On the structure of spatial infinity. II. Geodesically regular Ashtekar–Hansen structures

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The ambiguities in the differentiable structure of Ashtekar–Hansen completions satisfying a geodesic condition are analyzed. The results obtained imply, in particular, uniqueness up to a four-parameter family of "logarithmic transformations" of completions of asymptotically flat space-times stationary "in a neighborhood of $i_0$".

I. INTRODUCTION

In a previous paper of this series\(^1\) uniqueness up to a three-parameter family of logarithmic transformations of conformal one-point compactifications of three-dimensional asymptotically flat Riemannian manifolds was established. A corollary of that result is that within the Geroch framework\(^2\) the four-momentum of an initial data set for Einstein equations is unambiguously defined. In general relativity, which is a four-dimensional theory par excellence, one hopes to assign a four-momentum $p_\mu$ or, say, its invariant square $m^2 = -n_{\mu\nu}p^\mu p^\nu$ to a four-dimensional set, rather than to a three-dimensional subset thereof. It has been shown in Ref. 3 that one can, in a meaningful way, associate an invariant mass $m$ to a boost-type domain or, more generally, to a four-dimensional asymptotically flat end of a Lorentzian manifold defined by a collection of boost-type domains. This relatively satisfactory result suffers from the drawback that the somewhat arbitrary notion of the boost-type domain plays an essential role in the analysis. One would like to replace the statement that "two three-dimensional ends included in some boost-type domain of a vacuum space-time have the same mass" by something of the kind "two three-dimensional ends included in the same asymptotic region have the same mass," avoiding the use of some unnaturally preferred sets in some coordinate system as a primary concept of the construction. A reasonably natural setup in which one can define the notion of an asymptotic region has been proposed by Ashtekar and Hansen\(^4,5\) who describe the behavior of the gravitational field at spatial infinity by means of conformal completions of asymptotically flat four-dimensional manifolds in which spatial infinity is represented by a point $i_0$. The existence of an Ashtekar–Hansen completion—or some variation thereof, as considered in this paper—adds useful information about the global causal structure of space-time to the standard coordinate notion of asymptotic flatness,\(^6,3,7\) which seems difficult to describe in terms of asymptotically flat coordinates only. The main problem with the Ashtekar–Hansen completions is their potential nonuniqueness. In this paper we show that if a certain geodesic condition is satisfied by some completion, then there exists a four-parameter family of inequivalent completions only.

In Sec. II we introduce the notion of weak conformal completions and the geodesic regularity condition. We show that weak geodesically regular completions are unique. We also show that every completion of a no-radiation metric (in particular, of the Kerr metrics) is geodesically regular. In Sec. III we define strong completions (the conditions of this section are essentially those of Ashtekar and Hansen) and we show their uniqueness up to "logarithmic ambiguities" provided that geodesic regularity holds.

II. WEAK CONFORMAL COMPLETIONS

In order to give a motivation to the definitions of this section let us recall the fundamental result of Christodoulou and O’Murchadha\(^6\) (the "boost theorem"): Given asymptotically flat data for general relativity (cf. Ref. 6 for the appropriate definition of asymptotic flatness) and given any "boost slope" $\theta < 1$ there exists a metric $g_{\mu\nu}$, solution of the vacuum Einstein equations, the evolution of the Cauchy data, and positive constants $R$ and $T$ such that $g_{\mu\nu}$ is defined for all $x^a$ belonging to the boost-type domain $\Omega_{R,T}$ (Ref. 8):

$$\begin{align*}
\Omega_{R,T} &= \{x^a: x^\mu > R, |x^0|/|\theta r + T| \}, \\
\theta > 0, \quad R > 0, \quad T e^{-\infty - \infty},
\end{align*}$$

with $g_{\mu\nu}$ satisfying

$$|g_{\mu\nu} - \eta_{\mu\nu}| < C(1 + r)^{-\alpha}, \quad |\partial_\tau g_{\mu\nu}| < C(1 + r)^{-\alpha - 1}$$

(1)

for some constants $C(\theta, R, T, g_{\mu\nu})$, $\alpha > 0$, where $\eta_{\mu\nu}$ is the Minkowski metric. For $x^a x_a > 0$ (Ref. 9) (signature $++-\ldots$) let $\Phi$ denote the inversion $x^a \mapsto y^a = x^a/(x^a x_a)$. It is simple to check that for $\theta < 1$ and $T > 0$ the set $\Phi(\Omega_{R,T})$ contains the "wedge" $W_{R,T}$ (Ref. 10):

$$W_{\partial e} = \{y^a r(y) < x_a^0 | |\theta r| \}
$$
and Eq. (1) gives, for $y^a \in W_{R,T}$,

$$\begin{align*}
g_{\mu\nu} \frac{dx^\mu}{\partial x^a} \frac{dx^\nu}{\partial x^b} &= (y^a y_b) - 2 g_{\mu\nu} dy^a dy^b, \\
\partial_\tau g_{\mu\nu} - \eta_{\mu\nu} | &< C'(\theta, R, T, g_{\mu\nu}) r(y)\alpha, \\
\frac{\partial g_{\mu\nu}}{\partial y^a} | &< C'(\theta, R, T, g_{\mu\nu}) r(y)^{\alpha - 1}.
\end{align*}$$

(2)

Equation (2) displays the expected behavior of the metric under an inversion which brings "spatial infinity" to a point $y^a = 0$, say $i_0$. Following Ashtekar and Hansen\(^4,5\) we shall ask for some more structure than what follows in a straightforward manner from the "boost theorem."
Definition 1: Let \((M,g)\) be a space-time \((\equiv C^3 \text{ four-dimensional manifold})\) with a \(C^2\) Lorentzian metric and let \(\mathfrak{M}\) denote the disjoint union \(\bigcup \{i_0\}\) where \(i_0\) is a point. We shall say that \((\mathfrak{M},g_{i_0})\) is a weak \(\alpha\)-completion of \(M\), \(\alpha(0,1]\), if the following holds.

(i) In \(\mathfrak{M}\) there exists a coordinate system \(\{y^\mu\}\), \(y^\mu \in \mathcal{W}_{1,\alpha} \cup \{0\}\) such that \(i_0 = 0\).

(ii) There exists a function \(\Omega: \mathcal{W}_{1,\alpha} \to \mathbb{R}^+\) such that the metric \(\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}\) satisfies
\[
\forall \theta < 1, \quad \forall x \in \mathcal{W}_{\epsilon,\alpha} \tilde{g}_{\mu\nu} - \eta_{\mu\nu} \leq C(\theta)r(\eta)^\alpha,
\]
\[
|\partial_{\alpha} \tilde{g}_{\mu\nu}| \leq C(\theta) r(\eta)^{\alpha - 1}.
\]

(iii) \[
\lim_{\gamma^{- \infty}} \Omega = 0, \quad \lim_{\gamma^{- \infty}} \partial_{\gamma} \Omega = 0, \quad |\nabla_{\mu} \Omega - 2 \tilde{g}_{\mu\nu}| \leq C(\theta) r(\eta)^{\alpha - 1}.
\]

with some function \(\Omega, \theta \in (0,1)^{\infty}\). (Ref. 11)

(iv) For all \(p \in M\), there exists no timelike curve in \(\mathfrak{M}\) from \(p\) to \(i_0\).

\[B = \{p \in \Gamma^+; \quad \text{where} \quad \Gamma^+ = \text{complete spacelike geodesic}\} \quad \text{satisfying} \quad x^\pm(0) = R, n, \]
\[
\text{ne} S(1)^{\alpha} \quad \frac{dx^\pm}{ds}(0) = n, \quad (x^\pm)^0(0) = 0, \quad \lim_{s \to \infty} r(x^\pm(s)) = \infty, \quad (x^\pm)^0(s) > 0 \text{ on } \Gamma_+^+, \]
\[
(x^\pm)^0(s) < 0 \text{ on } \Gamma^-_+ \}
\]

[In other words, \(B^\pm\) are "sewn up" from geodesics starting from the sphere \(\{x = R, x^0 = 0\}\) which stay in \(\Omega_{\alpha,\alpha,\alpha}\) and remain either to the local future (with respect to the chronology of \(\Omega_{\alpha,\alpha,\alpha}\)) or to the local past of \(N^0_k\), \(\equiv \{x^\pm x^0 = 0, r(x) > R_1\}\). The \(B^\pm\) are bars over \(N_{\alpha,\alpha}^0\) because \(B^\pm\) are constructed over \(N_{\alpha,\alpha}^0\) and remain entire within the wedge \(W = \{x^\mu r(x) > R, 0 < x^\mu < \rho_0(x)\}\). Every future directed timelike curve starting at \(N_{\alpha,\alpha}^0\), either remains entirely within the wedge \(W^- = \{x^\mu r(x) > R^-, 0 < x^\mu \leq \rho_0(x)\}\) or meets \(B^+\). Similarly, every past directed timelike curve starting at \(N_{\alpha,\alpha}^0\) either remains entirely within the wedge \(W^- = \{x^\mu r(x) > R^-, 0 < x^\mu \leq \rho_0(x)\}\) or meets \(B^+\).

Proof: By Propositions B1 and B2 of Appendix B of Ref. 3 for \(R_1\) sufficiently large the family of geodesics \(x^\pm\) \(\text{satisfying} \quad \text{ne} S(1) \quad \text{defined by} \quad x^\pm(0) = R, n, \quad \text{ne} S(1), \quad (dx^\pm/ds)(0) = n, \quad (x^\pm)^0(0) = 0, \quad \text{and} \quad d(x^\pm)^0/ds(0) = \pm \theta_0 \equiv \pm \theta(0)\} \text{ will satisfy} \quad \text{ne} S(1), \quad (dx^\pm/ds)(0) = n, \quad (x^\pm)^0(0) = 0, \quad \text{and} \quad d(x^\pm)^0/ds(0) = \pm \theta_0 \equiv \pm \theta(0)\} \text{ will satisfy} \quad \forall s > 0, (x^\pm)^0(s) > 0, \quad (x^\pm)^0(s) > 0, \quad \text{and} \quad \theta_0 < 2 \leq d(x^\pm)^0/ds(0) \leq 2 \theta_0/2. \text{Let} \quad \mathcal{A} = \{p \in R; \forall x \in B(\rho) \cup \text{Int}[B(R,1)] \exists n \pm \in S(1) \text{ and} \quad x^0_\pm(x), \quad |x^0_\pm(x) < \theta(\rho) x| \quad \text{such that} \quad x^\pm_\pm \in S(1) \}
\]

where \(B(\rho)\) denotes a closed ball of radius \(\rho\). \(A\) is nonempty because \(R \geq \alpha, o(1)\). \(A\) is closed by standard properties of solutions of differential equations, and openness of \(A\) follows from the implicit function theorem and the fact that \(dx^\pm/ds\) is everywhere transversal to the sphere \(S(\rho)\). This implies \(A = \{p \in R; \rho > R_1\}\); so that for every \(x \in R^\infty \cup B(\rho)\) there exists
\[p^\pm \equiv \{p \in R; \rho > R_1\}\] which pass through \(p^\pm\).

Lemma 2: Let \((\mathfrak{M}_1, g_{i_1}), (\mathfrak{M}_2, g_{i_2})\) be two weak \(\alpha\)-completions of a space-time \((M,g)\), \(0 < \alpha < 1\); let \(x\) be the appropriate coordinate systems, \(x^\mu \in \mathcal{W}_{1,\alpha} \cup \{0\}\), \(y^\mu \in \mathcal{W}_{1,\alpha} \cup \{0\}\); let \(\Phi\) denote the coordinate transformation \(y^\mu(x^\nu)\) wherever defined; suppose that \(\Phi\) is differentiable; and define \(N_{\alpha,\alpha} = \{x^\mu x^0 = 0, r(x) < \epsilon_1\}\). If \(\Phi(N_{\alpha,\alpha})\) is contained in a wedge \(W_{1,\alpha} \epsilon, \theta < 1\), then there exists a Lorentz matrix \(A^\nu_{\mu}\) such that
\[y^\mu = A^\mu_{\nu} x^\nu + \xi^\mu, \quad (4)\]
where \(\xi^\mu\) satisfies
\[\forall \psi < 1, \quad \forall x \in W_{1,\alpha} \epsilon, (\psi^\mu < C(\psi)r^\alpha + \zeta, \quad \psi^\mu < C(\psi)^{\alpha + 1}, \quad \psi^\mu < C(\psi)r^\alpha + \zeta, \quad \psi^\mu < C(\psi)^{\alpha + 1}. \quad (5)\]

Proof: The image by \(\Phi\) of \(i_1\) must be \(i_2\), otherwise there would exist a timelike curve from some point \(p \in M\) to \(i_1\), contradicting point (iv) of Definition 1. Let \(\tilde{x}^\mu = x^\mu(x \times x_0), \quad \tilde{y}^\mu = y^\mu(y \times y_0, \quad \text{the hypersurface \(N_{1,\alpha} \equiv (\tilde{x}^\mu y \times y_0) \tilde{y}^\mu \equiv 0\)}\) is asymptotically flat and is included in some boost-type domain of coordinates \(y^\mu\) with slope \(\theta\) smaller than \(1\)---the result follows from Theorem 1 and Point 2 of Corollary 2 of Ref. 3.

Lemmas 1 and 2 lead to the following theorem.

Theorem 1 (uniqueness of geodesically regular weak completions, \(\alpha < 1\)): If a space-time \((M,g)\) admits one geodesically regular weak \(\alpha\)-completion \((M, g, \alpha, o(1))\), then the following holds.
(i) All weak $\alpha$-completions of $(M,g)$ are geodesically regular.

(ii) All weak $\alpha$-completions of $(M,g)$ are related to each other by coordinate transformations of the form (4) and (5).

Proof: Let $(\tilde{M},\tilde{g},\tilde{t})$ be some completion of $(M,g)$, let $\{x^{a}\}$ be the appropriate coordinate system for $(\tilde{M},\tilde{g})$, let $y^{a}$ be the coordinates of Definition 1 for $(\tilde{M},\tilde{g})$, and let $\Psi$ denote the transformation $y^{a}(x^{a})$. From Lemma 1 applied to the physical metric $\tilde{g}_{\mu\nu}$ in coordinates $\tilde{x}^{\mu} = x^{a}/(x^{a}x_{a})$ one infers the existence of two hypersurfaces $B^{\pm}$ sewn up from the geodesics $\Gamma_{\pm} \in \mathbb{S}(1)$. By geodesic regularity for all $\mathbb{S}(1)$ there exist $\theta_{\pm}^{\ast}$ such that $\Psi(\Gamma_{\pm}^{\ast}) \subset W_{\theta_{\pm}^{\ast},\epsilon}$. By compactness of $\mathbb{S}(1)$ we have $sup_{\Gamma_{1}(\mathbb{R})\in \mathbb{S}(1)}(\theta_{\pm}^{\ast}) = \theta_{0} < 1.15$. Since $N_{1/R}(x) = \{x^{a}, x^{b} = 0, r(x) < 1/R_{1}\}$ ($R_{1}$ given by Lemma 1) lies to the local future of $B^{-} \subset W_{\theta_{0},\epsilon}$ and to the local past of $B^{+} \subset W_{\theta_{0},\epsilon}$, we must have $N_{1/R}(x) \subset W_{\theta_{0},\epsilon}$, so that we can apply Lemma 2 to conclude that $y(x)$ is of the form (4), which establishes point (ii). It is not too difficult to show that $y(x)$ must be of the form (4) as well, so that the image $\Psi(\Gamma)$ of any spacelike geodesic $\Gamma = \{y^{a}(s), y^{b}(s) \in \mathbb{C} \Gamma_{1}(\mathbb{R})\}$ will be included in $\Psi(W_{\theta_{0},\epsilon}) \subset W_{\theta_{0}^{\ast},\epsilon}$, with some $\theta_{0}^{\ast} < 1$ and point (i) ensues.

A metric shall be called a non-radiation metric if there exist coordinates $x^{a} \in \Omega_{1}(\mathbb{R},\infty) \equiv \{(-\infty, \infty) \times [R^{3} \setminus B(R)]\}$ such that $\gamma_{\mu\nu} - \epsilon > 0$ for some positive $\epsilon$, $\gamma_{\mu\nu}$ is a positive definite matrix with eigenvalues separated from zero, and (3) holds throughout $\Omega_{1}(\mathbb{R},\infty)$ with some constant $C$. The Kerr metrics are non-radiation metrics in this sense, with $\alpha = 1$.

Proposition 1: Weak $\alpha$-completions of non-radiation metrics, $\alpha > 0$, are geodesically regular.

Proof: By point (i) of Theorem 1 it is sufficient to show the existence of a geodesically regular completion. Let $x^{a}$ be the coordinates satisfying (3) and let $\tilde{x}^{a} = x^{a}/(x^{a}x_{a})$. By Proposition B1 of Appendix B of Ref. 3 every spacelike geodesic meeting $i_{0}$ [i.e., such that $\dot{x} \to 0 \Rightarrow r(x) \to \infty$] behaves asymptotically as follows:

$$x^{a}(s) = \eta^{a}_{\mu} s + o(s)$$

for some constant vector $\eta^{a}_{\mu}$ satisfying $\eta^{a}_{\mu} \eta^{a}_{\nu} \eta_{\mu \nu} > 0$, where $s$ is an affine parameter, and we can normalize $\eta^{a}_{\mu}$ to satisfy $\eta^{a}_{\mu} \eta^{a}_{\nu} = 1 (\Rightarrow |\eta^{a}_{\mu}| < 1)$. We have

$$\tilde{x}^{a}(s) = [\eta^{a}_{\mu}/(1 - |\eta^{a}_{\mu}|^{2})]^{-1} + o(s^{-1}),$$

so that $\tilde{x}^{a}(s) = \eta^{a}_{\mu}(\tilde{x}(s)) + o[\tilde{x}(s)]$ and for $s$ large enough one obtains

$$[\tilde{x}^{a}(s)] < \theta \in (|\eta^{a}_{\mu}| + 1)/2 < 1.$$  

III. STRONG CONFORMAL COMPLETIONS

In Sec. II we have investigated the structure of the set of conformal completions in which the metric is allowed to blow up as one approaches "what would be the light cone of $i_{0}$." As has been shown by Schmidt and Walker (cf. Appendix C of Ref. 4) much better behaved completions can be obtained for Kerr metrics. To justify our conditions on the conformally rescaled metric, to be presented later, let us recall the Schmidt–Walker coordinates for the Schwarzschild metric: For $s > 2m$ let $f(s)$ be defined by

$$f(s) = s + 2m \ln(s/2m - 1)$$

and let us set

$$r^{\ast} = f(r), \quad r^\ast = [f(\tilde{r} - 1) + f(\tilde{r} - 1)]/2,$$

$$t = [f(\tilde{r} - 1) - f(\tilde{r} - 1)]/2,$$

$$\tilde{r} = (\tilde{t} + \tilde{u})/2, \quad \tilde{t} = (\tilde{t} - \tilde{u})/2,$$

where $r$ and $t$ are the standard Schwarzschild coordinates

$$ds^{2} = -(1 - 2m/r)dt^{2} + (1 - 2m/r)^{-1}dr^{2} + r^{2}d\Omega^{2}. $$

In the coordinates $\tilde{b}, \tilde{u}, \tilde{b} \tilde{u} = 0$, the Schwarzschild metric takes the form

$$ds^{2} = \frac{1}{(\tilde{b}\tilde{u})^{2}} \left\{ \frac{(1 - 2m/r)}{(1 - 2m\tilde{b})(1 - 2m\tilde{u})} d\tilde{b} d\tilde{u} + (\tilde{b}\tilde{u}^{2}) d\Omega^{2} \right\}.$$  

From Eqs. (6) and (7) one has

$$r = r^{\ast} - 2m \ln(r^{\ast}/2m) + \xi^{\ast},$$

with

$$\xi^{\ast} = 2m \ln \left[ \frac{1 + 2m \ln[r/2m - 1]/r}{1 - 2m/r} \right] = O(\frac{\ln[r^{\ast}]}{r^{\ast}})$$

for large $r^{\ast}$; therefore, from

$$r^{\ast} = \tilde{r}/\tilde{u}$$

one obtains, for small $\tilde{r}$,

$$\xi^{\ast} = (\tilde{b} \tilde{u} \tilde{r}) O(\ln[\tilde{r}] + \ln[\tilde{b} \tilde{u}]) = O(\tilde{r} \ln[\tilde{r}]), \quad \text{for } |\tilde{r}| < \tilde{r}$$

and one obtains

$$\omega = \tilde{b} \tilde{u} \tilde{r} = 1 + (2m \tilde{b} \tilde{u}) \ln(2m \tilde{b} \tilde{u}) + O(\tilde{r}^{2}).$$

so that

$$\omega - 1/\tilde{r} = m(1 - \tilde{r}^{2}/\tilde{r}^{2}) \ln(1 - \tilde{r}^{2}/\tilde{r}^{2}) + O(\tilde{r}).$$

The metric (9) can be written in the form

$$ds^{2} = \Omega^{-2} \tilde{g}_{\mu\nu} d\tilde{x}^{\mu} d\tilde{x}^{\nu} = \Omega^{-2} d\tilde{r}^{2}, \quad \Omega = \tilde{b} \tilde{u} \tilde{r} = \tilde{r}^{-2},$$

$$d^{2} = \left( \frac{(1 - 2m \tilde{b} \tilde{u})}{(1 - 2m \tilde{u})} \right) \left( \frac{d\tilde{r}^{2} + (d\tilde{x} d\tilde{x})^{2}}{\tilde{r}^{2}} \right)$$

$$+ \omega^{2} \left( d\tilde{r}^{2} + d\tilde{u}^{2} + d\tilde{b}^{2} - \frac{(d\tilde{x} d\tilde{x})^{2}}{\tilde{r}^{2}} \right),$$

and we have $\lim_{|\tilde{r}| \to 0} = \omega = 1$, so that $\tilde{g}_{\mu\nu}$ can be continuously extended to the set $0 < \tilde{r} < \tilde{r}_{0}$ with some $\tilde{r}_{0}$. The metric $\tilde{g}_{\mu\nu}$ is of the form

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(\theta, \tilde{r}^{2}) + o(\tilde{r})$$

and we have

$$\left| \tilde{g}_{\mu\nu} - \eta_{\mu\nu} \right| < C \tilde{r}$$

for some constant $C$; the derivatives of $\tilde{g}_{\mu\nu}$ are, however, not
bounded up to the light cone of \( i_0 = \{ x^\alpha = 0 \} \): One finds
\[
\partial \omega \sim \ln (1 - \hat{r}^2/\hat{r}^2),
\]
where \( \omega \) is defined in (10), so that one obtains
\[
| \partial_\alpha \tilde{g}_{\mu\nu} | < C_1(\hat{r}/\hat{r})
\]
with \( C_1(\eta) = C_1(\ln (1 - \eta^2)) + 1 \) for some constant \( C \). It must be emphasized that the singularity of \( C_1(\eta) \) at \( \eta = 1 \) is rather mild in the sense that
\[
\int_0^1 C_1(\eta)d\eta < \infty.
\]
We also have
\[
\hat{r} | \partial_\alpha \partial_\mu \tilde{g}_{\nu\rho} | < C_2(\hat{r}/\hat{r}), \quad C_2(\eta) \sim (1 - \eta^2)^{-1},
\]
with
\[
\int_0^\eta C_2(\eta)d\eta < C' \int_0^1 C_1(\eta)d\eta.
\]
This analysis motivates the following definition.

Definition 2: Here (\( \tilde{M}, \tilde{g}_{\mu\nu} \)) will be called a strongly completed space-time (\( \tilde{M}, \tilde{g}_{\mu\nu} \)) if \( \tilde{M} \) is the disjoint union \( M \cup \{ i_0 \} \) and the following holds.

(i) For all \( \rho \in \tilde{M} \) there exists no timelike curve in \( \tilde{M} \) from \( \rho \) to \( i_0 \).

(ii) There exists a coordinate system \( x^\alpha \in W_{1,\varepsilon} \cup \{ 0 \} \) such that \( i_0 = 0 \) and the points \( x^\alpha x_\alpha > 0 \) correspond to points in \( M \). On \( M \) there exists a function \( \Omega : \tilde{M} \rightarrow \Omega \), \( \Omega > 0 \) such that \( \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \) and there exists a constant \( C \) and nondecreasing functions \( C_1, C_2 : [0, 1) \rightarrow \mathbb{R}^+ \) such that
\[
\begin{align*}
\forall x \in \Omega W_{1,\varepsilon} : | \tilde{g}_{\mu\nu} - g_{\mu\nu} | < C, \\
| \partial_\alpha \tilde{g}_{\mu\nu} | < C_1(\eta), \\
r | \partial_\alpha \partial_\mu \tilde{g}_{\nu\rho} | < C_2(\eta), \quad \eta = t/r.
\end{align*}
\]

(iii) \( \Omega \) satisfies
\[
\lim_{r \to 0} \Omega = 0, \quad \lim_{r \to 0} \partial_\mu \partial_\nu \Omega = 0, \quad \text{for } x \neq 0
\]

(iv) For every \( |\eta| < 1 \) the limits
\[
\begin{align*}
\lim_{r \to 0} \hat{g}_{\mu\nu}(t = \eta r, r, \theta, \phi) & = \eta g_{\mu\nu}(t = \eta r, r, \theta, \phi), \\
\lim_{r \to 0} (r \partial_\alpha \tilde{g}_{\mu\nu}(t = \eta r, r, \theta, \phi)) & = 0.
\end{align*}
\]

exist and are continuous functions of \( (\eta, \theta, \phi) \), where \( \theta \) and \( \phi \) are standard spherical angles.

Let us note that (13) implies that the Ricci tensor \( \overline{R}_{\alpha\beta} \) of \( \tilde{g}_{\mu\nu} \) satisfies
\[
\overline{R}_{\alpha\beta} \sim \frac{1}{C_R} (\eta)^{-1},
\]
\[
\overline{R}_{\alpha\beta}(\theta) = \int_0^\theta C_R(\eta)d\eta, \quad \int_0^1 C_R(\theta)d\theta = C_R < \infty.
\]
\[
| C_R(\eta) | < | C_1(\eta) | + C_3(\eta^2), \quad \text{therefore, } \overline{R}_{\alpha\beta}(\theta) \sim | C_1(\theta) | + C_3(\theta^2) | C_R(\eta)d\eta | = C(1 + \overline{C_1(\theta)}) C_1(\theta) |
\]

with some numerical factor \( c \). Our main result is the following theorem.

Theorem 2 (quasiusuniqueness of geodesically regular strong completions): Let \( (\tilde{M}, \tilde{g}_{\mu\nu} \tilde{t}, \tilde{G}_{\nu\rho}) \) be two geodesically regular strong completions of a space-time \( (M, g) \) and let \( (x^\alpha, x_\alpha) \) be the appropriate coordinate systems in \( \tilde{M} \). There exists a Lorentz matrix \( \Lambda^\alpha_\mu \), a constant vector \( C^\alpha \), and a constant \( C \) such that for \( x \in W_{1,\varepsilon} \),
\[
y'^\alpha = \Lambda^\alpha_\mu x'^\mu + \left( C^\alpha x^\nu x_\mu - 2 x^\mu x'^\nu C_\mu \right) \ln r + \xi'^\alpha,
\]
\[
| \zeta'^\alpha | < C r, \quad | \partial_\alpha \zeta'^\alpha | < C r.
\]

Moreover, for all \( \eta = 1 \) the limits
\[
\lim_{r \to 0} \left[ | \zeta'^\alpha(\tau = r, r, \theta, \phi) \right] = 0,
\]
\[
\lim_{r \to 0} \left[ | \partial_\alpha \zeta'^\alpha(\tau = r, r, \theta, \phi) \right] = 0,
\]
\[
\lim_{r \to 0} \left[ \partial_\alpha \partial_\nu \zeta'^\alpha(\tau = r, r, \theta, \phi) \right] = 0,
\]
\[
\lim_{r \to 0} \left[ \partial_\alpha \partial_\nu \zeta'^\alpha(\tau = r, r, \theta, \phi) \right] = 0.
\]

exist and are continuous functions of \( (\eta, \theta, \phi) \).

Proof: By Theorem 1 there exists a Lorentz matrix \( \Lambda^\alpha_\mu \) such that \( x \in W_{1,\varepsilon} \),
\[
y^\alpha = \Lambda^\alpha_\mu x^\mu + \xi^\alpha, \quad | \zeta^\alpha | < C r, \quad | \partial_\alpha \zeta^\alpha | < C r^{-1},
\]
with any \( \varepsilon > 0 \) and, in fact, a straightforward extension of the estimates of Ref. 3 leads to
\[
\begin{align*}
\zeta'^\alpha & = (C^\alpha x^\nu - 2 x^\nu C_\alpha) \ln r + \xi'^\alpha(\theta, \phi) + o(r), \\
\partial_\alpha \zeta'^\alpha & = \partial_\alpha( (C^\alpha x^\nu - 2 x^\nu C_\alpha) \ln r + \xi'^\alpha(\theta, \phi) + o(r), \\
\partial_\alpha \partial_\alpha \zeta'^\alpha & = \partial_\alpha( (C^\alpha x^\nu - 2 x^\nu C_\alpha) \ln r + \xi'^\alpha(\theta, \phi) + o(1))
\end{align*}
\]

for some constant \( C^\alpha \). By a slight abuse of notation let us denote by \( y^\alpha \) the coordinates \( \Lambda^\alpha_\mu y^\mu \), so that we can set \( \Lambda^\alpha_\mu = \tilde{g}_{\mu\nu} \) in Eq. (16), and on \( N_{\varepsilon} \), we have
\[
\tau_{r=0} \equiv y_{r=0} = O(r^{-1} \ln r),
\]
\[
\eta y(x)|_{r=0} = O(r \ln r),
\]
\[
\partial_\alpha y^\alpha|_{r=0} = O(1),
\]
\[
\partial_\alpha \partial_\alpha y^\alpha|_{r=0} = O(1).
\]

Now \( \tilde{g}^\alpha_\mu \) and \( g^\alpha_\mu \) are conformally related to each other, so that by definition, there exists a function \( \Phi: W_{1,\varepsilon} \rightarrow \mathbb{R}^+ \) such that
\[
\begin{align*}
\forall x^\alpha x_\alpha > 0 \Phi^2(x) & = \tilde{g}^\alpha_\mu(y(x)) \frac{\partial y^\alpha}{\partial x^\nu} \frac{\partial y^\nu}{\partial x^\mu}.
\end{align*}
\]

Equations (13), (18), and (19) yield
\[
\Phi = \frac{\det \tilde{g}^\alpha_\mu}{\det g^\alpha_\mu} \frac{1}{1/8},
\]
\[
\Rightarrow \Phi|_{r=0} = 1 - 2 C \ln r + \xi(\theta, \phi) r + o(r),
\]
\[
\partial_\alpha \Phi|_{r=0} = -2 C \ln r + \chi(\theta, \phi) + o(1).
\]
for some functions \( \Phi(\theta, \phi), \chi_\alpha(\theta, \phi) \). The transformation law of the Christoffel symbols gives

\[
\delta^\mu_{\beta\gamma} = \left[ \Gamma^1_{\alpha\beta} + \Phi^{-1} (\delta^\mu_{\alpha} \partial_\beta + \delta^\mu_{\beta} \partial_\alpha \Phi) - \delta^\mu_{\beta\gamma} \partial_\alpha \Phi \right] \left( \frac{\partial y^\mu}{\partial x^\alpha} - \frac{\partial y^\mu}{\partial x^\beta} \right) - \delta^\mu_{\beta\gamma} \delta^\alpha_{\beta\gamma} \Phi(\theta, \phi) \]

(22)

where \( \Gamma^1 \) and \( \Gamma^2 \) are the Christoffel symbols of the metrics \( g^\mu_{\nu} \) and \( g^\alpha_{\beta} \). The formula for the transformation of the Ricci tensor under conformal transformations reads as

\[
\frac{\partial^2 \Phi}{\partial t^2} = \frac{\partial^2 \Phi}{\partial x^a \partial x^b} + \frac{1}{2} \left[ \Phi \left( R^1_{\mu\nu} - R^2_{\mu\nu} \right) \frac{\partial y^\mu}{\partial x^a} \frac{\partial y^\nu}{\partial x^b} \right] \\
+ 4 \Phi^{-1} \frac{\partial \Phi}{\partial x^a} \frac{\partial \Phi}{\partial x^b} + \frac{1}{6} \left[ \frac{\partial^2 y}{\partial x^a} \right] \frac{\partial y}{\partial x^b} - \frac{1}{6} \Phi \frac{\partial R_1}{\partial x^a} \\
- \Phi^{-1} \frac{\partial \Phi}{\partial x^a} \frac{\partial \Phi}{\partial x^b} \left( \frac{\partial y}{\partial x^a} \right) \left( \frac{\partial y}{\partial x^b} \right) \right].
\]

(23)

By (18) and (21) one can find \( c_0 \leq \min(\epsilon_s, e^{-1}) \) (\( e \) is the Euler number) small enough so that for all \( x \in N_e \), we have

\[
0 < \Phi_{\theta \phi} \leq \frac{3}{4}, \quad 3r(x)/4 < r(y(x))_{\theta \phi} \leq 3r(x)/4, \\
\left| \frac{\partial^2 y}{\partial x^a} \right|_{\theta \phi} < \frac{1}{4}, \quad \left| \frac{\partial r}{\partial t} - \frac{\tau(r(y(x)))}{r(y(x))} \right|_{\theta \phi} < \frac{3}{4}.
\]

Let

\[
C_{\phi}(\eta) = \sup_{\eta \in [0,1]} \left[ \ln r^{-1} \Phi \right],
\]

for \( \eta \in [0,1] \),

\[
C_{\phi}(\eta) = \sup_{\eta \in [0,1]} r(x) \left| R_{\mu\nu}^1(x) \right| \quad \text{and} \quad C_{\phi}(\eta) = \sup_{\eta \in [0,1]} r(x) \left| R_{\mu\nu}^2(x) \right|
\]

for \( \eta \in [0,1] \),

\[
C_{\phi}(\eta) = \sup_{\eta \in [0,1]} \left| g^\mu_{\nu}(x) \right| \quad \text{and} \quad C_{\phi}(\eta) = \sup_{\eta \in [0,1]} \left| g^\nu_{\mu}(x) \right|
\]

for \( a = 1,2 \),

\[
\overline{C}_{\phi}(\eta) = \int_0^\eta C_{\phi}(\eta) \, d\eta, \quad \overline{C}_{\alpha}(\eta) = \int_0^\eta C_{\alpha}(\eta) \, d\eta < \infty.
\]

Let \( \Omega \subset W_{t,e} \) be the set of points such that

\[
|\partial_\theta \Phi| < \overline{C}_{\phi}(x) + C_{\phi} \ln(1/\delta), \quad 1 < \Phi \leq 2, \\
r(x)/2 < r(y(x)) < 3r(x)/2, \\
\left| \frac{\partial y^\mu}{\partial x^a} \right| < \frac{1}{2}, \\
\left| \frac{\partial r}{\partial t} - \frac{\tau(r(y(x)))}{r(y(x))} \right| < \frac{1}{2},
\]

(24)

where

\[
\overline{C}_{\phi}(x) = \left( 2 + 2^3 \overline{C}_{\phi}(x) + 3 \Phi(x) \right) + 2^3 C_{\phi}(x) \left( \overline{C}_{\phi}(x) \right) + 1 = \overline{C}_{\phi}(x) + 2^3 C_{\phi}(x) \left( \overline{C}_{\phi}(x) \right) + 1.
\]

Let \( \Omega \subset \Omega \) be the set of points \( x^a \) such that the curve \( [0,t] \supset \rightarrow x(t) \) is included in \( \Omega \). We have \( N_e \subset \Omega \); let \( \Omega \) be the connected component of \( \Omega \) which contains \( N_e \). We shall show that there exists \( 0 < \epsilon_1 < \epsilon_2 \) such that \( \Omega \cap W_{t,e} \) is closed in \( W_{t,e} \). By (23) we have, for \( x^a \in \Omega \),

\[
\left| \frac{\partial \Phi}{\partial t} \right| < \left| \frac{\partial \Phi}{\partial x^a} \right|_{(0,x)} + \int_0^t \left| \frac{\partial^2 \Phi}{\partial t \partial x^a} \right|_{(s,x)} \, ds,
\]

and from

\[
\int_0^t \overline{C}_{\phi}(x) \, ds = \overline{C}_{\phi}(x), \\
\int_0^t \left[ \frac{\tau(r(y(x)))}{r(y(x))} \right] \, ds = \int_0^t \left[ \frac{\tau(r(y(x)))}{r(y(x))} \right] \, ds \\
< 3r(x) \int_0^t \overline{C}_{\phi}(x) \, ds < 3r(x) \overline{C}_{\phi}(x),
\]

\[
< 3r(x) \overline{C}_{\phi}(x) + 2^3 C_{\phi} \left( \overline{C}_{\phi}(x) \right) + 1 \left( \overline{C}_{\phi}(x) \right) \int_0^t \left| \frac{\partial r}{\partial t} - \frac{\tau(r(y(x)))}{r(y(x))} \right| \, ds
\]

we have used the facts that \( \overline{C}_{\phi} \) is nondecreasing along the curves \( [0,t] \supset \rightarrow x(t) \), and \( |t| < r \) in \( W_{t,e} \),

\[
\int_0^t \left| \frac{\partial \Phi}{\partial x^a} \right|_{(s,x)} \, ds < r \overline{C}_{\phi}(x), \quad \overline{C}_{\phi}(x) = \int_0^t \left| \frac{\partial \Phi}{\partial x^a} \right|_{(s,x)} \, ds
\]

one obtains

\[
|\partial_\theta \Phi(x)|_a < \overline{C}_{\phi}(x) + C_{\phi} \ln(1/\delta), \quad 1 < \Phi \leq 2,
\]

\[
< \overline{C}_{\phi}(x) + 2^3 C_{\phi} \left( \overline{C}_{\phi}(x) \right) + \left( \overline{C}_{\phi}(x) \right) \left( \overline{C}_{\phi}(x) \right) + 1.
\]

\[ + 2^{3} \mathcal{C}_{\phi} \int_{0}^{\infty} \mathcal{C}_{\phi}(x) \] 
\[ < \mathcal{C}_{\phi} \ln \left( \frac{1}{r} + \frac{3 \mathcal{C}_{\phi}(x)}{4} \right) \]
\[ \mathcal{C}_{\phi} \ln \left( \frac{1}{r} + \frac{5 \mathcal{C}_{\phi}(x)}{8} \right) \]
for \( \epsilon_i \) small enough. This gives
\[ |\Phi(t,x) - 1| = \int_{0}^{\infty} |\partial_n \Phi(s,x)| \, ds \]
\[ \leq \frac{1}{4} + (\mathcal{A}_{\rho} \mathcal{K} + 3 \mathcal{A}_{\rho} \mathcal{K} + 1) \epsilon_i \]
\[ + \mathcal{C}_{\phi} \epsilon_i \ln \left( \frac{1}{\epsilon_i} \right) \leq \frac{3}{8}, \]
decreasing \( \epsilon_i \) if necessary. One shows in a similar way that none of the inequalities in (24) can saturate for \( x^e \epsilon\Omega_1 \cap W_{1,1}, \) when \( \epsilon_i \) is small enough, so that \( \Omega_1 \cap W_{1,1} \) is both open and closed in \( W_{1,1}, \) therefore, \( \Omega_1 \cap W_{1,1} = W_{1,1}. \) It is not too difficult to show from (22) and (24) that \( \Phi \) and \( \partial^n \phi / \partial x^n \) uniformly tend to 1 and \( \mathcal{A}_{\rho} \) at \( \bar{r}_0 \) and (23) yields
\[ \lim_{r \to 0} \left[ \frac{\partial \mathcal{A}}{\partial x}(t = \mathcal{A}_1, r, \theta, \phi) \right] = \frac{\partial \mathcal{A}}{\partial x}(t = \mathcal{A}_1, r, \theta, \phi) \]
\[ = \int_{0}^{\infty} \frac{\partial \mathcal{A}}{\partial x}(0, r, \theta, \phi) \, ds : = A_{\lambda}(\theta, \phi, \phi), \]
with some continuous functions \( A_{\lambda}(\theta, \phi, \phi), \) so that from (21) and (25) one has
\[ \frac{\partial \mathcal{A}}{\partial x}(t = \mathcal{A}_1, r, \theta, \phi) = - 2 \mathcal{C}_{\phi} \ln r + \mathcal{A}_{\lambda}(\theta, \phi, \phi) + o(1) \]
for some continuous functions \( \mathcal{A}_{\lambda} \) and (17) and (22) imply
\[ \partial_{x} y^e = \delta_{e} + 2(\mathcal{A}_{\rho} x_{\rho} - \mathcal{C}_{\phi} x_{\phi} - \delta_{e} \mathcal{C}_{\phi} x_{\phi}) \]
\[ \times \ln r + r A_{\lambda}^{\phi}(\theta, \phi, \phi) + o(r), \]
with some continuous functions \( A_{\lambda}^{\phi}, \) a straightforward analysis establishes our remaining claims.

Two completions differing by a transformation of the form (15) with \( \mathcal{C} = 0 \) can be considered as equivalent. Theorem 2 and Proposition 1 imply that the following corollary.

**Corollary 1:** Strong conformal completions of no-radiation space-times are unique up to the four-parameter family of transformations (15).

It may be of some relevance to note that the logarithmic forms of the transformations (15) are not the ones given in Appendix 1 of Ref. 18: The latter introduce singularities in \( g_{\mu\nu} \) at the light cone of \( \bar{r}_0, \) while (15) do not.

It is natural to ask about the group properties of the transformations (15) since it is not Lorentz invariant. Under a Lorentz transformation \( y^e = \Lambda e x^e \) we have
\[ r(y) = \sqrt{1 \mathcal{C}_{\phi} x^e \Lambda^e \Lambda^e \mathcal{C}_{\phi} x^e} \]
\[ = r(x) \sqrt{1 \mathcal{C}_{\phi} (x^e / r) \Lambda^e (x^e / r)} = J(f, \theta, \phi) \, r(x), \]
so that
\[ \ln r(y) = \ln r(x) + \ln f \]
and the \( \ln f \) terms can be absorbed in \( \mathcal{C}_{\phi}, \) which shows that a composition of transformations (15) is still of the form (15).

Let us finally note the existence of a set of coordinates for the Schwarzschild metric in which the metric is slightly worse behaved than in (11) and (12); however, for \( r \neq 0 \) the first derivatives of the metric do not blow up as one approaches the light cone of \( \bar{r}_0. \) Let us set
\[ r^* = f(r), \]
\[ t^* = \frac{1}{2} \left[ f(t^*) - f(t^*) \right], \]
\[ \bar{r} = \frac{1}{2} \left[ f(t^*) + f(t^*) \right], \]
with \( r, t \) as in (8) [this choice of \( f \) cancels these terms in \( r^*, t^* \) which exhibit the worst behavior at the light cone of \( \bar{r}_0. \) One obtains
\[ ds^2 = \bar{r}^{-2} ds^2, \]
\[ \bar{r} = \bar{v} \bar{w}, \]
\[ d \bar{s}^2 = \bar{g}_{\mu\nu} \, d \bar{x}^\mu d \bar{x}^\nu, \]
\[ \bar{g}_{\mu\nu} = \eta_{\mu\nu} + \bar{f} \, \eta_{\mu\nu}(\eta, \theta, \phi) \ln \bar{r} \]
\[ + \bar{f} \, \eta_{\mu\nu}(\eta, \theta, \phi) \ln \bar{r}, \]
which is of simple and tractable form.

**IV. CONCLUSIONS.**

We have shown uniqueness and logarithmic ambiguities of Ashtekar-Hansen completions satisfying a geodesic condition. We conjecture that the geodesic regularity condition is unnecessary in the case of strong completions and that it cannot be removed without losing quasiuniqueness of the weak completions. It may be of some interest to mention that in Theorems 1 and 2 the geodesic regularity hypothesis may be replaced by the probably much weaker condition that there exists a spacelike geodesic \( \Gamma \) extending to \( \bar{r}_0 \) such that \( \Gamma \subset W_{1,1}, \Gamma \subset W_{1,1}, \) with some 0 < \( \theta_0, \theta_0 < 1, \) where \( W_{1,1}, \Gamma \subset W_{1,1}, \) are appropriate \( \rho \) and \( x \) coordinate wedges. This condition does not, however, characterize one completion, but pairs of completions. It is likely that a proof of quasiuniqueness of strong conformal completions without any further conditions can be obtained by showing that such geodesics always exist.

It must be stressed that the Kerr family of metrics exhausts the up-until-now known set of vacuum Einstein metrics admitting strong completions of spatial infinity, so that the results of Sec. III cover all actually known physically relevant examples: The metrics recently constructed by Carter and Wald and Christodoulou are "Schwarzschild in a neighborhood of \( \bar{r}_0, \) so clearly our theorems apply. It seems rather difficult to guess whether there exists some sufficiently large class of vacuum space-times admitting weak or strong completions, the existence of which would justify the need of a search for more general results than those presented here.

An important consequence of our results is that one can assign an invariant mass parameter (cf., e.g., Ref. 21) to every vacuum space-time admitting strong or weak geodesi-
cally regular completions, $\alpha > \frac{1}{2}$—this complements the results presented in Ref. 3.

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8Latin indices run from 1–3 and Greek indices run from 0–3, $\alpha(x) = (\Sigma, \zeta^2)^{\frac{1}{2}}$.
9Throughout, $x^a x_a = \eta_{ab} x^b x^c y^a y^b$.
10Note that with this definition $0 < W_{\text{KLR}}$.
11$C(\theta)$ can blow up to $\infty$ as $\theta$ goes to 1.
12A related stronger condition would be the requirement that there exists a parametrization $u(s)$ such that the limit $\lim_{s\to \infty} dy^{a}(s)/du = \eta^{a}$ exists, with $\eta^{a} \eta_{a} = 1$.
13By complete geodesic we mean a geodesic defined for all $s \geq s_0$ for some $s_0$, where $s$ is an affine parameter.
14$S(\rho) = $ sphere of radius $\rho$.
15It is simple, although a little tedious, to show that the assignment $a = \theta^{-1}$ can be made in a continuous way.
16A similar analysis (cf. Appendix C of Ref. 4) shows that the Kerr metrics also satisfy the requirements of the definition below.
17It should be borne in mind that some no-radiation metrics may admit no strong conformal completions—Corollary 1 classifies all of them if one exists. It has been pointed out to the author by B. Schmidt that it is likely that every no-radiation metric satisfying vacuum Einstein equations is a Kerr metric (which admits strong completions), but no field equations are assumed in Theorem 2 or Proposition 1.
19C. Cutler and R. M. Wald, Class. Quantum Grav. 6, 453 (1989); the Cutler–Wald construction of global solutions of Einstein–Maxwell equations can be generalized to Einstein–Yang–Mills equations (R. Bartnik, private communication).
20D. Christodoulou, Commun. Math. Phys. 105, 327 (1986); 106, 587 (1986); 109, 591 (1987); 109, 613 (1987). In these papers existence of global solutions is established only in the interior of the forward light cone of a point. It has, however, been pointed out to the author by D. Christodoulou that it is easy to prove that time symmetric, spherically symmetric, Schwarzschildian outside a compact set Cauchy data on a spacelike hypersurface can be evolved to a set that contains a complete light cone of the origin. If the norm of these data is small the norm of the resulting data on the light cone will be small as well, thus yielding, by the results proved in the above cited papers, a global solution of Einstein equations for a metric interacting with a scalar field.