alpha}$ conformal description of initial data slices is completely equivalent to the standard coordinate description (cf., e.g., Refs. 1,4,14,5), and the use of one or another seems to be a matter or personal taste, depending upon whether one prefers the symbol $\lim_{r\to 0}$ to the symbol $\lim_{r\to\infty}$.

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