On the structure of spatial infinity. II. Geodesically regular Ashtekar–Hansen structures

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The ambiguities in the differentiable structure of Ashtekar–Hansen completions satisfying a geodesic condition are analyzed. The results obtained imply, in particular, uniqueness up to a four-parameter family of "logarithmic transformations" of completions of asymptotically flat space-times stationary "in a neighborhood of i_0."

I. INTRODUCTION

In a previous paper of this series[1] uniqueness up to a three-parameter family of logarithmic transformations of conformal one-point compactifications of three-dimensional asymptotically flat Riemannian manifolds was established. A corollary of that result is that within the Geroch framework[2] the four-momentum of an initial data set for Einstein equations is unambiguously defined. In general relativity, which is a four-dimensional theory *par excellence*, one hopes to assign a four-momentum p_μ or, say, its invariant square $m^2 = -\eta_{\alpha\beta}p^\alpha p^\beta$ to a four-dimensional set, rather than to a three-dimensional subset thereof. It has been shown in Ref. 3 that one can, in a meaningful way, associate an invariant mass m to a boost-type domain or, more generally, to a four-dimensional asymptotically flat end of a Lorentzian manifold defined by a collection of boost-type domains. This relatively satisfactory result suffers from the drawback that the somewhat arbitrary notion of the boost-type domain plays an essential role in the analysis. One would like to replace the statement that "two three-dimensional ends included in some boost-type domain of a vacuum space-time have the same mass" by something of the kind "two three-dimensional ends included in the same asymptotic region have the same mass," avoiding the use of some unnaturally preferred sets in some coordinate system as a primary concept of the construction. A reasonably natural setup in which one can define the notion of an asymptotic region has been proposed by Ashtekar and Hansen[4,5] who describe the behavior of the gravitational field at spatial infinity by means of conformal completions of asymptotically flat four-dimensional manifolds in which spatial infinity is represented by a point i_0. The existence of an Ashtekar–Hansen completion—or some variation thereof, as considered in this paper—adds useful information about the global causal structure of space-time to the standard coordinate notion of asymptotic flatness,[6,3,7] which seems difficult to describe in terms of asymptotically flat coordinates only. The main problem with the Ashtekar–Hansen completions is their potential nonuniqueness. In this paper we show that if a certain geodesic condition is satisfied by some completion, then there exists a four-parameter family of inequivalent completions only.

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In Sec. II we introduce the notion of *weak conformal completions* and the *geodesic regularity* condition. We show that weak geodesically regular completions are unique. We also show that every completion of a *no-radiation* metric (in particular, of the Kerr metrics) is geodesically regular. In Sec. III we define *strong completions* (the conditions of this section are essentially those of Ashtekar and Hansen) and we show their uniqueness up to "logarithmic ambiguities" provided that geodesic regularity holds.

II. WEAK CONFORMAL COMPLETIONS

In order to give a motivation to the definitions of this section let us recall the fundamental result of Christodoulou and O'Murchadha[6] (the "boost theorem"): Given asymptotically flat data for general relativity (cf. Ref. 6 for the appropriate definition of asymptotic flatness) and given any "boost slope" $\theta < 1$ there exists a metric $g_{\mu\nu}$, solution of the vacuum Einstein equations, the evolution of the Cauchy data, and positive constants R and T such that $g_{\mu\nu}$ is defined for all x^μ belonging to the *boost-type domain* $\Omega_{\theta,R,T}$ (Ref. 8):

$$\Omega_{\theta,R,T} = \{x^\alpha : r \geqslant R, |x^0| \leqslant \theta r + T\},$$

$$\theta > 0, \quad R \geqslant 0, \quad T \in (-\infty, \infty],$$

with $g_{\mu\nu}$ satisfying

$$|g_{\mu\nu} - \eta_{\mu\nu}| \leqslant C(1+r)^{-\alpha}, \quad |\partial_\sigma g_{\mu\nu}| \leqslant C(1+r)^{-\alpha-1} \quad (1)$$

for some constants $C(\theta,R,T,g_{\mu\nu})$, $\alpha > 0$, where $\eta_{\mu\nu}$ is the Minkowski metric. For $x^\mu x_\mu > 0$ (Ref. 9) (signature $-+++$) let Φ denote the inversion $x^\mu \to y^\mu = x^\mu/(x^\alpha x_\alpha)$. It is simple to check that for $\theta < 1$ and $T > 0$ the set $\Phi(\Omega_{\theta,R,T})$ contains the "wedge" $W_{\theta,1/R}$ (Ref. 10):

$$W_{\theta,\epsilon} = \{y^\mu : r(y) < \epsilon, |y^0| < \theta r\}$$

and Eq. (1) gives, for $y^\mu \in W_{\theta,1/R}$,

$$\begin{aligned} &g_{\mu\nu}\, dx^\mu\, dx^\nu = (y^\alpha y_\alpha)^{-2} \bar g_{\mu\nu}\, dy^\mu\, dy^\nu, \\ &|\bar g_{\mu\nu} - \eta_{\mu\nu}| \leqslant C'(\theta,R,T,g_{\mu\nu})\, r(y)^\alpha, \\ &\left|\frac{\partial \bar g_{\mu\nu}}{\partial y^\sigma}\right| \leqslant C'(\theta,R,T,g_{\mu\nu})\, r(y)^{\alpha-1}. \end{aligned} \quad (2)$$

Equation (2) displays the expected behavior of the metric under an inversion which brings "spatial infinity" to a point $y^\alpha = 0$, say i_0. Following Ashtekar and Hansen[4,5] we shall ask for some more structure than what follows in a straightforward manner from the "boost theorem."

Definition 1: Let (M,g) be a space-time ($\equiv C^3$ four-dimensional manifold with a C^2 Lorentzian metric) and let $\mathscr{M}$ denote the disjoint union $M \cup \{i_0\}$ where i_0 is a point. We shall say that $(\mathscr{M},\bar{g},i_0)$ is a *weak α-completion* of M, $\alpha \in (0,1]$, if the following holds.

(i) In $\mathscr{M}$ there exists a coordinate system $\{y^\mu\}$, $y^\mu \in W_{1,\epsilon} \cup \{0\}$ such that $i_0 = 0$.

(ii) There exists a function Ω: $W_{1,\epsilon} \to \mathbb{R}^+$ such that the metric $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ satisfies

$$\forall \theta < 1, \quad \forall x \in W_{\theta,\epsilon} |\bar{g}_{\mu\nu} - \eta_{\mu\nu}| \leqslant C(\theta) r(y)^\alpha,$$
$$|\partial_\sigma \bar{g}_{\mu\nu}| \leqslant C(\theta) r(y)^{\alpha-1}.$$

(iii)

$$\lim_{y\to 0} \Omega = 0, \quad \lim_{y \to 0} \partial_\alpha \Omega = 0, \quad |\bar{\nabla}_\mu \bar{\nabla}_\nu \Omega - 2\bar{g}_{\mu\nu}| \leqslant C(\theta) r(y)^\alpha$$

with some function $[0,1) \ni \theta \to C(\theta) < \infty$ (Ref. 11).

(iv) For all $p \in M$ there exists no timelike curve in $\mathscr{M}$ from p to i_0.

Here $\mathscr{M}$ shall be equipped with the natural topology induced by the topology of M and the coordinates y^α. We shall say that a weak conformal completion is *geodesically regular* if for every affinely parametrized spacelike geodesic Γ of the physical metric $g_{\mu\nu}$, $\Gamma = \{y^\alpha(s), s \geqslant s_0\}$, which extends to i_0, there exist constants $s_1 \geqslant s_0$, $\Psi(\Gamma) < 1$ such that for all $s \geqslant s_1$ we have[12]

$$|y^0(s)| \leqslant \Psi(\Gamma) r(y(s)).$$

We shall show that weak geodesically regular α-completions are unique for $0 < \alpha < 1$.

Lemma 1: Let $g_{\mu\nu}$ be a C_2 metric in a boost-type domain $\Omega_{\theta,R,T}$ satisfying

$$|g_{\mu\nu} - \eta_{\mu\nu}| \leqslant C(1+r)^{-\alpha}, \quad |\partial_\sigma g_{\mu\nu}| \leqslant C(1+r)^{-\alpha-1} \quad (3)$$

with some constant C. There exist spacelike hypersurfaces $B^\pm \subset \Omega_{\theta,R,T}$ defined by

$$B^\pm = \{p \in \Gamma_{\mathbf{n}}^\pm, \quad \text{where } \Gamma_{\mathbf{n}}^\pm = \text{complete spacelike geodesic}^{13} \text{ satisfying } \mathbf{x}_{\mathbf{n}}^\pm(0) = R_1 \mathbf{n},$$
$$\mathbf{n} \in S(1)^{14}, \quad \frac{d\mathbf{x}_{\mathbf{n}}^\pm}{ds}(0) = \mathbf{n}, \quad (x_{\mathbf{n}}^\pm)^0(0) = 0, \quad \lim_{s\to\infty} r(x_{\mathbf{n}}^\pm(s)) = \infty, \quad (x_{\mathbf{n}}^\pm)^0(s) \geqslant 0 \text{ on } \Gamma_{\mathbf{n}}^+,$$
$$(x_{\mathbf{n}}^\pm)^0(s) \leqslant 0 \quad \text{on } \Gamma_{\mathbf{n}}^-\}$$

[in other words, $B^\pm$ are "sewn up" from geodesics starting from the sphere $\{r(x) = R_1, x^0 = 0\}$ which stay in $\Omega_{\theta,R,T}$ and remain either to the local future (with respect to the chronology of $\Omega_{\theta,R,T}$) or to the local past of $N^x_{R_1} \equiv \{x^\alpha : x^0 = 0, r(x) \geqslant R_1\}$]. The $B^\pm$ are graphs over $N^x_{R_1}: B^\pm = \{x^\mu : r(x) \geqslant R_1, x^0 = w^\pm(\mathbf{x})\}$. Every future directed timelike curve starting at $N^x_{R_1}$ either remains entirely within the wedge $W^+ = \{x^\mu : r(x) \geqslant R_1, 0 \leqslant x^0 \leqslant w^+(\mathbf{x})\}$ or meets B^+. Similarly, every past directed timelike curve starting at $N^x_{R_1}$ either remains entirely within the wedge $W^- = \{x^\mu : r(x) \geqslant R_1, w^-(\mathbf{x}) \leqslant x^0 \leqslant 0\}$ or meets B^-.

Proof: By Propositions B1 and B2 of Appendix B of Ref. 3 for R_1 sufficiently large the family of geodesics $\Gamma_{\mathbf{n}}^\pm$, $\mathbf{n} \in S(1)$ defined by $\mathbf{x}_{\mathbf{n}}^\pm(0) = R_1 \mathbf{n}$, $\mathbf{n} \in S(1)$, $(d\mathbf{x}_{\mathbf{n}}^\pm/ds)(0) = \mathbf{n}$, $(x_{\mathbf{n}}^\pm)^0(0) = 0$, and $d(x_{\mathbf{n}}^\pm)^0/ds(0) = \pm\theta_0 \equiv \pm \min(\theta,1)/2$ will satisfy $\forall s \geqslant 0 (x_{\mathbf{n}}^+)^0(s) \geqslant 0$, $(x_{\mathbf{n}}^-)^0(s) \leqslant 0$, $(\mathbf{x}_{\mathbf{n}}^\pm/r)(d\mathbf{x}_{\mathbf{n}}^\pm/ds)(s) \geqslant \frac{1}{2}$, and $\theta_0/2 \leqslant \pm d(x_{\mathbf{n}}^\pm)^0/ds \leqslant 3\theta_0/2$. Let

$$A = \{\rho \geqslant R_1 : \forall \mathbf{x} \in B(\rho) \setminus \text{Int}[B(R_1)] \exists \mathbf{n}_\pm(\mathbf{x}) \in S(1) \text{ and}$$
$$\mathbf{x}^0_\pm(\mathbf{x}), \quad |\mathbf{x}^0_\pm(\mathbf{x})| \leqslant \theta r(\mathbf{x})$$
$$+ T \text{ such that } (\mathbf{x}^0_\pm(\mathbf{x}), \mathbf{x}) \in \Gamma^\pm_{\mathbf{n}_\pm(\mathbf{x})}\},$$

where $B(\rho)$ denotes a closed ball of radius ρ. A is nonempty because $R_1 \in A$, A is closed by standard properties of solutions of differential equations, and openness of A follows from the implicit function theorem and the fact that $d\mathbf{x}_{\mathbf{n}}^\pm/ds$ is everywhere transversal to the spheres $S(\rho)$. This implies $A = \{\rho \in \mathbb{R} : \rho \geqslant R_1\}$, so that for every $\mathbf{x} \in \mathbb{R}^3 \setminus B(R_1)$ there exists $p^\pm = (w^\pm(\mathbf{x}) \equiv \mathbf{x}^0_\pm(\mathbf{x}), \mathbf{x}) \in \Omega_{\theta,R,T}$ and two geodesics $\Gamma^\pm_{\mathbf{n}_\pm}$ which pass through $p^\pm$.

Lemma 2: Let $(\mathscr{M}_1, g_1, i_1)$, $(\mathscr{M}_2, g_2, i_2)$ be two weak α-completions of a space-time (M,g), $0 < \alpha < 1$; let x and y be the appropriate coordinate systems, $x^\alpha \in W_{1,\epsilon_1} \cup \{0\}$, $y^\alpha \in W_{1,\epsilon_2} \cup \{0\}$; let Φ denote the coordinate transformation $y^\alpha(x^\mu)$ wherever defined; suppose that Φ be differentiable; and define $N_{\epsilon_1} = \{x^\alpha : x^0 = 0, r(x) \leqslant \epsilon_1\}$. If $\Phi(N_{\epsilon_1})$ is contained in a wedge W_{θ,ϵ_2}, $\theta < 1$, then there exists a Lorentz matrix Λ^μ_ν such that

$$y^\mu = \Lambda^\mu_\nu x^\nu + \zeta^\mu, \quad (4)$$

where ζ^μ satisfies

$$\forall \psi < 1, \quad \forall x \in W_{\psi,\epsilon_1} : |\zeta^\mu| \leqslant C(\psi) r^{1+\alpha},$$
$$|\partial_\mu \zeta^\nu| \leqslant C(\psi) r^\alpha, \quad |\partial_\mu \partial_\nu \zeta^\rho| \leqslant C(\psi) r^{\alpha-1}. \quad (5)$$

Proof: The image by Φ of i_1 must be i_2, otherwise there would exist a timelike curve from some point $p \in M$ to i_1, contradicting point (iv) of Definition 1. Let $\hat{x}^\mu = x^\mu/(x^\alpha x_\alpha)$, $\hat{y}^\mu = y^\mu/(y^\alpha y_\alpha)$; the hypersurface $\hat{N}^x_{1/\epsilon_1} = \{\hat{x}^\mu : r(\hat{x}) \geqslant 1/\epsilon_1, \hat{x}^0 = 0\}$ is asymptotically flat and is included in some boost-type domain of coordinates $\hat{y}^\mu$ with slope θ smaller than 1—the result follows from Theorem 1 and Point 2 of Corollary 2 of Ref. 3.

Lemmas 1 and 2 lead to the following theorem.

Theorem 1 (uniqueness of geodesically regular weak completions, $\alpha < 1$): If a space-time (M,g) admits one geodesically regular weak α-completion $(\bar{M},\bar{g},\bar{\imath})$, $\alpha \in (0,1)$, then the following holds.

(i) All weak α-completions of (M,g) are geodesically regular.

(ii) All weak α-completions of (M,g) are related to each other by coordinate transformations of the form (4) and (5).

Proof: Let $(\widetilde{M},\tilde{g},\bar{\imath})$ be some completion of (M,g), let $\{x^\alpha\}$ be the appropriate coordinate system for $(\widetilde{M},\tilde{g})$, let y^α be the coordinates of Definition 1 for $(\overline{M},\bar{g})$, and let Ψ denote the transformation $y^\alpha(x^\mu)$. From Lemma 1 applied to the physical metric $g_{\mu\nu}$ in coordinates $\hat{x}^\mu = x^\mu/(x^\alpha x_\alpha)$ one infers the existence of two hypersurfaces $B^\pm$ sewn up from the geodesics $\Gamma_{\mathbf{n}}^\pm$, $\mathbf{n}\in S(1)$. By geodesic regularity for all $\mathbf{n}\in S(1)$ there exist $\theta_{\mathbf{n}}^\pm$ such that $\Psi(\Gamma_{\mathbf{n}}^\pm)\subset W_{\theta_{\mathbf{n}}^\pm,\epsilon_2}$. By compactness of $S(1)$ we have $\sup_{\mathbf{n}\in S(1)}\theta_{\mathbf{n}}^\pm = \theta_0 < 1$.[15] Since $N_{1/R_1} = \{x^\alpha : x^0 = 0, r(x) \leqslant 1/R_1\}$ (R_1 given by Lemma 1) lies to the local future of $B^-\subset W_{\theta_0,\epsilon_2}$ and to the local past of $B^+\subset W_{\theta_0,\epsilon_2}$ we must have $N_{1/R_1}\subset W_{\theta_0,\epsilon_2}$, so that we can apply Lemma 2 to conclude that $y(x)$ is of the form (4), which establishes point (ii). It is not too difficult to show that $x(y)$ must be of the form (4) as well, so that the image $\Psi(\Gamma)$ of any spacelike geodesic $\Gamma = \{y^\alpha(s), s\geqslant s_0\}\subset\Omega_{\theta(\Gamma),\epsilon_2}$ will be included in $\Psi(W_{\theta(\Gamma),\epsilon_2})\subset W_{\theta',\epsilon_1'}$ with some $\theta' < 1$ and point (i) ensues.

A metric shall be called a *no-radiation metric* if there exist coordinates $x^\alpha\in\Omega_{R,\infty}\equiv(-\infty,\infty)\times[\mathbb{R}^3\setminus B(R)]$ such that $g_{00}\leqslant -\epsilon$ for some positive ϵ, g_{ij} is a positive definite matrix with eigenvalues separated from zero, and (3) holds throughout $\Omega_{R,\infty}$ with some constant C. The Kerr metrics are no-radiation metrics in this sense, with $\alpha = 1$.

Proposition 1: Weak α-completions of no-radiation metrics, $\alpha > 0$, are geodesically regular.

Proof: By point (i) of Theorem 1 it is sufficient to show the existence of one geodesically regular completion. Let x^α be the coordinates satisfying (3) and let $\hat{x}^\mu = x^\mu/(x^\alpha x_\alpha)$. By Proposition B1 of Appendix B of Ref. 3 every spacelike geodesic meeting i_0 [i.e., such that $\hat{x}\to 0 \Rightarrow r(x)\to\infty$] behaves asymptotically as follows:

$$x^\alpha(s) = \eta_\infty^\alpha s + o(s)$$

for some constant vector η_∞^α satisfying $\eta_{\mu\nu}\eta_\infty^\mu\eta_\infty^\nu > 0$, where s is an affine parameter, and we can normalize η_∞^α to satisfy $\eta_\infty^i\eta_\infty^i = 1 (\Rightarrow |\eta_\infty^0| < 1)$. We have

$$\hat{x}^\alpha(s) = [\eta_\infty^\alpha/(1-|\eta_\infty^0|^2)]s^{-1} + o(s^{-1}),$$

so that $\hat{x}^0(s) = \eta_\infty^0 r(\hat{x}(s)) + o[r(\hat{x}(s))]$ and for s large enough one obtains

$$|\hat{x}^0(s)|\leqslant\theta r(s),\ \theta = (|\eta_\infty^0| + 1)/2 < 1.$$

III. STRONG CONFORMAL COMPLETIONS

In Sec. II we have investigated the structure of the set of conformal completions in which the metric is allowed to blow up as one approaches "what would be the light cone of i_0." As has been shown by Schmidt and Walker (cf. Appendix C of Ref. 4) much better behaved completions can be obtained for Kerr metrics. To justify our conditions on the conformally rescaled metric, to be presented later, let us recall the Schmidt–Walker coordinates for the Schwarzschild metric: For $s > 2m$ let $f(s)$ be defined by

$$f(s) = s + 2m\ln(s/2m - 1) \tag{6}$$

and let us set

$$\begin{aligned} &r^* = f(r), \quad r^* = [f(\hat{v}^{-1}) + f(\hat{w}^{-1})]/2,\\ &t = [f(\hat{v}^{-1}) - f(\hat{w}^{-1})]/2, \\ &\hat{r} = (\hat{v}+\hat{w})/2, \quad \hat{t} = (\hat{v}-\hat{w})/2, \end{aligned} \tag{7}$$

where r and t are the standard Schwarzschild coordinates

$$ds^2 = -(1-2m/r)dt^2 + (1-2m/r)^{-1}dr^2 + r^2 d\Omega^2. \tag{8}$$

In the coordinates $\hat{v}$, $\hat{w}$, $\hat{v}\hat{w} > 0$, the Schwarzchild metric takes the form

$$ds^2 = \frac{1}{(\hat{v}\hat{w})^2} \times\left\{\frac{(1-2m/r)}{(1-2m\hat{v})(1-2m\hat{w})}d\hat{v}\,d\hat{w} + (\hat{v}\hat{w}r)^2 d\Omega^2\right\}. \tag{9}$$

From Eqs. (6) and (7) one has

$$r = r^* - 2m\ln(r^*/2m) + \zeta^*,$$

with

$$\zeta^* = 2m\ln\left\{\frac{1+2m\ln[r/2m-1]/r}{1-2m/r}\right\} = O\left(\frac{\ln[r^*]}{r^*}\right)$$

for large r^*; therefore, from

$$r^* = \hat{r}/\hat{v}\hat{w} + m\{\ln[(1-2m\hat{v})(1-2m\hat{w})] - \ln[4m^2\hat{v}\hat{w}]\}$$

one obtains, for small $\hat{r}$,

$$\zeta^* = (\hat{v}\hat{w}/\hat{r})O(|\ln[\hat{r}]| + |\ln[\hat{v}\hat{w}]|) = O(\hat{r}|\ln[\hat{r}]|), \quad \text{for } |\hat{t}|\leqslant\hat{r}$$

and one obtains

$$\omega\equiv\hat{v}\hat{w}r/\hat{r} = 1 + (m\hat{v}\hat{w}/\hat{r})\ln(\hat{v}\hat{w}/\hat{r}^2) + O(\hat{r}^2), \tag{10}$$

so that

$$(\omega - 1)/\hat{r} = m(1-\hat{t}^2/\hat{r}^2)\ln(1-\hat{t}^2/\hat{r}^2) + O(\hat{r}).$$

The metric (9) can be written in the form

$$ds^2 = \Omega^{-2}\hat{g}_{\mu\nu}\,d\hat{x}^\mu\,d\hat{x}^\nu = \Omega^{-2}\,d\hat{s}^2, \quad \Omega = \hat{v}\hat{w} = \hat{r}^2 - \hat{t}^2,$$

$$d\hat{s}^2 = \frac{(1-2m\hat{v}\hat{w}/(\omega\hat{r}))}{(1-2m\hat{v})(1-2m\hat{w})}\left(-d\hat{t}^2 + \frac{(\hat{x}^i\,d\hat{x}^i)^2}{\hat{r}^2}\right) + \omega^2\left[d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2 - \frac{(\hat{x}^i\,d\hat{x}^i)^2}{\hat{r}^2}\right],$$

and we have $\lim_{|\hat{t}|\nearrow\hat{r}}\omega = 1$, so that $\hat{g}_{\mu\nu}$ can be continuously extended to the set $\hat{t}^2\leqslant\hat{r}^2$, $0\leqslant\hat{r}\leqslant\hat{r}_0$ with some $\hat{r}_0$. The metric $\hat{g}_{\mu\nu}$ is of the form

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \hat{r}k_{\mu\nu}(\theta,\phi,\hat{t}/\hat{r}) + o(\hat{r}) \tag{11}$$

and we have

$$|\hat{g}_{\mu\nu} - \eta_{\mu\nu}|\leqslant C\hat{r} \tag{12}$$

for some constant C; the derivatives of $\hat{g}_{\mu\nu}$ are, however, not bounded up to the light cone of $i_0 = \{x^\alpha = 0\}$: One finds

$$\partial\omega \sim \ln(1 - \hat{t}^2/\hat{r}^2),$$

where ω is defined in (10), so that one obtains

$$|\partial_\sigma \hat{g}_{\mu\nu}| \leqslant C_1(\hat{t}/\hat{r}),$$

with $C_1(\eta) = C[|\ln(1-\eta^2)| + 1]$ for some constant C. It must be emphasized that the singularity of $C_1(\eta)$ at $\eta = 1$ is rather mild in the sense that

$$\int_0^1 C_1(\eta)d\eta < \infty.$$

We also have

$$\hat{r}|\partial_\sigma \partial_\rho \hat{g}_{\mu\nu}| \leqslant C_2(\hat{t}/\hat{r}), \quad C_2(\eta) \sim (1-\eta^2)^{-1},$$

with

$$\int_0^\theta C_2(\eta)d\eta \leqslant C' C_1(\theta).$$

This analysis[16] motivates the following definition.

Definition 2: Here $(\overline{M},\overline{g},i_0)$ will be called a *strong completion* of a space-time (M,g) if $\overline{M}$ is the disjoint union $M \cup \{i_0\}$ and the following holds.

(i) For all $p \in M$ there exists no timelike curve in $\overline{M}$ from p to i_0.

(ii) There exists a coordinate system $x^\alpha \in W_{1,\epsilon_x} \cup \{0\}$ such that $i_0 = 0$ and the points $x^\alpha x_\alpha > 0$ correspond to points in M. On M there exists a function $\Omega: M \to \mathbb{R}$, $\Omega > 0$ such that $\overline{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ and there exists a constant C and nondecreasing functions $C_1, C_2: [0,1) \to \mathbb{R}^+$ such that

$$\forall x \in \Omega W_{1,\epsilon_x}: |\overline{g}_{\mu\nu} - \eta_{\mu\nu}| \leqslant Cr, \quad |\partial_\sigma \overline{g}_{\mu\nu}| \leqslant C_1(|\eta|),$$
$$r|\partial_\sigma \partial_\rho \overline{g}_{\mu\nu}| \leqslant C_2(|\eta|), \quad \eta \equiv t/r, \tag{13}$$

with

$$\int_0^\theta C_2(\eta)d\eta \leqslant C_1(\theta), \quad \int_0^1 C_1(\eta)d\eta = \overline{C}_1 < \infty.$$

(iii) Ω satisfies

$$\lim_{x\to 0} \Omega = 0, \quad \lim_{x \to 0} \partial_\mu \Omega = 0, \text{ for } x \neq 0$$
$$|\partial_\mu \partial_\nu \Omega - 2\overline{g}_{\mu\nu}| \leqslant Cr.$$

(iv) For every $|\eta| < 1$ the limits

$$\lim_{r\to 0} [\overline{g}_{\mu\nu}(t = \eta r, r, \theta, \phi) - \eta_{\mu\nu}]/r,$$

$$\lim_{r \to 0} \partial_\sigma \overline{g}_{\mu\nu}(t = \eta r, r, \theta, \phi),$$

$$\lim_{r\to 0} [r \partial_\alpha \partial_\beta \overline{g}_{\mu\nu}(t = \eta r, r, \theta, \phi)] \tag{14}$$

exist and are continuous functions of (η,θ,ϕ), where θ and ϕ are standard spherical angles.

Let us note that (13) implies that the Ricci tensor $\overline{R}_{\alpha\beta}$ of $\overline{g}_{\mu\nu}$ satisfies

$$|\overline{R}_{\alpha\beta}| \leqslant C_{\rm R}(\eta) r^{-1},$$
$$\overline{C}_{\rm R}(\theta) = \int_0^{|\theta|} C_{\rm R}(\eta)d\eta, \quad \int_0^1 \overline{C}_{\rm R}(\theta)d\theta = \overline{\overline{C}}_{\rm R} < \infty$$

$(C_{\rm R}(\eta) \leqslant c[C_2(|\eta|) + C_1^2(|\eta|)]$; therefore, $\overline{C}_{\rm R}(\theta) \leqslant c[C_1(|\theta|) + C_1(|\theta|)\int_0^{|\theta|} C_1(\eta)d\eta] = c[1 + \overline{C}_1]C_1(|\theta|)$ with some numerical factor c). Our main result is the following theorem.

Theorem 2 (quasiuniqueness of geodesically regular strong completions): Let $(\mathscr{M}_1,g_1,i_1)$, $(\mathscr{M}_2,g_2,i_2)$ be two geodesically regular strong completions of a space-time (M,g) and let $\{x^\alpha\}$, $x \in W_{1,\epsilon_x} \cup \{0\}$ and $\{y^\alpha\}$, $y \in W_{1,\epsilon_y} \cup \{0\}$ be the appropriate coordinate systems in $\mathscr{M}_1$ and $\mathscr{M}_2$. There exists a Lorentz matrix Λ^μ_ν, a constant vector C^μ, and a constant C such that for $x \in W_{1,\epsilon_x}$,

$$y^\alpha = \Lambda^\alpha_\beta x^\beta + (C^\alpha x^\mu x_\mu - 2x^\alpha x^\mu C_\mu)\ln r + \zeta^\alpha,$$
$$|\zeta^\mu| \leqslant Cr^2, \quad |\partial_\nu \zeta^\mu| \leqslant Cr. \tag{15}$$

Moreover, for all $|\eta| < 1$ the limits

$$\lim_{r\to 0} [\zeta^\mu(t = r\eta, r, \theta, \phi) r^{-2}],$$

$$\lim_{r \to 0} [\partial_\alpha \zeta^\mu(t = r\eta, r, \theta, \phi) r^{-1}],$$

$$\lim_{r\to 0} \partial_\alpha \partial_\beta \zeta^\mu(t = r\eta, r, \theta, \phi),$$

$$\lim_{r \to 0} [r \partial_\alpha \partial_\beta \partial_\gamma \zeta^\mu(t = r\eta, r, \theta, \phi)]$$

exist and are continuous functions of (η,θ,ϕ).

Proof: By Theorem 1 there exists a Lorentz matrix Λ^μ_ν such that for $x \in N_{\epsilon_x} = \{x^\alpha : x^0 = 0, r \leqslant \epsilon_x\}$,

$$y^\alpha = \Lambda^\alpha_\beta x^\beta + \zeta^\alpha, \quad |\zeta^\alpha| \leqslant Cr^{2-\epsilon}, \quad |\partial_\beta \zeta^\alpha| \leqslant Cr^{1-\epsilon}, \tag{16}$$

with any $\epsilon > 0$ and, in fact, a straightforward extension of the estimates of Ref. 3 leads to

$\forall x \in N_{\epsilon_x}$:

$$\zeta^\mu = (C^\mu \mathbf{x}^2 - 2x^\mu \mathbf{Cx})\ln r + \mathring{\zeta}^\mu(\theta,\phi) r^2 + o(r^2),$$
$$\partial_\nu \zeta^\alpha = \partial_\nu[(C^\alpha x^\mu x_\mu - 2x^\alpha x^\mu C_\mu)\ln r + \mathring{\zeta}^\alpha r^2] + o(r),$$
$$\partial_\rho \partial_\sigma \zeta^\alpha = \partial_\rho \partial_\sigma [(C^\alpha x^\mu x_\mu - 2x^\alpha x^\mu C_\mu)\ln r$$
$$+ \mathring{\zeta}^\alpha r^2] + o(1) \tag{17}$$

for some constant vector C^μ. By a slight abuse of notation let us denote by y^α the coordinates $\Lambda^\alpha_\beta y^\beta$, so that we can set $\Lambda^\mu_\nu = \delta^\mu_\nu$ in Eq. (16), and on N_{ϵ_x} we have

$$\tau_{|t=0} \equiv y^0_{|t=0} = O(r^2 \ln r),$$
$$r(y(x))_{|t=0} = r(x) + O(r^2 \ln r),$$
$$\partial_\mu y^\alpha_{|t=0} = \delta^\alpha_\mu + O(r\ln r),$$
$$\partial_\alpha \partial_\beta y^\mu_{|t=0} = O(\ln r). \tag{18}$$

Now $g^1_{\mu\nu}$ and $g^2_{\mu\nu}$ are conformally related to each other, so that by definition, there exists a function Φ: $W_{1,\epsilon_x} \to \mathbb{R}^+$ such that

$$\forall x^\alpha, x^\alpha x_\alpha > 0: \Phi^2(x) g^1_{\mu\nu}(x) = g^2_{\alpha\beta}(y(x)) \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu}. \tag{19}$$

Equations (13), (18), and (19) yield

$$\Phi = \left\{\frac{\det g^2_{\mu\nu}}{\det g^1_{\mu\nu}}\right\}^{1/8} \left\{\det\left(\frac{\partial y^\beta}{\partial x^\nu}\right)\right\}^{1/4} \tag{20}$$

$$\Rightarrow \Phi_{|t=0} = 1 - 2\mathbf{Cx} \ln r + \mathring{\Phi}(\theta,\phi) r + o(r),$$
$$\partial_\mu \Phi_{|t=0} = -2C_\mu \ln r + \chi_\mu(\theta,\phi) + o(1) \tag{21}$$

for some functions $\mathring{\Phi}(\theta,\phi)$, $\chi_\mu(\theta,\phi)$. The transformation law of the Christoffel symbols gives

$$\partial_\alpha \partial_\beta y^\mu = [\Gamma^{1\lambda}_{\alpha\beta} + \Phi^{-1}(\delta^\lambda_\alpha \partial_\beta \Phi + \delta^\lambda_\beta \partial_\alpha \Phi - g^1_{\alpha\beta} g_1^{\lambda\sigma} \partial_\sigma \Phi)] \frac{\partial y^\mu}{\partial x^\lambda} - \Gamma^{2\mu}_{\rho\sigma}(y(x)) \frac{\partial y^\rho}{\partial x^\alpha} \frac{\partial y^\sigma}{\partial x^\beta}, \tag{22}$$

where Γ^1 and Γ^2 are the Christoffel symbols of the metrics $g^1_{\mu\nu}$ and $g^2_{\mu\nu}$. The formula for the transformation of the Ricci tensor under conformal transformations reads as

$$\frac{\partial^2\Phi}{\partial x^\mu \partial x^\nu} = \Gamma^{1\lambda}_{\mu\nu} \frac{\partial\Phi}{\partial x^\lambda} + \frac{1}{2}\left\{\Phi\left[R^{\,1}_{\mu\nu} - R^{\,2}_{\alpha\beta}(y(x)) \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu}\right] + 4\Phi^{-1} \frac{\partial\Phi}{\partial x^\mu}\frac{\partial\Phi}{\partial x^\nu} + \left[\frac{1}{6}\Phi^3 R_2(y(x)) - \frac{1}{6}\Phi R_1 - \Phi^{-1} g_1^{\alpha\beta} \frac{\partial\Phi}{\partial x^\alpha}\frac{\partial\Phi}{\partial x^\beta}\right] g^1_{\mu\nu}\right\}. \tag{23}$$

By (18) and (21) one can find $\epsilon_0 \leqslant \min(\epsilon_x, e^{-1})$ (e is the Euler number) small enough so that for all $x \in N_{\epsilon_0}$ we have

$$\tfrac{3}{4} \leqslant \Phi_{|t=0} \leqslant \tfrac{5}{4}, \quad 3r(x)/4 \leqslant r(y(x))_{|t=0} \leqslant 5r(x)/4,$$

$$\left|\frac{\partial y^\sigma}{\partial x^\beta} - \delta^\sigma_\beta\right|_{|t=0} \leqslant \frac{1}{4}, \quad \left|\frac{\partial \tau}{\partial t} - \frac{\tau(y(x))}{r(y(x))}\frac{\partial r(y)}{\partial t}\right|_{|t=0} \geqslant \frac{3}{4}.$$

Let

$$C_\Phi \equiv \sup_{\mu, x \in N_{\epsilon_0}} |[\ln r]^{-1} \partial_\mu \Phi|,$$

for $\eta \in [0,1)$,

$$C_{R_1}(\eta) \equiv \sup_{\mu,\nu,x \in W_{\eta,\epsilon_x}} r(x)|R^{\,1}_{\mu\nu}(x)| \quad \text{and} \quad C_{R_1}(-\eta) \equiv C_{R_1}(\eta),$$

for $\eta \in [0,1)$,

$$C_{R_2}(\eta) \equiv \sup_{\mu,\nu,y \in W_{\eta,\epsilon_y}} r(y)|R^{\,2}_{\mu\nu}(y)| \quad \text{and} \quad C_{R_2}(-\eta) \equiv C_{R_2}(\eta),$$

$$C_{g_1} \equiv \sup_{\mu,\nu,x \in W_{1,\epsilon_x}} |g^1_{\mu\nu}(x)| + \sup_{\mu,\nu,x \in W_{1,\epsilon_x}} |g_1^{\mu\nu}(x)|,$$

$$C_{g_2} \equiv \sup_{\mu,\nu,y \in W_{1,\epsilon_y}} |g^2_{\mu\nu}(y)| + \sup_{\mu,\nu,y \in W_{1,\epsilon_y}} |g_2^{\mu\nu}(y)|,$$

for $a = 1,2$,

$$\overline{C}_{R_a}(\theta) = \int_0^{|\theta|} C_{R_a}(\eta)\, d\eta, \quad \overline{\overline{C}}_{R_a} = \int_0^1 \overline{C}_{R_a}(\eta)\, d\eta \,(< \infty).$$

Let $\Omega \subset W_{1,\epsilon_0}$ be the set of points such that

$$\begin{aligned} &|\partial_\mu \Phi| < \widetilde{C}_\Phi(x) + C_\Phi \ln(1/r), \\ &\tfrac{1}{2} < \Phi < 2, \\ &r(x)/2 < r(y(x)) < 3r(x)/2, \\ &\left|\frac{\partial y^\mu}{\partial x^\nu} - \delta^\mu_\nu\right| < \frac{1}{2}, \\ &\left|\frac{\partial \tau}{\partial t} - \frac{\tau(y(x))}{r(y(x))}\frac{\partial r(y)}{\partial t}\right| > \frac{1}{2}, \end{aligned} \tag{24}$$

where

$$\widetilde{C}_\Phi(x) := (2 + 2^4 3^{-1} C^2_{g_1})\overline{C}_{R_1}(\eta) + (2^4 3^3 + 2^7 C_{g_1} C_{g_2})\overline{C}_{R_2}(\tilde{\eta}(x)) + 1 =: a\overline{C}_{R_1}(\eta) + b\overline{C}_{R_2}(\tilde{\eta}(x)) + 1, \quad \tilde{\eta}(x) \equiv \tau(x)/r(y(x)).$$

Let $\Omega_1' \subset \Omega$ be the set of points x^α such that the curve $[0,t] \ni s \to (s,\mathbf{x})$ is included in Ω. We have $N_{\epsilon_0} \subset \Omega_1'$; let Ω_1 be the connected component of Ω_1' which contains N_{ϵ_0}. We shall show that there exists $0 < \epsilon_1 \leqslant \epsilon_0$ such that $\Omega_1 \cap W_{1,\epsilon_1}$ is closed in W_{1,ϵ_1}. By (23) we have, for $x^\alpha \in \Omega_1$,

$$\left|\frac{\partial}{\partial t}\frac{\partial\Phi}{\partial x^\mu}\right| \leqslant [(1 + 2^3 3^{-1} C^2_{g_1}) C_{R_1}(\eta) + (2^3 3^2 + 2^6 3^{-1} C_{g_1} C_{g_2}) C_{R_2}(\tilde{\eta})] r(x)^{-1} + (2^2 + 2^4 C^2_{g_1})\left\{\widetilde{C}_\Phi(x) + C_\Phi \ln\left[\frac{1}{r(x)}\right]\right\}^2 + 2^3 3 C_{g_1} C^x_1(\eta)\left\{\widetilde{C}_\Phi(x) + C_\Phi \ln\left[\frac{1}{r(x)}\right]\right\},$$

where C^x_1 is the function C_1 in Eq. (13) for the coordinate system x^α, extended to negative η by $C^x_1(-\eta) \equiv C^x_1(\eta)$. From

$$\left|\frac{\partial\Phi}{\partial x^\mu}(t,\mathbf{x})\right| \leqslant \left|\frac{\partial\Phi}{\partial x^\mu}(0,\mathbf{x})\right| + \int_0^t \left|\frac{\partial^2\Phi}{\partial t\, \partial x^\mu}(s,\mathbf{x})\right| ds,$$

and from

$$\int_0^t C_{R_1}\left(\frac{s}{r}\right) ds = r\overline{C}_{R_1}\left(\frac{t}{r}\right),$$

$$\int_0^t C_{R_2}\left[\frac{\tau(s,\mathbf{x})}{r(y(s,\mathbf{x}))}\right] ds = \int_0^{\tilde{\eta}(x)} C_{R_2}(\eta) \frac{r(y)}{\partial\tau/\partial t - [\tau/r(y)][\partial r(y(x))/\partial t]}\, d\eta \leqslant 3r(x)\int_0^{\tilde{\eta}(x)} C_{R_2}(\eta)\, d\eta = 3r(x)\overline{C}_{R_2}\left(\frac{\tau(x)}{r(y(x))}\right),$$

$$\int_0^t \left[\widetilde{C}_\Phi(s,\mathbf{x}) + C_\Phi \ln\left(\frac{1}{r}\right)\right]^2 ds \leqslant 2\int_0^t \left[\widetilde{C}^2_\Phi(s,\mathbf{x}) + C^2_\Phi \ln^2\left(\frac{1}{r}\right)\right] ds \leqslant 2\widetilde{C}_\Phi(x)(a\overline{\overline{C}}_{R_1} + 3b\overline{\overline{C}}_{R_2} + 1)r(x) + 2C^2_\Phi r \ln^2\left(\frac{1}{r}\right)$$

we have used the facts that $\widetilde{C}_\Phi$ is nondecreasing along the curves $[0,t) \ni s \to (s,\mathbf{x})$, and $|t| < r$ in W_{1,ϵ_1}],

$$\int_0^t \widetilde{C}_\Phi(s,\mathbf{x}) C^x_1\left(\frac{s}{r}\right) ds \leqslant r\overline{C}^x_1 \widetilde{C}_\Phi(x), \quad \overline{C}^x_1 \equiv \int_0^1 C^x_1(\eta)\, d\eta,$$

one obtains

$$|\partial_\mu \Phi(t,\mathbf{x})| < C_\Phi \ln\left(\frac{1}{r}\right) + (2^3 + 2^5 C^2_{g_1})\epsilon_1 \ln^2\left(\frac{1}{\epsilon_1}\right) C^2_\Phi + 2^3 3 C_{g_1} \overline{C}^x_1 \epsilon_1 \ln\left(\frac{1}{\epsilon_1}\right) C_\Phi + \left\{\frac{1}{2} + [(2^3 + 2^5 C^2_{g_1})(a\overline{\overline{C}}_{R_1} + 3b\overline{\overline{C}}_{R_2} + 1)\right.$$

$$+ 2^3 3C_{g_1}\overline{C}_1^x]\epsilon_1\Big\}\widetilde{C}_\Phi(x)$$
$$\leqslant C_\Phi \ln\left(\frac{1}{r}\right)+\frac{1}{8}+\frac{5\widetilde{C}_\Phi(x)}{8}$$
$$\leqslant C_\Phi \ln\left(\frac{1}{r}\right)+\frac{3\widetilde{C}_\Phi(x)}{4}$$

for ϵ_1 small enough. This gives

$$|\Phi(t,\mathbf{x})-1|\leqslant|\Phi(0,\mathbf{x})-1|+\int_0^t|\partial_t\,\Phi(s,\mathbf{x})|ds$$
$$\leqslant\frac{1}{4}+(a\overline{\overline{C}}_{R_1}+3b\overline{\overline{C}}_{R_2}+1)\epsilon_1$$
$$+C_\Phi\epsilon_1\ln\left(\frac{1}{\epsilon_1}\right)\leqslant\frac{3}{8},$$

decreasing ϵ_1 if necessary. One shows in a similar way that none of the inequalities in (24) can saturate for $x^\alpha\in\Omega_1\cap W_{1,\epsilon_1}$ if ϵ_1 is small enough, so that $\Omega_1\cap W_{1,\epsilon_1}$ is both open and closed in W_{1,ϵ_1}; therefore, $\Omega_1\cap W_{1,\epsilon_1}=W_{1,\epsilon_1}$. It is not too difficult to show from (22) and (24) that Φ and $\partial y^\mu/\partial x^\nu$ uniformly tend to 1 and δ^μ_ν at i_0 and (23) yields

$$\lim_{r\to0}\left[\frac{\partial\Phi}{\partial x^\lambda}(t=r\eta,r,\theta,\phi)-\frac{\partial\Phi}{\partial x^\lambda}(0,r,\theta,\phi)\right]$$
$$=\lim_{r\to0}\int_0^{\eta r}\frac{\partial^2\Phi}{\partial t\,\partial x^\lambda}ds=:A_\lambda(\eta,\theta,\phi), \tag{25}$$

with some continuous functions $A_\lambda(\eta,\theta,\phi)$, so that from (21) and (25) one has

$$\frac{\partial\Phi}{\partial x^\lambda}(t=r\eta,r,\theta,\phi)=-2C_\lambda\ln r+\widetilde{A}_\lambda(\eta,\theta,\phi)+o(1)$$

for some continuous functions $\widetilde{A}_\lambda$ and (17) and (22) imply

$$\partial_\beta\,y^\alpha=\delta^\alpha_\beta+2(C^\alpha x_\beta-C_\beta x^\alpha-\delta^\alpha_\beta C^\mu x_\mu)$$
$$\times\ln r+rA^{\,\alpha}_{\,\beta}(\theta,\phi,\eta)+o(r),$$

with some continuous functions $A^{\,\alpha}_{\,\beta}$; a straightforward analysis establishes our remaining claims.

Two completions differing by a transformation of the form (15) with $C^\mu=0$ can be considered as equivalent. Theorem 2 and Proposition 1 imply[17] the following corollary.

Corollary 1: Strong conformal completions of no-radiation space-times are unique up to the four-parameter family of transformations (15).

It may be of some relevance to note that the logarithmic transformations (15) are not the ones given in Appendix 1 of Ref. 18: The latter introduce singularities in $g_{\mu\nu}$-$\eta_{\mu\nu}$ at the light cone of i_0, while (15) do not.

It is natural to ask about the group properties of the transformations (15) since r is not Lorentz invariant: Under a Lorentz transformation $y^\alpha=\Lambda^\alpha_\beta x^\beta$ we have

$$r(y)=\sqrt{\Lambda^i_\mu x^\mu\Lambda^i_\nu x^\nu}$$
$$=r(x)\sqrt{\Lambda^i_\mu(x^\mu/r)\Lambda^i_\nu(x^\nu/r)}=:f(\eta,\theta,\phi)r(x),$$

so that

$$\ln r(y)=\ln r(x)+\ln f$$

and the $\ln f$ terms can be absorbed in ζ^μ, which shows that a composition of transformations (15) is still of the form (15).

Let us finally note the existence of a set of coordinates for the Schwarzschild metric in which the metric is slightly worse behaved at i_0 than in (11) and (12); however, for $r\neq0$ the first derivatives of the metric do not blow up as one approaches the light cone of i_0. Let us set

$$r^*=f(r),\quad r^*=\tfrac12[\tilde f(\tilde v^{-1})+\tilde f(\tilde w^{-1})],$$
$$t=\tfrac12[\tilde f(\tilde v^{-1})-\tilde f(\tilde w^{-1})],\quad \tilde r=\tfrac12[\tilde v+\tilde w],$$
$$\tilde t=\tfrac12[\tilde v-\tilde w],\quad \tilde f(s)=s+4m\ln(s/2m),$$

with r, t as in (8) [this choice of $\tilde f$ cancels these terms in $r(\tilde v,\tilde w)$ which exhibit the worst behavior at the light cone of i_0]. One obtains

$$ds^2=\widetilde\Omega^{-2}\,d\tilde s^2,\quad\widetilde\Omega=\tilde v\tilde w,\quad d\tilde s^2=\tilde g_{\mu\nu}\,d\tilde x^\mu\,d\tilde x^\nu,$$
$$|\tilde g_{\mu\nu}-\eta_{\mu\nu}|\leqslant\widetilde C\tilde r\ln\tilde r,\quad|\partial_\sigma\,\tilde g_{\mu\nu}|\leqslant\widetilde C\ln\tilde r$$

for some constant $\widetilde C$. An unpleasant feature of these coordinates is the logarithmic blowing up of $(\tilde g_{\mu\nu}-\eta_{\mu\nu})/\tilde r$ at $\tilde r=0$; however, we have

$$\tilde g_{\mu\nu}=\eta_{\mu\nu}+\tilde r[\tilde h^{\,1}_{\mu\nu}(\tilde\eta,\theta,\phi)\ln\tilde r$$
$$+\tilde h^{\,2}_{\mu\nu}(\tilde\eta,\theta,\phi)]+o(\tilde r),\quad\tilde\eta=\tilde t/\tilde r,$$

which is of simple and tractable form.

IV. CONCLUSIONS

We have shown uniqueness "up to logarithmic ambiguities" of Ashtekar–Hansen[4,5] completions satisfying a geodesic condition. We conjecture that the geodesic regularity condition is unnecessary in the case of strong completions and that it cannot be removed without losing quasiuniqueness of the weak completions. It may be of some interest to mention that in Theorems 1 and 2 the geodesic regularity hypothesis may be replaced by the probably much weaker condition that there exists a spacelike geodesic Γ extending to i_0 such that $\Gamma\subset W^y_{\theta_y,\epsilon_y}$, $\Gamma\subset W^x_{\theta_x,\epsilon_x}$ with some $0\leqslant\theta_x$, $\theta_y<1$, where $W^y_{\theta_y,\epsilon_y}$ and $W^x_{\theta_x,\epsilon_x}$ are appropriate y and x coordinate wedges. This condition does not, however, characterize some completion, but pairs of completions. It is likely that a proof of quasiuniqueness of strong conformal completions without any further conditions can be obtained by showing that such geodesics always exist.

It must be stressed that the Kerr family of metrics exhausts the up-until-now known set of vacuum Einstein metrics admitting strong completions of spatial infinity, so that the results of Sec. III cover all actually known physically relevant examples: The metrics recently constructed by Cutler and Wald[19] or Christodoulou[20] are "Schwarzschild in a neighborhood of i_0," so clearly our theorems apply. It seems rather difficult to guess whether there exists some sufficiently large class of vacuum space-times admitting weak or strong completions, the existence of which would justify the need of a search for more general results than those presented here.

An important consequence of our results is that one can assign an invariant mass parameter (cf., e.g., Ref. 21) to every vacuum space-time admitting strong or weak geodesically regular completions, $\alpha > \frac{1}{2}$—this complements the results presented in Ref. 3.

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[8]Latin indices run from 1–3 and Greek indices run from 0–3, $r(x) \equiv \{\Sigma_i (x^i)^2\}^{1/2}$.

[9]Throughout, $x^\alpha x_\alpha \equiv \eta_{\mu\nu} x^\mu x^\nu$, $y^\alpha y_\alpha \equiv \eta_{\mu\nu} y^\mu y^\nu$.

[10]Note that with this definition $0 \notin W_{\theta,1/R}$.

[11]$C(\theta)$ can blow up to ∞ as θ goes to 1.

[12]A related stronger condition would be the requirement that there exists a parametrization $u(s)$ such that the limit $\lim_{s\to\infty} dy^\mu(s(u))/du = \eta^\mu$ exists, with $\eta_{\mu\nu}\eta^\mu\eta^\nu > 0$.

[13]By complete geodesic we mean a geodesic defined for all $s \geqslant s_0$ for some s_0 where s is an affine parameter.

[14]$S(r)$ = sphere of radius r.

[15]It is simple, although a little tedious, to show that the assignment $\mathbf{n} \to \theta_{\mathbf{n}}^{\pm}$ can be made in a continuous way.

[16]A similar analysis (cf. Appendix C of Ref. 4) shows that the Kerr metrics also satisfy the requirements of the definition below.

[17]It should be borne in mind that some no-radiation metrics may admit no strong conformal completions—Corollary 1 classifies all of them if one exists. It has been pointed out to the author by B. Schmidt that it is likely that every no-radiation metric satisfying vacuum Einstein equations is a Kerr metric (which admits strong completions), but no field equations are assumed in Theorem 2 or Proposition 1.

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