SOLUTIONS OF THE CONSTRAINT EQUATIONS IN GENERAL RELATIVITY SATISFYING "HYPERBOLOIDAL BOUNDARY CONDITIONS"

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Abstract. We prove existence of the solutions of the constraint equations satisfying "hyperboloidal boundary conditions" using the Choquet–Bruhat—Lichnerowicz—York conformal method and we analyze in detail their differentiability near the conformal boundary. We show that generic "hyperboloidal initial data" display asymptotic behaviour which is *not* compatible with Penrose's hypothesis of smoothness of $\mathscr{I}$. We also show that a large class of "non–generic" initial data satisfying Penrose smoothness conditions exists. The results are established by developing a theory of regularity up–to–boundary for a class of linear and semilinear elliptic systems of equations uniformly degenerating at the boundary.

Contents

CHAPTER 1

Introduction

A couple of years ago the first semi–global (*i.e.*, global to the future of the initial data hypersurface) existence theorem in general relativity was proved by Friedrich [**35**], for a class of "small" initial data satisfying some asymptotic conditions (*cf.* [**22**] for a different class of "small" data results). An important problem related to Friedrich's theorem remained open: it was not clear "how many" initial data satisfying the required "hyperboloidal" asymptotic conditions existed. The classes of metrics which were known to satisfy the appropriate asymptotic conditions (*cf. e.g.* [**12**] for a review of previous work on this subject), namely the boost–rotation symmetric space–times and the Robinson–Trautman space–times (*cf.* also [**67**]) were of a rather special kind, and semi–global existence results have been proved for these space–times by different methods [**13, 23**]. More recently, a new class of space–times satisfying the hyperboloidal asymptotic conditions has been constructed by Friedrich [**34**], using a clever trick of "exchanging i_o with $i^{\pm}$", it is, however, still unclear how large a class of metrics can be obtained in this way, especially since the space–times constructed in [**34**] are analytic.

In [**6**] in collaboration with H. Friedrich we have constructed a large class of "hyperboloidal" initial data using the conformal Choquet–Bruhat—Lichnerowicz—York method under the assumption of smooth background fields and of pure–trace extrinsic curvature of the initial data hypersurface. In this work we generalize the results of [**6**] in two directions: We construct solutions of the constraint equations with an extrinsic curvature tensor which is *not* pure–trace. Moreover only a finite degree of differentiability of the "background fields" is assumed. We analyze exhaustively the asymptotic regularity of the fields thus constructed in the "asymptotically hyperboloidal" setting. We show that for generic "backgrounds" the Cauchy data constructed by the conformal method starting from a "conformally smoothly compact" Riemannian manifold will *not* possess the asymptotic regularity compatible with Penrose's smoothness requirements for $\mathscr{I}$. On the other hand, we also show that there exists a large class of non–generic "backgrounds" for which the solutions will display the required regularity.

We shall analyze the asymptotic behaviour of the solutions of the constraint equations under various differentiability conditions on the "background" metric on the conformally compactified manifold. We wish to stress that the issue of the differentiability hypotheses is not an academic one, because "physical" fields are obtained by an infinite stretching of the "unphysical" ones and thus imposing too strong differentiability conditions at the conformal boundary may result in a set–up inadequate for describing sufficiently general physical situations; some indications that it might be unrealistic to expect smoothness at the conformal boundary in generic situations may be found in [**22, 21, 70, 26**] (*cf. e.g.* [**27**] for a review of previous results on this subject). To obtain a real understanding of the behaviour of the gravitational field at null infinity it clearly is necessary to establish what asymptotic conditions are appropriate from a physical point of view. One might try to put forward various criteria which might be considered as physically desirable:

(i) existence of a local in time evolution of the data,
(ii) existence of a notion of total energy
(iii) and finiteness thereof,
(iv) existence of a notion of angular momentum
(v) and finiteness thereof,
(vi) existence of a development $({}^4\mathcal{M},\gamma)$ of the initial data set which admits a $\mathscr{I}$ (incomplete, half–complete (*i.e.* complete in one time–direction only) or complete ?) of some differentiability class ($C^0({}^4\bar{\mathcal{M}})$, $C^{2+\lambda}({}^4\bar{\mathcal{M}})$, $C^3({}^4\bar{\mathcal{M}})$, $H_6({}^4\bar{\mathcal{M}}\cap{}^3\bar{M})$, $C^\infty({}^4\bar{\mathcal{M}})$?)
(vii) existence of a development of the data up to i^+,
(viii) one might wish to add the requirement that the function spaces considered include those data sets which arise by evolution from generic initial data which are asymptotically flat at spatial infinity, and finally
(ix) one might ask for various mixtures of the above.

$C^8({}^3\bar{M})\oplus C^6({}^3\bar{M})$ differentiability of (g,K) after compactification together with the vanishing of "shear of the conformal boundary" guarantee that most of the requirements (i)–(vii) are satisfied [**5**]. Recall that Friedrich's theorem, as applied in [**5**], guarantees, loosely speaking, a) an incomplete $\mathscr{I}$ for general $C^8({}^3\bar{M})\oplus C^6({}^3\bar{M})$ data as above, and b) a half–complete $\mathscr{I}$ for those such data which are close enough to the Minkowski data in $C^8({}^3\bar{M})\oplus C^6({}^3\bar{M})$ norm. Of course, one has the known angular momentum ambiguities in the BMS group. In any case, point (viii) is not known to hold. We would like to emphasize that *it is not known* what degree of differentiability up to the conformal boundary is a *necessary* condition[1] for any of criteria (ii)–(viii) to hold in generic situations. Let us note that given initial data $({}^3M,g,K)$ (satisfying the, say vacuum, constraint equations) with (g,K) *locally* in $H_k({}^3\bar{M})\oplus H_{k-1}({}^3\bar{M})$, $k>5/2$, there exists a vacuum developement $({}^4\mathcal{M},\gamma)$ regardless of any asymptotic behaviour of the

[1] This should be contrasted with some of the results concerning space–times asymptotically flat at spacelike infinity, where *e.g.* sharp results are known for the well–posedness of the notion of energy–momentum, etc.

fields (this follows from the results of [**48**] by causality arguments as presented, *e.g.*, in [**43**]). Moreover requirements of maximality and global hyperbolicity render $({}^4\mathcal{M},\gamma)$ unique, at least when k is large enough[2]. It follows that perhaps the most fundamental criterion (i) above *imposes no restrictions* on the asymptotics at the conformal boundary. For this reason we have found it of interest to construct Cauchy data for which there exists a conformal compactification in some sense, under conditions which arise naturally from the mathematical analysis of the problem. We are planning to investigate the problem of weakest possible hypotheses on the asymptotic behaviour of (g,K) for criteria (ii)–(vi) in the future.

It should be emphasized that we construct initial data sets which admit conformal compactifications of various degree of differentiability, leaving aside the problem of compactifiability of the Cauchy developments thereof (criterion (vi) above). Under the condition that the "background fields" used to construct the Cauchy data are in, say, $C^\infty({}^3\bar{M})$ we show that the resulting Cauchy data set is generically in $C^{2+\lambda}({}^3\bar{M})$, $\lambda\in(0,1)$, but *not* in $C^3({}^3\bar{M})$. However, we also show that there exists a "big" (infinite–dimensional) subset of non–generic smooth background fields such that the resulting Cauchy data are smoothly compactifiable. It follows from the Penrose's vanishing Weyl curvature theorem [**64**] that only a small subset (still infinite dimensional) of the above smoothly compactifiable Cauchy data will give rise to a space–time with a smooth $\mathscr{I}$ (*cf.* [**6**] for some results concerning this problem in the pure trace extrinsic curvature case, and [**5**] in the general case). One can envisage the possibility that the existence of any kind of compactification of the space–time necessarily leads one to, say, $C^k({}^3\bar{M})$ compactifiable hyperboloidal initial data sets only with some $k\geq 3$, though this seems rather unlikely to the authors.

To deal with the problem of regularity at the conformal boundary we had to introduce a large number of function spaces, probably more than seems reasonable at first sight: we wish to argue that this is not the case. As usual in PDE problems the appropriate function spaces are Hölder– and Sobolev– type spaces: to capture decay of the solutions one has to consider weighted versions of those. The motivation for considering various Sobolev–type weighted spaces, as *e.g.* $x^\alpha W^p_k$, is their appropriateness for studying the evolution of the initial data, a problem which we plan to consider in the future. On the other hand Hölder–type weighted spaces $x^\alpha C_{k+\lambda}$ are intuitively more transparent, and the results in these spaces are easier to understand for non–experts; these spaces also seem to be somewhat better suited for proving results which can be used to draw conclusions about the interesting case of a smooth–up–to–boundary background — this is especially apparent in the case of the vector constraint equation. $C^\alpha_{k+\lambda}$ ($\neq x^\alpha C_{k+\lambda}$!) and $W^{\alpha;p}_k$ ($\neq x^\alpha W^p_k$!) arise as spaces on which the operators we consider are isomorphisms, for appropriate ranges of the exponents α. It

[2] *cf.* [**17**] in the smooth case, and [**43**] for a reasonably complete proof for $k\geq 4$; for $k>5/2$ this has been claimed essentially without proof in [**20**].

turns out that for some critical values $\mu_\pm$ of the exponent α the solutions of the equations we consider "pick up" log terms, even though the source term did not have any. Since in the non–linear problem we need to iterate the results of the linear theory, we are forced to consider equations where the log weighted terms appear as sources: this leads us to introduce the spaces $C^{\alpha,\beta}_{k+\lambda}$ and $W^{\alpha,\beta;p}_k$ of functions weighted by a factor $x^\alpha|\log x|^\beta$. The "conormal–type" spaces $x^\alpha C_{k+\lambda|m}$ appear naturally when investigating in more detail the behaviour of the solutions near the boundary. Finally the spaces $\mathcal{A}^{\mathrm{phg}}$ of "polyhomogeneous" functions are the spaces to which solutions of the equations considered here belong, in the case of smooth coefficients and smooth sources.

This paper is organized as follows: in Section 2.1 we briefly introduce the "hyperboloidal initial data problem"; for the sake of the reader interested mainly in the C^∞ case we state in detail our main existence and regularity theorems under the hypothesis of a smooth background; results under various different differentiability hypotheses are stated in detail and proved in chapters 6 and 7. In Section 2.2 we shortly mention some generalizations of our results from the vacuum case to the case of matter fields. In Chapter 3 we establish our notations, describe the various function spaces, and prove or review some function–analytic results. Lemma 3.3.1 and Corollary 3.3.2 proved in Section 3.3 turn out to be very useful in our applications; we believe that these results are new.

In Chapter 4 various results concerning the regularity near the boundary are proved for a class of "edge–type" (*cf.* [**58**]) linear operators. More precisely, we consider elliptic systems of equations, with the ellipticity constants degenerating in a uniform way near the boundary, *cf.* eqs. (4.1.3)–(4.1.4). The results of that chapter generalize similar results of Mazzeo [**58**] because we do not need to assume smoothness of the "background" metric; the methods seem also to be rather simpler than those of ref. [**58**] (note, however, that the theory of [**58**] applies to more general operators and addresses more issues than what we do here). The main results there are Theorems 4.2.6, 4.2.9, 4.2.10 and 4.2.11. Those theorems are stated in a somewhat abstract form, but we show that their hypotheses are satisfied in the applications we have in mind. More precisely, Theorems 4.2.6, 4.2.9 and 4.2.10 are used in Chapter 7 to prove (a form of) classical regularity at the boundary of solutions to a class of scalar equations, *cf.* Corollary 7.2.3. Theorem 4.2.11 is used in Section 6.3 to prove (a form of) classical regularity at the boundary for the solutions of the equation considered there, *cf.* Theorem 6.3.11.

In Chapter 5.1 we prove polyhomogeneity of solutions of some fully non–linear elliptic equations, under a rather heavy set of hypotheses; these hypotheses are however satisfied for our (semi–linear) problem at hand, *cf.* Theorem 7.4.1 in Chapter 7. In Chapter 6 we construct classes of solutions of the vector constraint equation under various differentiability and decay conditions. We use two different approaches to do that; these approaches give the same space of solutions of the vector constraint equation in the polyhomogeneous or smooth case. We do not know whether or not this is true in the finite differentiability case, but we suspect that this is not the case. In Section 6.2 the problem is formulated in terms of an equation which does *not* degenerate at the boundary. The main results there are Theorems 6.2.7, 6.2.8 and 6.2.10. In Section 6.3 the problem is formulated in terms of an equation uniformly degenerating at the conformal boundary, of the type considered in Chapter 4. The main results there are Theorems 6.3.8, 6.3.9, 6.3.10 and 6.3.11.

In Chapter 7 solutions of the scalar constraint equation are constructed and their asymptotic properties are established. More precisely, in Corollary 7.2.3 we verify that the hypotheses made in the regularity theory developed in Chapter 4 are satisfied for Laplace–type scalar equations uniformly degenerating at the boundary. In Proposition 7.3.1 we prove existence of solutions to a class of semi-linear equations uniformly degenerating at the boundary. Existence of solutions of the Lichnerowicz equation, Theorem 7.3.2, is a corollary of this result. As already mentioned above, in Theorem 7.4.1 we apply the results of Chapter 5.1 to prove polyhomogeneity of solutions to a class of semi–linear equations; the Lichnerowicz equation is again a special case of the equations considered in Theorem 7.4.1 (*cf.* Corollary 7.4.2). The finite differentiability counterpart of Theorem 7.4.1 is Theorem 7.4.5, which is perhaps the main result of this paper. Theorem 7.4.7 is a specialization of Theorem 7.4.5 to the case of the Lichnerowicz equation. In Appendix A we prove that log terms arise generically in the solutions of the constraint equations constructed by our method. The main results there are Theorems A.2.2 and A.2.3. Finally, in Appendix B we show how to construct coordinate systems near the boundary ∂M of M which, for many purposes, are as useful as Gauss coordinates, and in which no differentiability of the metric is lost[3]. This result is a rather straightforward application of the extension results proved in Section 3.3. Due to the large number of function spaces involved, for the convenience of the reader we have included an index (both for terminology and for notation) at the end of the paper.

There is a non–empty intersection between some of our results and those of refs. [**42, 57, 58, 59, 60**], some similar methods have been used in [**42**]. Some of the arguments we use are standard for this type of problems: we believe, however, that the overall approach[4] is new. Our methods involve only elementary techniques — scaling, commutation, difference quotients, and an analysis of properties of solutions of an ODE (with parameters). A key for a relatively simple proof of boundary regularity is an extension technique for extending finitely differentiable functions on the boundary to locally smooth functions defined on the whole manifold, *cf.* Section 3.3. The method is somewhat reminiscent of that

[3]Recall that one loses two degrees of differentiability of the metric when going to Gauss coordinates.

[4]Once most of this paper was written we have been informed that some of our arguments are somewhat similar to those of [**53**]. It should, however, be stressed that most of the work in this paper is done to handle the finite differentiability hypotheses, while in [**53**] smoothness of the coefficients of the equations is assumed.

used by Hörmander [**45**, Vol. III, Appendix B] in a Sobolev spaces setting. It must however be admitted that the proofs of our main results are somewhat complicated: the intricacies arise mainly from the finite differentiability hypotheses, and from rather general hypotheses on α, $\alpha_\pm$, $\mu_\pm$ (*cf.* Chapter 4 for details). The proofs can be considerably simplified if one assumes smoothness of coefficients and sources ($k = \infty$; *cf. e.g.* [**6**]).

In the case when the conformal background metric is smooth, a special case of our main results — the boundary regularity of solutions of the Yamabe equation — has been independently established by R. Mazzeo [**59**] using quite different techniques[5]. See also [**32, 62**] for some related results; *cf.* [**54**] for a survey of the Yamabe problem on compact manifolds and *cf.* [**65**] and references therein for some results about the Yamabe problem with non–constant prescribed curvature.

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[5]It should be pointed out, however, that the work in [**59**] is done using stronger *a–priori* assumptions, *cf.* [**32**] for a discussion.

CHAPTER 2

The hyperboloidal initial value problem.

2.1. Conformal compactifications and Cauchy data

In this section we will recall the basic notions of the conformal framework introduced by Penrose [**64**] to describe the behaviour of physical fields at null infinity.

By a spacetime, we mean a connected C^∞ 4-dimensional manifold endowed with a Lorentzian metric. Given a "physical" space-time $(\hat{\mathcal{M}}, \hat{\gamma})$ one associates to it a smooth "unphysical space-time" $(\mathcal{M}, \gamma)$ and a smooth function Ω on $\mathcal{M}$, such that $\hat{\mathcal{M}}$ is a subset of $\mathcal{M}$ and

$$\Omega\Big|_{\hat{\mathcal{M}}} > 0\,, \quad \gamma_{\mu\nu}\Big|_{\hat{\mathcal{M}}} = \Omega^2 \hat{\gamma}_{\mu\nu}\ , \tag{2.1.1}$$

$$\Omega\Big|_{\partial\hat{\mathcal{M}}} = 0\ , \tag{2.1.2}$$

$$d\Omega(p) \neq 0 \quad \text{for} \quad p \in \partial\hat{\mathcal{M}}\,, \tag{2.1.3}$$

where $\partial\hat{\mathcal{M}}$ is the topological boundary of $\hat{\mathcal{M}}$ in $\mathcal{M}$. It is common usage in general relativity to use the symbol $\mathscr{I}$ for $\partial\hat{\mathcal{M}}$, and we shall sometimes do so.

We will use the convention that geometric objects in the physical spacetime $\hat{\mathcal{M}}$ defined w.r.t. the metric $\hat{\gamma}$ are decorated with a "$\hat{\ }$", for example $\hat{R}$ denotes the scalar curvature defined w.r.t. the connection $\hat{\nabla}$ and the metric $\hat{\gamma}$.

For simplicity, we will only consider the case when $(\hat{\mathcal{M}}, \hat{\gamma})$ is vacuum, i.e. $\hat{R}_{\mu\nu} = 0$. In general, this assumption may be replaced by fall–off conditions on the matter fields. The hypothesis of smoothness of $(\mathcal{M}, \gamma, \Omega)$ and the assumption that $(\hat{\mathcal{M}}, \hat{\gamma})$ is vacuum imposes severe restrictions on the geometry of $(\mathcal{M}, \gamma, \Omega)$. If one defines (*cf.* [**64**])

$$P_{\mu\nu} = \tfrac{1}{2}\left(R_{\mu\nu} - \tfrac{1}{6}\, R\gamma_{\mu\nu}\right), \tag{2.1.4}$$

with an analogous definition for the quantities defined w.r.t. $(\hat{\mathcal{M}}, \hat{\gamma})$, one has

$$0 = \hat{P}_{\mu\nu} = P_{\mu\nu} + \tfrac{1}{\Omega}\nabla_\mu \nabla_\nu \Omega - \tfrac{1}{2\Omega^2}\nabla^\alpha \Omega\, \nabla_\alpha \Omega\ \gamma_{\mu\nu}\ , \tag{2.1.5}$$

where ∇_μ is the covariant derivative of the metric $\gamma_{\mu\nu}$. Throughout this paper the following conventions on curvature are used:

$$\nabla_\alpha \nabla_\beta X^\gamma - \nabla_\beta \nabla_\alpha X^\gamma = R^\gamma{}_{\mu\alpha\beta} X^\mu \,,$$
$$R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta} \,, \qquad R = R^\gamma{}_\gamma \,.$$

Note that the signs in (2.1.5) are opposite to those of ref. [**64**] because of different curvature conventions used. Equations (2.1.2) and (2.1.5) imply

$$\nabla^\alpha \Omega \nabla_\alpha \Omega\Big|_{\partial\hat{\mathcal{M}}} = 0 \,, \tag{2.1.6}$$

$$\left(\nabla_\mu \nabla_\nu \Omega - \tfrac{1}{4} \nabla^\alpha \nabla_\alpha \Omega \gamma_{\mu\nu}\right)\Big|_{\partial\hat{\mathcal{M}}} = 0 \,. \tag{2.1.7}$$

Let M be a spacelike hypersurface with boundary ∂M in $(\mathcal{M}, \gamma)$ and assume that $M \subset \hat{\mathcal{M}}$ and $\partial M \subset \partial\hat{\mathcal{M}}$. Then we may write $\bar{M} = M \cup \partial M$ so that $\bar{M}$ is the topological closure of M in $\mathcal{M}$. Let us stress that M is a spacelike hypersurface in physical space–time, and this is the only exception to our rule that physically relevant objects are denoted with a hat. Alternatively, we have $\hat{M} = M$, but we never refer to $\hat{M}$, only to M.

Let g_{ij}, K_{ij}, respectively $\hat{g}_{ij}, \hat{K}_{ij}$, be the induced metric and extrinsic curvature of M in $(\mathcal{M}, \gamma)$, respectively in $(\hat{\mathcal{M}}, \hat{\gamma})$. If we denote by L^{ij} and $\hat{L}^{ij}$ the traceless part of $K^{ij} = g^{ik} g^{j\ell} K_{k\ell}$, $\hat{K}^{ij} = \hat{g}^{ik} \hat{g}^{j\ell} \hat{K}_{k\ell}$,

$$\begin{aligned} L^{ij} &= K^{ij} - \tfrac{1}{3} K g^{ij} \,, \quad K = g^{ij} K_{ij} \,, \\ \hat{L}^{ij} &= \hat{K}^{ij} - \tfrac{1}{3} \hat{K} \hat{g}^{ij} \,, \quad \hat{K} = \hat{g}^{ij} \hat{K}_{ij} \,, \end{aligned} \tag{2.1.8}$$

one finds

$$\begin{aligned} \hat{L}^{ij} &= \Omega^3 L^{ij} \,, \quad |\hat{L}|_{\hat{g}} = \Omega\, |L|_g \,, \\ \hat{K} &= \Omega K - 3\, n^\alpha\, \Omega_{,\alpha} \,, \end{aligned} \tag{2.1.9}$$

where n^α is the unit normal to M for the metric γ, and $|\cdot|_h$ denotes the tensor norm in a Riemannian metric h. Since n^α is timelike and, by (2.1.6), $\nabla\Omega(p)$ is null for $p \in \partial M$ we have

$$\hat{K}\Big|_{\partial M} = -3 n^\alpha\, \Omega_{,\alpha}\Big|_{\partial M} \gtrless 0 \,, \tag{2.1.10}$$

because the scalar product of two non–vanishing non–spacelike vectors cannot change sign. From (2.1.2) we also have

$$g_{ij}\Big|_M = \Omega^2\, \hat{g}_{ij} \,,$$

and since $\nabla\Omega$ is null non–vanishing at ∂M the equations (2.1.9)–(2.1.10) imply

$$D^i\, \Omega\, D_i\, \Omega\Big|_{\partial M} = \left(\frac{\hat{K}}{3}\right)^2 \Big|_{\partial M} > 0 \,, \tag{2.1.11}$$

where D_i is the Riemannian connection of the metric g_{ij}. To summarize, necessary conditions for an initial data set $(M, \hat{g}_{ij}, \hat{K}^{ij})$ to arise from an "extended initial data set (M, g_{ij}, K^{ij}) intersecting a smooth $\mathcal{J}$" are

C1. There exists a Riemannian manifold (M, g) with smooth boundary ∂M and compact closure $\bar{M} = M \cup \partial M$ and a Riemannian metric g on $\bar{M}$ with $g \in C^k(\bar{M})$ for some $k \geq 2$ (*cf.* Chapter 3 for precise definitions of function spaces). Moreover there exists a function $\Omega \in C^k(\bar{M})$ such that

$$g_{ij} = \Omega^2 \hat{g}_{ij} \, , \tag{2.1.12}$$

$$\Omega\Big|_{\partial M} = 0 \, , \qquad |D\Omega|_g\Big|_{\partial M} > 0 \, . \tag{2.1.13}$$

C2. The symmetric tensor field $\hat{K}^{ij}$ satisfies, for some $\hat{K} \in C^{k-1}(\bar{M})$ and $L^{ij} \in C^{k-1}(\bar{M})$,

$$\hat{K}^{ij} = \Omega^3 L^{ij} + \tfrac{1}{3}\hat{K}\hat{g}^{ij} \, , \qquad \hat{K} = \hat{g}_{ij}\hat{K}^{ij} \, , \tag{2.1.14}$$

$$\hat{K}\Big|_{\partial M} \text{ is nowhere vanishing} \, . \tag{2.1.15}$$

The above conditions are necessary but far from sufficient, *cf.* [**5, 4**] for a detailed discussion. If there existed "a lot" of space-times satisfying the Penrose conformal conditions, there should exist "a lot" of initial data satisfying C1–C2. It is therefore natural to ask the question, can one construct such data sets? This involves constructing solutions of the scalar constraint equation,

$$\hat{R} + \hat{K}^2 - \hat{K}_{ij}\hat{K}^{ij} = 0 \, , \tag{2.1.16}$$

where $\hat{R}$ denotes the Ricci scalar of the metric $\hat{g}$, and the vector constraint equation,

$$\hat{D}_i(\hat{K}^{ij} - \hat{K}\,\hat{g}^{ij}) = 0 \, , \tag{2.1.17}$$

where $\hat{D}$ is the Riemannian connection of the metric $\hat{g}$, under appropriate asymptotic conditions. No general method of producing solutions of (2.1.16)–(2.1.17) is known, unless one assumes

$$\text{C3.} \qquad \hat{D}_i\hat{K} \equiv 0 \tag{2.1.18}$$

(*cf.*, however, [**18, 50, 28, 14**] for some results). Under (2.1.18) the scalar and the vector constraint equations decouple, and the well known Choquet–Bruhat — Lichnerowicz — York conformal procedure [**20**] allows one to construct solutions of (2.1.16)–(2.1.17). An initial data set satisfying C1–C3 will be called a C^k *hyperboloidal initial data set* (smooth if $k = \infty$), while conditions C1– C2 will be called Penrose's C^k conditions. Without loss of generality we may normalize $\hat{K}$ so that

$$\hat{K} = 3 \, , \tag{2.1.19}$$

and (2.1.16)–(2.1.17) can be rewritten as

$$\hat R + 6 = \hat L_{ij}\,\hat L^{ij} \tag{2.1.20}$$

$$\hat D_i\,\hat L^{ij} = 0\,. \tag{2.1.21}$$

To construct solutions of (2.1.20)–(2.1.21) one can proceed as follows: fix a Riemannian manifold (M, g) with smooth boundary ∂M and compact closure, and let x be any defining function for ∂M (by definition,

$$x\Big|_{\partial M} = 0\,, \qquad |dx|_g\Big|_{\partial M} > 0\,,$$

and $x(p) = 0 \Rightarrow p \in \partial M$), set

$$\tilde g_{ij} = x^{-2}\, g_{ij}\,.$$

Given a smooth traceless symmetric tensor field $\tilde B^{ij}$ on M satisfying

$$\tilde D_i(\tilde B^{ij}) = 0\,, \tag{2.1.22}$$

where $\tilde D$ is the Riemannian connection of the metric $\tilde g$, it is not too difficult to check that the fields

$$\hat g_{ij} = \phi^4 \tilde g_{ij}\,,$$
$$\hat L^{ij} = \phi^{-10}\tilde B^{ij}$$

will satisfy (2.1.20)–(2.1.21) if

$$8\tilde\Delta\phi - \tilde R\phi + \lambda\phi^{-7} - 6\phi^5 = 0\,, \tag{2.1.23}$$

where

$$\lambda \equiv |\tilde B|^2_{\tilde g} \equiv \tilde g_{ij}\,\tilde g_{k\ell}\,\tilde B^{ik}\,\tilde B^{j\ell}\,,$$

and where $\tilde\Delta = \tilde D^i \tilde D_i$ is the Laplacian of the metric $\tilde g_{ij}$. If ϕ and $\tilde B$ are smooth up to boundary and if moreover it holds that

$$\phi\Big|_{\partial M} = 1\,, \tag{2.1.24}$$

$$\lambda\Big|_{\partial M} = 0\,, \tag{2.1.25}$$

then $(M, \hat g_{ij}, \hat K^{ij})$ will satisfy C1–C3 (with $k = \infty$).

Note that (2.1.25) forces the coordinate components of $\tilde B^{ik}$ to vanish to third order at ∂M, in coordinates regular near ∂M. It is therefore natural to introduce a new tensor field B^{ij} defined as

$$B^{ij} \equiv x^{-3}\tilde B^{ij}\,. \tag{2.1.26}$$

It turns out that the condition that ϕ and $\tilde B$ be smooth up to boundary leads to non–trivial restrictions. One of the main results of this paper is the following (*cf.* Corollary 7.4.2, Chapter 7 and Theorem A.2.3 of Appendix A; *cf.* also [**5**] for some related results):

THEOREM 2.1.1. For any smooth (M, g_{ij}, x, B^{ij}) as above there exists a unique solution of (2.1.23)–(2.1.24). Further,

(i) For given M and x there exists an open dense set (in the $C^\infty(\bar M)$ topology) of (g_{ij}, B^{ij})'s for which the function ϕ^{-2} can be extended to a C^2 function from M to $\bar M$, but *not* to a C^3 function on $\bar M$ (the third derivatives of any extension of ϕ will blow up logarithmically as one approaches ∂M); in particular for generic (in the above sense) triples (g_{ij}, B^{ij}) the initial data set $(\hat g_{ij}, \hat K^{ij})$ will display asymptotic behaviour *incompatible* with Penrose's C^3 conditions.

(ii) There exists a "large set" of *non-generic* (g_{ij}, B^{ij}) for which $\Omega \equiv \phi^{-2}\, x$ satisfies $\Omega \in C^\infty(\bar M)$.

It should be emphasized that in Theorem 2.1.1 no hypotheses on the topology of M, or $\partial\hat M$ are made, thus the resulting space-time may have a conformal boundary consisting of several connected components of varying topology (recall that *e.g.* some Robinson–Trautman space-times admit a smooth $\mathscr{I}$ the "spatial" topology of which is not a sphere [**33, 24**]). Let us also note that even considering only those data sets for which $B = 0$, or for which B^{ij} vanishes on ∂M to some desired order, point (i) above will still hold in the sense that for generic g and B's vanishing to some prescribed order (or even *e.g.* identically vanishing) no C^3 extensions of ϕ from $\hat M$ to M will exist.

To complete the construction of initial data sets one also has to produce solutions of (2.1.22), the standard approach proceeds as follows: Let $A^{ij} \in C^\infty(\bar M)$ be a smooth trace–free symmetric tensor field and set $\tilde A^{ij} = x^3 A^{ij}$. Let X^i solve the equation

$$\tilde D_j(\tilde D^i X^j + \tilde D^j X^i - \frac{2}{3}\tilde D_k X^k \tilde g^{ij}) = -\tilde D_j \tilde A^{ij}\,. \tag{2.1.27}$$

Then the tensor field defined by

$$\tilde B^{ij} = \tilde A^{ij} + \tilde D^i X^j + \tilde D^j X^i - \frac{2}{3}\tilde D_k X^k \tilde g^{ij} \tag{2.1.28}$$

will satisfy (2.1.22).

In Chapter 6 the existence and regularity of solutions to the equation (2.1.27) is studied. In fact, in that chapter we present two different methods of constructing solutions of (2.1.27). In particular, the following is established:

THEOREM 2.1.2. Let (M, g) be a smooth Riemannian manifold, with boundary ∂M and with compact closure and let x be a smooth defining function of ∂M. Then, given a trace–free symmetric two–tensor $A^{ij} \in C^\infty(\bar M)$, there is a unique solution X^i to the equation (2.1.27) of the form

$$X^i = x^2 X^i_1 + x^4 \log(x)\, X^i_2\ ,$$
$$X^i_1,\ X^i_2 \in C^\infty(\bar M)\,.$$

It should be pointed out that for generic A^{ij}, the source term in (2.1.27) will be generic and thus the corresponding solution X will have log terms, consequently

B^{ij} given by (2.1.26) with $\tilde{B}^{ij}$ given by (2.1.28) will be C^1 but *not* C^2 extendible from M to $\bar{M}$. If on the other hand A^{ij} vanishes to order two or higher at the boundary, then no log terms occur in the solution of (2.1.27), *cf.* Appendix A. This condition is sufficient but *not* necessary for non–existence of log terms in X^i, *cf. e.g.* [**5**] for details.

In order to obtain initial data which can be used in Friedrich's stability theorem [**35**] (*cf.* also [**19**] for a somewhat different approach) further restrictions on $(\hat{g}_{ij}, \hat{K}^{ij})$ are needed, among others the assumption that both the tensor field

$$e_{\alpha\beta} \equiv \nabla_\alpha \nabla_\beta \Omega - \tfrac{1}{4}\gamma^{\mu\nu}\nabla_\mu\nabla_\nu\Omega\gamma_{\alpha\beta}$$

and the Weyl tensor $C^\alpha{}_{\beta\gamma\delta}$ vanish on ∂M. Here $e_{\alpha\beta}$ and $C^\alpha{}_{\beta\gamma\delta}$ are evaluated formally from the Cauchy data $(\hat{g}_{ij}, \hat{K}^{ij})$ assuming vacuum Einstein equations. The vanishing at ∂M of the tensor $e_{\alpha\beta}$ corresponds to the condition that $\mathcal{J}$ is "shear free", *cf. e.g.* [**5**]. The vanishing of $C^\alpha{}_{\beta\gamma\delta}$ on ∂M implies some further conditions on both L^{ij} and the metric g_{ij}. In the case when $\mathcal{J}$ is shear free and L^{ij} vanishes on ∂M, it turns out [**5**] that the Weyl tensor vanishes on ∂M precisely when the restriction of $\Omega^{-1}L^{ij}$ to ∂M is proportional to the induced metric on ∂M.

Point 1 of Theorem 2.1.1 thus shows that generic data constructed by the conformal method will not be regular enough to be used in Friedrich's existence theorems. In fact the problem here is much more serious than just being one or two degrees of differentiability away from a threshold, because one of the fields used in Friedrich's "conformally regular system" is $d^\alpha{}_{\beta\gamma\delta} \equiv x^{-1}C^\alpha{}_{\beta\gamma\delta}$. Whenever $C^\alpha{}_{\beta\gamma\delta}(p) \not\equiv 0$ for $p \in \partial\hat{M}$, the field $d^\alpha{}_{\beta\gamma\delta}$ blows up at $\partial\hat{M}$ as $1/x$, and is thus not even in $L^1(M)$. It should be stressed that nevertheless point 2 of Theorem 2.1.1 establishes existence of a large class of non–trivial data with asymptotic behaviour compatible with the Penrose–Friedrich conditions.

2.2. Some remarks on non–vacuum initial data sets

In addition to the vacuum case discussed in Section 2.1, two matter models have been studied from the point of view of the "conformal Einstein equations": The Einstein – Yang–Mills system has been analyzed in the conformal setting in [**36**]. The Einstein – scalar field model, with a massless scalar field "minimally coupled" to the metric, has been recently studied by P. Hübner [**46, 47**]. In those last references P. Hübner has generalized[1] Friedrich's results concerning existence of time–developments "with a piece of $\mathcal{J}$", or with a semi–global $\mathcal{J}$, to the scalar field case. In this section we wish to describe shortly how our results on vacuum initial data generalize to non–vacuum models.

First, it should be noted that the construction given here for the vacuum, zero cosmological constant case yields immediately initial data for Einstein equations

[1] P. Hübner [**46**] has also numerically analyzed the evolution problem at Scri for such a spherically symmetric model. Some related numerical results have been obtained by R. Gómez and J. Winicour [**41**]; *cf.* also M. W. Choptuik [**16, 15**].

with a non–vanishing cosmological constant with the appropriate sign. This requires only a reinterpretation of what the physical initial data are, once the solutions of the constraint equations, as described in Section 2.1, have been obtained. (We have, however, not analyzed in detail the question, under what conditions the initial data so obtained will satisfy all the conditions needed for the well posedness of the evolution problem in the conformal setting; *cf.* [**37**, Section 5.1] for partial results concerning this question.)

Next, let us point out that the results of Chapter 6 clearly allow for the introduction of sources in the vector constraint equation. Similarly, the existence Theorem 7.3.2 proved in Chapter 7 holds for rather quite a large class of couplings of the gravitational field with some matter sources (*cf. e.g.* [**51**] for a discussion of the conformal method for Einsteins equations with matter). In particular, in the Einstein – Yang–Mills case the scalar constraint equation takes the form

$$\frac{4(n-1)}{n-2}\Delta_{\tilde g}\phi - \tilde R\phi + \sum_{i=1}^{I}\xi_i\phi^{-\gamma_i} - n(n-1)\phi^{\frac{n+2}{n-2}} = 0\,, \tag{2.2.1}$$

with $I = 2, \gamma_1 = 3, \gamma_2 = 7, \xi_i \geq 0$, $n = 3$. The method of proof of Theorem 7.3.2 proves existence of solutions of equation (2.2.1) satisfying the asymptotic condition (7.3.3) when the initial data for the Yang-Mills field are suitably behaved, *cf.* also Theorem 7.3.1. Similarly, the arguments of Corollary 7.4.2, which asserts polyhomogeneity of solutions of the Lichnerowicz equation, carry over immediately to the Einstein – Yang–Mills case, so that if the initial data for the Yang–Mills fields are smooth or polyhomogeneous on $\bar M$, then the corresponding solution of the Lichnerowicz equation will be polyhomogeneous, *cf.* Theorem 7.4.1.

As discussed in the proof of the existence Theorem 7.3.2, one can always choose a conformal gauge in which $\tilde R = -n(n-1) = -6$. Then, as made explicit in the statement of Proposition 7.3.1, the existence argument applies to equations of the form

$$\Delta_{\tilde g}\phi + F(y,\phi) = 0\,,$$

with a large class of functions F. In particular the assertions of Theorem 7.3.2 hold in the case of a scalar field ψ minimally coupled to the gravitational field, in the following sense: For this model, assuming for simplicity that the extrinsic curvature is pure trace, the Lichnerowicz equation reads

$$8\Delta_{\tilde g}\phi + (6 - \frac{\kappa}{2}|d\psi|^2_{\tilde g})\phi - (6 - \frac{\kappa}{2}(\nabla_0\psi)^2)\phi^5 = 0\,, \tag{2.2.2}$$

where $\nabla_0\psi$ is the derivative of the scalar field ψ in the direction normal to the initial hypersurface, κ is the gravitational constant and we have assumed that $\tilde R = -6$. Suppose that there exists $c > 0$ such that

$$6 - \frac{\kappa}{2}|d\psi|^2_{\tilde g} \geq c\,, \qquad 6 - \frac{\kappa}{2}(\nabla_0\psi)^2 \geq c \tag{2.2.3}$$

on $\bar{M}$, and suppose further that there exist constants $C, \alpha > 0$ such that

$$|d\psi|^2_{\tilde{g}} + (\nabla_0 \psi)^2 \leq C x^\alpha$$

for x small enough. Then Proposition 7.3.1 together with the arguments of the proof of Theorem 7.3.2 guarantees existence of a solution ϕ of (2.2.2) satisfying the asymptotic boundary condition (7.3.3). (Here any sufficiently small constant can be used as the constant C_- of Proposition 7.3.1, and any sufficiently large constant can be used as the constant C_+ of that Proposition.) This, together with the analysis of [**63**] proves existence of solutions of the scalar constraint equation with appropriate asymptotic conditions for a large class of couplings of the scalar field ψ to the gravitational field, in particular for the conformally invariant coupling. The arguments which we present in Chapter 7 can be used to infer further regularity properties of ϕ. More precisely, if the initial data for the scalar field decay at ∂M and are smooth (or polyhomogeneous) on $\bar{M}$, then the corresponding solution ϕ of the Lichnerowicz equation will be polyhomogeneous. Let us also mention that (2.2.3), which is needed in our existence proof, is *not* needed for the regularity argument to go through.

CHAPTER 3

Definitions, Preliminary Results.

3.1. Function spaces

Let M be a smooth, paracompact, Hausdorff, n-dimensional manifold with compact smooth boundary ∂M. We will denote by $\bar{M} = M \cup \partial M$ the closure of M. Unless otherwise indicated, $\bar{M}$ is *not* assumed to be compact. Note that with these conventions M is open.

Throughout this paper x will denote a defining function for ∂M, *i.e.* a smooth (up to boundary) function satisfying $x\big|_{\partial M} = 0$, $x \geq 0$, $|dx|_g\big|_{\partial M} > 0$, and the implication $x(p) = 0 \Rightarrow p \in \partial M$ holds. The symbol $i_{\partial M}$ denotes the natural embedding of ∂M in $\bar{M}$, $i_{\partial M} : \partial M \to \bar{M}$.

We can always choose a finite number of coordinate charts

$$\phi_i : \mathcal{O}_i \to \mathbb{R}^{n,+} \equiv \{y \in \mathbb{R}^n : y^1 \geq 0\}\,, \qquad i = 1, \ldots, I\,,$$

covering a neighbourhood of ∂M such that

(i) $y^1 = x$.
(ii) $\phi_i(\mathcal{O}_i) = [0, x_0) \times \mathcal{U}_i$, for some $\mathcal{U}_i \subset \mathbb{R}^{n-1}$, $0 < x_0 \leq 1$.
(iii) The transition functions $\phi_i^{-1} \circ \phi_j$ are x-independent.
(iv) For every $p \in \bigcup_{i=1}^{I} \mathcal{O}_i$ such that $x(p) < \frac{x_0}{2}$ there exists $i(p)$ such that the coordinate ball $B\big(p, \frac{x(p)}{2}\big)$ centered at p of radius $x(p)/2$ is contained in $\mathcal{O}_{i(p)}$.

The symbol $\partial\Omega$ will always denote the topological boundary of the set Ω: $\partial\Omega \equiv \bar{\Omega} \setminus \Omega$, where $\bar{\Omega}$ is the closure of Ω. $\overset{\circ}{\Omega}$ will sometimes be used to denote the interior of Ω. We set $\tilde{\partial}\Omega \equiv \partial\Omega \setminus \partial M$. We shall write $\Omega' \subset\subset \Omega$ if $\bar{\Omega}' \subset \Omega$.

For $0 < \sigma \leq x_0$ we set $M_\sigma = \{p \in M : 0 < x(p) < \sigma\}$, $CM_\sigma = M \setminus \bar{M}_\sigma \equiv \{p \in M : x(p) > \sigma\}$. We thus have $\tilde{\partial} M_\sigma = \{p \in M : x(p) = \sigma\}$, $\partial M_\sigma = \tilde{\partial} M_\sigma \cup \partial M$. For $0 < \sigma < \rho \leq x_0$ we also define $M_{\sigma,\rho} \equiv \{p \in M : \sigma < x(p) < \rho\}$. Decreasing x_0 if necessary we may assume that for $0 < \sigma \leq x_0$ the sets $\tilde{\partial} M_\sigma$ are smooth manifolds.

When referring to coordinates on M_{x_0} we shall implicitly assume that points (i)–(iv) above hold; we shall use the letter v to denote the coordinates $y^2, \ldots, y^n$;

$$v^A = y^A\,, \quad A = 2, \ldots, n\,.$$

Thus

$$y = (x, v) \ .$$

Using the above coordinate system, we shall often identify functions on ∂M with functions on M_{x_0}, or on $\bar{M}_{x_0}$, similarly for tensors, etc. If useful in the context, functions f on ∂M can also be extended to functions on M by assigning to f the function $\phi(x/x_0)f(v)$, where $\phi \in C_\infty(\mathbb{R})$ is any function satisfying $\phi(x) \in [0,1]$, $\phi(x) = 1$ for $x \in [0, 1/2]$, $\mathrm{supp}(\phi) \subset [-1, 1]$, where here and throughout $\mathrm{supp}(f)$ denotes the support of f. Sometimes it is, however, useful to use better behaved extensions of f to M, *cf.* Lemma 3.3.1 and Corollary 3.3.2.

The standard Schwartz multi-index notation is used throughout, thus if $\alpha = (\alpha_1, \ldots \alpha_n)$, then $|\alpha| = \sum_{i=1}^n \alpha_i$ denotes the length of α and

$$\partial^\alpha = \partial_y^\alpha = \partial_{y^1}^{\alpha_1} \cdots \partial_{y^n}^{\alpha_n} = \partial_x^{\alpha_1} \partial_{y^2}^{\alpha_2} \cdots \partial_{y^n}^{\alpha_n} = \partial_x^{\alpha_1} \partial_v^{\beta} \ ,$$

where $\beta = (\alpha_2, \ldots, \alpha_n)$. Further, we will write $(x\partial_y)^\alpha = x^{|\alpha|}\partial_y^\alpha$.

$\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$, $\mathbb{N}_0^\infty = \mathbb{N}_0 \cup \{\infty\}$. An integer, without further qualification, is taken to be a number in $\mathbb{N}_0$. However, sometimes integers are defined as numbers in $\mathbb{N}$. For this reason when talking about an integer in $\mathbb{N}_0$ we shall indicate this explicitly, whenever the distinction matters and ambiguities are likely to occur.

Let F_1, F_2 be two function spaces. f will be said to belong to $F_1 + F_2$ if there exist functions $f_i \in F_i$, $i = 1, 2$, such that $f = f_1 + f_2$.

3.1.1. (Non–weighted) Hölder spaces. Let Ω be an open subset of M. For $k \in \mathbb{N}_0$ the spaces $C_k(\Omega)$, $C_{k+\lambda}(\Omega)$ are the spaces of k times differentiable functions on Ω; and in the $C_{k+\lambda}(\Omega)$ case the derivatives of order k satisfy a Hölder continuity condition with exponent $\lambda \in (0, 1]$. To emphasize the local character of the spaces $C_{k+\lambda}(\Omega)$ we shall sometimes write $C^{\mathrm{loc}}_{k+\lambda}(\Omega)$ for $C_{k+\lambda}(\Omega)$, whenever confusion is likely to occur. Following standard notation, (*cf.* *e.g.* [**1**]), for $k \in \mathbb{N}_0$ the symbol $C^k(\bar{\Omega})$ denotes the space of those functions in $C_k(\Omega)$ the derivatives of which up to order k can be extended by continuity to continuous functions on $\bar{\Omega}$; we equip $C^k(\bar{\Omega})$ with the supremum norm. The spaces $C_k(\bar{\Omega})$, $C_{k+\lambda}(\bar{\Omega})$ are the spaces of functions differentiable k-times on Ω and equipped with the supremum norm; and in the $C_{k+\lambda}(\bar{\Omega})$ case the derivatives of order k satisfy a *uniform* Hölder continuity condition with exponent $\lambda \in (0, 1]$. For $\lambda > 0$ we have $C_{k+\lambda}(\bar{\Omega}) = C^{k+\lambda}(\bar{\Omega})$, where $C^{k+\lambda}(\bar{\Omega})$ is defined in the usual way as in *e.g.* [**1**]; however $C_k(\bar{\Omega}) \neq C^k(\bar{\Omega})$ in general, because functions in $C_k(\bar{\Omega})$ do not necessarily extend to the boundary of Ω together with all derivatives: In particular $C_0(\bar{\Omega}) \neq C(\bar{\Omega})$, where $C(\bar{\Omega})$ is the usual space of continuous functions on $\bar{\Omega}$ (an example is given by the function $\sin(1/x)$ which is in $C_0(\bar{M}_{x_0})$, while it is *not* in $C(\bar{M}_{x_0})$). Our notation in which the differentiability index is put "downstairs" is mainly motivated by the fact that it is natural to put "upstairs" those indices which correspond to fall–off properties of functions near ∂M. As shown in the above discussion, this convention on the differentiability index also helps to avoid a possible confusion between the spaces $C_k(\bar{M})$ and $C^k(\bar{M})$. Note also that a function in $C_k(\Omega)$ or $C_{k+\lambda}(\Omega)$ is not necessarily bounded up to the boundary $\partial\Omega$, as opposed to a function in $C_k(\bar{\Omega})$ or $C_{k+\lambda}(\bar{\Omega})$. $\mathring{C}_k(\Omega)$, $\mathring{C}_{k+\lambda}(\Omega)$, etc., denotes the space of C_k, $C_{k+\lambda}$, etc., functions on Ω which vanish in a neighbourhood of $\partial\Omega$. When $\lambda = 1$ the space $C_{k+\lambda}$ will always be denoted by "$C_{k+\lambda}, \lambda = 1$", to avoid possible confusions with the space C_{k+1}. "$C_{k+\lambda}, \lambda = 0$" will sometimes be used to denote the space C_k, similarly for "$C^{\alpha,\beta}_{k+\lambda}$, $\lambda = 0$" (the spaces $C^{\alpha,\beta}_{k+\lambda}$ are defined below), etc. We set $C_\infty(\bar{\Omega}) \equiv \cap_k C_k(\bar{\Omega})$.

From the Whitney extension theorem (*cf. e.g.* [**40**, Lemma 6.37]) any $f \in C_{k+\lambda}(\bar{M})$, $\lambda \in (0,1]$ can be $C_{k+\lambda}$ extended across ∂M; this is however not necessarily so if $f \in C_k(\bar{M})$ $(\lambda = 0)$.

3.1.2. Weighted Hölder spaces. Let Ω be an open subset of M. For $\alpha, \beta \in \mathbb{R}$ and $k \in \mathbb{N}_0$ we denote by $C^{\alpha,\beta}_k(\Omega)$ the space of those functions in $C_k(\Omega)$ for which the norm

$$\|f\|_{C^{\alpha,\beta}_k(\Omega)} = \|f\|_{C_k(\overline{\Omega\setminus M_{x_0/2}})} + \sup_{\substack{0\le|\gamma|\le k\\ (x,v)\in\Omega\cap M_{x_0}}} (1+|\ln x|)^{-\beta}\, x^{-\alpha+|\gamma|}\, |\partial^\gamma f(x,v)|$$

is finite. For $\lambda \in (0,1]$ we denote by $C^{\alpha,\beta}_{k+\lambda}(\Omega)$ the space of those functions in $C^{\alpha,\beta}_k(\Omega)$ for which the norm

$$\begin{aligned}\|f\|_{C^{\alpha,\beta}_{k+\lambda}(\Omega)} &= \|f\|_{C^{\alpha,\beta}_k(\Omega)} + \|f\|_{C_{k+\lambda}(\overline{\Omega\setminus M_{x_0/2}})}\\ &\quad + \sup_{\substack{y\in M_{x_0}\cap\Omega\\ |\gamma|=k}} \sup_{\substack{y'\in B\left(y,\frac{x(y)}{2}\right)\cap\Omega\\ y'\neq y}} (1+|\ln x|)^{-\beta}\, x^{-\alpha+k+\lambda}\, \frac{|\partial^\gamma f(y) - \partial^\gamma f(y')|}{|y-y'|^\lambda}\end{aligned}$$

is finite. We set $C^{\alpha}_{k+\lambda}(\Omega) = C^{\alpha,0}_{k+\lambda}(\Omega)$. Although we shall generally avoid such a notation, we shall occasionally write C^α_k, $C^\alpha_{k+\lambda}$, C_k, $C_{k+\lambda}$, etc., for $C^\alpha_k(M)$, $C^\alpha_{k+\lambda}(M)$, $C_k(M)$, $C_{k+\lambda}(M)$, when confusion is unlikely to occur. Note that $C^0_k(M) \neq C_k(M)$, $C^0_{k+\lambda}(M) \neq C_{k+\lambda}(M)$, and that for $\sigma > 0$ functions in $C^{\alpha,\beta}_{k+\lambda}(M_\sigma)$ are, together with their derivatives up to order k, bounded near $\tilde{\partial} M_\sigma$ (*i.e.* that part of the boundary of M_σ which does not coincide with ∂M).

Let F be a function space, and let ρ be a function. By ρF we shall denote the space of functions f such that $\rho^{-1} f \in F$. If F has a norm, we set

$$\|f\|_{\rho F} = \|\rho^{-1} f\|_F \ .$$

We have *e.g.* $x^\alpha(1+|\log x|^2)^{\beta/2}\, C^0_{k+\lambda} = C^{\alpha,\beta}_{k+\lambda}$ (this is easily seen by scaling a ball around (x,v) of radius $x/2$ to a ball of radius $1/2$).

We shall say that $f \in x^\infty F$ if for all $i \in \mathbb{N}$, $f \in x^i F$.

3.1.3. Sobolev spaces. For $p \in [1,\infty)$, $L^p(\Omega, d\mu)$ denotes the space of μ–measurable functions defined μ-almost everywhere, the p-th power of which is integrable on Ω with the measure $d\mu$. $d\mu_g$ will always denote the measure associated with a Riemannian metric g (in local coordinates, $d\mu_g = \sqrt{\det g_{ij}}\, dy^1 \dots dy^n$).

The symbol $L^p(\Omega)$ or L^p will be used to denote $L^p(\Omega, d\mu)$ when $d\mu$ is the Lebesgue measure $d^n y = dx\, d^{n-1}v$ in local coordinates as described at the beginning of this chapter (or a measure equivalent to it in an L^∞ sense, e.g. $d\mu_g$, where g is a metric uniformly elliptic in the above coordinates).

Let Ω be an open subset of M, let g_{ij} be a metric which is uniformly elliptic in local coordinates near ∂M, let $d\mu$ be a measure on M. For $\alpha, \beta \in \mathbb{R}$, $p \in [1, \infty)$, $k \in \mathbb{N}_0$ we define $W^p_k(\Omega, g, d\mu)$ (respectively $\mathring{W}^p_k(\Omega, g, d\mu)$) as the completion of $\mathring{C}_\infty(M)$ (respectively of $\mathring{C}_\infty(\Omega)$) in the norm

$$||f||^p_{W^p_k(\Omega,g,d\mu)} = \sum_{0\le\ell\le k} \int_\Omega \left(|D^\ell f|_g\right)^p d\mu \ ,$$

where D is the Riemannian connection of the metric g, and $|\ |_g$ denotes the norm of a tensor with respect to the metric g. We shall sometimes write $W^p_k(\Omega)$ for $W^p_k(\Omega, g, d\mu)$ when $d\mu$ is uniformly equivalent to the Lebesgue measure in local coordinates near the boundary, as described at the beginning of this chapter. We shall say that $f \in W^{p,\mathrm{loc}}_k(\Omega, g, d\mu)$ if $f \in W^p_k(\Omega', g, d\mu)$ for all $\Omega' \subset\subset \Omega$. Similarly we define $W^{\alpha,\beta;p}_k(\Omega, g, d\mu)$ (respectively $\mathring{W}^{\alpha,\beta;p}_k(\Omega, g, d\mu)$) as the completion of $\mathring{C}_\infty(M)$ (respectively of $\mathring{C}_\infty(\Omega)$) in the norm

$$||f||^p_{W^{\alpha,\beta;p}_k(\Omega,g,d\mu)} = \sum_{0\le\ell\le k} \int_\Omega \left(x^{-\alpha+\ell}(1+|\ln x|)^{-\beta}\, |D^\ell f|_g\right)^p d\mu \ .$$

Note that if the metric g is in $C_k(\bar M)$, then in local coordinates we have

$$0 \le |\gamma| \le k \qquad \int_0^{x_0} dx \int dv \left(x^{-\alpha+|\gamma|}(1+|\ln x|)^{-\beta}\, |\partial^\gamma_y f|\right)^p \le C ||f||^p_{W^{\alpha,\beta;p}_k(M_{x_0},g,d\mu_g)} \ ,$$

for some (f-independent) constant C. The space $W^{\alpha,\beta;p}_k(\Omega, g, x^{-n}d\mu_g)$ will sometimes be denoted by $W^{\alpha,\beta;p}_k(\Omega, g)$, similarly $\mathring{W}^{\alpha,\beta;p}_k(\Omega, g) \equiv \mathring{W}^{\alpha,\beta;p}_k(\Omega, g, x^{-n}d\mu_g)$, $W^{\alpha;p}_k \equiv W^{\alpha,0;p}_k$ etc. We shall also write $L^{\alpha,\beta;p}$ for $W^{\alpha,\beta;p}_0$.

The inclusion of the factor x^{-n} in the definition of $W^{\alpha,\beta;p}_k(M, g)$ has the unfortunate consequence that we have $W^{\alpha,\beta;p}_0(M, g) = x^{\alpha+n/p}(1+|\ln x|)^\beta L^p(M, d^n y) \neq x^\alpha(1 + |\ln x|^\beta)\, L^p(M, d^n y)$ when g is uniformly elliptic in local coordinates near the boundary. Even though the latter would seem more natural, this is however more than compensated by the simplicity of the Hölder inclusion:

$$W^{\alpha,\beta;p}_k(M, g) \subset C^{\alpha,\beta}_{k-n/p}(M) \ , \qquad 0 < k - n/p \notin \mathbb{N} \tag{3.1.1}$$

(this is easily seen by a scaling argument (*cf. e.g.* the proof of Lemma 4.1.1) and the standard Hölder embedding, *cf.* also [**3**]).

If $\tilde g_{ij} = x^{-2} g_{ij}$, then $d\mu_{\tilde g} = x^{-n}\, d\mu_g$, so that $W^{\alpha,\beta;p}_k(M, g) = W^{\alpha,\beta;p}_k(M, g, d\mu_{\tilde g})$. For $\tilde g$ of this form we shall also define $H^{\alpha,\beta}_k(\Omega, \tilde g)$ as the completion of $\mathring{C}_\infty(M)$

with respect to the norm

$$\|X\|_{H_k^{\alpha,\beta}(\Omega,\tilde g)} = \left\{ \sum_{0\le \ell\le k} \int_\Omega \left(|\tilde D^\ell X|_{\tilde g}\, x^{-\alpha}(1+|\ln x|)^{-\beta}\right)^2 d\mu_{\tilde g} \right\}^{\frac12},$$

where $\tilde D$ is the Riemannian covariant derivative operator of the metric $\tilde g$; with $H_k^\alpha \equiv H_k^{\alpha,0}$, similarly $\mathring{H}_k^{\alpha,\beta}(\Omega,\tilde g)$ is the completion of $\mathring{C}_\infty(\Omega)$ in this norm. It is easy to show that for functions

$$H_k^{\alpha,\beta}(M,\tilde g) = W_k^{\alpha,\beta;2}(M,g)\,,$$

this will however not be the case for general tensor fields.

3.1.4. Nonisotropic function spaces. Let Ω be an open subset of M, let $k,m\in\mathbb{N}_0$, $\lambda\in[0,1]$. We shall say that $f\in C_{k+\lambda|m}(\bar\Omega)$, if $f\in C_{k+\lambda}(\bar\Omega)\cap C^{\mathrm{loc}}_{k+m+\lambda}(\Omega)$ and if for all $0\le i+|\gamma|\le m$ we have $(x\partial_x)^i\,\partial_v^\gamma f\in C_{k+\lambda}(\overline{\Omega\cap M_{x_0}})$. We set

$$\|f\|_{C_{k+\lambda|m}(\bar\Omega)} = \|f\|_{C_{k+\lambda}(\bar\Omega)} + \sum_{0\le i+|\gamma|\le m} \|(x\partial_x)^i\,\partial_v^\gamma f\|_{C_{k+\lambda}(\overline{\Omega\cap M_{x_0}})}\,.$$

Note that $C_{k+m+\lambda}(\bar\Omega)\subset C_{k+\lambda|m}(\bar\Omega)$, $C_{k+\lambda|0}(\bar\Omega)=C_{k+\lambda}(\bar\Omega)$.

Let Ω be an open subset of M. We define $C^{\alpha,\beta}_{k+\lambda,0}(\Omega)\equiv C^{\alpha,\beta}_{k+\lambda}(\Omega)$, with the appropriate norm. For $\mu\in(0,1]$ we shall say that $f\in C^{\alpha,\beta}_{k+\lambda,0+\mu}(\Omega)$ if $f\in C^{\alpha,\beta}_{k+\lambda}(\Omega)$ and if moreover there exists a constant C such that for $0\le|\gamma|\le k$ and for all $(x,v),(x,v')\in\Omega\cap\{[0,x_0)\times\mathcal{U}_i\}$, $|v-v'|>0$, $i=1,\dots,I$, we have

$$\left|(x\partial_y)^\gamma(f(x,v)-f(x,v'))\right| \le Cx^\alpha(1+|\ln x|)^\beta\,|v-v'|^\mu\,.$$

We define

$$\|f\|_{C^{\alpha,\beta}_{k+\lambda,0+\mu}(\Omega)} = \|f\|_{C^{\alpha,\beta}_{k+\lambda}(\Omega)} + \sup x^{-\alpha}(1+|\ln x|)^{-\beta}\frac{\left|(x\partial_y)^\gamma(f(x,v)-f(x,v'))\right|}{|v-v'|^\mu}\,,$$

where the sup is taken over $(x,v),(x,v')\in\Omega\cap\{[0,x_0)\times\mathcal{U}_i\}$, $|v-v'|>0$, $i=1,\dots,I$, and $0\le|\gamma|\le k$. f will be said to belong to $C^{\alpha,\beta}_{k+\lambda,m+\mu}(\Omega)$ if $f\in C^{\alpha,\beta}_{k+\lambda,0+\mu}(\Omega)$ and if for $0\le|\gamma|\le m$ we have $\partial_v^\gamma f\in C^{\alpha,\beta}_{\max(0,k-|\gamma|+\lambda),0+\mu}(\Omega\cap M_{x_0})$. Note that if $m\le k$ this is equivalent to the condition that for $0\le i+|\gamma|\le m$ we have $(x\partial_x)^i\partial_v^\gamma f\in C^{\alpha,\beta}_{k-|\gamma|-i+\lambda,0+\mu}(\Omega\cap M_{x_0})$. We set

$$\|f\|_{C^{\alpha,\beta}_{k+\lambda,m+\mu}(\Omega)} = \|f\|_{C^{\alpha,\beta}_{k+\lambda,0+\mu}(\Omega)} + \sum_{0\le|\gamma|\le m} \|\partial_v^\gamma f\|_{C^{\alpha,\beta}_{\max(0,k-|\gamma|+\lambda),0+\mu}(\Omega\cap M_{x_0})}\,,$$

and we define $C^\alpha_{k+\lambda,m+\mu}(\Omega)\equiv C^{\alpha,0}_{k+\lambda,m+\mu}(\Omega)$, with the obvious norm.

Although they will *not* be used any further, we would like to point out that the following spaces of functions arise rather naturally in the context of the problems considered here: Let $k,m\in\mathbb{N}_0$ and $\lambda,\mu\in(0,1]$ be such that $m+\mu\le k+\lambda$.

Then $f \in C^{\mathrm{loc}}_{k+\lambda}(\Omega)$ will be said to belong to $A^{\alpha,\beta}_{k+\lambda,m+\mu}(\Omega)$ if the norm defined by

$$\|f\|_{A^{\alpha,\beta}_{k+\lambda,m+\mu}(\Omega)} = \|f\|_{C^{\alpha,\beta}_{k+\lambda,m}(\Omega)} + \sup \frac{\|f(x,v+z)-f(x,v)\|_{C^{\alpha,\beta}_{k+\lambda-\mu,m}(\Omega\cap M_{x_0})}}{|z|^\mu}$$

is finite. Here the supremum is taken over $z \in \mathbb{R}^{n-1}$ such that

$$(x,v),(x,v+z) \in \phi_i(\Omega\cap\mathcal{O}_i), \qquad |z|>0\,.$$

[Strictly speaking, $f(x,v+z)-f(x,v)$ is not a function on M, so that the expression $\|f(x,v+z)-f(x,v)\|_{C^{\alpha,\beta}_{k+\lambda-\mu,m}(\Omega\cap M_{x_0})}$ above has to be understood in the sense of coordinate patches in the obvious way. Alternatively one could use the difference–quotients construction of the beginning of Chapter 4.] Indeed, the spaces $A^{\alpha,\beta}_{k+\lambda,m+\mu}$ possess better interpolation properties than the spaces $C^{\alpha,\beta}_{k+\lambda,m+\mu}$. While the use of the latter spaces introduces some complications in some of our arguments, some other arguments get simpler. Moreover, one can prove an isomorphism theorem in the $A^{\alpha,\beta}_{k+\lambda,m+\mu}$ spaces similar to Theorem 7.2.1; this is not the case in the $C^{\alpha,\beta}_{k+\lambda,m+\mu}$ spaces. There are, however, no essential overall simplifications gained in the proof by the usage of the $A^{\alpha,\beta}_{k+\lambda,m+\mu}$ spaces. Further, the results for $C^{\alpha,\beta}_{k+\lambda,m+\mu}$ spaces are somewhat more general than those in the $A^{\alpha,\beta}_{k+\lambda,m+\mu}$ spaces, the latter being proper subsets of the former. For those reasons we have decided to use the $C^{\alpha,\beta}_{k+\lambda,m+\mu}$ spaces in our constructions.

Proposition 3.1.1 (Taylor formula). Let $f \in C_{k+\lambda}(\bar M_{x_0})$, $\lambda \in (0,1]$. There exist functions f_i, r_i, $i=0,\dots,k$, such that for $0\le \ell\le k$ we have

$$\begin{aligned} f &= \sum_{i=0}^{\ell} f_i x^i + r_\ell\,,\\ f_i &\in C_{k-i+\lambda}(\partial M)\,, \end{aligned}$$

$$\forall\quad 0\le i\le \ell \quad \partial^i_x r_\ell \in x^{\ell-i}\, C_{k-\ell+\lambda}(\bar M_{x_0}) \cap x^{\ell-i+\lambda} C_{0|k-\ell}(\bar M_{x_0})\,, \tag{3.1.2}$$

Proof: We have $f_i(v) = \frac{1}{i!}\partial^i_x f(0,v)$, $r_0 = f(x,v)-f(0,v)$, and for $\ell\ge 1$

$$r_\ell = \int_0^x dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{\ell-1}} dx_\ell \big(\partial^\ell_x f(x_\ell,v) - \partial^\ell_x f(0,v)\big)\,. \tag{3.1.3}$$

For $0\le i\le\ell$ the property $\partial^i_x r_\ell \in x^{\ell-i}\, C_{k-\ell+\lambda}$ is established by straightforward estimations, $\partial^i_x r_\ell \in x^{\ell-i+\lambda}\, C_{0|k-\ell}$ follows from a change of variables $x_i \to s_i$ given by $x_i = xs_1\dots s_i$:

$$\partial^i_x r_\ell = x^{\ell-i}\int_0^1 ds_1 \dots \int_0^1 ds_{\ell-i}\Big[s_1^{\ell-i-1} s_2^{\ell-i-2}\cdots s_{\ell-i-1}\big(\partial^\ell_x f(s_1\cdots s_{\ell-i}x,v) - \partial^\ell_x f(0,v)\big)\Big]\,.$$

□

Let F be a function space over a set Ω, let $\mathcal{B}$ be a tensor bundle over Ω. A tensor field $X=(X^A{}_\beta)$, where A,β are some multi-indices, $|A|=r$, $|\beta|=s$, will be said to belong to $\Gamma_F(\mathcal{B})$ if in local coordinates as described at the beginning of this chapter the components $X^A{}_\beta$ of X are in F. Whenever confusion is unlikely to occur we shall write $X \in F$ rather than $X \in \Gamma_F(\mathcal{B})$. Let us note that for tensor fields $X \in C^{\alpha,\beta}_{k\ldots}$ or $X \in W^{\alpha,\beta;p}_{k\ldots}(M,g)$, $k > n/p$, where g is a metric uniformly elliptic in local coordinates near ∂M, the decay index α, β always corresponds to the decay of $|X|_g$ in our notation, *cf.* (3.1.1). This, in turn, corresponds to the behaviour of the components of X in the naturally preferred coordinate systems near ∂M, as discussed at the beginning of this chapter. It should be pointed out that this is not true for $X \in H^\alpha_k(M,\tilde{g})$ unless X is a function (*cf.* the beginning of Section 6.3).

3.1.5. Polyhomogeneous functions. Let f_i be a sequence of functions, $f_i \in C_\infty(M)$, such that for every $N \in \mathbb{N}$ and for all $|\alpha| \le N$ we have

$$\left|\partial^\alpha_y f_i\right| \le C_{i,N} x^{s_i} ,$$

for some sequence $s_i \to_{i\to\infty} \infty$ and some constants $C_{i,N}$. We shall write

$$f \sim \sum_{i=0}^{\infty} f_i$$

if for every $n, m \in \mathbb{N}$ there exists $N \in \mathbb{N}$ and a constant $C(n,m)$ such that for $0 < x \le x_0$ and for all $|\alpha| \le m$

$$\left|\partial^\alpha_y \left(f - \sum_{i=0}^{N} f_i\right)\right| \le C(n,m)\, x^n .$$

Let $I \in \mathbb{N}_0^\infty$, let the sequence $\{(s_i, \{N_{ij}\}_{j=0}^\infty)\}_{i=0}^I$, $s_i \in \mathbb{R}$, $N_{ij} \in \mathbb{N}_0$, satisfy $s_{i+1} > s_i$. f will be said to be polyhomogeneous, $f \in \mathcal{A}^{\{(s_i,N_{ij})\}_{i=0}^I}$, if there exists a sequence of functions $f_{ijk} \in C_\infty(\bar{M})$ such that, for $0 < x \le \min(x_0, \frac{1}{2})$,

$$f \sim \sum_{i=0}^{I} \sum_{j=0}^{\infty} \sum_{k=0}^{N_{ij}} f_{ijk}\, x^{s_i+j} \ln^k x . \tag{3.1.4}$$

We set $\mathcal{A}^{\{s_i\}_{i=0}^I} \equiv \cup_{\{N_{ij}\}} \mathcal{A}^{\{(s_i,\{N_{ij}\})\}_{i=0}^I}$, $\mathcal{A}^{\mathrm{phg}} \equiv \cup_{s_i,I} \mathcal{A}^{\{s_i\}_{i=0}^I}$. In what follows we shall need the following Lemma, the proof of which is a straightforward generalization of the proof of Borel's Lemma (*cf. e.g.* [**45**, Volume 1]):

LEMMA 3.1.2 (BOREL LEMMA). Let $I \in \mathbb{N}_0^\infty$, $\{(s_i, \{N_{ij}\}_{j=0}^\infty)\}_{i=0}^I$, $s_i \in \mathbb{R}$, $N_{ij} \in \mathbb{N}_0$, with $s_{i+1} > s_i$, suppose that $\tilde{f}_{ijk} \in C_\infty(\partial M)$, let $f_{ijk} \in C_\infty(\bar{M})$ be any functions such that $(f_{ijk} - \tilde{f}_{ijk})\big|_{\mathcal{O}_{ijk}} = 0$, for some neighbourhoods $\mathcal{O}_{ijk} \subset \cup_i \mathcal{O}_i$ of ∂M. (Here the sets $\mathcal{O}_i$ are as defined at the beginning of this Chapter, and the f_{ijk}'s have been extended to the neighbourhood $\cup_i \mathcal{O}_i$ of ∂M by setting $\partial f_{ijk}/\partial x = 0$ in some coordinate system as considered at the beginning of this Chapter.) There exists a function $f \in C^{s_0,N_{00}}_{\infty,\infty} \cap \mathcal{A}^{\mathrm{phg}}$ such that (3.1.4) holds. If $N_{ij} = 0, \forall i,j$, then $f \in C_\infty(\bar{M})$.

3.2. Some embeddings

Proposition 3.2.1. Suppose that $\Omega = M$, or $\Omega = M_\tau$, $0 < \tau \le x_0$, let $f \in x^\alpha\, C_{k+\lambda|m}(\bar\Omega)$.

(i)

$$0 \le |\beta_1| \le m\,,\ 0 \le |\beta_2| \le k \quad \partial_v^{\beta_1+\beta_2} f \in x^\alpha\, C_{k-|\beta_2|+\lambda|m-|\beta_1|}(\bar\Omega)\,. \tag{3.2.1}$$

(ii) Let $g \in x^{\alpha_1}\, C_{k_1+\lambda_1|m_1}(\bar\Omega)$, then

$$gf \in x^{\alpha+\alpha_1}\, C_{k_2+\lambda_2|m_2}(\bar\Omega)\,, \tag{3.2.2}$$

where $*_2 = \min(*, *_1)$, $* = k, m, \lambda$, in particular

$$g \in x^{\alpha_1}\, C_{k+m+\lambda}(\bar\Omega) \Rightarrow fg \in x^{\alpha+\alpha_1}\, C_{k+\lambda|m}(\bar\Omega)\,. \tag{3.2.3}$$

Suppose that $\Omega = M$, or $\Omega = M_\tau$, $0 < \tau \le x_0$, let $f \in C^{\alpha,\beta}_{k+\lambda,m+\mu}(\Omega)$.

(iii)

$$0 \le |\gamma| \le k \qquad (x\partial_y)^\gamma f \in C^{\alpha,\beta}_{k-|\gamma|+\lambda,\min(m,k-|\gamma|)+\mu}(\Omega)\,. \tag{3.2.4}$$

(iv)

$$0 \le |\gamma| \le m \qquad \partial_v^\gamma f \in C^{\alpha,\beta}_{\max(0,k-|\gamma|+\lambda),m-|\gamma|+\mu}(\Omega)\,. \tag{3.2.5}$$

(v) Let $g \in C^{\alpha_1,\beta_1}_{k_1+\lambda_1,m_1+\mu_1}(\Omega)$, then

$$fg \in C^{\alpha+\alpha_1,\beta+\beta_1}_{k_2+\lambda_2,m_2+\mu_2}(\Omega)\,. \tag{3.2.6}$$

with $*_2 = \min(*, *_1)$, $* = k, m, \lambda, \mu$, in particular

$$g \in x^{\alpha_1}\, C_{k+\lambda}(\Omega) \Rightarrow fg \in C^{\alpha+\alpha_1,\beta}_{k+\lambda,\min(k+\lambda,m+\mu)}(\Omega)\,. \tag{3.2.7}$$

Proof: All the results here are elementary, let us simply point out that Eq. (3.2.1) and (3.2.5) are a consequence of

$$\left[\partial_v^\alpha, \partial_x^i\right] = \left[\partial_v^\alpha, (x\partial_x)^i\right] = 0\,,$$
$$\left[\partial_x^i, (x\partial_x)^j\right] = \textstyle\sum_{0\le\ell\le j-1} d_{ij\ell}\, \partial_x^i (x\partial_x)^\ell\,,$$

with some constants $d_{ij\ell}$. Equation (3.2.3) follows from the inclusion

$$x^{\alpha_1}\, C_{k+m+\lambda}(\bar\Omega) \subset x^{\alpha_1}\, C_{k+\lambda|m}(\bar\Omega)\,.$$

Eq. (3.2.7) follows from (3.2.6) and the inclusion

$$x^{\alpha_1}\, C_{k+\lambda}(\bar\Omega) \subset C^{\alpha_1}_{k+\lambda,k+\lambda}(\Omega)\,.$$

□

Proposition 3.2.2. Let $\Omega = M$, or $\Omega = M_\tau$, $0 < \tau \le x_0$. The following inclusions are continuous

(i) $\mu, \lambda \in [0,1]$, $\alpha \geq 0$,

$$C^{\alpha}_{0+\mu,0+\lambda}(\Omega) \hookrightarrow C_{0+\sigma}(\bar\Omega), \qquad \sigma = \min(\alpha,\mu,\lambda)\,. \tag{3.2.8}$$

(ii) $\mu, \lambda \in [0,1]$, $\alpha \geq 0$, $k, \ell, m \in \mathbb{N}_0$, $k-\ell \geq 0$

$$C^{\ell+\alpha}_{k+m+\mu,k-\ell+\lambda}(M) \hookrightarrow \cap_{i=0}^{m} x^{-i} C_{\ell+i+\sigma|k-\ell}(\bar M)\,, \qquad \sigma = \min(\alpha,\mu,\lambda)\,. \tag{3.2.9}$$

(iii) $j \in \mathbb{N}_0$

$$x^j\, C_{k+\lambda|m+j}(\bar\Omega) \hookrightarrow C_{k+j+\lambda|m}(\bar\Omega)\,. \tag{3.2.10}$$

PROOF:
(3.2.10):

$$\begin{aligned}
f \in x^j\, C_{k+\lambda|m+j} &\Longrightarrow \quad 0 \leq i + |\gamma| \leq m+j \quad \partial_v^{\gamma}\, x^i\, \partial_x^i x^{-j} f \in C_{k+\lambda}\\
&\Longrightarrow \quad 0 \leq i_1 + |\gamma_1| \leq m\,,\ 0 \leq i_2 + |\gamma_2| \leq j \quad \partial_v^{\gamma_1} x^{i_1} \partial_x^{i_1} \partial_v^{\gamma_2} x^{i_2} \partial_x^{i_2} x^{-j} f \in C_{k+\lambda}\\
&\Longrightarrow \quad 0 \leq i_1 + |\gamma_1| \leq m\,,\ 0 \leq i_2 + |\gamma_2| \leq j \quad \partial_v^{\gamma_1} x^{i_1} \partial_x^{i_1} \partial_v^{\gamma_2} \partial_x^{i_2} f \in C_{k+\lambda}\\
&\Longrightarrow \quad f \in C_{k+j+\lambda|m}
\end{aligned}$$

(3.2.8): From the inclusion

$$C^{\alpha}_{0+\lambda,0+\mu} \subset C^{\alpha_1}_{0+\lambda_1,0+\mu}\,, \quad 0 \leq \lambda_1 \leq \lambda\,,\ \alpha_1 \leq \alpha\,, \tag{3.2.11}$$

it follows that without loss of generality we can suppose $f \in C^{\alpha_1}_{0+\lambda_1,0+\mu}$, with $0 < \alpha_1 \leq 1$, $\lambda_1 = \min(\alpha_1,\lambda)$. Let $x \leq y$, we have

$$|f(x,v) - f(y,w)| \leq A + B\,,$$

$$\begin{aligned}
A &= |f(x,v) - f(y,v)|\\
B &= |f(y,v) - f(y,w)|\,.
\end{aligned}$$

B is estimated in a straightforward way,

$$B \leq C y^{\alpha_1} |v-w|^{\mu} \leq C {x_0}^{\alpha_1} |v-w|^{\mu}\,.$$

To estimate A, suppose first that $x \in (y/2, y)$, which leads to

$$A \leq C x^{\alpha_1-\lambda_1} |x-y|^{\lambda_1} \leq C {x_0}^{\alpha_1-\lambda_1} |x-y|^{\lambda_1}\,.$$

Let now $x \in \big[y/2^{n+1}, y/2^n\big)$, $n \geq 1$; note that

$$\frac{y}{2^n} \leq y\Big(1 - \frac{1}{2^n}\Big) \leq y - x \leq y\Big(1 - \frac{1}{2^{n+1}}\Big)\,. \tag{3.2.12}$$

We have

$$\begin{aligned}
A &= A_1 + A_2\,,\\
A_1 &= \left| f(x,v) - f\Big(\frac{y}{2^n}, v\Big)\right|\\
A_2 &= \left| f(y,v) - f\Big(\frac{y}{2^n}, v\Big)\right|
\end{aligned}$$

and from (3.2.12) it follows that

$$\begin{aligned} A_1 &= C\left(\frac{y}{2^n}\right)^{\alpha_1-\lambda_1}\left|x-\frac{y}{2^n}\right|^{\lambda_1} \\ &\le C\left(\frac{y}{2^n}\right)^{\alpha_1-\lambda_1}\left|\frac{y}{2^{n+1}}-\frac{y}{2^n}\right|^{\lambda_1} \\ &= 2^{-\lambda_1}C\left(\frac{y}{2^n}\right)^{\alpha_1} \le 2^{-\lambda_1}C|x-y|^{\alpha_1}\,. \end{aligned}$$

On the other hand

$$\left|f(y,v)-f\left(\frac{y}{2^n},v\right)\right| \le \sum_{i=0}^{n-1}\left|f\left(\frac{y}{2^i},v\right)-f\left(\frac{y}{2^{i+1}},v\right)\right| \tag{3.2.13}$$

$$\le C\sum_{i=0}^{n-1}\left(\frac{y}{2^i}\right)^{\alpha_1-\lambda_1}\left|\frac{y}{2^i}-\frac{y}{2^{i+1}}\right|^{\lambda_1} \tag{3.2.14}$$

$$= \frac{2^{-\lambda_1}C}{1-2^{-\alpha}}\left(1-\frac{1}{2^{n\alpha_1}}\right)y^{\alpha_1}\,. \tag{3.2.15}$$

Recall that for $\alpha_1 \in (0,1]$ we have

$$a \ge b \ge 0 \quad \Longrightarrow \quad a^{\alpha_1}-b^{\alpha_1} \le (a-b)^{\alpha_1} \tag{3.2.16}$$

(a simple proof of (3.2.16) can be given as follows: for $0<\mu<1$ and $0 \le x \le y$ consider $\phi(x,y)=(y-x)^\mu - y^\mu + x^\mu$, we have $\phi(x,x)=0$ and for $0<x<y$, $\frac{\partial\phi}{\partial y} \ge 0$, thus $\phi \ge 0$). The inequality (3.2.16) gives

$$1-\frac{1}{2^{n\alpha_1}} \le (1-\frac{1}{2^n})^{\alpha_1}\;, \tag{3.2.17}$$

which together with (3.2.12) and (3.2.15) implies

$$\begin{aligned} A_2 &\le C'\left(1-\frac{1}{2^{n\alpha_1}}\right)y^{\alpha_1} \\ &\le C'\left[\left(1-\frac{1}{2^n}\right)y\right]^{\alpha_1} \le C'|y-x|^{\alpha_1}\,, \end{aligned}$$

and (3.2.8) follows. (3.2.9) is a straightforward consequence of (3.2.8). $\square$

3.3. Extensions of functions defined on ∂M

We shall use the following Lemma concerning extensions of functions defined on ∂M (*cf.* [**45**, Vol. III, Appendix B] for a similar approach in Sobolev spaces).

LEMMA 3.3.1 (EXTENSION LEMMA:). *Let $k \in \mathbb{N}_0$, $\lambda \in [0,1]$, consider $\psi \in C^{k+\lambda}(\partial M)$. For all $m \in \mathbb{N}_0$ there exists a function $\tilde\psi_m \in C^{k+m+\lambda}(\bar M) \cap C^m_{\infty,k+\lambda}(M)$ ($\Longrightarrow \tilde\psi_m \in C^{\rm loc}_\infty(M)$) satisfying*

$$0 \le i \le m \qquad \partial^i_x\tilde\psi_m\Big|_{\partial M} = \begin{cases} 0, & i \le m, \\ \psi, & i = m. \end{cases}$$

Moreover for all multi–indices α we have

$$(x\partial_y)^\alpha \tilde\psi_m \in C^{k+m+\lambda}(\bar M)\cap C^m_{\infty,k+\lambda}(M)\ , \tag{3.3.1}$$

and it also holds that

$$(x\partial_y)^\alpha \tilde\psi_m - \psi\,(x\partial_y)^\alpha x^m \in C^{m+\lambda}_{k,k}(M)\ , \tag{3.3.2}$$

$$x\partial_v\tilde\psi_m \in C^{m+\lambda}_{\infty,k}(M)\ . \tag{3.3.3}$$

PROOF: A simple partition of unity argument shows that it is sufficient to establish the result for $\partial M = \mathbb{R}^{n-1}$ and ψ — compactly supported. Let $\phi\in C^\infty(\mathbb{R}^{n-1})$ be any compactly supported function satisfying

$$\int_{\mathbb{R}^{n-1}}\phi(v)d^{n-1}v = 1\ .$$

Let $\chi\in C^\infty(\mathbb{R})$ be any function satisfying supp $\chi\subset(-x_0,x_0)$, $\chi|_{[-x_0/2,x_0/2]}=1$. Set

$$E[\psi](x,v) = x^{-(n-1)}\int_{\mathbb{R}^{n-1}}\phi(\frac{v-w}{x})\psi(w)d^{n-1}w, \tag{3.3.4}$$

$$\tilde\psi_m(x,v) = \frac{x^m}{m!}\chi(x)\,E[\psi](x,v)\ . \tag{3.3.5}$$

From (3.3.4) one immediately has $E[\psi]\in C^0_{\infty,k+\lambda}(M_{x_0})$, so that (3.3.5) gives $\tilde\psi_m\in C^m_{\infty,k+\lambda}(M)\subset C^{\rm loc}_\infty(M)$. Changing the integration variable $w\to z=\frac{v-w}{x}$ in (3.3.4) we have

(3.3.6)
$$0\le i+|\gamma|\le k\quad \partial^i_x\partial^\gamma_v E[\psi](x,v) = \int_{\mathbb{R}^{n-1}}\phi(z)\,\partial^i_x\partial^\gamma_v\psi(v-xz)d^{n-1}z.$$

Reverting to the integration variable w in (3.3.6) one concludes $\tilde\psi_m\in C^m_{\infty,k+\lambda}(M)$, and Proposition 3.2.2 gives $\tilde\psi_m\in C_{k+m}(\bar M)$. To obtain the stronger statement $\tilde\psi_m\in C^{k+m+\lambda}(\bar M)$ some more work is needed. Assume that $0\le i+|\gamma|\le m$; from what has been said it follows that it is sufficient to consider the case $k=0$. From (3.3.4)–(3.3.5) we obtain by direct differentiation

(3.3.7)
$$0<x\le x_0/2,\ \partial^i_x\partial^\gamma_v\tilde\psi_m(x,v) = \int_{\mathbb{R}^{n-1}}x^{m-i-|\gamma|-(n-1)}\chi_{i,\gamma}(\frac{v-w}{x})\psi(w)d^{n-1}w\ ,$$

for some compactly supported functions $\chi_{i,\gamma}\in C^\infty(\mathbb{R}^{n-1})$. Changing integration variables (3.3.7) becomes

$$\partial^i_x\partial^\gamma_v\tilde\psi_m(x,v) = x^{m-i-|\gamma|}\int\chi_{i,\gamma}(z)\psi(v-xz)d^{n-1}z\ . \tag{3.3.8}$$

The property $\partial_x^i\partial_v^\gamma\tilde\psi_m\in C^0(\bar M)$ follows from Lebesgue dominated convergence theorem. If $\lambda>0$ the further property $\partial_x^i\partial_v^\gamma\tilde\psi_m\in C^{0+\lambda}(\bar M)$ follows from (3.3.8) by straightforward estimations. For $0<x\le x_0/2$ we also have, for all multi–indices α,

$$\begin{aligned}
(x\partial_y)^\alpha\tilde\psi_m(x,v) &= \int_{\mathbb{R}^{n-1}}\psi(w)(x\partial_y)^\alpha\{x^{m-(n-1)}\phi(\frac{v-w}{x})\}d^{n-1}w\\
&= \int_{\mathbb{R}^{n-1}}\{\psi(w)-\psi(v)+\psi(v)\}(x\partial_y)^\alpha\{x^{m-(n-1)}\phi(\frac{v-w}{x})\}\,d^{n-1}w\\
&= \int_{\mathbb{R}^{n-1}}\{\psi(w)-\psi(v)\}(x\partial_y)^\alpha\{x^{m-(n-1)}\phi(\frac{v-w}{x})\}\,d^{n-1}w\\
&\qquad +\psi(v)\int_{\mathbb{R}^{n-1}}(x\partial_y)^\alpha\{x^{m-(n-1)}\phi(\frac{v-w}{x})\}\,d^{n-1}w\\
&= \int_{\mathbb{R}^{n-1}}\{\psi(w)-\psi(v)\}(x\partial_y)^\alpha\{x^{m-(n-1)}\phi(\frac{v-w}{x})\}\,d^{n-1}w\\
&\qquad +\psi(v)(x\partial_y)^\alpha\int_{\mathbb{R}^{n-1}}x^{m-(n-1)}\phi(\frac{v-w}{x})\,d^{n-1}w\\
&= \int_{\mathbb{R}^{n-1}}\{\psi(w)-\psi(v)\}(x\partial_y)^\alpha\{x^{m-(n-1)}\phi(\frac{v-w}{x})\}\,d^{n-1}w\\
&\qquad +\psi(v)(x\partial_y)^\alpha\,x^m\ ,
\end{aligned}$$

and (3.3.2)–(3.3.3) easily follow. (3.3.1) is established by a similar simpler calculation. □

Corollary 3.3.2. Let $k\in\mathbb{N}_0$, $\lambda\in[0,1]$.

(i) For $0\le i\le k$ let $f_i\in C^{k-i+\lambda}(\partial M)$. There exists $f\in C^{k+\lambda}(\bar M)\cap C^{\mathrm{loc}}_\infty(M)$ such that

$$\partial_x^i f\Big|_{\partial M}=f_i\ .$$

Moreover for all multi–indices α we have

$$(x\partial_y)^\alpha f\in C^{k+\lambda}(\bar M)\ . \tag{3.3.9}$$

(ii) Let $\hat g\in C^{k+\lambda}(\bar M)$. There exists $f\in C^{k+\lambda}(\bar M)\cap C^0_{\infty,k+\lambda}(M)$ such that

$$f-g\in C^k_{k+\lambda,0+\lambda}(M)\cap C^{k+\lambda}_{k+\lambda}(M)\ . \tag{3.3.10}$$

Proof: 1. We shall proceed by induction. Suppose thus that we have proved the existence of a function f satisfying our claims for $0\le i\le \ell<k$. To start the induction, for $\ell=-1$ set $f\equiv 0$. For $\ell\ge -1$ define

$$\psi_{\ell+1}=\partial_x^{\ell+1}f\Big|_{\partial M}\in C^{k-\ell-1+\lambda}(\partial M)\ .$$

The induction step is obtained by replacing f with

$$f+\tilde f_{\ell+1}-\tilde\psi_{\ell+1}\ ,$$

where $\tilde f_{\ell+1}$, $\tilde\psi_{\ell+1}$ are obtained from $f_{\ell+1}$ and of $\psi_{\ell+1}$ using Lemma 3.3.1 with $m=\ell+1$.

2. Let f be obtained from part 1 of this Corollary using the functions $f_i = \partial_x^i g|_{\partial M} \in C^{k-i+\lambda}(\partial M)$. Now (3.3.10) is equivalent to

$$0 \le i + |\gamma| \le k \qquad \partial_x^i \partial_v^\gamma (f-g) \in C^{k-i-|\gamma|}_{0+\lambda,0+\lambda}(M) \cap C^{k-i-|\gamma|+\lambda}_{0+\lambda}(M) \ .$$

Choose some γ satisfying $0 \le |\gamma| \le k$; since $f-g$ has vanishing Taylor coefficients at ∂M the result follows for $\lambda > 0$ from eq. (3.1.2) with $\ell = k - |\gamma|$ of Proposition 3.1.1. If $\lambda = 0$ the result follows in a similar way from (3.1.3). □

Let us recall that Whitney's extension Lemma is usually proved in a $C^{k+\lambda}(\bar M)$ context, with $\lambda > 0$. Corollary 3.3.2 can be used to prove the equivalent of the Whitney Lemma for functions in $C^k(\bar M)$ when the boundary ∂M is a sufficiently differentiable manifold (as is the case here); it also gives a rather elementary proof of that Lemma in our context.

As an application of the results here, in Appendix B we present a construction of "almost Gaussian" coordinates near ∂M. The construction leads to coordinates which for many purposes are as convenient as the "real" ones. Moreover, even[1] in the case where finite differentiability of the metric is assumed, the construction leads to a coordinate system in which the coefficients of the metric tensor are of the same differentiability class as in the original coordinates.

3.4. Mapping properties of some integral operators

A significant role in our approach to boundary regularity in Section 4.2 is played by the mapping properties of the integral operators

$$I^\mu_a(f)(x,v) = x^\mu \int_a^x s^{-1-\mu} f(s,v)\, ds\,, \tag{3.4.1}$$

$$x \in (0, x_0]\,, \quad v \in \partial M\,, \quad a \in [0,x_0]\,, \quad \mu \in \mathbb{R}\,,$$

$$I^\mu(f)(v) = x^\mu \int_0^{x_0} s^{-1-\mu} f(s,v)\, ds\,,$$

$$G_a = \frac{1}{\mu_+ - \mu_-}\left(I_a^{\mu_+} - I_0^{\mu_-}\right)\,, \quad \mu_- < \mu_+\,, \quad \mu_\pm \in \mathbb{R}\,,$$

which arise from fundamental solutions for some ordinary differential equations, *cf.* the beginning of Section 4.2. We have the following results:

LEMMA 3.4.1. The following maps are continuous:

(i) $\alpha \le \mu, \lambda, \lambda' \in [0,1]$

(3.4.2)
$$I^\mu_{x_0} : C^{\alpha,\beta}_{0+\lambda,0+\lambda'}(M_{x_o}) \to C^{\alpha,\beta'}_{0+\lambda,0+\lambda'}(M_{x_o})\,, \quad \beta' = \beta \ \text{ if } \ \alpha < \mu\,;$$

$\beta' = \beta + 1$ if $\alpha = \mu$ and $\beta \ne -1$; $\beta' > 0$ arbitrary if $\alpha = \mu$ and $\beta = -1$.

[1]Recall that one loses two degrees of differentiability of the coefficients of the metric tensor when transforming the metric to the *exact* Gauss coordinate system.

(ii) $\alpha > \mu\,, \lambda, \lambda' \in [0,1]$

(3.4.3)
$$I^\mu_{x_0} : C^{\alpha,\beta}_{0+\lambda,0+\lambda'}(M_{x_o}) \to C^{\mu}_{0+\lambda_1,0+\lambda'}(M_{x_o})\ , \quad \lambda_1 = \lambda \text{ for } \lambda \neq \alpha - \mu\ ,$$

and if $\lambda = \alpha - \mu$ then λ_1 is arbitrary in $[0, \lambda)$.

(iii) $\alpha > \mu\,, \lambda, \lambda' \in [0,1]$

(3.4.4)
$$I^\mu_{0} : C^{\alpha,\beta}_{0+\lambda,0+\lambda'}(M_{x_o}) \to C^{\alpha,\beta}_{0+\lambda'',0+\lambda'}(M_{x_o})\ , \quad \lambda'' = \min(\lambda, \lambda')\ .$$

(iv) $\alpha > \mu\,, \lambda \in [0,1]\,,$

(3.4.5)
$$I^\mu : C^{\alpha,\beta}_{0,0+\lambda}(M_{x_o}) \to x^\mu\, C_{0+\lambda}(\partial M)\,.$$

(v) $\mu \geq 0\,, \lambda \in [0,1]\,, \mu + \lambda > 0\ ,$

(3.4.6)
$$I^\mu_{x_0} : \{f \in C_{0+\lambda}(\bar M_{x_o}), f\big|_{\partial M} = 0\} \to C_{0+\lambda'}(\bar M_{x_o})\,,$$

for some $\lambda'(\mu, \lambda) \in [0, \lambda]$, $\lambda' > 0$ if $\lambda > 0$.

(vi) $\mu \leq 0\,, \lambda \in [0,1]\,, -\mu + \lambda > 0\ ,$

(3.4.7)
$$I^\mu_{0} : \{f \in C_{0+\lambda}(\bar M_{x_o})\,, f\big|_{\partial M} = 0\} \to C_{0+\lambda'}(\bar M_{x_o})\,,$$

for some $\lambda'(\mu, \lambda) \in [0, \lambda]$, $\lambda' > 0$ if $\lambda > 0$.

PROOF: For $\lambda = 0$ the proof of (3.4.2)–(3.4.4) is a straightforward estimation; in that case continuity of $I^\mu_0(f)$ in v in (3.4.4) follows from Lebesgue's dominated convergence theorem. To establish Hölder continuity when $\lambda > 0$, let us note that

$$I^\mu_a(f)(x,v) - I^\mu_a(f)(x',v') = A + B\ , \quad a = 0, x_0\,, \tag{3.4.8}$$
$$A = I^\mu_a(f)(x,v) - I^\mu_a(f)(x',v)\,,$$
$$B = I^\mu_a(f)(x',v) - I^\mu_a(f)(x',v')\,.$$

The estimation of B is straightforward, while A can be estimated using e.g. a scaling argument. To prove (3.4.6), the not entirely trivial point is to estimate the term A from (3.4.8). We have, for $0 < x \leq y \leq x_0$

$$I^\mu_{x_0}(f)(y,v) - I^\mu_{x_0}(f)(x,v) = A_1 + A_2$$
$$A_1 = -(y^\mu - x^\mu) \int_y^{x_0} s^{-1-\mu} f(s,v)\, ds\,,$$
$$A_2 = x^\mu \int_x^y s^{-1-\mu} f(s,v)\, ds\,. \tag{3.4.9}$$

Since f vanishes at ∂M, we have $|f(x,v)| \leq ||f||_{C_{0+\lambda}(\bar M_{x_o})} x^\lambda$. Suppose first that $0 \leq \mu \leq 1$, recall that

$$y^\mu - x^\mu \leq (y-x)^\mu\ , \tag{3.4.10}$$

(*cf.* (3.2.16)). For $\mu = 0$, $A_1 = 0$ and the result follows immediately from (3.4.9). If $0 < \mu \le 1$, setting $\lambda' = \min(\mu, \lambda)$ we have

$$\begin{aligned} |A_1| &\le C\big(x_0^{\lambda-\mu} + y^{\lambda-\mu}\big)(y-x)^{\mu} \\ &= C\big(x_0^{\lambda-\mu} + y^{\lambda-\mu}\big)(y-x)^{\mu-\lambda'}(y-x)^{\lambda'} \\ &\le C'\big(y^{\mu-\lambda'} + y^{\lambda-\lambda'}\big)(y-x)^{\lambda'} \\ &\le C'\big(x_0^{\lambda-\lambda'} + x_0^{\mu-\lambda'}\big)(y-x)^{\lambda'} . \end{aligned}$$

On the other hand if $\mu > 1$ then

$$\begin{aligned} |A_1| &\le C\big(x_0^{\lambda-\mu} + y^{\lambda-\mu}\big)(y^{\mu} - x^{\mu}) \\ &\le C\big(x_0^{\lambda-\mu} + y^{\lambda-\mu}\big)\mu y^{\mu-1}(y-x) \\ &\le \mu C\big(x_0^{\lambda-1} + y^{\lambda-1}\big)(y-x)^{1-\lambda}(y-x)^{\lambda} \\ &\le 2\mu C(y-x)^{\lambda} . \end{aligned}$$

If $\lambda - \mu > 0$ the estimation $|A_2| \le C(y-x)^{\lambda-\mu}$ follows from (3.4.10); if $\lambda - \mu \le 0$ we can without loss of generality assume $\lambda - \mu < 0$. If $-1 \le \lambda - \mu < 0$ we have, with $\lambda' = \min(\mu - \lambda, \lambda)$,

$$\begin{aligned} |A_2| &\le Cx^{\mu}\big(x^{\lambda-\mu} - y^{\lambda-\mu}\big) = Cx^{\lambda}y^{\lambda-\mu}\big(y^{\mu-\lambda} - x^{\mu-\lambda}\big) \\ &\le Cx^{\lambda}y^{\lambda-\mu}(y-x)^{\lambda'}(y-x)^{\mu-\lambda-\lambda'} \le Cx^{\lambda}y^{-\lambda'}(y-x)^{\lambda'} \\ &\le Cx_0^{\lambda-\lambda'}(y-x)^{\lambda'} . \end{aligned}$$

while in the case $\lambda - \mu < -1$ the estimation proceeds as follows:

$$\begin{aligned} |A_2| &\le Cx^{\lambda}y^{\lambda-\mu}(y^{\mu-\lambda} - x^{\mu-\lambda}) \\ &\le C(\mu-\lambda)x^{\lambda}y^{-1}(y-x) \\ &\le C(\mu-\lambda)x^{\lambda}y^{-\lambda}(y-x)^{\lambda} \le C(\mu-\lambda)(y-x)^{\lambda} . \end{aligned}$$

(3.4.7) is proved by similar methods, (3.4.5) is straightforward. □

For the purposes of this paper only the information contained in the Lemma above and in equation (3.4.14) below will be needed. For completeness, and for future reference, we wish however to point out the following consequences of Lemma 3.4.1:

Corollary 3.4.2. Let $i, k, m \in \mathbb{N}_0$. The following maps are continuous

(i) $\alpha \le \mu$, $\lambda, \lambda' \in [0, 1]$

$$I^{\mu}_{x_0} : C^{\alpha,\beta}_{k+\lambda,m+\lambda'}(M_{x_o}) \to C^{\alpha,\beta'}_{k+\lambda,m+\lambda'}(M_{x_o}) \tag{3.4.11}$$

$\beta' = \beta$ if $\alpha < \mu$; $\beta' = \beta + 1$ if $\alpha = \mu$ and $\beta \ne -1$; $\beta' > 0$ arbitrary if $\alpha = \mu$ and $\beta = -1$.

(ii) $\alpha > \mu$, $\lambda, \lambda' \in [0, 1]$

$$I^{\mu}_{x_0} : C^{\alpha,\beta}_{k+\lambda,m+\lambda'}(M_{x_o}) \to C^{\mu}_{k+\lambda'',m+\lambda'}(M_{x_o}) , \tag{3.4.12}$$

with some $\lambda'' \in [0, \lambda]$, $\lambda'' > 0$ if $\lambda > 0$.

(iii) $\alpha > \mu$, $\lambda, \lambda' \in [0,1]$, let k_0 be any integer such that $0 \le k_0 \le k$ and $k_0 - m < \alpha - \mu$,

$$I_0^\mu : C^{\alpha,\beta}_{k+\lambda,m+\lambda'}(M_{x_o}) \to C^{\alpha,\beta}_{k_0+\lambda_1,m+\lambda'}(M_{x_o}) \tag{3.4.13}$$

for some $\lambda_1 \in [0, \min(\lambda,\lambda')]$; $\lambda_1 = 0$ if $k_0 > m$, $\lambda_1 > 0$ if $\lambda\lambda' > 0$ and $k_0 \le m$.

(iv) $\alpha > \mu$, $\lambda, \lambda' \in [0,1]$, let k_0 be any integer such that $0 \le k_0 \le k$ and $k_0 - m < \alpha - \mu$,

$$I^\mu : C^{\alpha,\beta}_{k,m+\lambda}(M_{x_o}) \to x^\mu C_{m+\lambda}(\partial M) \cap x^\mu C_{k_0}(\partial M)\,. \tag{3.4.14}$$

(v) $\alpha < \mu$, $\lambda \in [0,1]$ let ℓ_0 be any integer such that $\mu - \alpha > \ell_0 \ge 0$

$$I^\mu_{x_0} : x^\alpha C_{k+\lambda|m}(\bar M_{x_o}) \to x^\alpha C_{\min(k,\ell_0)+\lambda'|m}(\bar M_{x_o}) \tag{3.4.15}$$

for some $\lambda' \in [0,\lambda]$, $\lambda' > 0$ if $\lambda > 0$.

(vi) $\alpha > \mu$, $\lambda \in [0,1]$

$$I_0^\mu : x^\alpha C_{k+\lambda|m}(\bar M_{x_o}) \to x^\alpha C_{k+\lambda'|m}(\bar M_{x_o})\,, \tag{3.4.16}$$

for some $\lambda' \in [0,\lambda]$, $\lambda' > 0$ if $\lambda > 0$.

(vii)

$$I^\mu_{x_0} : \mathcal{A}^{\mathrm{phg}} \to \mathcal{A}^{\mathrm{phg}}\,. \tag{3.4.17}$$

(viii) $\alpha > \mu$

$$I_0^\mu : \mathcal{A}^{\mathrm{phg}} \cap C_0^\alpha(M_{x_0}) \to \mathcal{A}^{\mathrm{phg}}\,. \tag{3.4.18}$$

PROOF: (3.4.11)–(3.4.14) follow by straightforward induction from (3.4.2)–(3.4.5). To prove (3.4.15), suppose first that $k \le \ell_0$, choose γ, δ satisfying $0 \le |\gamma| \le m$, $0 \le |\delta| \le k$, set $\beta = \delta+\gamma$, define $\tilde f \equiv \partial_v^\beta x^{-\alpha} f$, thus $\tilde f \in C_{k-|\delta|+\lambda|\,|\gamma|}$. Taylor expanding $\tilde f$ up to order $\ell = k - |\delta|$ we have from Proposition 3.1.1

$$\tilde f(x,v) = \sum_{i=0}^{\ell} x^i \tilde f_i(v) + \tilde r_\ell\,, \quad f_i \in C_{k-i-|\delta|+\lambda}(\partial M)\,,$$

$$\forall\, 0 \le i \le \ell \quad \partial_x^i r_\ell \in x^{\ell-i} C_{0+\lambda} \cap x^{\ell-i+\lambda} C_0\,, \tag{3.4.19}$$

so that

$$I^{\mu-\alpha}_{x_0}(\tilde f) = \sum_{i=0}^{\ell} \frac{x^i \tilde f_i(v)}{\mu-\alpha-i} + x^{\mu-\alpha} \sum_{i=0}^{\ell} \frac{x_0^{i-\mu+\alpha}\tilde f_i(v)}{i-\mu+\alpha} + I^{\mu-\alpha}_{x_0}(\tilde r_\ell)\,.$$

(3.4.6) and (3.4.19) imply by induction

$$0 \le p \le \ell \qquad \partial_x^p I^{\mu-\alpha}_{x_0}(\tilde r_\ell) \in C_{0+\lambda'}\,,$$

for some $\lambda' \in [0,\lambda]$, $\lambda' > 0$ if $\lambda > 0$, which gives $\partial_x^p I_{x_0}^{\mu-\alpha}(\tilde f) \in C_{0+\lambda'}$ for $0 \le p \le \ell$, therefore for $0 \le |\gamma| \le m$ we have $\partial_v^\gamma I_{x_0}^\mu(f) \in x^\alpha C_{k+\lambda'}$. From

$$(x\partial_x)^j I_a^\mu = \mu^j I_a^\mu + \sum_{p+q=j-1} (x\partial_x)^p \mu^q f \tag{3.4.20}$$

the result follows for $k \le \ell_0$. The inclusion

$$k \ge \ell_0 \qquad C_{k+\lambda|m}(\bar M_{x_o}) \subset C_{\ell_0+\lambda|m}(\bar M_{x_o})$$

reduces the case $k \ge \ell_0$ to the previous one. (3.4.16) is proved in a similar way using (3.4.7). (3.4.17) and (3.4.18) are elementary. □

Proposition 3.4.3. Let $i,k \in \mathbb{N}_0$, $\lambda \in [0,1]$. There exists $\lambda' \in [0,\lambda]$, $\lambda' > 0$ unless $\lambda = 0$, such that the following maps are continuous.

(i) $-k \le j \in \mathbb{Z}$, $\max(\mu_-, \mu_- + j) < \alpha \le \mu_+$,

$$G_{x_0} : C^{\alpha,\beta}_{k+j+\lambda,k+\lambda}(M_{x_o}) \to C^{\alpha,\beta'}_{k+j+\lambda',k+\lambda}(M_{x_o}) \ , \tag{3.4.21}$$

$\beta' = \beta$ if $\alpha < \mu_+$; $\beta' = \beta + 1$ if $\alpha = \mu_+$ and $\beta \ne -1$; $\beta' > 0$ arbitrary if $\alpha = \mu_+$, $\beta = -1$.

(ii) $\alpha > \mu_+$, $\alpha > \mu_- + j$, $j \in \mathbb{N}_0$,

$$G_{x_0} : C^{\alpha,\beta}_{k+j+\lambda,k+\lambda}(M_{x_o}) \to C^{\mu_+}_{k+j+\lambda',k+\lambda}(M_{x_o})\,. \tag{3.4.22}$$

(iii) $j = 0,1$

$$G_0 : C^{\mu_++1}_{j+\lambda}(M_{x_o}) \to C^{\mu_++1}_0(M_{x_o})\,. \tag{3.4.23}$$

(iv) $j = 1,2$

$$G_0 : C^{\mu_++j}_{k+j+\lambda,k+\lambda}(M_{x_o}) \to C^{\mu_++j}_{k+j-1+\lambda',k+\lambda}(M_{x_o})\,. \tag{3.4.24}$$

(v) $j \in \mathbb{N}_0$, $\mu_+ + j < \alpha$,

$$G_0 : C^{\alpha,\beta}_{k+j+\lambda,k+\lambda}(M_{x_o}) \to C^{\alpha,\beta}_{k+j+\lambda',k+\lambda}(M_{x_o}) \ . \tag{3.4.25}$$

(vi) $\mu_- < \alpha < \mu_+$, let ℓ_0 be any integer such that $\mu_+ - \alpha > \ell_0$,

$$G_{x_0} : x^\alpha C_{k+\lambda|m}(\bar M_{x_o}) \to x^\alpha C_{\min(k,\ell_0)+\lambda'|m}(\bar M_{x_o})\,. \tag{3.4.26}$$

(vii) $\alpha > \mu_+$,

$$G_0 : x^\alpha C_{k+\lambda|m}(\bar M_{x_o}) \to x^\alpha C_{k+\lambda'|m}(\bar M_{x_o})\,. \tag{3.4.27}$$

(viii)

$$G_a : \mathcal{A}^{\{s_i\}_{i=0}^I} \to \mathcal{A}^{\mathrm{phg}}\,, \quad a = 0, x_0 \ , \tag{3.4.28}$$

provided that $s_0 > \mu_-$ if $a = x_0$, and $s_0 > \mu_+$ if $a = 0$.

PROOF: The results follow from Corollary 3.4.2 and from

$$G_{x_0} = \frac{1}{\mu_+ - \mu_-}\left(I_{x_0}^{\mu_+} - I_0^{\mu_-}\right) = \frac{1}{\mu_+ - \mu_-}\left(I_0^{\mu_+} - I_0^{\mu_-} - I^{\mu_+}\right),$$

$$x\frac{\partial}{\partial x} \cdot G_{x_0} = \frac{1}{\mu_+ - \mu_-}\left(\mu_+ I_{x_0}^{\mu_+} - \mu_- I_{x_0}^{\mu_-}\right).$$

$\square$

CHAPTER 4

Regularity at the boundary: the linear problem.

4.1. Tangential regularity below the threshold

Throughout this paper the letter C denotes a constant, the value of which may change from line to line. We use the summation convention unless specified otherwise.

We shall consider systems of equations which in the local coordinates as described in Chapter 3 can be written in the form

$$L^a{}_b\, u^b = f^a\,, \qquad a, b = 1, \dots, N\,, \tag{4.1.1}$$

Here

$$L^a{}_b = \sum_{|\alpha|\le m} a^a{}_{b\alpha}(y)(x\partial_y)^\alpha\,, \tag{4.1.2}$$

where we use the notation

$$(x\partial_y)^\alpha \equiv (x\partial_x)^{\alpha_1}(x\partial_{y^2})^{\alpha_2}\dots(x\partial_{y^n})^{\alpha_n} \quad \text{for } \alpha = (\alpha_1, \dots, \alpha_n)\,.$$

Dropping indices we can write

$$L = \sum_{|\alpha|\le m} a_\alpha(x\partial_y)^\alpha\,, \tag{4.1.3}$$

or briefly

$$Lu = f\,.$$

Let F be a function space. We shall say that $L \in OP^m_F$ if $a^a{}_{b\alpha} \in F$. We set

$$\mathbf{a} = (a_\alpha)\,, \qquad \|\mathbf{a}\|_F \equiv \sum_{a,b,\alpha} \|a^a{}_{b\alpha}\|_F\,.$$

The operator L will be called *elliptic on* M_{x_0} if there exist constants $c_1, c_2 > 0$ such that on M_{x_0} it holds that

$$c_1|\zeta|^{mN} \le \det\Big(\sum_{|\alpha|=m} a^a{}_{b\alpha}\zeta^\alpha\Big) \le c_2|\zeta|^{mN}\,, \tag{4.1.4}$$

$$|\zeta| = \{(\zeta^1)^2 + \dots + (\zeta^n)^2\}^{\frac{1}{2}}\,.$$

The operator L will be called *elliptic* if it is elliptic on M_{x_0}, and if moreover it is elliptic in any standard sense in CM_{x_0}; no uniformity conditions on CM_{x_0} are assumed. In some of the results in our paper, the notion of ellipticity on M_{x_0} can be weakened to a suitably weighted (in the sense of (4.1.2)) version of the definition of Douglis–Nirenberg [**30**].

The main results of this chapter – Theorems 4.2.6, 4.2.9, 4.2.10 and 4.2.11 – establish some form of "boundary regularity" for solutions of (4.1.1), under the hypothesis of the existence of a "regularity interval" for weighted Hölder or for weighted Sobolev spaces, as defined below. Thus the results of this chapter reduce the problem of boundary behaviour to that of existence of a "regularity interval" (which may be quite difficult to prove for specific operators). In section 6.3 we shall prove the existence of a regularity interval for weighted Sobolev spaces for the "conformal vector Laplacian" (in fact, of a *strong* regularity interval), while in Chapter 7 the existence of a (strong) regularity interval for weighted Hölder spaces will be established for the Laplace operator (the existence of a regularity interval for weighted Sobolev spaces for the Laplacian follows from Corollary 3.13 of [**3**] and from Proposition 4.1.2 below).

Let $F_*^{\alpha,\beta}(\Omega)$ be one of the spaces $C_*^{\alpha,\beta}$, $W_*^{\alpha,\beta;p}$ or $H_*^{\alpha,\beta}$ introduced in Chapter 3, where $*$ stands for a regularity index, e.g. $* = k$, or $* = k+\lambda, m+\mu$, etc. We shall always be interested in $\Omega = M$ or $\Omega = M_{x_0}$, with x_0 — as described at the beginning of Chapter 3; note that if $\Omega = M_\sigma$ with $0 < \sigma < x_0$, one can always replace x_0 with σ to reduce this to the case $\Omega = M_{x_0}$ — our results and methods are "stable" under such replacements. We shall say that an interval $(\alpha_-, \alpha_+) \subset \mathbb{R}$ is a *regularity interval* for $F_*^{\alpha,\beta}(\Omega)$ except for $\beta \in B \subset \mathbb{R}$ for an operator L of order m if the following implication holds, with $F_m^{\mathrm{loc}}(\Omega)$ – space of functions which are locally in F_m,

$$\left(\begin{array}{c} Lu = f, \quad f \in F_*^{\alpha,\beta}(\Omega)\,, \quad \alpha \in (\alpha_-, \alpha_+), \quad \beta \notin B, \\ u \in F_0^{\alpha_- + \epsilon}(\Omega) \cap F_m^{\mathrm{loc}}(\Omega), \qquad \epsilon > 0 \end{array}\right)$$

$$\Longrightarrow$$

$$\left(\ u \in F_0^{\alpha,\beta}(\Omega)\ \right) \tag{4.1.5}$$

Usually we shall consider $B = \emptyset$, $B = \{-1\}$, or $B = \mathbb{R}\backslash\{0\}$; in that last case we shall say that (α_-, α_+) is a regularity interval for F_*^{α}.

We shall say that an interval (α_-, α_+) is a *strong regularity interval* for $F_*^{\alpha,\beta}(\Omega)$, $\beta \notin B$, if (α_-, α_+) is a regularity interval for $F_0^{\alpha,\beta}(\Omega)$ and if moreover there exist constants $0 < x_1(\alpha,\beta) \le x_0$, $C(\alpha,\beta)$ such that the following implication holds,

$$\left(\begin{array}{c} Lu = f, \quad f \in F_0^{\alpha,\beta}(\Omega)\,, \quad \alpha \in (\alpha_-, \alpha_+), \quad \beta \notin B, \\ u \in F_0^{\alpha_- + \epsilon}(\Omega) \cap F_m^{\mathrm{loc}}(\Omega), \qquad \epsilon > 0 \end{array}\right)$$

$$\Longrightarrow$$

$$\left(\ \|u\|_{F_0^{\alpha,\beta}(M_{x_0})} \leq C\Big(\|f\|_{F_0^{\alpha,\beta}(M_{x_0})} + \|u\|_{F_m(\overline{M_{x_1,x_0}})}\Big) \ \right) \tag{4.1.6}$$

(recall that $M_{x_1,x_0} \equiv CM_{x_1} \cap M_{x_0} \equiv \{p \in M : x_1 < x(p) < x_0\}$). For instance, in the weighted Hölder spaces $C_0^{\alpha,\beta}(M_{x_0})$ case the inequality (4.1.6) reads

$$\|u\|_{C_0^{\alpha,\beta}(M_{x_0})} \leq C_H \left(\|f\|_{C_0^{\alpha,\beta}(M_{x_0})} + \|u\|_{C_m(\overline{M_{x_1,x_0}})}\right) . \tag{4.1.7}$$

In the weighted Sobolev spaces $L^{\alpha,\beta;p}(M_{x_0}, x^{-n}d\mu_g)$ case the inequality (4.1.6) reads

(4.1.8)
$$\|u\|_{L^{\alpha,\beta;p}(M_{x_0},x^{-n}d\mu_g)} \leq C_S \left(\|f\|_{L^{\alpha,\beta;p}(M_{x_0},x^{-n}d\mu_g)} + \|u\|_{W^p_m(M_{x_1,x_0},g,d\mu_g)}\right) .$$

In what follows we shall use difference–quotient arguments, writing expressions of the type "$u(x,v') - u(x,v)$", by which we mean the following construction: we can always find a covering of ∂M by coordinate charts $(\mathcal{U}_i, \Phi_i)$, $i = 1, \ldots, I$, with $\mathcal{U}_i = \Phi_i(\mathcal{B}_0^{n-1}(3\delta_i))$, where $\mathcal{B}_0^{n-1}(r)$ is a ball of radius r in $\mathbb{R}^{n-1}$ centered at the origin, such that $\mathcal{U}_i = \Phi_i(\mathcal{B}_0^{n-1}(\delta_i))$ is also a covering of ∂M, let φ_i be a function such that $\varphi_i\big|_{\Phi_i(\mathcal{B}_0^{n-1}(2\delta_i))} = 1$, $\operatorname{supp}\varphi_i \subset \mathcal{U}_i$, $0 \leq \varphi_i \leq 1$. Let $\Delta \in S(1)$, where $S(r)$ is a sphere of radius r in $\mathbb{R}^{n-1}$, for $(x,v) \in \bar{M}_{x_0}$ ($v \in \partial M$) define vector fields X_j, $j = 1, \ldots, I$, as follows

$$X_j(x,v) = \begin{cases} \varphi_j(v)\chi(x)\Delta^k\partial_{v^k} & v \in \mathcal{U}_j, \quad x(p) \leq x_0 \ , \\ 0 & \text{otherwise}, \end{cases}$$

where $\chi \in C_\infty(\mathbb{R})$, $\chi(x) = 1$ for $0 \leq x \leq x_0/2$, $0 \leq \chi(x) \leq 1$, $\chi(0) = 0$ for $x \geq 3x_0/4$. Set

$$\Psi_{\Delta,h,j} = \exp\{hX_j\} \ ,$$

where $\exp\{hX_j\}$, $h \in \mathbb{R}$, is the one parameter group of diffeomorphisms generated by X_j on $\bar{M}_{x_0}$. $\Psi_{\Delta,h,j}$ can be extended to a smooth map from $\bar{M}$ to $\bar{M}$ by setting $\Psi_{\Delta,h,j}\big|_{CM_{x_0}} = \mathrm{id}$. Note that

(4.1.9)
$$0 \leq x \leq x_0/2 \, , \quad v \in \mathcal{U}_j, \quad |h| \leq \delta_j : \qquad \Psi_{\Delta,h,j}(x,v) = (x, v + h\Delta) \, .$$

In all further considerations we shall always assume that the fields u^a, f^a are *geometric objects*, by which we mean that 1) one can define an action ${\Psi_{\Delta,h,j}}^* f$, ${\Psi_{\Delta,h,j}}^* u$, and 2) that the equation (4.1.1) is geometric in the sense that

$${\Psi_{\Delta,h,j}}^*(Lu) = L_{\Delta,h,j}{\Psi_{\Delta,h,j}}^*u = {\Psi_{\Delta,h,j}}^* f \tag{4.1.10}$$

with some operator $L_{\Delta,h,j}$ of the form (4.1.2):

$$L_{\Delta,h,j} = \sum_{|\alpha| \leq m} a_{\Delta,h,j,\alpha}(x\partial_y)^\alpha \, .$$

We shall also assume that for $0 \le x \le x_0/2$, $|h| \le \delta_j$ and $v \in \mathcal{U}_j$ we have

$$\Psi_{\Delta,h,j}{}^* \, u(x,v) = u(x, v + h\Delta) \ , \tag{4.1.11}$$

$$a_{\Delta,h,j,\alpha}(x,v) = a_\alpha(x, v+h\Delta), \qquad \Psi_{\Delta,h,j}{}^* \, f(x,v) = f(x, v+h\Delta) \, , \tag{4.1.12}$$

and that there exists a constant C such that for $0 \le \ell \le k$, $|h| \le \max_j \delta_j$ we have

$$\|\Psi_{\Delta,h,j}{}^* f - f\|_{C^{\alpha,\beta}_{k,\ell}(M_{x_0})} \le C\|f\|_{C^{\alpha,\beta}_{k,\ell+\mu}(M_{x_0})} h^\mu \ , \tag{4.1.13}$$

$$\|\Psi_{\Delta,h,j}{}^* f\|_{C^{\alpha,\beta}_{k+\lambda,\ell}(M_{x_0})} \le C\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell}(M_{x_0})} \ , \tag{4.1.14}$$

$$\|\Psi_{\Delta,h,j}{}^* f - f\|_{W^{\alpha,\beta;p}_{k,\ell-1}(M_{x_0})} \le C\|f\|_{W^{\alpha,\beta;p}_{k,\ell}(M_{x_0})} h \ , \tag{4.1.15}$$

$$\|\Psi_{\Delta,h,j}{}^* f\|_{W^{\alpha,\beta;p}_{k,\ell}(M_{x_0})} \le C\|f\|_{W^{\alpha,\beta;p}_{k,\ell}(M_{x_0})} \ , \tag{4.1.16}$$

$$\|a_{\Delta,h,j,\alpha} - a_\alpha\|_{C^{\alpha,\beta}_{k,\ell}(M_{x_0})} \le C\|\mathbf{a}\|_{C^{\alpha,\beta}_{k,\ell+\mu}(M_{x_0})} h^\mu \ , \tag{4.1.17}$$

$$\|a_{\Delta,h,j,\alpha}\|_{C^{\alpha,\beta}_{k+\lambda,\ell}(M_{x_0})} \le C\|\mathbf{a}\|_{C^{\alpha,\beta}_{k+\lambda,\ell}(M_{x_0})} \ . \tag{4.1.18}$$

Note that (4.1.11) implies that we have

$$\|u\|_{C^{\alpha,\beta}_{k+\lambda,\ell+\mu}(M_{x_0})} \le \|u\|_{C^{\alpha,\beta}_{k+\lambda,\ell}(M_{x_0})} + \sum_{i=0}^{I} \sup_{0<h\le\delta_i, \ \Delta\in S(1)} h^{-\mu} \|\Psi_{\Delta,h,j}{}^* \, u - u\|_{C^{\alpha,\beta}_{k,\ell}(M_{x_0})} \ . \tag{4.1.19}$$

Similarly the well known property of difference quotients in Sobolev spaces (*cf. e.g.* [**40**, Lemma 7.24]) also gives

$$\|u\|_{W^{\alpha,\beta;p}_{k,\ell}(M_{x_0})} \le \|u\|_{W^{\alpha,\beta;p}_{k,\ell-1}(M_{x_0})} + \sum_{i=0}^{I} \sup_{0<h\le\delta_i, \ \Delta\in S(1)} h^{-1} \|\Psi_{\Delta,h,j}{}^* \, f - f\|_{W^{\alpha,\beta;p}_{k,\ell-1}(M_{x_0})} \ . \tag{4.1.20}$$

If u, f are tensor fields and L is an invariant operator, (4.1.10)–(4.1.18) are easily seen to hold (if u is *e.g.* a covariant tensor field, then $\Psi_{\Delta,h,j}{}^*$ is the pull–back operation by $\Psi_{\Delta,h,j}$; if u is *e.g.* a contravariant tensor field, then $\Psi_{\Delta,h,j}{}^*$ is the push–forward operation by $\Psi^{-1}_{\Delta,h,j}$). In all subsequent arguments involving difference quotients the reader should assume that we have this construction in mind. We shall say that L is a *geometric operator*, or that eq. (4.1.1) is a *geometric equation*, whenever the above "coordinate invariance" hypotheses hold.

The scaling technique, illustrated in the proof of the following Lemma, will be used throughout:

LEMMA 4.1.1 (SCALING ESTIMATES). (i) Let $L \in OP^m_{C^0_k(M_{x_0})}$, $k \geq 0$, be elliptic on M_{x_0}, let $g \in C^0_k(M_{x_0})$ be a Riemannian metric on $\bar{M}$. There exists a constant C such that for all $u \in W^{\alpha,\beta;p}_0(M_{x_0}, d\mu_g) \cap W^{p,\mathrm{loc}}_m(M_{x_0}, g, d\mu_g)$, $p \in (1,\infty)$, we have

(4.1.21)
$$\|u\|_{W^{\alpha,\beta;p}_{k+m}(M_{x_0/2},g,d\mu_g)} \leq C\Big\{\|Lu\|_{W^{\alpha,\beta;p}_k(M_{x_0},g,d\mu_g)} + \|u\|_{W^{\alpha,\beta;p}_0(M_{x_0},d\mu_g)}\Big\}.$$

(ii) Let $L \in OP^m_{C^0_{k+\lambda}(M_{x_0})}$, $k \geq 0$, $\lambda \in (0,1)$, be elliptic on M_{x_0}. There exists a constant C such that for all $u \in C^{\alpha,\beta}_0(M_{x_0}) \cap C^{\mathrm{loc}}_m(M_{x_0})$ we have

(4.1.22)
$$\|u\|_{C^{\alpha,\beta}_{k+m+\lambda}(M_{x_0/2})} \leq C\Big\{\|Lu\|_{C^{\alpha,\beta}_{k+\lambda}(M_{x_0})} + \|u\|_{C^{\alpha,\beta}_0(M_{x_0})}\Big\}.$$

PROOF: Let $\hat{L}$ be an elliptic operator of order m on $\mathcal{B}^n_0(1)$, the coefficients of which are k times continuously differentiable, where $\mathcal{B}^n_0(s)$ is a ball in $\mathbb{R}^n$ of radius s centered at the origin. From [**2**, Theorem 10.3] and the argument of the proof of Theorem 9.11 in [**40**] for $p \in (1,\infty)$ one has

(4.1.23)
$$\|\hat{u}\|_{W^p_{k+m}(\mathcal{B}^n_0(1/2),\delta_{ij},d^ny)} \leq C\Big\{\|\hat{L}\hat{u}\|_{W^p_k(\mathcal{B}^n_0(1),\delta_{ij},d^ny)} + \|\hat{u}\|_{L^p(\mathcal{B}^n_0(1),d^ny)}\Big\}.$$

(4.1.21) follows from (4.1.23) applied to the functions $\hat{u} \equiv u_{\hat{y}}$ defined by

$$u_{\hat{y}}(y) = u\Big(\frac{\hat{x}}{2}\,y + \hat{y}\Big)\hat{x}^{-\alpha}(|\ln\hat{x}| + 1)^{-\beta}, \quad y \in \mathcal{B}^n_0(1), \quad \hat{y} = (\hat{x},\hat{v}) \in M_{x_0/2}$$

by a Whitney cube (*cf. e.g.* [**68**, Chapter 6]) decomposition argument. (4.1.22) is obtained by applying the interior Schauder estimates [**30**, Theorem 1].

$$\|\hat{u}\|_{C_{k+m+\lambda}(\mathcal{B}^n_0(1/2))} \leq C\Big\{\|\hat{L}\hat{u}\|_{C_{k+\lambda}(\mathcal{B}^n_0(1))} + \|\hat{u}\|_{C_0(\mathcal{B}^n_0(1))}\Big\}$$

to the functions $u_{\hat{y}}$. □

The following Proposition (*cf.* [**3**, Proposition 2.6]) provides a criterion for I to be a regularity interval for second order operators:

PROPOSITION 4.1.2. Let $k \in \mathbb{N}_0$, $\lambda \in (0,1)$, and let x be a defining function for ∂M, with x_0 — as described at the beginning of Chapter 3, recall that $\tilde{\partial} M_s = \{p \in M : x(p) = s\}$, suppose that $B \subset \mathbb{R}$ satisfies $0 \notin B$.

(i) Let $L \in OP^2_{C^0_{k+\lambda}(M_{x_0})}$, suppose that for all $\alpha \in (\alpha_-, \alpha_+)$ and $\beta \notin B$ there exists $0 < x_1 \leq x_0$ (possibly depending upon α and β) such that for all $\psi \in C_{k+2+\lambda}(\tilde{\partial} M_{x_1})$ L is an isomorphism between $\{u \in C^{\alpha,\beta}_{k+2+\lambda}(M_{x_1}) : u|_{\tilde{\partial} M_{x_1}} = \psi\}$ and $C^{\alpha,\beta}_{k+\lambda}(M_{x_1})$. Then L has (α_-, α_+) as a regularity interval for $C^{\alpha,\beta}_{k+\lambda}(M)$, $\beta \notin B$.

(ii) Let $L \in OP^2_{C^0_k(M_{x_0})}$, suppose that for all $\alpha \in (\omega_-, \omega_+)$ and $\beta \notin B$ there exists $0 < x_1 \le x_0$ (possibly depending upon α and β) such that for all $\psi \in W^{\alpha,\beta;p}_{k+2}(M_{x_1}, g, x^{-n}d\mu_g)$, L is an isomorphism between $\{u : u - \psi \in \mathring{W}^{\alpha,\beta;p}_1 \cap W^{\alpha,\beta;p}_{k+2}(M_{x_1}, g, x^{-n}d\mu_g)\}$ and $W^{\alpha,\beta;p}_k(M_{x_1}, g, x^{-n}d\mu_g)$. Then L has (ω_-, ω_+) as a regularity interval for $W^{\alpha,\beta;p}_k(M, g, x^{-n}d\mu_g)$, $\beta \notin B$. Moreover if the above holds with $k = 0$, there exists a constant C such that for $u \in W^{\omega_- + \epsilon;p}_0(M_{x_1}, x^{-n}d\mu_g) \cap W^{\rm loc}_2(M_{x_1})$, $\epsilon > 0$ for which $Lu \in W^{\alpha,\beta;p}_0(M_{x_1}, x^{-n}d\mu_g)$, $\alpha \in (\omega_-, \omega_+)$, $\beta \notin B$, we have

(4.1.24)
$$\|u\|_{W^{\alpha,\beta;p}_2(M_{x_1},g,x^{-n}d\mu_g)} \le C(\|Lu\|_{W^{\alpha,\beta;p}_0(M_{x_1},x^{-n}d\mu_g)} + \|u\|_{W^p_2(M_{x_1/2,x_1},g,d\mu_g)})$$

(recall that $M_{x_1/2,x_1} \equiv M_{x_1} \setminus \overline{M_{x_1/2}}$), in particular (ω_-, ω_+) is a *strong* regularity interval for $W^{\alpha,\beta;p}_0(M, g, x^{-n}d\mu_g)$, $\beta \notin B$.

Remark: Let us point out, that if the hypotheses of point (i) above hold with $k = 0$, then we certainly have the inequality

$$\|u\|_{C^{\alpha,\beta}_0(M_{x_0})} \le C_H\left(\|f\|_{C^{\alpha,\beta}_{0+\lambda}(M_{x_0})} + \|u\|_{C_{m+\lambda}(\overline{M_{x_1,x_0}})}\right) . \tag{4.1.25}$$

This does *not* lead to a *strong* regularity interval because of the Hölder continuity exponent λ in the norm of f above. It should be mentioned that at the price of some supplementary complications most of our results would go through if in the definition of a *strong regularity interval* the inequality (4.1.7) were replaced by (4.1.25). Since, however, (4.1.7) is sufficient for our purposes (*cf.* Chapter 7) we shall not consider that possibility.

Proof: (i): Let $Lu = f$, $f \in C^{\alpha,\beta}_{k+\lambda}(M)$, $u = O(x^{\alpha_- + \epsilon})$, decreasing ϵ if necessary we may assume $\gamma = \alpha_- + \epsilon < \alpha$. By Lemma 4.1.1 $u \in C^{\gamma}_{k+2+\lambda}(M_{x_0})$. Set $\psi = u|_{\tilde\partial M_{x_1}}$, thus $\psi \in C_{k+2+\lambda}(\tilde\partial M_{x_1})$. For $\alpha \in (\alpha_-, \alpha_+)$, $\beta \notin B$, L is surjective there exists therefore $\chi \in C^{\alpha,\beta}_{k+2+\lambda}(M_{x_1})$, $\chi|_{\tilde\partial M_{x_1}} = \psi$, such that $L\chi = f|_{M_{x_1}}$. We have $u|_{M_{x_1}}, \chi \in C^{\gamma}_{k+2+\lambda}(M_{x_1})$, $\gamma \in (\alpha_-, \alpha_+)$, by injectivity of L on $C^{\gamma}_{k+2+\lambda}(M_{x_1})$ it follows that $u|_{M_{x_1}} = \chi$, thus $u \in C^{\alpha,\beta}_{k+2+\lambda}(M_{x_0})$.

(i): Let $\varphi \in C_\infty(M)$ be any function satisfying $\varphi\big|_{M_{x_1/2}} = 1$, $\varphi\big|_{CM_{3x_1/4}} = 0$, set $u_1 = \varphi u$, $u_2 = (1-\varphi)u$, define $Lu = f$. We have $u_1 \in W^{\omega_- + \epsilon;p}_0(M_{x_1}, x^{-n}d\mu_g) \cap W^{\rm loc}_2(M_{x_1})$, $u_1 = 0$ in $M_{3x_1/4,x_1}$, from $\varphi\big|_{M_{x_1/2}} = 1$ it follows that $Lu_1\big|_{M_{x_1/2}} = f\big|_{M_{x_1/2}}$, and an argument as in the proof of point (i) shows that

$$u_1 \in W^{\alpha,\beta;p}_2(M_{x_1}, g, x^{-n}d\mu_g)\,,$$

thus $u \in W^{\alpha,\beta;p}_2(M_{x_1}, g, x^{-n}d\mu_g)$ and (ω_-, ω_+) is a regularity interval. If $k = 0$ then $L : \mathring{W}^{\alpha,\beta;p}_1 \cap W^{\alpha,\beta;p}_2(M_{x_1}, g, x^{-n}d\mu_g) \to W^{\alpha,\beta;p}_0(M_{x_1}, g, x^{-n}d\mu_g)$ is a bijection, and the open mapping theorem shows that there exists a constant C such that for all $w \in \mathring{W}_1^{\alpha,\beta;p} \cap W_2^{\alpha,\beta;p}(M_{x_1}, g, x^{-n}d\mu_g)$ we have

$$\|w\|_{W_2^{\alpha,\beta;p}(M_{x_1},g,x^{-n}d\mu_g)} \le C\|Lw\|_{W_0^{\alpha,\beta;p}(M_{x_1},x^{-n}d\mu_g)} .$$

This inequality applied to $w = u_1$ gives

$$\|u_1\|_{W_2^{\alpha,\beta;p}(M_{x_1},g,x^{-n}d\mu_g)} \le C(\|f\|_{W_0^{\alpha,\beta;p}(M_{x_1},x^{-n}d\mu_g)} + \|u\|_{W_1^p(M_{x_1/2,x_1},g,d\mu_g)})$$

(it is easily seen that for $\sigma > 0$ we have $W_k^p(M_{\sigma,x_1}, g, d\mu_g) = W_k^{\alpha,\beta;p}(M_{\sigma,x_1}, g, x^{-n}d\mu_g)$ with equivalent norms), and (4.1.24) immediately follows. □

The main result of this section is the following:

THEOREM 4.1.3 (TANGENTIAL REGULARITY, $\alpha \in (\alpha_-, \alpha_+)$). (i) Let (α_-, α_+) be a strong regularity interval for $C_0^{\alpha,\beta}(M)$, $\beta \notin B$, for a geometric elliptic operator $L \in OP^m_{C^0_{k+\lambda,\ell+\mu}(M)}$, $\lambda \in (0,1)$, $\mu \in [0,1]$, $0 \le \ell + \mu \le k + \lambda$, let $\alpha_- < \alpha < \alpha_+$, suppose that $u \in C_m^{\mathrm{loc}}(M)$ satisfies

$$Lu = f\,, \quad f \in C^{\alpha,\beta}_{k+\lambda,\ell+\mu}(M)\,, \quad u = O(x^{\alpha_-+\epsilon})\,, \quad \epsilon > 0\,, \quad \beta \notin B\,.$$

Then for all $\lambda' \in (0,1)$, $\mu' \in [0,\mu)$ if $\mu > 0$, $\mu' = 0$ otherwise we have

$$u \in C^{\alpha,\beta}_{k+m-1+\lambda',\ell+\mu}(M_{x_0}) \cap C^{\alpha,\beta}_{k+m+\lambda,\ell+\mu'}(M_{x_0})\,. \tag{4.1.26}$$

Moreover for all $\lambda' \in (0,1)$, $\mu' \in [0,\mu)$ if $\mu > 0$, $\mu' = 0$ otherwise there exists a constant C depending only upon α, β, n=$dim M$, N (*cf.* eq. (4.1.1)), the ellipticity constants c_1, c_2 of eq. (4.1.4), the constant C_H of (4.1.7), the constants in (4.1.13)–(4.1.18), $\|\mathbf{a}\|_{C^0_{k+\lambda,\ell+\mu}(M)}$, λ' and μ' such that we have

$$\begin{aligned} &\|u\|_{C^{\alpha,\beta}_{k+m-1+\lambda',\ell+\mu}(M_{x_0})} + \|u\|_{C^{\alpha,\beta}_{k+m+\lambda,\ell+\mu'}(M_{x_0})} \\ &\quad \le C\left(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell+\mu}(M_{x_0})} + \|u\|_{C_{k+m+\lambda}(\overline{M_{\tilde{x}_1,x_0}})}\right)\,, \end{aligned} \tag{4.1.27}$$

$\tilde{x}_1 = \min(x_1, x_0/2)$. If (α_-, α_+) is only a regularity interval for $C^{\alpha,\beta}_{0+\lambda}(M)$, then (4.1.26) holds with $\mu' = \mu = 0$ provided that $\alpha_- + 1 < \alpha < \alpha_+$ and $u = O(x^{\alpha_-+1+\epsilon})$, $\epsilon > 0$.

(ii) Let (ω_-, ω_+) be a strong regularity interval for $W_0^{\alpha,\beta;p}(M, g, x^{-n}d\mu_g)$, $\beta \notin B$ for a geometric elliptic operator $L \in OP^m_{C^0_{k,\ell}(M)}$, $0 \le \ell \le k$, let $\omega_- < \alpha < \omega_+$, $p \in (1,\infty)$, suppose that $u \in W_m^{p,\mathrm{loc}}(M, g, d\mu_g)$ satisfies

$$Lu = f\,, \quad f \in W^{\alpha,\beta;p}_{k,\ell}(M, g, x^{-n}d\mu_g)\,, \quad u \in W_0^{\omega_-+\epsilon;p}(M, g, x^{-n}d\mu_g)\,,$$

for some $\epsilon > 0$, $\beta \notin B$. Then

$$u \in W^{\alpha,\beta;p}_{k+m,\ell}(M_{x_0}, g, x^{-n}d\mu_g)\,. \tag{4.1.28}$$

Moreover there exists a constant C depending only upon α, β, n=dim$M$, N (*cf.* eq. (4.1.1)), the ellipticity constants c_1, c_2 of eq. (4.1.4), the constant C_S of (4.1.8), the constants in (4.1.13)–(4.1.18), $\|\mathbf{a}\|_{C^0_{k,\ell}(M)}$

and on the modulus of continuity of $(x\partial_y)^\gamma a^a_{b\alpha}(y)$, $|\gamma| = k$, $y \in M_{x_0}$, such that

(4.1.29)
$$\begin{aligned}&\|u\|_{W^{\alpha,\beta;p}_{k+m,\ell}(M_{x_0},g,x^{-n}d\mu_g)}\\ &\le C\left(\|f\|_{W^{\alpha,\beta;p}_{k,\ell}(M_{x_0},g,x^{-n}d\mu_g)} + \|u\|_{W^p_{k+m}(M_{\tilde x_1,x_0},g,d\mu_g)}\right),\end{aligned}$$

$\tilde x_1 = \min(x_1, x_0/2)$. If (ω_-, ω_+) is only a regularity interval for $W^{\alpha,\beta;p}_0(M)$, then (4.1.28) holds provided that $\omega_- + 1 < \alpha < \omega_+$ and

$$u \in W^{\omega_-+1+\epsilon;p}_0(M, g, x^{-n}d\mu_g), \qquad \epsilon > 0\,.$$

REMARK: If $L \in OP^m_{C^0_\infty}$ and (α_-, α_+) is only a regularity interval, the restriction $\alpha > \alpha_- + 1$ can be relaxed to $\alpha > \alpha_-$ by commuting the equation satisfied by u with a pseudodifferential operator [**25**], we shall however not discuss this here.

PROOF: We shall prove point (i), point (ii) is proved in a similar (and simpler) way. We shall proceed by induction on ℓ. Since (α_-, α_+) is a regularity interval we have $u \in C^{\alpha,\beta}_{k+m+\lambda}$ by the scaling estimates, thus the result holds for $\ell + \mu = 0$. Let $f \in C^{\alpha,\beta}_{k+\lambda,\ell_0+\mu}$, $\mu \in (0,1]$, $0 \le \ell_0 \le k$, or $f \in C^{\alpha,\beta}_{k+\lambda,\ell_0+1+\mu}$, $\mu = 0$, $0 \le \ell_0 + 1 \le k$, and suppose that $u \in C^{\alpha,\beta}_{k+m+\lambda,\ell_0}$. If (α_-, α_+) is a strong regularity interval suppose moreover that we have

(4.1.30)
$$\|u\|_{C^{\alpha,\beta}_{k+m+\lambda,\ell_0}(M_{x_0})} \le C\left(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell_0+\mu}(M_{x_0})} + \|u\|_{C_{k+m+\lambda}(\overline{M_{x_1,x_0}})}\right).$$

Let Y_i, $i = 1, \ldots, \ell_1$, be any smooth vector fields of the form $\chi(x)Y^A_i(v)\frac{\partial}{\partial v^A}$, $Y^A_i \in C_\infty(\partial M)$, $\chi \in C_\infty(\mathbb{R})$, $\chi = 1$ in a neighbourhood of 0, $\chi = 0$ outside of $[0, x_0/2]$, where $\ell_1 = \ell_0 + 1$ when (α_-, α_+) is a regularity interval and $\ell_1 = \ell_0$ when (α_-, α_+) is a strong regularity interval. We have

(4.1.31)
$$\begin{aligned} L\tilde u &= \tilde\varphi, \quad \tilde\varphi = \tilde f + \tilde\rho\,,\\ \tilde u &= Y_1 \ldots Y_{\ell_1} u\,,\\ \tilde f &= Y_1 \ldots Y_{\ell_1} f \in C^{\alpha,\beta}_{k-\ell_1+\lambda,0+\mu}\,,\\ \tilde\rho &= [L, Y_1 \ldots Y_{\ell_1}]u\,.\end{aligned}$$

If (α_-, α_+) is a regularity interval and $\mu = 0$, by the inductive hypothesis we have $\tilde\rho \in C^{\alpha,\beta}_{k-\ell_0+\lambda}$ and from $\tilde u \in C^{\alpha-1,\beta}_{k+m-\ell_0-1+\lambda}$, $\alpha - 1 > \alpha_-$, it follows that $\tilde u \in C^{\alpha,\beta}_{k-\ell_0-1+m+\lambda}$, thus $u \in C^{\alpha,\beta}_{k+m+\lambda,\ell_0+1}$ and the inductive step is completed. On the other hand if (α_-, α_+) is a strong regularity interval and $\mu \in (0,1]$ then $\tilde u \in C^{\alpha,\beta}_{k-\ell_0+m+\lambda}$, $\tilde\rho \in C^{\alpha,\beta}_{k-\ell_0+1+\lambda,1} \subset C^{\alpha,\beta}_{k-\ell_0+1+\lambda,0+\mu}$, therefore $\tilde\varphi \in C^{\alpha,\beta}_{k-\ell_0+\lambda,0+\mu}$ and for $0 < x \le x_0$ we have , with $v' = v + v''$, for some fixed v'' (*cf.* the beginning of this chapter for the meaning of $\tilde u(x,v) - \tilde u(x,v')$, etc.)

$$L(x,v)(\tilde u(x,v) - \tilde u(x,v')) = \psi(x,v)\,,$$

$$\begin{aligned} |\psi(x,v)| &\equiv \Big|\Big(L(x,v')-L(x,v)\Big)\tilde u(x,v')+\tilde\varphi(x,v)-\tilde\varphi(x,v')\Big| \\ &\le C\Big(\|f\|_{C^{\alpha,\beta}_{\ell_0,\ell_0+\mu}(M_{x_0})}+\|\mathbf{a}\|_{C^0_{\ell_0,\ell_0+\mu}(M_{x_0})}\|u\|_{C^{\alpha,\beta}_{m+\ell_0,\ell_0-1+\mu}(M_{x_0})}\Big)\times \\ &\qquad \times x^\alpha(1+|\ln x|^\beta)|v-v'|^\mu\ , \end{aligned}$$

and similarly, for higher derivatives,

(4.1.32)

$$\|\psi\|_{C^{\alpha,\beta}_{k-\ell_0}(M_{x_0})}\le C\Big(\|f\|_{C^{\alpha,\beta}_{k,\ell_0+\mu}(M_{x_0})}+\|\mathbf{a}\|_{C^0_{k,\ell_0+\mu}(M_{x_0})}\|u\|_{C^{\alpha,\beta}_{k+m,\ell_0}(M_{x_0})}\Big)\ |v-v'|^\mu\ ,$$

(4.1.33)

$$\|\psi\|_{C^{\alpha,\beta}_{k-\ell_0+\lambda}(M_{x_0})}\le C\Big(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell_0}(M_{x_0})}+\|\mathbf{a}\|_{C^0_{k+\lambda,\ell_0}(M_{x_0})}\|u\|_{C^{\alpha,\beta}_{k+m+\lambda,\ell_0}(M_{x_0})}\Big)$$

(*cf.* (4.1.13)–(4.1.18)). The definition of the strong regularity interval and (4.1.30) imply

(4.1.34)

$$\begin{aligned} |\tilde u(x,v)-\tilde u(x,v')| &\le C\Big(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell_0+\mu}(M_{x_0})}+\|u\|_{C_{k+m+\lambda}(\overline{M_{x_1,x_0}})}\Big)\times \\ &\qquad \times x^\alpha(1+|\ln x|)^\beta\ |v-v'|^\mu\ . \end{aligned}$$

If $\ell_0<k$, $\mu=1$, passing to the limit $v\to v'$ one obtains from (4.1.34)

$$|\partial_v\tilde u|\le C\Big(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell_0+\mu}(M_{x_0})}+\|u\|_{C_{k+m+\lambda}(\overline{M_{x_1,x_0}})}\Big)\ x^\alpha(1+|\ln x|)^\beta$$

because $\tilde u\in C^{\alpha,\beta}_{k+m-\ell_0}(M_{x_0})\subset C^{\rm loc}_{k+m-\ell_0}(M_{x_0})\subset C^{\rm loc}_1(M_{x_0})$, which gives $\partial_v\tilde u\in C^{\alpha,\beta}_0(M_{x_0})$. The scaling estimates applied to the equation (4.1.31) with $\ell_1=\ell_0+1$ give $\partial_v\tilde u\in C^{\alpha,\beta}_{k+m-\ell_0-1+\lambda}(M_{x_0})$, together with the inequality

$$\|\partial_v\tilde u\|_{C^{\alpha,\beta}_{k+m-\ell_0-1+\lambda}(M_{x_0})}\le C\Big(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell_0+1}(M_{x_0})}+\|u\|_{C_{k+m+\lambda}(\overline{M_{x_1,x_0}})}\Big)\ ,$$

therefore $u\in C^{\alpha,\beta}_{k+m+\lambda,\ell_0+1}(M_{x_0})$, and it holds that

$$\|u\|_{C^{\alpha,\beta}_{k+m+\lambda,\ell_0+1}(M_{x_0})}\le C\Big(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell_0+1}(M_{x_0})}+\|u\|_{C_{k+m+\lambda}(\overline{M_{x_1,x_0}})}\Big)\ ,$$

which completes the induction step.

We have thus shown that $u\in C^{\alpha,\beta}_{k+m+\lambda,\ell}(M_{x_0})$, and it remains to show the tangential Hölder continuity of the $\ell_0=\ell$'th tangential derivatives of u. Set $\hat u(x,v)=\tilde u(x,v)-\tilde u(x,v+v'')$, for $\hat y=(\hat x,\hat v)\in M_{x_0/2}$ consider the functions

$$\mathcal{B}^n_0(1/2)\ni y\longrightarrow \hat u_{\hat y}(y)=\tilde u(\hat x y+\hat y)\ \hat x^{-\alpha}(1+|\log\hat x|)^{-\beta}\ ,$$

$$\mathcal{B}^n_0(1/2)\ni y\longrightarrow \psi_{\hat y}(y)=\psi(\hat x y+\hat y)\ \hat x^{-\alpha}(1+|\log\hat x|)^{-\beta}\ ,$$

where $\mathcal{B}^n_0(r)$ is an open ball in $\mathbb{R}^n$ of radius r centered at the origin. From (4.1.32)–(4.1.33) we have

(4.1.35)
$$\|\psi_{\hat y}\|_{C_{k-\ell_0}(\overline{\mathcal{B}^n_0(1/2)})} \le C\Big(\|f\|_{C^{\alpha,\beta}_{k,\ell_0+\mu}(M_{x_0})} + \|u\|_{C^{\alpha,\beta}_{k+m,\ell_0}(M_{x_0})}\Big)|v''|^\mu \ ,$$

(4.1.36)
$$\|\psi_{\hat y}\|_{C_{k-\ell_0+\lambda}(\overline{\mathcal{B}^n_0(1/2)})} \le C\Big(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell_0}(M_{x_0})} + \|u\|_{C^{\alpha,\beta}_{k+m+\lambda,\ell_0}(M_{x_0})}\Big) \ .$$

(4.1.35), (4.1.34) and the L^p interior elliptic estimates (*cf.* (4.1.23)) give
$$\begin{aligned}\|\hat u_{\hat y}\|_{W^p_{k-\ell_0+m}(\mathcal{B}^n_0(1/4),d^ny)} &\le C(\|\psi_{\hat y}\|_{W^p_{k-\ell_0}(\mathcal{B}^n_0(1/2),d^ny)} + \|\hat u_{\hat y}\|_{L^p(\mathcal{B}^n_0(1/2),d^ny)})\\ &\le C\Big(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell_0+\mu}(M_{x_0})} + \|u\|_{C^{\alpha,\beta}_{k+m,\ell_0}(M_{x_0})}\Big)|v''|^\mu \ ,\end{aligned}$$
so that from Sobolev's embedding one gets, for any $\lambda' \in [0,1)$,
$$\|\hat u_{\hat y}\|_{C_{k-\ell_0+m-1+\lambda'}(\overline{\mathcal{B}^n_0(1/4)})} \le C\Big(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell_0+\mu}(M_{x_0})} + \|u\|_{C^{\alpha,\beta}_{k+m,\ell_0}(M_{x_0})}\Big)|v''|^\mu \quad \Longrightarrow$$
$$\|\hat u\|_{C^{\alpha,\beta}_{k-\ell_0+m-1+\lambda'}(M_{x_0/2})} \le C\Big(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell_0+\mu}(M_{x_0})} + \|u\|_{C^{\alpha,\beta}_{k+m,\ell_0}(M_{x_0})}\Big)|v''|^\mu \quad \Longrightarrow$$
$$\begin{aligned}\|u\|_{C^{\alpha,\beta}_{k+m-1+\lambda',\ell_0+\mu}(M_{x_0})} \quad \le \quad & C\Big(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell_0+\mu}(M_{x_0})}\\ &+\|u\|_{C^{\alpha,\beta}_{k+m,\ell_0}(M_{x_0})} + \|u\|_{C_{k+m-1+\lambda'}(\overline{M_{x_0/2,x_0}})}\Big) \ .\end{aligned}$$
To get control of the $k-\ell_0+m$'th derivatives of $\tilde u$ some more work is required. By interpolation (*cf. e.g.* [**44**, Theorem A.5]) from (4.1.35)–(4.1.36) for any $\sigma \in (0,1)$ one obtains
$$\begin{aligned}\|\psi_{\hat y}\|_{C_{k-\ell_0+\sigma\lambda}(\overline{\mathcal{B}^n_0(1/2)})} &\le C\|\psi_{\hat y}\|^\sigma_{C_{k-\ell_0}(\overline{\mathcal{B}^n_0(1/2)})}\|\psi_{\hat y}\|^{1-\sigma}_{C_{k-\ell_0+\lambda}(\overline{\mathcal{B}^n_0(1/2)})}\\ &\le C\Big(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell_0+\mu}(M_{x_0})} + \|u\|_{C^{\alpha,\beta}_{k+m+\lambda,\ell_0}(M_{x_0})}\Big)|v''|^{(1-\sigma)\mu} \ ,\end{aligned}$$
and elliptic interior Hölder estimates give
$$\begin{aligned}\|\hat u_{\hat y}\|_{C_{k-\ell_0+m+\sigma\lambda}(\overline{\mathcal{B}^n_0(1/4)})} &\le C(\|\psi_{\hat y}\|_{C_{k-\ell_0+\sigma\lambda}(\overline{\mathcal{B}^n_0(1/2)})} + \|\hat u_{\hat y}\|_{C_0(\overline{\mathcal{B}^n_0(1/2)})})\\ &\le C\Big(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell_0+\mu}(M_{x_0})} + \|u\|_{C^{\alpha,\beta}_{k+m+\lambda,\ell_0}(M_{x_0})}\Big)|v''|^{(1-\sigma)\mu} \ ,\end{aligned}$$
so that $\tilde u \in C^{\alpha,\beta}_{k+m-\ell_0+\sigma\lambda,(1-\sigma)\mu}$, and for all $|\gamma| \le k+m-\ell_0$ we have
$$\begin{aligned}&\Big|(x\partial_y)^\gamma\Big(\tilde u(x,v) - \tilde u(x,v')\Big)\Big|\\ &\le C\Big(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell_0+\mu}(M_{x_0})} + \|u\|_{C^{\alpha,\beta}_{k+m+\lambda,\ell_0}(M_{x_0})}\Big)\ x^\alpha(1+|\ln x|)^\beta\,|v-v'|^{(1-\sigma)\mu} \ .\end{aligned}$$
Since $u \in C^{\alpha,\beta}_{k+m+\lambda,\ell_0}$ it follows that $u \in C^{\alpha,\beta}_{k+m+\lambda,\ell_0+\mu'}$, with

(4.1.37)
$$\|u\|_{C^{\alpha,\beta}_{k+m+\sigma\lambda,\ell_0+(1-\sigma)\mu}(M_{x_0})} \le C\Big(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell_0+\mu}(M_{x_0})} + \|u\|_{C^{\alpha,\beta}_{k+m+\lambda,\ell_0}(M_{x_0})}\Big) \ ,$$

and (4.1.27) follows by (4.1.30). □

Note that the argument of the last part of the proof of Theorem 4.1.3 proves the following Lemma, which shall be needed later on:

LEMMA 4.1.4. Let $L \in OP^m_{C^0_{k+\lambda,\ell+\mu}(M_{x_0})}$ be a geometric elliptic operator, $\lambda \in (0,1)$, $\mu \in (0,1]$, $0 \le \ell+\mu \le k+\lambda$, suppose that $u \in C^{\mathrm{loc}}_m(M_{x_0}) \cap C^{\alpha,\beta}_{0,\ell+\mu}(M_{x_0})$ satisfies

$$Lu \in C^{\alpha,\beta}_{k+\lambda,\ell+\mu}(M_{x_0}) .$$

Then for all $\lambda' \in (0,1)$, $\mu' \in [0,\mu)$ we have

$$u \in C^{\alpha,\beta}_{k+m-1+\lambda',\ell+\mu}(M_{x_0/2}) \cap C^{\alpha,\beta}_{k+m+\lambda,\ell+\mu'}(M_{x_0/2}) .$$

Moreover for all $\lambda' \in (0,1)$, $\mu' \in [0,\mu)$ there exists a constant C depending only upon n=$dimM$, N (*cf.* eq. (4.1.1)), the ellipticity constants c_1, c_2 of eq. (4.1.4), the constant C_H of (4.1.7), the constants in (4.1.13)–(4.1.18), $\|\mathbf{a}\|_{C^0_{k+\lambda,\ell+\mu}(M_{x_0})}$, λ' and μ' such that we have

$$\begin{aligned} &\|u\|_{C^{\alpha,\beta}_{k+m-1+\lambda',\ell+\mu}(M_{x_0/2})} + \|u\|_{C^{\alpha,\beta}_{k+m+\lambda,\ell+\mu'}(M_{x_0/2})} \\ &\quad \le C\left(\|Lu\|_{C^{\alpha,\beta}_{k+\lambda,\ell+\mu}(M_{x_0})} + \|u\|_{C^{\alpha,\beta}_{0,\ell+\mu}(M_{x_0})}\right) . \end{aligned}$$

4.2. Boundary regularity for a class of second order systems

Theorem 4.1.3 gives some more information about the solutions than Lemma 4.1.1, however only for exponents $\alpha \in (\alpha_-, \alpha_+)$. To be able to cover α's in the interval (α_-, ∞), or to obtain more regularity, some other techniques are required, and we shall from now on restrict our consideration to *second order* systems[1], for which L can be written in the form

$$L = L_{ab} + \tilde{L} , \tag{4.2.1}$$

$$L_{ab} = (x^2\partial_x^2 + ax\partial_x + b) \otimes id_{\mathbb{R}^N} , \tag{4.2.2}$$

$$\begin{aligned} \tilde{L} = {} & x^2 a^{xA}\frac{\partial}{\partial x}\frac{\partial}{\partial v^A} + x^2 a^{AB}\frac{\partial}{\partial v^A}\frac{\partial}{\partial v^B} + xa^A\frac{\partial}{\partial v^A} + x^3\psi\frac{\partial^2}{\partial x^2} \\ & + x^2\varphi\frac{\partial}{\partial x} + x\chi , \qquad \text{with } x\psi,\ x\varphi \text{ and } x\chi - o(1) , \end{aligned} \tag{4.2.3}$$

where $id_{\mathbb{R}^N}$ is the $N \times N$ identity matrix, and a and b are real *constants* satisfying

$$\frac{(1-a)^2}{4} - b > 0 . \tag{4.2.4}$$

Moreover a^{xA}, a^{AB}, a^A, $x\psi$, $x\varphi$ and $x\chi$ are $N \times N$ matrices the entries of which are bounded functions of y. Note that one can redefine L by multiplying it by an appropriate (non–degenerate) matrix from the left (this might perhaps necessitate a decrease of x_0) to obtain $\psi \equiv 0$ in M_{x_0}. We shall often write L_{ab} in the form $L_{ab} = x^2\partial_x^2 + ax\partial_x + b$, hoping that no confusion will occur. As will be

[1]It should, however, be noted that several of the arguments given here would apply to higher order systems as well.

seen in the following chapters, (4.2.1)–(4.2.3) are sufficient for the applications we have in mind — the constraint equations of general relativity. All subsequent results are based on an analysis of the ODE[2]

$$L_{ab}u = f\,,$$

solutions of which are given by

$$u = Ax^{\mu_-} + Bx^{\mu_+} + G_{x_0}(f)\,, \tag{4.2.5}$$

$$G_{x_0}(f)(x) = \frac{1}{\mu_+ - \mu_-}\left\{ - x^{\mu_+}\int_x^{x_0} s^{-1-\mu_+}f(s)\,ds - x^{\mu_-}\int_0^x s^{-1-\mu_-}f(s)\,ds\right\}, \tag{4.2.6}$$

assuming that the integrals in (4.2.6) converge. Here the *indicial roots* $\mu_\pm$ are given by

$$\mu_\pm = \frac{1-a}{2} \pm \sqrt{\frac{(1-a)^2}{4} - b}\,, \tag{4.2.7}$$

and we will assume throughout that (4.2.4) holds. If $u = o(x^{\mu_-})$, $f = O(x^\alpha)$, $\alpha > \mu_-$, then $A = 0$. For $\alpha > \mu_+$ one can replace $G_{x_0}(f)$ in (4.2.5) by

$$G_0(f) = \frac{1}{\mu_+ - \mu_-}\left\{x^{\mu_+}\int_0^x s^{-1-\mu_+}f(s)\,ds - x^{\mu_-}\int_0^x s^{-1-\mu_-}f(s)\,ds\right\}\,, \tag{4.2.8}$$

and it will be convenient to do so.

Let us discuss shortly the relationship between $\mu_\pm$ and a weighted Hölder space regularity interval (α_-, α_+). When $\psi = 0$, $\varphi = \varphi(x)$, $\chi = \chi(x)$, by considering v-independent sources and solutions it follows immediately that $(\alpha_-, \alpha_+) \subset (-\infty, \mu_+)$, and that $\mu_- \notin (\alpha_-, \alpha_+)$. For the standard Laplacian on hyperbolic space (which is an operator in the class considered here) it is known that $(\alpha_-, \alpha_+) \not\subset (-\infty, \mu_-)$ (in fact $(\alpha_-, \alpha_+) = (\mu_-, \mu_+)$, *cf.* Section 7), and therefore we shall only consider the case

$$\mu_- \le \alpha_- < \alpha_+ \le \mu_+\,.$$

It will be seen that for the equations considered in Section 7 we have $\alpha_\pm = \mu_\pm$, and we expect this to always hold for operators of the form (4.2.1)–(4.2.4). In fact Theorem 4.2.1 in the next Section shows that one may (in some sense, made precise there) assume that $\alpha_+ = \mu_+$ for sufficiently smooth sources.

Let us emphasize that the reason for considering here the possibility that $\alpha_- \ne \mu_-$, or $\alpha_+ \ne \mu_+$, or both, is not because we expect this to be a real feature of some operators. Rather, this is motivated by the fact that for some

[2]In the general case the corresponding ODE is usually analyzed by Mellin transform techniques (*cf. e.g.* **[58]**); for the special class of equations considered in this paper we have found it simpler and more elementary to use the fundamental solution representation (4.2.5). We are grateful to G. Lysik and B. Ziemian for pointing out the Mellin transform approach.

equations it might be possible to prove the existence of a (perhaps strong) regularity interval (α_-, α_+) which is a proper subset of (μ_-, μ_+), while the known proofs do not allow one to obtain the full expected range (μ_-, μ_+) (this is actually what happens in our analysis of the vector constraint equation in weighted Sobolev spaces, Section 6.3). We shall nevertheless show that the existence of any strong regularity interval, or of a regularity interval of length larger than one, is sufficient to conclude some (perhaps not optimal) classical regularity of solutions at the boundary.

4.2.1. "Tangential regularity" above the threshold. For α's above the threshold α_+ we have the following equivalent of Theorem 4.1.3:

THEOREM 4.2.1 (TANGENTIAL REGULARITY, $\alpha \geq \alpha_+$). Let (α_-, α_+) be a strong regularity interval for $C^{\alpha,\beta}_{0+\lambda}(M)$, $\beta \notin B$, for a geometric elliptic operator $L \in OP^2_{C^0_{k+\lambda,\ell+\mu}(M)}$, $\lambda \in (0,1)$, $\mu \in (0,1]$, $1 \leq \ell+\mu \leq k+\lambda$. Suppose that L is of the form (4.2.1)–(4.2.4) with ψ, ϕ, $\chi \in C^0_{k-1+\lambda,\ell-1+\mu}(M)$. Suppose that

$$\mu_- \leq \alpha_- < \alpha_+ \leq \mu_+ \,, \quad \alpha_+ \leq \alpha \,.$$

Let

$$Lu = f \,, \quad f \in C^{\alpha,\beta}_{k+\lambda,\ell+\mu}(M) \,, \quad u = O(x^{\alpha_- + \epsilon}) \,, \quad \epsilon > 0 \,, \quad \beta \notin B \,.$$

If $\ell \geq \ell_0$, where $\ell_0 \in \mathbb{N}$ is the smallest integer such that $\alpha_+ + \ell_0 > \min(\alpha, \mu_+)$, then there exist $\sigma \in (0,1)$ and $\delta > 0$ such that for $\mu' \in [0,\mu)$ and $\epsilon > 0$ we have

(i) $\alpha < \mu_+$

$$u \in C^{\alpha,\beta}_{k+2+\lambda,\ell-\ell_0+\mu'}(M_{x_0}) \cap C^{\alpha_+ + \ell_0 - 1 + \delta}_{k+2+\lambda,\ell-\ell_0+1+\sigma}(M_{x_0}) \,. \tag{4.2.9}$$

(ii) $\alpha = \mu_+$, $\beta \neq -1$

$$u \in C^{\mu_+,\beta+1}_{k+2+\lambda,\ell-\ell_0+\mu'}(M_{x_0}) \cap C^{\alpha_+ + \ell_0 - 1 + \delta}_{k+2+\lambda,\ell-\ell_0+1+\sigma}(M_{x_0}) \,. \tag{4.2.10}$$

(iii) $\alpha > \mu_+$

$$u \in C^{\mu_+}_{k+2+\lambda,\ell-\ell_0+\mu'}(M_{x_0}) \cap C^{\alpha_+ + \ell_0 - 1 + \delta}_{k+2+\lambda,\ell-\ell_0+1+\sigma}(M_{x_0}) \,. \tag{4.2.11}$$

(iv) If $\alpha > \mu_+$ and $\ell \geq \ell_1$, where $\ell_1 \in \mathbb{N}$ is the smallest integer such that $\alpha_+ + \ell_1 > \alpha$, then

$$u = o(x^{\mu_+}) \Longrightarrow u \in C^{\alpha,\beta}_{k+2+\lambda,\ell-\ell_1+\mu'}(M_{x_0}) \cap C^{\alpha_+ + \ell_1 - 1 + \delta}_{k+2+\lambda,\ell-\ell_1+1+\sigma}(M_{x_0}) \,. \tag{4.2.12}$$

The constants σ and δ depend only upon α_+, μ_+, α, μ, λ, ℓ and k. Moreover if F denotes any of the spaces appearing in eqs. (4.2.9)–(4.2.12), then there exists a (u and f independent) constant C such that

$$\|u\|_F \leq C \left(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell+\mu}(M)} + \|u\|_{C^{\alpha_- + \epsilon}_0(M)} \right) \,. \tag{4.2.13}$$

REMARK: If (α_-, α_+) is a regularity interval for $C^{\alpha,\beta}_{0+\lambda}(M)$ only, then the following modifications are needed above: One has to make the supplementary assumptions that $\alpha_- + 1 < \alpha_+$ and that $u = O(x^{\alpha_- + 1 + \epsilon})$ for some $\epsilon > 0$; on the other hand one can also allow $\mu = 0$. Then eqs. (4.2.9)–(4.2.12) hold with $\mu' = 0$ and *without* the claim about the appropriate decay of the transverse derivatives of order $\ell - \ell_* + 1$, $\ell_* = \ell_0$ or $\ell_* = \ell_1$.

PROOF: In what follows we shall assume that $\mu = 0$, $\mu' = 0$, etc., if (α_-, α_+) is *not* a strong regularity interval. From (4.2.1) it follows that u satisfies the equation

$$L_{ab} u = f - \tilde{L} u \,, \tag{4.2.14}$$

and by Theorem 4.1.3 for any $\epsilon > 0$ and $\mu' \in [0, \mu)$ if $\mu > 0$, $\mu' = 0$ otherwise we have $u \in C^{\alpha_+ - \epsilon}_{k+2+\lambda, \ell+\mu'}$, thus $\tilde{L} u \in C^{1+\alpha_+ - \epsilon}_{k-1+\lambda, \ell-1+\mu'} \cap C^{\alpha_+ - \epsilon}_{k+\lambda, \ell+\mu'}$ (*cf.* (3.2.4)–(3.2.6)). Solving the ODE (4.2.14) and choosing ϵ appropriately we obtain

$$u = \psi x^{\mu_+} + G_{x_0}(f - \tilde{L} u) = O(x^\sigma (1 + |\ln x|)^\rho) \,, \tag{4.2.15}$$

for some function $\psi(v)$, with

$$\begin{aligned}
\alpha_+ \le \alpha < 1 + \alpha_+ \,, \quad \alpha < \mu_+ \quad &\Longrightarrow \quad \sigma = \alpha, \, \rho = \beta \,, \\
\alpha_+ \le \alpha < 1 + \alpha_+ \,, \quad \alpha = \mu_+, \;\beta \ne -1 \quad &\Longrightarrow \quad \sigma = \mu_+, \, \rho = \beta + 1 \,, \\
\alpha_+ \le \alpha < 1 + \alpha_+ \,, \quad \alpha > \mu_+ \quad &\Longrightarrow \quad \sigma = \mu_+, \, \rho = 0 \,, \\
1 + \alpha_+ \le \alpha \quad &\Longrightarrow \quad \sigma = \min(\mu_+, 1 + \alpha_+ - \epsilon), \, \rho = 0 \,.
\end{aligned}$$

The scaling estimates, Lemma 4.1.1, imply $u \in C^{\sigma,\rho}_{k+2+\lambda}$. Setting $x = x_0/2$ in (4.2.15) from $\tilde{L} u \in C^{\alpha_+ - \epsilon}_{k+\lambda, \ell+\mu'}$ we get $G_{x_0}(f - \tilde{L} u)\big|_{x = x_0/2} \in C_{\ell+\mu'}(\partial M)$ (*cf.* (3.4.14)), which together with $u(x_0/2, \cdot) \in C_{k+2+\lambda}(\partial M)$ shows that $\psi \in C_{\ell+\mu'}(\partial M)$. This in turn shows that we can v-differentiate (4.2.15): for $0 \le i \le \ell - 1$ let $X_i = X_i{}^A(v) \frac{\partial}{\partial v^A}$ be vector fields with $X_i{}^A \in C_\infty(\partial M)$. (4.2.15) gives

$$\begin{aligned}
0 \le i \le \ell - 1 \quad X_1 \cdots X_i u \;&=\; X_1 \cdots X_i \, \psi \, x^{\mu_+} + G_{x_0}(X_1 \cdots X_i (f - \tilde{L} u)) \\
&=\; O(x^\sigma (1 + |\ln x|)^\rho) \,,
\end{aligned} \tag{4.2.16}$$

$$\Big(X_1 \cdots X_i u\Big)(x, v) - \Big(X_1 \cdots X_i u\Big)(x, v') = O(x^\sigma (1 + |\ln x|)^\rho |v - v'|^{\mu'}) \,, \tag{4.2.17}$$

this last inequality being uniform in $|v - v'|$. The equations

$$L X_1 \cdots X_i u = X_1 \cdots X_i f + [L, X_1 \cdots X_i] \, u \tag{4.2.18}$$

interior elliptic estimates and a scaling argument yield $X_1 \cdots X_i u \in C^{\sigma,\rho}_{k+2-i+\lambda}$, $i \le \ell - 1$ and thus $u \in C^{\sigma,\rho}_{k+2+\lambda, \ell-1}$. If (α_-, α_+) is a strong regularity interval, then Lemma 4.1.4 and eq. (4.2.17) show that $u \in C^{\sigma,\rho}_{k+2+\lambda, \ell-1+\mu'}$, for any $\mu' \in [0, \mu)$.

One can now repeat the whole argument $\ell_0 - 1$ times, where ℓ_0 is the smallest integer such that $\alpha_+ + \ell_0 > \min(\alpha, \mu_+)$ (note that at each iteration one loses one degree of tangential differentiability of u) to obtain $u \in C^{\tilde\sigma,\rho}_{k+\lambda,\ell-\ell_0+\mu'}$, with $\tilde\sigma = \min(\alpha,\mu_+)$, provided $\ell \geq \ell_0$. Finally if $\alpha > \mu_+$ and $u = o(x^{\mu_+})$ then eq. (4.2.15) holds with G_{x_0} replaced by G_0 and with $\psi = 0$. Repeating the argument $\ell_1 - \ell_0$ times more, where ℓ_1 is the smallest integer such that $\alpha_+ + \ell_1 > \alpha$, one obtains $u \in C^{\alpha,\beta}_{k+2+\lambda,\ell-\ell_1+\mu'}$.

To obtain the claimed decay of the $\ell - i_0 + 1$ transverse derivatives in the case of a strong regularity interval, where $i_0 = \ell_0$ or $i_0 = \ell_1$, note that at the last step of the iteration above we shall have, for any $\epsilon > 0$ and $\mu' \in (0,\mu)$,

$$u \in C^{\min(\alpha,\mu_+),\beta+1}_{k+2+\lambda,\ell-i_0+\mu'}(M_{x_0}) \cap C^{\alpha_++i_0-1-\epsilon}_{k+2+\lambda,\ell-i_0+1+\mu'}(M_{x_0}) \,. \tag{4.2.19}$$

We have the following:

LEMMA 4.2.2. Under the hypotheses of Theorem 4.2.1, suppose that

$$u \in C^{\sigma_1,\rho}_{k+2+\lambda,\ell+\mu'}(M_{x_0}) \cap C^{\sigma_0,\rho}_{k+2+\lambda,\ell-1+\mu'}(M_{x_0})\,, \quad \sigma_1 < \sigma_0\,, \quad 0 < \mu' < \mu\,,$$

with

$$0 < x \leq x_0 \qquad x^\alpha(1 + |\ln x|)^\beta \leq C x^{\sigma_0}(1 + |\ln x|)^\rho\,.$$

Then for all $0 < \theta < 1$, $0 \leq t < \ell + \mu' - \theta$ we have

$$u \in C^{\theta\sigma_0 + (1-\theta)\sigma_1,\rho}_{k+2+\lambda,t}(M_{x_0})\,.$$

PROOF: Set $u_x(v) = u(x,v)$, and for $0 \leq t_0 < t_1$ define $t = \theta t_0 + (1-\theta)t_1$. We have the interpolation inequality (*cf. e.g.* [**44**] [Appendix A] or [**69**, p. 236])

$$\|u_x\|_{C_t(\partial M)} \leq C\|u_x\|^\theta_{C_{t_0}(\partial M)}\|u_x\|^{1-\theta}_{C_{t_1}(\partial M)}\,.$$

Setting $t_0 = \ell - 1 + \mu'$, $t_1 = \ell + \mu'$ we obtain

$$u \in C^{\theta\sigma_0 + (1-\theta)\sigma_1,\rho}_{0,\ell+\mu'-\theta}(M_{x_0})\,,$$

and the result follows from Lemma 4.1.4. $\square$

Returning to the proof of Theorem 4.2.1, let θ be any number smaller than μ' and let $2\sigma = \mu' - \theta > 0$. Eq. (4.2.19) and Lemma 4.2.2 give

$$u \in C^{\alpha_++i_0-1+\delta}_{k+2+\lambda,\ell-i_0+1+\sigma}(M_{x_0})\,,$$

with

$$\delta = \theta(\alpha - i_0 + 1 - \alpha_+ + \mu') - (1-\theta)\epsilon\,.$$

As ϵ can be chosen arbitrarily small we have $\delta > 0$, and the result follows. $\square$

In Proposition 4.1.2 we have proved that if L is an isomorphism in weighted Hölder spaces "close to the boundary ∂M" for an interval of weights (α_-,α_+), then (α_-,α_+) is a regularity interval for weighted Hölder spaces. The following result shows, that an isomorphism property "close to the boundary ∂M" in weighted Sobolev spaces for an interval (ω_-,ω_+) implies that $\Big(\max(\omega_-,\mu_-) +$

$(n-1)/p, \min(\omega_+ + (n-1)/p, \mu_+)\Big)$ is a regularity interval for weighted Hölder spaces with tangential regularity:

THEOREM 4.2.3. Let $p \in (1,\infty)$, and let $L \in OP^2_{C^0_{k+\lambda,\ell+\mu}(M)}$ be a geometric elliptic operator, $(n-1)/p < \ell \le k$, $\lambda \in (0,1)$, $\mu \in [0,1]$. Suppose that (4.2.1)–(4.2.4) hold with ψ, ϕ, $\chi \in C^0_{k-1+\lambda,\ell-1+\mu}(M)$. Assume that for all $\omega \in (\omega_-,\omega_+)$ there exists $0 < x_1 \le x_0$ (possibly depending upon ω) such that

$$L : \mathring{W}^{\omega;p}_1 \cap W^{\omega;p}_2(M_{x_1}, g, x^{-n}d\mu_g) \to W^{\omega;p}_0(M_{x_1}, x^{-n}d\mu_g)$$

is an isomorphism, with $\mu_- \le \omega_- < \min\left(\omega_+, \mu_+ - \frac{n-1}{p}\right)$.

(i) Let $u \in W^{p,\mathrm{loc}}_2(M)$ satisfy

$$Lu = f\,, \qquad f \in C^{\alpha,\beta}_{k+\lambda,\ell+\mu}(M)\,, \tag{4.2.20}$$

with $\alpha \in (\alpha_-,\alpha_+)$, where

$$(\alpha_-,\alpha_+) = \left(\omega_- + \frac{n-1}{p}, \min\left(\mu_+, \omega_+ + \frac{n-1}{p}\right)\right)\,, \tag{4.2.21}$$

suppose that either

$$u = O(x^{\alpha_- + \epsilon})\,,\ \epsilon > 0\ , \tag{4.2.22}$$

or

$$u \in W^{\omega;p}_0(M_{x_1}, x^{-n}d\mu_g)\,, \quad \omega > \omega_-\,. \tag{4.2.23}$$

Then for all $\mu' \in [0,\mu)$ if $\mu > 0$, $\mu' = 0$ otherwise we have

$$u \in C^{\alpha,\beta}_{k+2+\lambda,\ell_1+\mu'}(M_{x_1})\,, \tag{4.2.24}$$

where $\ell_1 \in \mathbb{N}_0$ is the largest integer such that $\ell_1 < \ell - (n-1)/p$. Moreover there exists a (u and f independent) constant C such that

$$\|u\|_{C^{\alpha,\beta}_{k+2+\lambda,\ell_1+\mu}(M_{x_1})} \le C(\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell+\mu}(M)} + \|u\|_{C_0(\overline{CM_{x_1/2}})})\ . \tag{4.2.25}$$

(ii) For any $u_0 \in C_{k+\lambda}(\partial M)$ there exists a solution of (4.2.20) satisfying (4.2.24) such that

$$u(x_1,v) = u_0(v)\,. \tag{4.2.26}$$

u is unique in the class of solutions satisfying either (4.2.22) or (4.2.23).

REMARKS:

(i) The results proved below are unsatisfactory in two respects: 1) it would be more natural to assume $\mu_- - (n-1)/p \le \omega_-$ rather then $\mu_- \le \omega_-$; 2) u is "less tangentially regular" then f.

(ii) Let us note that under the hypotheses of Theorem 4.2.3 we must have $\mu_+ - \frac{n-1}{p} \notin (\omega_-, \omega_+)$: suppose to the contrary that

$$L : \mathring{W}_1^{\omega;p} \cap W_2^{\omega;p}(M_{x_1}, g, d\mu_g) \to W_0^{\omega;p}(M_{x_1}, d\mu_g)$$

is an isomorphism for $\omega \in (\omega_-, \omega_+)$ and that $\mu_+ - \frac{n-1}{p} \in (\omega_-, \omega_+)$. Set $\rho = L(\varphi x^{\mu_+})$, where $\varphi(x)$ is a $C_\infty(\mathbb{R})$ function satisfying $\varphi = 1$ for $0 \le x \le x_1/2$, $\varphi = 0$ for $x \ge 3x_1/4$. If L is of the form (4.2.1)–(4.2.4), then for any $0 \le \epsilon < 1$ we have $\rho \in W_{k,\ell}^{\mu_+ - \frac{n-1}{p} + \epsilon;p}$, and if $\mu_+ - \frac{n-1}{p} \in (\omega_-, \omega_+)$ we can choose ϵ so that $\omega_\epsilon = \mu_+ - \frac{n-1}{p} + \epsilon \in (\omega_-, \omega_+)$, therefore there exists $u_\epsilon \in \mathring{W}_1^{\omega;p} \cap W_2^{\omega_\epsilon;p}$ such that $Lu_\epsilon = \rho\big|_{M_{x_1}}$. Now for any $\epsilon > 0$, $\varphi x^{\mu_+} \notin W_2^{\mu_+ - \frac{n-1}{p} + \epsilon;p}$, therefore $u_\epsilon \ne \varphi x^{\mu_+}$. From $\varphi x^{\mu_+} \in \mathring{W}_1^{\mu_+ - \frac{n-1}{p} - \delta;p} \cap W_2^{\mu_+ - \frac{n-1}{p} - \delta;p}$ for all $\delta > 0$ it follows that $0 \ne u_\epsilon - \varphi x^{\mu_+} \in \mathring{W}_1^{\mu_+ - \frac{n-1}{p} - \delta;p} \cap W_2^{\mu_+ - \frac{n-1}{p} - \delta;p}$, but $L(u_\epsilon - \varphi x^{\mu_+}) = 0$, which contradicts injectivity of L on $\mathring{W}_1^{\mu_+ - \frac{n-1}{p} - \delta;p} \cap W_2^{\mu_+ - \frac{n-1}{p} - \delta;p}$, when δ is chosen small enough so that $\mu_+ - \frac{n-1}{p} - \delta \in (\omega_-, \omega_+)$. Similarly one can show that if *e.g.* $L \in OP^2_{C_{k+\lambda}(\bar{M})}$ with k large enough, then $\mu_- - \frac{n-1}{p}$ cannot be larger than ω_-.

PROOF: (i): It follows from Lemma 4.1.1 that for $\omega \in (\omega_-, \omega_+)$ the map $L : \mathring{W}_1^{\omega;p} \cap W_{k+2}^{\omega;p} \to W_k^{\omega;p}$ is an isomorphism, and Proposition 4.1.2 shows that (ω_-, ω_+) is a strong regularity interval. Let $\varphi \in C_\infty(M)$ be any function satisfying $\varphi\big|_{M_{x_1/2}} = 1$, $\varphi\big|_{CM_{x_1}} = 0$, replacing u by φu if necessary we may without loss of generality assume $u(x_1, v) = 0$. It is convenient to structure the proof in several steps:

Step 1: Uniqueness of solutions in weighted Hölder spaces: Let

$$Lu = 0\,, \quad u \in C_2^{\alpha_1,\beta_1}(M_{x_1})\,, \quad \alpha_1 > \omega_- + (n-1)/p\,, \quad u(x_1, v) = 0\ .$$

For all $\omega < \alpha_1 - (n-1)/p$ we have $u \in \mathring{W}_1^{\omega;p} \cap W_2^{\omega;p}$, and if $\omega_- < \alpha_1 - (n-1)/p$ we can choose ω so that $\omega \in (\omega_-, \omega_+)$ — $u = 0$ follows from injectivity of L as an operator from $\mathring{W}_1^{\omega;p} \cap W_2^{\omega;p}$ to $W_0^{\omega;p}$.

Step 2: Existence of a solution in a weighted Sobolev space with some decay exponent: since $f \in C_{k+\lambda}^{\alpha,\beta}$, for all $\omega < \alpha - (n-1)/p$ we have $f \in W_k^{\omega;p}$, thus if $\omega_- + (n-1)/p < \alpha$ we can choose $\omega \in (\omega_-, \omega_+)$. Since L as an operator from $\mathring{W}_1^{\omega;p} \cap W_{k+2}^{\omega;p}$ to $W_k^{\omega;p}$ is surjective, there exists $u \in \mathring{W}_1^{\omega;p} \cap W_{k+2}^{\omega;p}$ which solves (4.2.20).

Step 3: Let $u \in \mathring{W}_1^{\omega;p} \cap W_2^{p,\mathrm{loc}}$, $\omega_- < \omega < \omega_+$, satisfy (4.2.20). By the scaling estimates of Lemma 4.1.1 we have $u \in W_{k+2}^{\omega;p}$, and by point (ii) of Theorem 4.1.3 u is in $W_{k+2,\ell}^{\omega;p}$. By Step 2 for any $\omega_- < \omega' < \alpha - (n-1)/p$ there exists $u' \in \mathring{W}_1^{\omega';p} \cap W_{k+2}^{\omega';p}$ which satisfies (4.2.20), uniqueness of solutions in $\mathring{W}_1^{\omega';p} \cap W_{k+2}^{\omega';p}$

implies $u' = u$, and therefore for any $\delta > 0$ we can choose ω such that we have $\alpha - (n-1)/p - \delta < \omega$, and $\omega \neq \alpha$. Similarly we may assume that the exponent ω in (4.2.23) satisfies $\omega > \alpha - (n-1)/p - \delta$, $\omega \neq \alpha$. For $y = (x, v) \in M_{x_1/2}$ consider the function

$$\tilde{u}(\tilde{y}) = x^{-\omega}\, u\Big(y + \frac{x}{2}\,\tilde{y}\Big) \ , \quad \tilde{y} \in \mathcal{B}^n_0(1) \ ,$$

where $\mathcal{B}^n_0(\sigma)$ is an open ball in $\mathbb{R}^n$ of radius σ centred at the origin. From the interior $C_{k+\lambda}$ and L^p elliptic estimates combined with Sobolev's embedding it is easily shown that there exists a constant C such that

$$(4.2.27) \quad \|\tilde{u}\|_{C_{k+2+\lambda}(\overline{\mathcal{B}^n_0(1/2)})} \leq C\{\|\widetilde{Lu}\|_{C_{k+\lambda}(\overline{\mathcal{B}^n_0(1)})} + \|\tilde{u}\|_{L^p(\mathcal{B}^n_0(1), d\tilde{\mu})}\} \ ,$$

$$\widetilde{Lu}(\tilde{y}) = x^{-\omega}\, Lu\Big(y + \frac{x}{2}\,\tilde{y}\Big) \ , \qquad d\tilde{\mu}(\tilde{y}) = \sqrt{\det g_{ij}(y + \frac{x}{2}\,\tilde{y})}\ d^n\tilde{y} \ .$$

From the straightforward inequality

$$\|\tilde{u}\|_{L^p(\mathcal{B}^n_0(1), d\tilde{\mu})} \leq C\|u\|_{W^{\omega;p}_0(M_{x_1}, x^{-n}d\mu_g)}$$

for some constant C, combined with (4.2.27), it follows that $u \in C^{\omega}_{k+2+\lambda}$.

Step 4: Let us show by induction that:

(i) $u \in C^{\omega}_{k+2+\lambda,\ell}(M_{x_1})$.

(ii) Let ℓ_* be an integer in $[1, \ell]$. If $u \in C^*_{k+2+\lambda}(M_{x_1})$ and if $\partial^\gamma_v u = O(\text{“}x^*\text{”})$, $0 \leq |\gamma| \leq \ell_*$ (*cf.* Step 6), then $u \in C^*_{k+2+\lambda,\ell_*}(M_{x_1})$. Here $*$ is either equal to some number $\hat{\omega}$ (and then “x^*”$= x^{\hat{\omega}}$, or $* = \alpha, \beta$ (and then “x^*”$= x^\alpha(1 + |\ln x|)^\beta$). We also assume that $x^* \geq x^\alpha(1 + |\ln x|)^\beta$.

Indeed, suppose that $u \in C^*_{k+2+\lambda,\ell_0}$. In case (i) this holds for $* = \omega$ and $\ell_0 = 0$ by Step 3; in case (ii) this holds by the inductive assumption. Let $X_1, \ldots, X_i$ be any smooth vector fields of the form $X^A(v)\frac{\partial}{\partial v^A}$, $X^A(v) \in C_\infty(\partial M)$. Let $u_i = X_1 \cdots X_i u$, $f_i = X_1 \cdots X_i f$. In case (i) for $\ell_0 + 1 \leq \ell$ and in case (ii) for $\ell_0 + 1 \leq \ell_*$ we have

$$Lu_{\ell_0+1} = f_{\ell_0+1} + [L, X_1 \cdots X_{\ell_0+1}]u \ \in \ C^{\alpha,\beta}_{k-\ell_0-1+\lambda} + C^*_{k+1-\ell_0+\lambda} \ .$$

Note that in case (i) we have $u \in W^{\omega;p}_{k+2,\ell}$ (*cf.* Step 3), so that for $\ell_0 + 1 \leq \ell$ we have $u_{\ell_0+1} \in W^{\omega;p}_{k-\ell_0-1}$; setting $\tilde{u}_{\ell_0+1}(\tilde{y}) \equiv$“$x^{-*}$” $u_{\ell_0+1}\Big(y + \frac{x}{2}\,\tilde{y}\Big)$, $\tilde{y} \in \mathcal{B}^n_0(1)$ we obtain

$$\|\tilde{u}_{\ell_0+1}\|_{L^p(\mathcal{B}^n_0(1), d\tilde{\mu})} \leq C\|u_{\ell_0+1}\|_{W^{\omega;p}_0(M_{x_1}, x^{-n}d\mu_g)} \ .$$

Applying now the estimate (4.2.27) to the function $\tilde{u}_{\ell_0+1}$ ends the induction.

Step 5: Suppose that $u = O(x^{\alpha_- + \epsilon})$. By the scaling estimates of Lemma 4.1.1 we have $u \in C^{\alpha_- + \epsilon}_{k+2+\lambda}$, by existence of a solution as considered in step 4 and by uniqueness of solutions in weighted Hölder spaces, step 1, it follows that decreasing ϵ if necessary we can assume $\omega = \alpha_- + \epsilon > \omega_- \geq \mu_-$, $\omega \neq \alpha$.

Step 6: The functions u_i defined in step 4 satisfy the ODE's

$$L_{ab}\, u_i = f_i - X_1 \cdots X_i\, \tilde{L}u \ ,$$

and since $u_i = O(x^\omega)$, $\omega > \mu_-$, we have

$$u_i = \psi_i\, x^{\mu_+} + G(f_i) - G(X_1 \cdots X_i\, \tilde{L} u)\,, \tag{4.2.28}$$

$G \equiv G_{x_1}$, for some functions $\psi_i(v)$. Since $u \in C^\omega_{k+2+\lambda,\ell}(M_{x_1}) \subset C^{\mathrm{loc}}_\ell(M_{x_1})$, $X_1 \cdots X_i\, \tilde{L} u \in C^{\mathrm{loc}}_{\ell-i}(M_{x_1})$ (*cf.* (3.2.4)–(3.2.6)), $G(X_1 \cdots X_i\, \tilde{L} u) \in C^{\mathrm{loc}}_{\ell-i}(M_{x_1})$, (*cf.* (3.4.2)–(3.4.4)), $G(f_i) \in C^{\mathrm{loc}}_{\ell-i}(M_{x_1})$, setting $x = x_1/2$ in (4.2.28) we obtain $\psi_i \in C_{\ell-i}(\partial M)$, in particular ψ_i is bounded. We thus have

$$\begin{array}{rcl} \psi_i\, x^{\mu_+} & = & O(x^{\mu_+})\ , \\ G(f_i) & = & O(x^\alpha (1 + |\ln x|)^\beta)\ , \\ 0 \le i \le \ell - 1 & & G(X_1 \cdots X_i \tilde{L} u) = O(x^{\omega+1})\ , \end{array}$$

which gives for $0 \le i \le \ell - 1$

$$\text{a)} \qquad \omega + 1 > \alpha \Longrightarrow u_i = O(x^\alpha (1 + |\ln x|)^\beta) \tag{4.2.29}$$

$$\text{b)} \qquad \omega + 1 < \alpha \Longrightarrow u_i = O(x^{\omega+1}) \tag{4.2.30}$$

From (4.2.27) we obtain $u \in C^*_{k+2+\lambda}$, where $* = \omega + 1$ or $* = \alpha, \beta$.

Step 7 (and final): We can go through step 4 again to conclude either $u \in C^{\alpha,\beta}_{k+2+\lambda,\ell-1}$ in case a), or $u \in C^{\omega+1}_{k+2+\lambda,\ell-1}$ in case b). After going through steps 6 and 4 at most k_1 times, where k_1 is the smallest integer larger than $\frac{n-1}{p}$, one obtains $u \in C^{\alpha,\beta}_{k+2+\lambda,\ell-k_1}$; if $\mu > 0$ the above considerations applied to difference quotients together with Lemma 4.1.4 give $u \in C^{\alpha,\beta}_{k+2+\lambda,\ell-k_1+\mu'}$ for all $\mu' \in [0,\mu)$. The inequality (4.2.25) follows by noting that every step above "comes equipped" with an appropriate inequality, the details are left to the reader.

(ii): Existence of some solution follows from Step 2 above in a standard way; uniqueness follows from Steps 1 and 5; regularity follows from point (i) of the Theorem. □

Combining the results of Theorem 4.2.3 with the arguments of Theorem 4.2.1 we obtain:

THEOREM 4.2.4. Let $p \in (1,\infty)$, and let $\hat{\ell}$ be the smallest integer larger than $(n-1)/p$. Let $L \in OP^2_{C^0_{k+\lambda,\ell+\hat{\ell}+\mu}(M)}$ be a geometric elliptic operator and suppose that $\hat{\ell} + \ell \le k$, $\lambda \in (0,1)$, $\mu \in (0,1]$, $\ell \ge 1$. Suppose moreover that (4.2.1)–(4.2.4) hold with ψ, ϕ, $\chi \in C^0_{k-1+\lambda,\ell+\hat{\ell}-1+\mu}(M)$, and that for all $\omega \in (\omega_-,\omega_+)$ there exists $0 < x_1 \le x_0$ (possibly depending upon ω) such that

$$L : \mathring{W}^{\omega;p}_1 \cap W^{\omega;p}_2(M_{x_1}, g, x^{-n} d\mu_g) \to W^{\omega;p}_0(M_{x_1}, x^{-n} d\mu_g)$$

is an isomorphism, with $\mu_- \le \omega_- < \min\left(\omega_+, \mu_+ - \frac{n-1}{p}\right)$. Let $u \in W^{p,\mathrm{loc}}_2(M)$ satisfy

$$Lu = f\,, \qquad f \in C^{\alpha,\beta}_{k+\lambda,\hat{\ell}+\ell+\mu}(M)\,, \tag{4.2.31}$$

with $\alpha \geq \alpha_+$, where

$$(\alpha_-, \alpha_+) = \left(\omega_- + \frac{n-1}{p}, \min\left(\mu_+, \omega_+ + \frac{n-1}{p}\right)\right) . \tag{4.2.32}$$

Suppose finally that either

$$u = O(x^{\alpha_- + \epsilon}) , \ \epsilon > 0 , \tag{4.2.33}$$

or

$$u \in W_0^{\omega;p}(M_{x_1}, x^{-n} d\mu_g) , \quad \omega > \omega_- . \tag{4.2.34}$$

If $\ell \geq \ell_0$, where ℓ_0 is the smallest integer such that $\alpha_+ + \ell_0 > \min(\alpha, \mu_+)$, then there exist $\sigma \in (0,1)$ and $\delta > 0$ such that for $\mu' \in [0, \mu)$ eqs. (4.2.9)–(4.2.12) hold. The constants σ and δ depend only upon ω_+, μ_+, α, μ, λ, ℓ, $(n-1)/p$ and k. Moreover the *a–priori* estimate (4.2.13) holds with a (u and f independent) constant C provided that the norm $\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell+\mu}(M)}$ there is replaced by $\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell+\hat{\ell}+\mu}(M)}$.

4.2.2. Classical regularity at the boundary. In this section we shall establish boundary regularity of solutions of the problem

$$Lu = f, \qquad f \in x^\alpha C^{k+\lambda}(\bar{M}) . \tag{4.2.35}$$

The following Lemma reduces this problem to that of regularity of solutions of

$$Lu = f, \qquad f \in C^{k+\alpha,\beta}_{k+\lambda,0+\lambda}(M) \cap C^{k+\alpha+\lambda,\beta}_{k+\lambda}(M), \qquad \beta = 0 \quad \text{or} \quad \beta = 1 .$$

LEMMA 4.2.5. Let $k \in \mathbb{N}_0$, $\lambda \in [0,1]$, and let an operator $L \in OP^2_{C^{k+\lambda}(\bar{M})}$ satisfy (4.2.1)–(4.2.4). Suppose that

$$f \in x^\alpha C^{k+\lambda}(\bar{M}), \qquad \alpha > \mu_- .$$

(i) Suppose moreover that $\alpha + k < \mu_+$ or that $\mu_+ - \alpha \notin \mathbb{N}_0$. There exists a function $\hat{u} \in C^\alpha_\infty(M) \cap_{i \in \mathbb{N}_0} x^{\alpha - i} C^{k+i+\lambda}(\bar{M})$ such that

$$L\hat{u} - f \in C^{k+\alpha}_{k+\lambda,0+\lambda}(M) \cap C^{k+\alpha+\lambda}_{k+\lambda}(M) . \tag{4.2.36}$$

(ii) Suppose alternatively that $\alpha + k > \mu_+$ and $\mu_+ - \alpha \in \mathbb{N}_0$, set $k_1 = \mu_+ - \alpha$. There exist functions $\hat{u} \in C^\alpha_\infty(M) \cap_{i \in \mathbb{N}_0} x^{\alpha - i} C^{k+i+\lambda}(\bar{M})$ and $\hat{u}_{\log} \in C^{\mu_+}_\infty(M) \cap_{i \in \mathbb{N}_0} x^{\mu_+ - i} C^{k - k_1 + i + \lambda}(\bar{M})$ such that

$$L(\hat{u} + \hat{u}_{\log} \log x) - f \in C^{k+\alpha,1}_{k+\lambda,0+\lambda}(M) \cap C^{k+\alpha+\lambda,1}_{k+\lambda}(M) . \tag{4.2.37}$$

PROOF: We shall prove point (i) assuming $\lambda > 0$; point (i) with $\lambda = 0$ and point (ii) are proved by a similar argument. Suppose that for some ℓ satisfying $-1 \leq \ell \leq k-1$ we have found a function $u_\ell \in C^\alpha_\infty$ such that $x^{-\alpha}(Lu_\ell - f)$ is in $C_{k+\lambda}(\bar{M})$ and has vanishing Taylor expansion at ∂M up to order ℓ (*cf.* Proposition 3.1.1); the result is true for $\ell = -1$ if one sets $u_{-1} \equiv 0$. Let $x^{-\alpha} w_\ell$ be obtained from Lemma 3.3.1 with $m = \ell + 1$ as an extension of a function

$\psi_\ell \in C_{k-\ell-1+\lambda}(\partial M)$ to be specified below. By eqs. (3.3.1)–(3.3.2) and (4.2.1)–(4.2.3) we have

$$Lw_\ell - L_{ab}w_\ell \in C_0^{\alpha+\ell+1+\lambda}(M)$$

(note that the second term in eq. (3.3.2) drops out when at least one of the derivatives there is a v–derivative). By hypothesis it holds that $\mu_+ - \ell - 1 \neq \alpha$ and a straightforward calculation (*cf. e.g.* the proof of Theorem 6.2.8) shows that we can choose ψ_ℓ so that the function $x^{-\alpha}Lu_{\ell+1} \equiv x^{-\alpha}L(u_\ell + w_\ell)$ is in $C_{k+\lambda}(\bar{M})$ and vanishes to order $\ell+1$ at the boundary. The remaining claims follow by properties of the extension operator E of Lemma 3.3.1 (*cf.* eq. (3.3.4)), using arguments similar to those of the proof of Corollary 3.3.2. □

We are ready now to prove the following:

THEOREM 4.2.6 (CLASSICAL BOUNDARY REGULARITY, $\alpha \in (\alpha_-, \alpha_+)$). Let (α_-, α_+) be a strong regularity interval for $C^\alpha_{0+\lambda}(M)$ for a geometric elliptic operator $L \in OP^2_{C_{k+\lambda}(\bar{M})}$, $\lambda \in (0,1)$, suppose that L is of the form (4.2.1)–(4.2.4), and that

$$Lu = f \in x^\alpha C_{k+\lambda}(\bar{M}), \qquad \alpha \in (\alpha_-, \alpha_+)\,, \qquad u = O(x^{\alpha_-+\epsilon})\,, \quad \epsilon > 0\,.$$

Let $\ell_0 \in \mathbb{N}_0$ be the largest integer such that $\alpha + \ell_0 < \alpha_+$. Then there exists $\sigma > 0$ such that

$$\alpha + k < \alpha_+ \quad \Longrightarrow \quad u \in \cap_{i=0}^2 x^{\alpha-i} C_{k+i+\sigma}(\bar{M}_{x_0})\,, \tag{4.2.38}$$

$$k > \ell_0 \quad \Longrightarrow \quad u \in \cap_{i=0}^2 x^{\alpha-i} C_{\ell_0+i+\sigma|k-\ell_0}(\bar{M}_{x_0})\,. \tag{4.2.39}$$

The constant σ depends only upon α_+, μ_+, α, λ and k. Moreover when F denotes any of the spaces appearing in eqs. (4.2.38)–(4.2.39), then the *a–priori* estimate (4.2.13) holds with a (u and f independent) constant C provided that the norm $\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell+\mu}(M)}$ there is replaced by $\|f\|_{x^\alpha C_{k+\lambda}(\bar{M})}$.

REMARKS: 1. The above results are sharp when $\alpha_+ = \mu_+$, except perhaps for the value of the modulus of Hölder continuity which we leave unspecified.

2. For $\alpha_+ < \mu_+$ define $\hat{\ell}_0 \in \mathbb{N}_0$ to be the largest integer such that $\alpha + \hat{\ell}_0 < \mu_+$. The reader will notice that for $\alpha \le \alpha_+$ the arguments of the proof of Theorem 4.2.9 can be used to show that the right–hand–side of the implication (4.2.38) holds for all $k \le \hat{\ell}_0$; similarly for $k > \hat{\ell}_0$ the right–hand–side of the implication (4.2.39) will hold with ℓ_0 replaced by $\hat{\ell}_0$. This is again sharp except for the value of the modulus of Hölder continuity of u.

3. Under the conditions of (4.2.39) in the special case $\mu_+ - \alpha \in \mathbb{N}_0$ one can obtain some more information about u. This, however, requires different techniques and will be considered separately in Theorem 4.2.10 below.

PROOF: Suppose first that $\alpha + k < \alpha_+$, let $\hat{u}$ be given by Lemma 4.2.5, by point (i) of that Lemma and by interpolation (*cf.* the arguments of proof of Lemma 4.2.2) there exists $0 < 2\sigma < \lambda$ such that

$$L(u - \hat{u}) \in C^{\alpha+k+2\sigma}_{k+2\sigma,0+2\sigma}(M)\,.$$

By Theorem 4.1.3 point (i) we have

$$u-\hat{u}\in C^{\alpha+k+2\sigma}_{k+2+2\sigma,0+\sigma}(M)\subset\cap_{i=0}^{2}x^{\alpha-i}C_{k+i+\sigma}(\bar{M}_{x_0})$$

(this last inclusion following from Proposition 3.2.2), and (4.2.38) follows. Equation (4.2.39) follows from the inclusions $C^{\alpha+k+2\sigma}_{k+2\sigma,0+2\sigma}(M)\subset C^{\alpha+\ell_0+2\sigma}_{k+2\sigma,k-\ell_0+2\sigma}(M)$ and $C^{\alpha+\ell_0+2\sigma}_{k+2+2\sigma,k-\ell_0+\sigma}(M)\subset\cap_{i=0}^{2}x^{\alpha-i}C_{\ell_0+i+\sigma|k-\ell_0}(\bar{M}_{x_0})$ by the same argument. □

To obtain classical regularity at the boundary for α's beyond the threshold α_+ some more work is required. The next Lemma will allow us to reduce the problem to an analysis of those solutions, the decay rate of which is faster than the one corresponding to the larger critical exponent. The proof proceeds along the lines of proof of Lemma 4.2.5.

LEMMA 4.2.7. Under the hypotheses of Lemma 4.2.5 let $\psi\in C^{k+\lambda}(\partial M)$. There exists $\hat{\psi}\in\cap_{i\in\mathbb{N}_0}x^{\mu_+-i}\,C^{k+i+\lambda}(\bar{M})$ such that

$$x^{-\mu_+}\hat{\psi}\Big|_{\partial M}=\psi\ ,$$
$$L\hat{\psi}\in C^{k+\mu_+}_{k+\lambda,0+\lambda}(M)\cap C^{k+\mu_++\lambda}_{k+\lambda}(M)\ .$$

Arguing as in the proof of Theorem 4.2.1 one obtains the following:

LEMMA 4.2.8. Let $L\in OP^2_{C^0_{k+\lambda,k+\lambda}(M)}$, $\lambda\in(0,1)$, be a geometric elliptic operator with non–empty strong regularity interval (α_-,α_+) for $C^{\alpha}_{0+\lambda}(M)$. Suppose that L is of the form (4.2.1)–(4.2.4) with ψ, ϕ, $\chi\in C^0_{k-1+\lambda,k-1+\lambda}(M)$. Let m, $\ell\in\mathbb{N}_0$ and assume that $\beta\geq\mu_-$ and $m\leq\ell\leq k$. If $\beta+m\geq\mu_+$ assume further that $u=o(x^{\mu_+})$. If

$$Lu=f\ ,\quad f\in C^{\beta+m+\lambda}_{k+\lambda,0+\lambda}(M)\ ,\quad u\in C^{\beta+\lambda}_{k+2+\lambda,\ell+\lambda}(M)\ ,$$

then there exists $\sigma>0$ such that

$$u\in C^{\beta+m+\sigma}_{k+2+\sigma,\ell-m+\sigma}(M_{x_0})\ .$$

We are ready now to prove classical regularity at the boundary above the α_+ treshhold in the presence of a regularity interval, or of a strong regularity interval. To avoid a somewhat tedious and not very enlightning discussion of various cases we shall only give the proofs when $\alpha_+=\mu_+$. It should be clear from the proofs below how to generalize the argument to the case $\alpha_+<\mu_+$, *cf.* also Remark 1 below.

THEOREM 4.2.9 (CLASSICAL BOUNDARY REGULARITY, $\alpha>\alpha_+$). Let (α_-,α_+) be a strong regularity interval for $C^{\alpha}_{0+\lambda}(M)$ for a geometric elliptic operator $L\in OP^2_{C_{k+\lambda}(\bar{M})}$, $\lambda\in(0,1)$, $k\geq1$, suppose that L is of the form (4.2.1)–(4.2.4), and that

$$Lu=f\in x^{\alpha}C_{k+\lambda}(\bar{M}),\qquad\alpha>\alpha_+=\mu_+\ ,\qquad u=O(x^{\alpha_-+\epsilon})\ ,\quad\epsilon>0\ .$$

Then there exists $\sigma > 0$ such that

$$u \in \cap_{i=0}^{2} x^{\mu_+ - i} C_{k+i+\sigma}(\bar{M}_{x_0}) + \cap_{i\in\mathbb{N}_0} x^{\alpha-i} C_{k+i+\lambda}(\bar{M}_{x_0}) . \tag{4.2.40}$$

In particular, if we denote by $\ell_0 \in \mathbb{N}_0$ the largest integer such that $\mu_+ + \ell_0 < \alpha$, then we have

(i) If $\mu_+ + k < \alpha$ or if $\alpha - \mu_+ \in \mathbb{N}$ (or both)

$$u \in \cap_{i=0}^{2} x^{\mu_+ - i} C_{k+i+\sigma}(\bar{M}_{x_0}) , \tag{4.2.41}$$

(ii) $k > \ell_0$

$$u \in \cap_{i=0}^{2} x^{\mu_+ - i} C_{\ell_0+i+\sigma|k-\ell_0}(\bar{M}_{x_0}) . \tag{4.2.42}$$

If moreover $u = o(x^{\mu_+})$ then it also holds that

$$u \in \cap_{i=-k}^{2} x^{\mu_+ - i} C_{k+i+\sigma}(\bar{M}_{x_0}) + \cap_{i\in\mathbb{N}_0} x^{\alpha-i} C_{k+i+\lambda}(\bar{M}_{x_0}) , \tag{4.2.43}$$

and for $k \geq \tilde{\ell}$, where $\tilde{\ell}$ is the smallest integer such that $\tilde{\ell} \geq \alpha - \mu_+$, we also have

$$u \in \cap_{i=0}^{\tilde{\ell}+2} x^{\alpha-i} C_{k-\tilde{\ell}+i+\sigma}(\bar{M}_{x_0}) . \tag{4.2.44}$$

The constant σ depends only upon α_+, μ_+, α, λ and k. Moreover when F denotes any of the spaces appearing in eqs. (4.2.41)–(4.2.42) or (4.2.44), then the *a–priori* estimate (4.2.13) holds with a (u and f independent) constant C provided that the norm $\|f\|_{C^{\alpha,\beta}_{k+\lambda,\ell+\mu}(M)}$ there is replaced by $\|f\|_{x^\alpha C_{k+\lambda}(\bar{M})}$. Similarly if F_1, F_2 denote any of the summand spaces in eqs. (4.2.40) and (4.2.43), we have the *a priori* estimate

$$\|u\|_{F_1\oplus F_2} \leq C\left(\|f\|_{C_{x^\alpha C_{k+\lambda}(\bar{M})}} + \|u\|_{C_0^{\alpha_-+\epsilon}(M)}\right) . \tag{4.2.45}$$

Remarks: 1. If $\alpha_+ < \mu_+$ one can still obtain some classical regularity of the solutions, the results being somewhat worse than in the case $\alpha_+ = \mu_+$. More precisely, we have the following: If $\alpha_+ < \alpha < \mu_+$ and if $k \geq \ell_1$, where ℓ_1 is the smallest integer larger than $\alpha - \alpha_+$, then the implications (4.2.38)–(4.2.39) hold when k is replaced by $k - \ell_1$ at both sides of those implications. Similarly if $\alpha > \mu_+$ and $k \geq \ell_2 + 1$, where ℓ_2 is the smallest integer larger than or equal to $\mu_+ - \alpha_+$, then eqs. (4.2.40)–(4.2.44) hold with, however, k replaced by $k - \ell_2$; this substitution should be done in both members of the implication (4.2.41). For the validity of (4.2.42), respectively of (4.2.44), the condition $k > \ell_0$, respectively $k \geq \tilde{\ell}$, should of course be replaced by $k > \ell_0 + \ell_2$, respectively $k \geq \tilde{\ell} + \ell_2$. In all cases discussed here one has the corresponding *a–priori* estimates.

2. If (α_-, α_+) is only a regularity interval for $C^{\alpha}_{0+\lambda}(M)$ and if $\alpha_+ = \mu_+$, then eqs. (4.2.40)–(4.2.44) hold if we moreover assume $\alpha_- + 1 < \alpha_+$, $u = o(x^{\alpha_- + 1 + \epsilon})$ for some $\epsilon > 0$ with, however, $\sigma = 0$ and k replaced by $k - 1$; in (4.2.42) the condition $k > \ell_0$ should of course be replaced by $k > \ell_0 + 1$; similarly for (4.2.44). If $\alpha_+ < \mu_+$ then the results discussed in Remark 1 will hold with $\sigma = 0$ and with a value of k further decreased by 1 as compared to the value there.

PROOF: In what follows the symbol σ denotes a real number in the interval $(0,\lambda)$ the value of which may vary from one expression to the next. Let $\hat{u}$ be given by Lemma 4.2.5, by point (i) of that Lemma and by interpolation (*cf.* the arguments of proof of Lemma 4.2.2) there exists $0<\sigma<\lambda$ such that

$$\tilde{f} \equiv L(u-\hat{u}) \in C^{\alpha+k+\sigma}_{k+\sigma,0+\sigma}(M) \subset C^{\mu_++\sigma}_{k+\sigma,k+\sigma}(M)\,.$$

By Theorem 4.1.3 point (i) for all $\epsilon>0$ we have $u-\hat{u} \in C^{\mu_+-\epsilon}_{k+2+\sigma,k+\sigma}(M_{x_0})$ so that $\tilde{L}(u-\hat{u}) \in C^{\mu_++1-\epsilon}_{k-1+\sigma,k-1+\sigma}(M_{x_0}) \cap C^{\mu_+-\epsilon}_{k+\sigma,k+\sigma}(M_{x_0})$; interpolation gives $\tilde{L}(u-\hat{u}) \in C^{\mu_++\sigma}_{k+\sigma,k+\sigma}(M_{x_0})$. We have

$$u-\hat{u} = \psi x^{\mu_+} + G_0[\tilde{f}] - G_0[\tilde{L}(u-\hat{u})] \tag{4.2.46}$$

with some function $\psi(v)$, and it follows from this equation at $x=x_0/2$ and from (3.4.14) that $\psi \in C_{k+\sigma}(\partial M)$. Let $\hat{\psi}$ be given by Lemma 4.2.7, set $\tilde{u} \equiv u-\hat{u}-\hat{\psi}$, we then have

$$L\tilde{u} \in C^{\mu_++k+\sigma}_{k+\sigma,0+\sigma}(M_{x_0}), \qquad \tilde{u} = o(x^{\mu_+})\,.$$

so that we can write

$$\tilde{u} = G_0[L\tilde{u}] - G_0[\tilde{L}\tilde{u}]\,. \tag{4.2.47}$$

It follows that $\tilde{u} \in C^{\mu_++\sigma}_{0,k+\sigma}(M_{x_0})$ and Lemma 4.1.4 gives $\tilde{u} \in C^{\mu_++\sigma}_{k+2+\sigma,k+\sigma}(M_{x_0})$. Lemma 4.2.8 with $\beta=\mu_+$, λ there replaced by σ, and $\ell=m=k$ yields

$$u-\hat{u}-\hat{\psi} \in C^{\mu_++k+\sigma}_{k+2+\sigma,0+\sigma}(M_{x_0})\,.$$

Eqs. (4.2.40)–(4.2.43) follow now from Proposition 3.2.2. $\square$

Similar arguments give the following, no details will be given (note that below we are *not* assuming that $\alpha_+=\mu_+$):

THEOREM 4.2.10 (CLASSICAL BOUNDARY REGULARITY, $\mu_+-\alpha\in\mathbb{N}_0$). Let (α_-,α_+) be a strong regularity interval for $C^{\alpha}_{0+\lambda}(M)$ for a geometric elliptic operator $L\in OP^2_{C_{k+\lambda}(\bar{M})}$, $\lambda\in(0,1)$, suppose that L is of the form (4.2.1)–(4.2.4) and that

$$\alpha_+ \le \mu_+\,, \qquad \alpha_- < \alpha \le \mu_+\,, \qquad \mu_+-\alpha\in\mathbb{N}_0\,,$$
$$Lu = f \in x^{\alpha}C_{k+\lambda}(\bar{M}), \qquad u = O(x^{\alpha_-+\epsilon})\,, \quad \epsilon>0\,.$$

For $\alpha\le\alpha_+$ set $\ell=0$, otherwise let ℓ be the smallest integer such that $\alpha_++\ell>\alpha$. If $\alpha+k\ge\mu_++\ell+1$, then there exists $\sigma>0$ and $u_{\log}\in\cap_{i=0}^{\mu_+-\alpha+2}x^{\mu_+-i}\,C_{k-(\mu_+-\alpha)+i+\lambda}(\bar{M})$ such that

$$u-u_{\log}\log x \in \cap_{i=0}^{2} x^{\alpha-i}C_{k-\ell+i+\sigma}(\bar{M}_{x_0})\,. \tag{4.2.48}$$

If $u_{\log}|_{\partial M}=0$, then $u_{\log}\equiv 0$. The constant σ depends only upon α_+, μ_+, α, λ and k. Moreover when F_1, F_2 denote any of the summand spaces in eq. (4.2.48), then the *a–priori* estimate (4.2.45) holds with a (u and f independent) constant C.

REMARK: If (α_-,α_+) is only a regularity interval for $C^{\alpha}_{0+\lambda}(M)$ and if we moreover assume that $\alpha_- + 1 < \alpha_+$ and $u = O(x^{\alpha_-+1+\epsilon})$ for some $\epsilon > 0$, then eq. (4.2.48) holds with $\sigma = 0$ and with k replaced by $k-1$.

Theorems 4.2.6, 4.2.9 and 4.2.10 were proved assuming the existence of a (strong) regularity interval in weighted Hölder spaces. In weighted Sobolev spaces we have the following corresponding result:

THEOREM 4.2.11. Let $p \in (1,\infty)$, and let $\hat\ell$ be the smallest integer larger than or equal to $(n-1)/p$ (recall that $n = dim\, M$). Let $L \in OP^2_{C_{k+\hat\ell+\lambda}(\bar M)}$, $\lambda \in (0,1)$ be a geometric elliptic operator of the form (4.2.1)–(4.2.4). Suppose that for all $\omega \in (\omega_-,\omega_+)$ there exists $0 < x_1 \le x_0$ (possibly depending upon ω) such that

$$L : \mathring{W}^{\omega;p}_1 \cap W^{\omega;p}_2(M_{x_1}, g, x^{-n}d\mu_g) \to W^{\omega;p}_0(M_{x_1}, x^{-n}d\mu_g)$$

is an isomorphism, with $\mu_- \le \omega_- < \min\left(\omega_+, \mu_+ - \frac{n-1}{p}\right)$. Define

$$(\alpha_-,\alpha_+) = \left(\omega_- + \frac{n-1}{p}, \min\left(\mu_+, \omega_+ + \frac{n-1}{p}\right)\right). \tag{4.2.49}$$

Let $u \in W^{p,\mathrm{loc}}_2(M)$ satisfy

$$Lu = f\,, \qquad f \in x^{\alpha}C_{k+\hat\ell+\lambda}(\bar M)\,, \tag{4.2.50}$$

with $\alpha > \alpha_-$. Suppose finally that either

$$u = O(x^{\alpha_-+\epsilon})\,, \quad \epsilon > 0\ , \tag{4.2.51}$$

or

$$u \in W^{\omega;p}_0(M_{x_1}, x^{-n}d\mu_g)\,, \quad \omega > \omega_-\,. \tag{4.2.52}$$

Then:

(i) Consider $\alpha \in (\alpha_-,\alpha_+)$, let $\ell_0 \in \mathbb{N}_0$ be the largest integer such that $\alpha + \ell_0 < \mu_+$. There exists $\sigma > 0$ such that

$$\alpha + k < \mu_+ \quad \Longrightarrow \quad u \in \cap_{i=0}^{2+\hat\ell} x^{\alpha-i}C_{k+i+\sigma}(\bar M_{x_0})\,, \tag{4.2.53}$$

$$k > \ell_0 \quad \Longrightarrow \quad u \in \cap_{i=0}^{2+\hat\ell} x^{\alpha-i}C_{\ell_0+i+\sigma|k-\ell_0}(\bar M_{x_0})\,. \tag{4.2.54}$$

(ii) If $\alpha_+ < \mu_+$ consider $\alpha_+ \le \alpha < \mu_+$. Define ℓ_1 to be the smallest integer larger than $\alpha - \alpha_+$. If $k - \ell_1 \ge 1$, then the implications (4.2.53)–(4.2.53) hold with $k-\ell_1$ substituted for k at both sides of those implications.

(iii) If $\mu_+ - \alpha \in \mathbb{N}_0$, then the conclusions of Theorem 4.2.10 hold.

(iv) Let $\ell_2 \in \mathbb{N}_0$ be the smallest number larger than or equal to $\mu_+ - \alpha_+$. If $\alpha > \mu_+$ and if in Theorem 4.2.9 we replace the conditions on k by conditions on $k - \ell_2$ (in particular the condition $k - \ell_2 \ge 1$ must hold), then the conclusions of Theorem 4.2.9 will hold with k replaced by $k - \ell_2$.

Moreover the *a–priori* estimates of those theorems hold modified in the obvious way.

PROOF: When $(n-1)/p$ is not an integer the result follows from Theorems 4.2.3, 4.2.4 and Lemma 4.2.8 using the arguments given in the proofs of Theorems 4.2.6 and 4.2.9. Here Lemma 4.2.8 is used with $\ell = k$ or $\ell = k - \ell_1$, and with k there replaced by $k + \hat{\ell}$. Note, however, that for $(n-1)/p \in \mathbb{N}$ a *verbatim* application of Theorems 4.2.3 and 4.2.4 would lead to a result worse by one as far as differentiability of u is concerned. This can be improved by noting that, in the notation[3] of Theorem 4.2.3 and under the hypotheses there, for $\alpha < \alpha_+$ the inclusion (4.2.24) in Theorem 4.2.3 can be replaced by

$$\forall\, \epsilon > 0 \qquad u - \hat{u} \in C^{\alpha-\epsilon}_{k+2+\lambda,k-\hat{\ell}+\lambda}(M_{x_1})\,. \tag{4.2.55}$$

In eq. (4.2.55) the constant $\hat{\ell}$ is as defined in the statement of Theorem 4.2.11. This is sufficient for our purposes and leads to the result here with $\hat{\ell}$ — as described in the statement of Theorem 4.2.11. Similarly for $(n-1)/p \in \mathbb{N}$ the constant δ in Theorem 4.2.4 can be chosen so that $\alpha_+ + \ell_0 - 1 + \delta$ is arbitrarily close to α. The result follows then by the same arguments as indicated above. □

[3] It should be stressed that k in eq. (4.2.55) does *not* coincide with k used in the statement of Theorem 4.2.11.

CHAPTER 5

Nonlinear equations with polyhomogeneous coefficients.

5.1. Polyhomogeneity of solutions of some fully nonlinear equations

In this chapter we shall show, under appropriate hypotheses, that solutions of a class of (uniformly degenerating) fully nonlinear second order systems of equations are polyhomogeneous near ∂M. Roughly speaking, to prove polyhomogeneity of solutions we shall need

(i) existence of a polyhomogeneous "approximate solution" ϕ_0,

(ii) existence of a "regularity interval" for the equation linearized at ϕ_0,

(iii) boundedness of the solution in some weighted Hölder class.

More precisely, consider a system of equations of the form

$$F^A[y, \phi^B, x\partial_i\phi^B, x^2\partial_i\partial_j\phi^B] = 0, \qquad A = 1, \ldots, N, \tag{5.1.1}$$

or, shortly,

$$F[\phi] = 0.$$

We shall assume that both the fields ϕ and the equations (5.1.1) are geometric in the sense of Chapter 4, we leave to the reader the easy task of formalizing this notion along the lines of Chapter 4. We shall assume that we have an approximate solution ϕ_0 of equation (5.1.1) satisfying

$$\phi_0 \in \mathcal{A}_{\mathrm{phg}} \cap C_0(\bar{M})\,, \tag{5.1.2}$$

$$F[\phi_0] \in \mathcal{A}_{\mathrm{phg}} \cap C_0^{\alpha_0}(M), \qquad \alpha_0 > 0\,. \tag{5.1.3}$$

The functions $F^A(y, z^B, p_i^B, q_{ij}^B)$ will be assumed to be continuous with respect to all the variables, smooth with respect to the variables z, p, q, and polyhomogeneous w.r.t. x in the following sense: for all multiindices ν_1, ν_2, ν_3 we have

$$[\partial_z^{\nu_1}\partial_p^{\nu_2}\partial_q^{\nu_3}F]\Big|_{\phi=\phi_0} \in \mathcal{A}_{\mathrm{phg}} \cap C_0(\bar{M})\,, \tag{5.1.4}$$

and for all i and for all multiindices μ, ν_1, ν_2, ν_3 it holds that

$$(x\partial_x)^i\partial_v^{\mu}\partial_z^{\nu_1}\partial_p^{\nu_2}\partial_q^{\nu_3}F \in C^0(\bar{M} \times \mathbb{R}^{N(1+n+n^2)}). \tag{5.1.5}$$

REMARK: It should be clear from the proof below that it is sufficient to assume (5.1.5) in a neighbourhood of the graph of ϕ_0 over $\bar{M}_{x_0}$ defined as

$$\{(y, \phi_0, x\partial\phi_0, x^2\partial_i\partial_j\phi_0), y \in \bar{M}_{x_0}\} \subset \bar{M}_{x_0} \times \mathbb{R}^{N(1+n+n^2)}\,.$$

Let $F'[\chi]$ be the linearization of (5.1.1) at $\phi = \chi$:

$$F'[\chi]\psi = \frac{\partial F^A}{\partial z^B}\Big|_{\phi=\chi}\psi^B + x\frac{\partial F^A}{\partial p_i^A}\Big|_{\phi=\chi}\frac{\partial \psi^B}{\partial y^i} + x^2\frac{\partial F^A}{\partial q_{ij}^B}\Big|_{\phi=\chi}\frac{\partial^2\psi^B}{\partial y^i\partial y^j}\,. \tag{5.1.6}$$

Our hypotheses above are motivated by the harmonic map equation (under appropriate conditions; *cf.* [**55, 56, 31**] for some related results), and by the equation for hypersurfaces with prescribed mean curvature (*cf.* [**49, 7**]). Indeed, for metrics on the source space of the form $\tilde{g}_{ij} = x^{-2}g_{ij}$ as considered in Chapter 6.3 or in Chapter 7 the harmonic map equation will be of the form considered here. Similarly, in a space–time with a smooth or polyhomogeneous Scri, spacelike hypersurfaces intersecting Scri transversally will satisfy an equation of the form (5.1.1). Our theorem below together with the remarks following it reduce the question of boundary regularity of solutions to those equations to an *a–priori* estimate in $C_1^\alpha(M_{x_0})$ spaces, together with the proof of existence of the strong regularity interval. Note that for the prescribed mean curvature equation the existence of a strong regularity interval for appropriate approximate solutions required in Theorem 5.1.1 below follows from the results in Section 7.2.

We have the following:

THEOREM 5.1.1. Suppose that (5.1.2)–(5.1.5) hold, assume that $F'[\phi_0]$ is a geometric elliptic operator of the form (4.2.1)–(4.2.4) considered in Section 4.2 and that $F'[\phi_0]$ has a strong regularity interval (α_-, α_+) with

$$\alpha_+ > 0, \qquad \alpha_0 > \alpha_-\,. \tag{5.1.7}$$

Let $\phi \in C_\infty^{\mathrm{loc}}(M)$ be a solution of (5.1.1) satisfying

$$\phi - \phi_0 \in C_\infty^\alpha(M_{x_0}), \qquad \alpha > \max(0, \alpha_-)\,. \tag{5.1.8}$$

Then

$$\phi \in \mathcal{A}_{\mathrm{phg}} \cap C^0(\bar{M}_{x_0})\,.$$

REMARKS:

(i) In the case when eq. (5.1.1) is *quasi–linear*, *i.e.*, linear in second derivatives, then eq. (5.1.8) can be further weakened to $\phi - \phi_0 \in C_1^\alpha(M_{x_0})$, $\alpha > \max(0, \alpha_-)$. Finally, if eq. (5.1.1) is linear both in second and first derivatives of ϕ, then eq. (5.1.8) can be replaced by $\phi - \phi_0 \in C_0^\alpha(M_{x_0})$, $\alpha > \max(0, \alpha_-)$; *cf.* Proposition 5.1.4 below.

(ii) If (α_-, α_+) is a *weak* regularity interval, the result remains true if we add the conditions $\alpha_+ - \alpha_- > 1, \alpha_0 > \alpha_- + 1, \alpha > \max(0, \alpha_- + 1)$, *cf.* the Remark following Lemma 5.1.3 below.

PROOF: We shall prove that there exists a sequence $s_i \to_{i\to\infty} \infty$ and a sequence of functions $\phi_i \in \mathcal{A}_{\rm phg} \cap C_0(\bar M)$ such that

$$\phi_i - \phi \in C^{s_i}_{\infty,\infty}(M) + \mathcal{A}_{\rm phg} \cap C^{\alpha_0}_0(M) \tag{5.1.9}$$

from which the result follows. We shall proceed recursively; suppose thus that ϕ_i has already been defined for some $i \geq 0$: Then ϕ satisfies the equation

$$L_i(\phi-\phi_i) \equiv F'[\phi_i](\phi-\phi_i) = G_i(\phi-\phi_i) + F(\phi_i), \tag{5.1.10}$$

where

$$G_i(\psi) := F[\phi_i+\psi] - F'[\phi_i]\psi - F[\phi_i] \tag{5.1.11}$$

We shall need the following two Lemmata:

LEMMA 5.1.2. Let $\phi_i - \phi_0 \in \mathcal{A}_{\rm phg} \cap C^{\alpha_0}_0(M)$. For $k \in \mathbb{N}_0^\infty$, $\lambda \in [0,1]$, $\delta > 0$ we have:

(i) $k \in \mathbb{N}_0^\infty$, $\lambda \in [0,1]$, $\psi \in C^\delta_{\infty,k+\lambda}(M_{x_0})$,

$$G_i(\psi) \in C^{2\delta}_{\infty,k+\lambda}(M_{x_0})\,.$$

(ii) $\psi \in \mathcal{A}_{\rm phg} \cap C^\delta_0(M_{x_0})$,

$$F(\phi_0+\psi) \in \mathcal{A}_{\rm phg} \cap C_0^{\min(\alpha_0,\delta)}(M_{x_0})$$

(iii) $\psi \in \mathcal{A}_{\rm phg} \cap C^\delta_0(M_{x_0})$, $\chi \in C^\gamma_{\infty,k+\lambda}(M_{x_0})$,

$$(F'[\phi_0+\psi] - F'[\phi_0])\chi \in C^{\gamma+\min(\alpha_0,\delta)}_{\infty,k+\lambda}(M_{x_0})\,.$$

PROOF: Point (ii) follows from (5.1.4)–(5.1.5) using the Taylor expansion. Points (i) and (iii) follow by a straightforward analysis of the remainder term (3.1.3) in the Taylor expansion. $\square$

We can find $s_0 > \max(0,\alpha_-)$ satisfying $s_0 \leq \min(\alpha,\alpha_0)$ and $s_0 < \alpha_+$. Decreasing α_0 if necessary, we may assume that $s_0 = \alpha_0$. We have

LEMMA 5.1.3. $\phi - \phi_0 \in C^{s_0}_{\infty,\infty}(M_{x_0})$.

PROOF: Let $u = \phi - \phi_0$ and consider

$$G[u] = F[\phi_0+u] - F'[\phi_0]u - F[\phi_0]. \tag{5.1.12}$$

We will argue by induction. Fix an irrational $\theta \in (0,1)$ so that $2s_0 - \theta \geq s_0$. It follows from our assumptions that $u \in C^{s_0}_{\infty,0}$ which is the starting point of the induction.

Now assume that $u \in C^{s_0}_{\infty,k\theta}$ for some $k \in \mathbb{N}$. By point (i) of Lemma 5.1.2, $G[u] \in C^{2s_0}_{\infty,k\theta}$. For any $f \in C^{2s_0}_{\infty,k\theta}$ and any multiindex γ we have

$$(x\partial_y)^\gamma f \in C^{2s_0}_{\infty,k\theta} \cap C^{2s_0-1}_{\infty,k\theta+1}.$$

Applying interpolation as in the proof of Lemma 4.2.2 gives with the above choice of θ,

$$(x\partial_y)^\gamma f \in C^{s_0}_{0,(k+1)\theta}$$

but γ was arbitrary so we get $f \in C^{s_0}_{\infty,(k+1)\theta}$. Applying this result to $G[u]$ and noting that $F[\phi_0] \in C^{s_0}_{\infty,\infty}$ and $F[\phi_0 + u] = 0$ by assumption, we find

$$F'[\phi_0]u \in C^{s_0}_{\infty,(k+1)\theta}.$$

By assumption (α_-, α_+) is a strong regularity interval for $F'[\phi_0]$ and $s_0 \in (\alpha_-, \alpha_+)$. We can now apply Theorem 4.1.3 to conclude that $u \in C^{s_0}_{\infty,(k+1)\theta}$. It follows by induction that $u \in C^{s_0}_{\infty,\infty}$. □

REMARK: In the case when (α_-, α_+) is only a weak regularity interval, the above argument is valid only under restrictions on s_0. Instead the appropriate version of Lemma 5.1.3 is proved under the conditions outlined under point (ii) of the remark following Theorem 5.1.1 by considering the identity

$$F'[\phi_0]\frac{\partial \phi}{\partial v^A} = \underbrace{\Big[F'[\phi_0] - F'[\phi]\Big]\frac{\partial \phi}{\partial v^A}} + \underbrace{\frac{\partial F}{\partial v^A}[\phi_0] - \frac{\partial F}{\partial v^A}[\phi]} - \frac{\partial F}{\partial v^A}[\phi_0] \ .$$

By eq. (5.1.3) and by point (iii) of Lemma 5.1.2 each group at the right hand side of this equation is in $C^{s_0}_{\infty}$. From $\phi \in C^{s_0}_{\infty}$ we have $\partial\phi/\partial v^A \in C^{s_0-1}_{\infty}$, and since (α_-, α_+) is a regularity interval and both s_0 and $s_0 - 1$ are in (α_-, α_+) we find $\partial\phi/\partial v^A \in C^{s_0}_{\infty}$, hence $\phi \in C^{s_0}_{\infty,1}$. The result follows by an inductive repetition of this argument; *cf.* the proof of Theorem 7.4.4 in Chapter 7 below for a more detailed exposition in a similar context. □

Returning to the proof of Theorem 5.1.1, Lemma 5.1.3 shows that (5.1.9) is satisfied with $i = 0$. Consider again equation (5.1.10). By Lemma 5.1.2, point (i), we have $G_i(\phi - \phi_i) \in C^{2s_i}_{\infty,\infty}(M)$. Now point (ii) of Lemma 5.1.2 gives $F[\phi_i] \in \mathcal{A}_{\rm phg} \cap C^{\alpha_0}_0(M)$, thus

$$L_i(\phi - \phi_i) \in C^{2s_i}_{\infty,\infty} + \mathcal{A}_{\rm phg} \cap C^{\alpha_0}_0(M).$$

This and point (iii) of Lemma 5.1.2 imply

$$L_0(\phi - \phi_i) = (L_0 - L_i)(\phi - \phi_i) + L_i(\phi - \phi_i) \in C^{s_i+\alpha_0}_{\infty,\infty} + C^{2s_i}_{\infty,\infty} + \mathcal{A}_{\rm phg} \cap C^{\alpha_0}_0(M)\,.$$

A straightforward extension of Lemma 4.2.5 shows[1] that we can find $\hat\phi \in \mathcal{A}_{\rm phg} \cap C^{\alpha_0}_0(M)$ such that

$$L_0(\phi - \phi_i - \hat\phi) \in C^{s_i+\alpha_0}_{\infty,\infty} + C^{2s_i}_{\infty,\infty}\,. \tag{5.1.13}$$

Similarly, for $s_i + \alpha_0 > \mu_+$ it follows from a straightforward extension[1] of Lemma 4.2.7 that $\hat\phi$ can moreover be chosen so that

$$\phi - \phi_i - \hat\phi = o(x^{\mu_+})\,.$$

[1]The result needed here can actually be proved in a rather simpler way as compared to the proof of Lemma 4.2.7: Indeed, all the functions appearing in the polyhomogeneous expansions are smooth in the v variable, so that in the proof one does not need to use the extension operators of Lemma 3.3.1.

Here μ_+ is the larger indicial exponent of L_0, as defined at the beginning of Chapter 4.1. Lemma 4.2.8 gives

$$\phi - \phi_i \in \mathcal{A}_{\mathrm{phg}} \cap C_0^\alpha(M) + C_{\infty,\infty}^{s_i+\delta}, \qquad \delta = \min(s_i, \alpha_0) \geq s_0 > 0 .$$

ϕ_{i+1} is now defined as ϕ_i plus the sum of those terms in $\phi - \phi_i$ which are polyhomogeneous and which decay slower than $s_{i+1} = s_i + \delta$. This completes the induction step, and the result follows. □

Now we specialize to consider the cases when F is quasi– or semi–linear. For such equations we shall show that the hypothesis $\phi - \phi_0 \in C_\infty^\alpha(M_{x_0})$ can be weakened.

DEFINITION 5.1.1.

- We will say that the system F is quasi–linear if F is of the form

$$F[\phi] = x^2 a^{ij}\partial_i\partial_j\phi + c \tag{5.1.14}$$

where $a^{ij} \equiv a^{ij}[y, \phi, x\partial_k\phi]$ and similarly for c.
- We will say that the system F is semi–linear if F is of the form

$$F[\phi] = x^2 a^{ij}[y,\phi]\partial_i\partial_j\phi + xb^i[y,\phi]\partial_i\phi + c[y,\phi] \tag{5.1.15}$$

PROPOSITION 5.1.4. Assume that F is of the form (5.1.14) with F satisfying (5.1.4)–5.1.5). Further let ϕ_0 be an approximate solution to $F[\phi] = 0$ satisfying (5.1.2)–(5.1.3). Assume that $F'[\phi_0]$ is an elliptic operator on M in the sense of Section 4.1, that

$$F[\phi] = 0 ,$$

and that

$$\phi - \phi_0 \in C_1^\alpha(M_{x_0}) \cap C_2^{\mathrm{loc}}(M), \qquad \alpha > 0 . \tag{5.1.16}$$

Then there exists $x_1 > 0$ such that $\phi - \phi_0 \in C_\infty^{s_0}(M_{x_1})$, where $s_0 = \min(\alpha_0, \alpha)$. If moreover F is of the form (5.1.15), then (5.1.16) can be replaced by $\phi - \phi_0 \in C_0^\alpha(M_{x_0}) \cap C_2^{\mathrm{loc}}(M)$, $\alpha > 0$.

PROOF: Let L_ϕ be the operator defined by

$$L_\phi\psi = x^2 a^{ij}(y, \phi, x\partial_k\phi)\partial_i\partial_j\psi + xb^i(y,\phi,x\partial_k\phi)\partial_i\psi$$

and let

$$u = \phi - \phi_0 .$$

From the regularity assumptions on F, a straightforward argument using Taylors theorem shows the existence of some $x_1 > 0$ so that $L_\phi \in OP^2_{C_0^0(M_{x_1})}$ is elliptic in the sense of Chapter 4.1 and

$$L_\phi u \in C_0^{s_0}(M_{x_1}) .$$

Now an application of interior estimates and scaling gives $u \in C_{1+\lambda}^{s_0}(M_{x_1/2})$ for $\lambda \in (0,1)$. From the above we have that $u \in C_{k+\lambda}^{s_0}(M_{x_1/2})$ with $k = 1$ and, since by assumption $u \in C_2^{\mathrm{loc}}(M)$, it follows that $u \in C_{k+\lambda}^{s_0}(M_{x_1})$ with $k = 1$.

We wish to show by induction that $u \in C^{s_0}_{k+\lambda}(M_{x_1})$ for all k. We have shown this to be true for $k = 1$. Assuming that $u \in C^{s_0}_{k+\lambda}(M_{x_1})$ it follows from $F[\phi] = 0$ and (5.1.14), using Taylors theorem, that

$$F'[\phi_0]u \in C^{s_0}_{k-1+\lambda}(M_{x_1}).$$

An application of the scaling estimate Lemma 4.1.1 gives $u \in C^{s_0}_{k+1+\lambda}(M_{x_1})$. This achieves the induction step and proves the Proposition. $\square$

CHAPTER 6

The vector constraint equation.

6.1. Introductory remarks

Let $(\bar M, g)$ be a compact Riemannian manifold with boundary ∂M, let x be any defining function for ∂M, for $x \in M \equiv \bar M \setminus \partial M$ let

$$\tilde g_{ij} = x^{-2} g_{ij} \,. \tag{6.1.1}$$

As discussed in Chapter 2, to construct solutions of the vector constraint equation one makes use of the conformal invariance of the system

$$D_i \, P^{ij} = 0 \,, \qquad D_k \left(g_{ij} \, P^{ij} \right) = 0 \,.$$

There are at least two ways to proceed:

Method 1. Let A^{ij} be a symmetric, traceless tensor field ($g_{ij} \, A^{ij} = 0$), suppose that $\beta \in \mathbb{R}$, let the vector field X^i satisfy

$$D_i \left(D^i X^j + D^j X^i - \frac{2}{n} \left(D_k X^k \right) g^{ij} \right) = -D_i \left(x^\beta \, A^{ij} \right) \,, \tag{6.1.2}$$

where D is the Riemannian connection of g_{ij}. Then the tensor field

$$B^{ij} \equiv D^i X^j + D^j X^i - \frac{2}{n} \left(D_k X^k \right) g^{ij} + x^\beta \, A^{ij}$$

is symmetric, traceless and transverse for the metric g_{ij} ($D_j B^{ij} = 0$). Moreover the tensor field $\tilde B^{ij} \equiv x^{-(n+2)} B^{ij}$ is transverse for the metric $\tilde g_{ij}$ ($\tilde D_j \tilde B^{ij} = 0$).

Method 2. Let $\tilde A^{ij}$ be a symmetric, traceless tensor field ($\tilde g_{ij} \, \tilde A^{ij} = 0$), suppose that $\beta \in \mathbb{R}$, let the vector field $\tilde X^i$ satisfy

$$\tilde D_i \left(\tilde D^i \tilde X^j + \tilde D^j \tilde X^i - \frac{2}{n} \left(\tilde D_k \tilde X^k \right) \tilde g^{ij} \right) = -\tilde D_i \left(x^\beta \, \tilde A^{ij} \right) \tag{6.1.3}$$

where $\tilde D$ is the Riemannian connection of $\tilde g_{ij}$. Then the tensor field

$$\tilde B^{ij} = \tilde D^i \tilde X^j + \tilde D^j \tilde X^i - \frac{2}{n} \left(\tilde D_k \tilde X^k \right) g^{ij} + x^\beta \, \tilde A^{ij}$$

is symmetric, traceless and transverse for the metric $\tilde{g}_{ij}$ ($\tilde{D}_j \tilde{B}^{ij} = 0$). Moreover the tensor field $B^{ij} \equiv x^{(n+2)} \tilde{B}^{ij}$ is transverse for the metric g_{ij} ($D_j B^{ij} = 0$).

Because of the singular character (recall that $x = 0$ on ∂M), of the relationship (6.1.1) between g_{ij} and $\tilde{g}_{ij}$, the methods turn out to be quite different. Since the operator on the right-hand-side of (6.1.2) is a regular elliptic operator on M (*i.e.*, elliptic *up–to–boundary* in the standard sense), the first method turns out to be more convenient for proving existence of solutions of the vector constraint which are smoothly extendable across ∂M; this is most easily achieved using (standard) non–weighted Hölder spaces on the compactified manifold, as shown in Section 6.2. On the other hand the evolution problem is most directly formulated in terms of Sobolev-type spaces, in the non-compactified picture, it is therefore natural to use the second method to do that: this analysis is carried out in Section 6.3.

6.2. (Non–weighted) Hölder spaces on the compactified manifold

In this section we shall prove existence of solutions of the system of equation

$$\Delta_{L,g} X = Y \,, \tag{6.2.1}$$

$$(\Delta_{L,g} X)^i \equiv D_j L^{ij}(X) \,, \tag{6.2.2}$$

$$L^{ij}(X) \equiv \frac{1}{2}(D^i X^j + D^j X^i) - \frac{1}{n} D_k X^k g^{ij} \,, \tag{6.2.3}$$

under various hypotheses on the source term Y, where g_{ij} is a Riemannian metric on a compact manifold $\bar{M}$ with boundary ∂M; by this we mean that $(\bar{M} \backslash \partial M, g|_{\bar{M}\backslash\partial M})$ is a Riemannian manifold and in local coordinates $\{y^i\}$ near ∂M as described at the beginnig of Chapter 3 the matrix $g_{ij} \equiv g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)$ is strictly positive definite up to ∂M, thus:

$$\forall\ X^i \in \mathbb{R}^n \quad C^{-1} \sum_i (X^i)^2 \le g_{ij} X^i X^j \le C \sum_i (X^i)^2 \tag{6.2.4}$$

for some constant C. D_i always denotes the Riemannian connection of g_{ij}.

Definition 6.2.1. The metric g_{ij} will be said to be of class $M^\alpha_{k+\lambda}$, $\alpha \ge 0$, if (6.2.4) holds and if for $0 \le |\gamma| \le k$ we have $\partial^\gamma_y g_{ij} \in C^{\min(0,\alpha-|\gamma|)}_{0+\lambda}(M)$ (thus $g_{ij} \in C^{\rm loc}_{k+\lambda}(M)$, and in local coordinates near ∂M (cf. the beginning of Chapter 3) we have

$$|g_{ij}| + |g^{ij}| + (1 + x^{\alpha-1})^{-1} \left|\frac{\partial}{\partial y^k} g_{ij}\right| + \ldots + (1 + x^{\alpha-k})^{-1} \left|\frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_k}} g_{ij}\right| \le C \,, \tag{6.2.5}$$

for some constant C, together with an appropriate weighted Hölder condition if $\lambda > 0$).

Note that a metric $g \in C_{k+\lambda}(\bar{M})$ is in $M^\alpha_{k+\lambda}$ for all $\alpha > 0$.

PROPOSITION 6.2.2. Let g be a metric of $M^{\alpha}_{k+\lambda}$ class on M, $2 < \dim M$, $\lambda \in [0,1]$, $\alpha > 0$, $k \geq 3$, let $X \in C^{\tilde\alpha}_1(M) \cap W^{p;\mathrm{loc}}_3(M, g, d\mu_g)$, $\tilde\alpha > 0$, $p \in [1,\infty)$, satisfy

$$L(X) = \frac{1}{2}(D^iX^j + D^jX^i) - \frac{1}{n}\left(D_kX^k\right) g^{ij} = 0\,. \tag{6.2.6}$$

Then

$$X = 0\,.$$

To prove Proposition 6.2.2 we shall need the following Lemmata:

LEMMA 6.2.3. Let g be of class M^{α}_k, $k \geq 3$, then for $x \leq x_0$ in local coordinates we have

$$\left|\Gamma^i_{jk}\right|_\delta + x|R|_g + x^2|DR|_g \leq C(1 + x^{\alpha-1})\,,$$

for some constant C, where $R = \left(R^i{}_{jk\ell}\right)$ is the Riemann tensor, $DR = \left(D_kR^i{}_{jmn}\right)$, $|\,|_h$ stands for the Riemannian norm of a tensor with respect to the metric h.

PROOF: Simple computation. □

The result that follows is well known (we use the convention $D_kD_jX^i - D_jD_kX^i = R^i{}_{\ell kj}X^\ell$):

LEMMA 6.2.4. Let $\delta X \equiv D_iX^i$, $X_i \equiv g_{ij}X^j$. Under the hypotheses of Proposition 6.2.2 we have $X \in C^{\mathrm{loc}}_3(M)$, and the following hold ($n = \dim M > 2$):

$$\begin{aligned} D_mD_kD_iX_j &= D_mR^\ell{}_{kij}X_\ell + R^\ell{}_{kij}D_mX_\ell \\ &\quad + \frac{1}{n}\left\{D_mD_k(\delta X)g_{ij} + D_mD_i(\delta X)g_{jk} - D_mD_j(\delta X)g_{ik}\right\}\,, && (6.2.7) \\ D_iD_k\delta X &= \frac{n}{n-2}\Big\{ -2R_{\ell(i}D_{k)}X^\ell + \frac{R}{n(n-1)}\,\delta X g_{ik} \\ &\quad + X^\ell D_\ell\left(\frac{1}{2(n-1)}\,Rg_{ik} - R_{ik}\right)\Big\}\,, && (6.2.8) \end{aligned}$$

where $R_{ij} = R^k{}_{ikj}$ is the Ricci tensor of g_{ij}, $R = g^{ij}R_{ij}$ the Ricci scalar.

PROOF: It is an easy exercise in distributional differentiation to show from (6.2.6) and $X \in W^{p,\mathrm{loc}}_3(M, g, d\mu_g)$ that the equations (6.2.7)–(6.2.8) hold in a distributional sense, which together with $X \in C^{\mathrm{loc}}_1(M)$ implies that $X \in C^{\mathrm{loc}}_3(M)$, and shows that (6.2.7)–(6.2.8) hold pointwise. □

LEMMA 6.2.5. Under the hypotheses of Proposition 6.2.2 for any $\sigma > 0$ there exists a constant $C(\sigma)$ such that

$$|X|_g + x|DX|_g + x^2|DDX|_g \leq Cx^\sigma\,, \tag{6.2.9}$$

where $DX = (D_iX_j)$, $DDX = (D_iD_jX_k)$.

PROOF: Equations (6.2.7)–(6.2.8) and $X \in C^{\tilde\alpha}_1$ show that $DDDX \in C^{\tilde\alpha-3}_0$, which easily (*e.g.*,by scaled interpolation) implies $X \in C^{\tilde\alpha}_3$. Let

$$I = \{\sigma \in \mathbb{R} : \exists C \quad \text{such that (6.2.9) holds}\}$$

since $X \in C_3^{\tilde\alpha}$, $\tilde\alpha > 0$, it follows that $(-\infty, \tilde\alpha] \subset I$, thus $I \neq \emptyset$. Let

$$\tilde\sigma = \limsup I \,,$$

suppose that $\tilde\sigma \neq \infty$, set $\sigma_1 = \tilde\sigma - \frac{1}{2}\min(\alpha, 1)$; by definition of $\tilde\sigma$ we have $\sigma_1 \in I$, thus

$$|X|_g + x|DX|_g + x^2|DDX|_g \le Cx^{\sigma_1} \,. \tag{6.2.10}$$

Decreasing σ_1 slightly if necessary we may assume $-\sigma_1 \notin \mathbb{N}$. Equations (6.2.7)–(6.2.8) imply an equation of the form

$$\begin{aligned} \partial_x DDX &= \text{“}\Gamma\, DDX + RDX + DR\,X\text{”} \\ &= O(x^{\sigma_1+\min(1,\alpha)-3}) \,. \end{aligned} \tag{6.2.11}$$

Suppose first that $\sigma_1 + \min(1, \alpha) < 1$; from

$$f(x_1, v) = f(x_2, v) - \int_{x_1}^{x_2} \frac{\partial f}{\partial x}(s, v)\, ds \tag{6.2.12}$$

one has, setting $f = DDX$, $x_1 = x$, $x_2 = x_0$,

$$DDX = O(x^{\sigma_1+\min(1,\alpha)-2}) \,, \tag{6.2.13}$$

similarly from

$$\partial_x DX = \text{“}\Gamma DX + DDX\text{”} \tag{6.2.14}$$

and from (6.2.12)–(6.2.13) it follows

$$DX = O(x^{\sigma_1+\min(1,\alpha)-1}) \,. \tag{6.2.15}$$

Passing to the limit $x_1 \to 0$ in the equation

$$X(x, v) = X(x_1, v) + \int_{x_1}^{x} \text{“}(\Gamma X + DX)\text{”}(s, v)\, ds \tag{6.2.16}$$

(the integral at the right-hand-side of (6.2.16) converges because $\sigma_1 + \min(1, \alpha) > \tilde\sigma > \tilde\alpha > 0$), making use of $X(0, v) = 0$, one obtains

$$X = O(x^{\sigma_1+\min(1,\alpha)}) \,, \tag{6.2.17}$$

which together with (6.2.13) and (6.2.15) contradicts the definition of $\tilde\sigma$, thus $\sigma_1 + \min(1, \alpha) > 1$; suppose that $\sigma_1 + \min(1, \alpha) < 2$. (6.2.13) still holds so that (6.2.12) with $f = DX$ shows (*cf.* (6.2.14)) that the limit

$$A_{ij}(v) = \lim_{x\to 0} D_i X_j(x, v)$$

exists, and we have

$$D_i X_j - A_{ij} = O(x^{\sigma_1+\min(1,\alpha)-1}) \,, \tag{6.2.18}$$

while (6.2.16) gives

$$X_j = O(x) \,. \tag{6.2.19}$$

(6.2.13), (6.2.18) and (6.2.19) yield

$$\partial X = O(1) \; ,$$

$$\partial\partial X = O(x^{\epsilon-1}), \quad \epsilon = \min(0, \sigma_1 - 1) + \min(1, \alpha) > 0 \,. \tag{6.2.20}$$

Let $v_0 \in \partial M$; (6.2.12) with $f = g_{ij}$ together with $g \in M^\alpha_3$ implies that $f_{ij} \equiv \lim_{x\to 0} g_{ij}(x, v_0)$ exists. Passing to the limit $x \to 0$ in (6.2.6) one has

$$A_{ij} + A_{ji} = \frac{2}{n}\, f^{\ell m}\, A_{\ell m}\, f_{ij} \; , \tag{6.2.21}$$

where $f^{\ell m} \equiv (f_{ij})^{-1}$ (*cf.* (6.2.4)). The interpolation inequality (*cf. e.g.* [**44**, Appendix A]:

$$\|f\|_{C_1(\partial M)} \leq C \|f\|^{1/2}_{C_2(\partial M)} \|f\|^{1/2}_{C_0(\partial M)} \tag{6.2.22}$$

applied to $f(v) = X_i(x, v)$ together with (6.2.19) and (6.2.20) gives

$$\left|\frac{\partial X_i}{\partial v}\right| (v, x) \leq C x^{\epsilon/2} \, , \tag{6.2.23}$$

thus $A_{Ai} = 0$. Let $e_{(i)} = e_{(i)}{}^j\, \partial_j$ be an orthonormal basis at $(0, v_0)$ for f_{ij}, with $e_{(A)} = e_{(A)}{}^B\, \partial_{v^B}$, $e_{(1)} \perp \partial M$, set $A_{(i)(j)} = A_{ij}\, e^i{}_{(i)}\, e^j{}_{(j)}$; we have, for $A = 2, \ldots, n$,

$$e^1_{(A)} = 0 \quad \Longrightarrow \quad A_{(A)(j)} \equiv e^k_{(A)}\, e^\ell_{(j)}\, A_{k\ell} = 0 \quad \Longrightarrow \quad f^{ij}\, A_{ij} = A_{(1)(1)} \,.$$

(6.2.21) rewritten in the orthonormal frame $e_{(i)}$,

$$A_{(i)(j)} + A_{(j)(i)} = \frac{2}{n}\, A_{(1)(1)}\, \delta_{(i)(j)} \, ,$$

with $(i)(j) = (1)(1)$, $(1)(A)$ gives

$$A_{ij} = 0 \, ,$$

thus (6.2.15) and (6.2.17) hold, contradicting again the definition of $\tilde\sigma$, thus $\sigma_1 + \min(1, \alpha) > 2$. Going through the whole argument once again one similarly shows that the limit

$$A_{ijk} = \lim_{x\to 0} D_i D_j X_k(x, v_0)$$

exists, and $A_{Ajk} = 0$. Algebraic manipulations with the equations

$$A_{kij} + A_{kji} - \frac{2}{n}\, f^{\ell m}\, A_{\ell m k}\, f_{ij} = \lim_{x\to 0} D_k \left(D_i X_j + D_j X_i - \frac{2}{n}\, g^{k\ell}\, D_k X_\ell\, g_{ij} \right) = 0$$

lead to

$$A_{ijk} = 0 \, .$$

This implies that (6.2.13), (6.2.15) and (6.2.17) hold again, contradicting the definition of $\tilde\sigma$, thus $\tilde\sigma = \infty$, which had to be established. □

Proof of Proposition 6.2.2: Let

$$f(x) = (|X|_g^2 + x^2|DX|_g^2 + x^4|DDX|_g^2)\,.$$

We have

$$\frac{\partial f}{\partial x} = \frac{2}{x}\langle X, xD_xX\rangle + \frac{2}{x}\,x^2|DX|_g^2 + \frac{2}{x}\langle xDX, x^2D_xDX\rangle + \frac{4}{x}\,x^4|DDX|_g^2 + F\,,$$

where F is given by

$$F = 2\langle x^2DDX, x^2D_xDDX\rangle\,.$$

Thus, using $2ab \le a^2+b^2$,

$$\frac{\partial f}{\partial x} \le \frac{4+|\frac{\partial}{\partial x}|_g}{x}\,f + F\,.$$

From Lemmata 6.2.3 and 6.2.4 it follows

$$\begin{aligned}
|F| &\le Cx^2|DDX|(x^{\min(1,\alpha)-1}|X| + x^{\min(1,\alpha)-1}x|DX|)\\
&\le \frac{Cx^{\min(1,\alpha)}}{x}\,(x^2|DDX|\,|X| + x^2|DDX|\,x|DX|)\\
&\le \frac{Cx^{\min(1,\alpha)}}{x}\,f\,.
\end{aligned}$$

Therefore, for $0 \le x \le x_0$, we have

$$\frac{\partial f}{\partial x} \le \frac{4+|\frac{\partial}{\partial x}|_g + C}{x}\,f \le \frac{\sigma f}{x}\,,$$

for some constant σ, so that for $0 < x_1 \le x \le x_0$ one has

$$\ln\left[\frac{f(x)x_1^\sigma}{f(x_1)x^\sigma}\right] \le 0\,,$$

which implies

$$f(x) \le \frac{f(x_1)}{x_1^\sigma}\,x^\sigma\,.$$

Passing to the limit $x_1 \to 0$ one obtains from Lemma 6.2.5

$$f(x) \equiv 0\,,$$

thus X vanishes in a neighbourhood of ∂M, which by well known results on conformal Killing vector fields (*cf.* Lemma 6.2.4) implies $X \equiv 0$. □

Proposition 6.2.6. Let g be a $C_{\ell+\lambda}(\bar M) \cap M_3^\alpha$ metric, $\ell \ge 2$, $\alpha > 0$.

(i) For $p \in [2,\infty)$ if $k=0$ and $p \in (1,\infty)$ if $1 \le k \le \ell-2$ there exists a constant C such that for all $X \in \mathring{W}{}_1^p \cap W_{k+2}^p(M,g,d\mu_g)$ we have

$$\|X\|_{W^p_{k+2}(M,g,d\mu_g)} \le C\|\Delta_{L,g}X\|_{W^p_k(M,g,d\mu_g)}\,. \tag{6.2.24}$$

(ii) For $0 \leq k \leq \ell - 2$, $\lambda \in (0,1)$, there exists a constant C such that for all $X \in C_{k+2+\lambda}(\bar{M})$ satisfying $X(p) = 0$ for $p \in \partial M$,

$$\|X\|_{C_{k+2+\lambda}(\bar{M})} \leq C \|\Delta_{L,g} X\|_{C_{k+\lambda}(\bar{M})} . \tag{6.2.25}$$

PROOF: To prove (6.2.24), recall that since $\Delta_{L,g}$ is elliptic we have the estimate [**2**, Theorem 10.5]

$$\|X\|_{W^p_{k+2}} \leq C \left(\|\Delta_{L,g} X\|_{W^p_k} + \|X\|_{L^1} \right) . \tag{6.2.26}$$

Suppose that (6.2.24) does not hold; thus for $i \in \mathbb{N}$ there exists X_i such that $\|X_i\|_{L^1} = 1$ and

$$\|\Delta_{L,g} X_i\|_{W^p_k} \leq \frac{1}{i} \|X\|_{W^p_{k+2}} .$$

(6.2.26) implies

$$\|X_i\|_{W^p_{k+2}} \leq \frac{C}{i} \|X_i\|_{W^p_{k+2}} + C \quad \Longrightarrow \quad \|X_i\|_{W^p_{k+2}} \leq 2C \text{ for } i \geq 2C ,$$

therefore

$$\|\Delta_{L,g} X_i\|_{W^p_{k+2}} \leq \frac{2C}{i} .$$

By the Rellich-Kondrakov theorem a subsequence, still denoted by X_i, can be chosen converging strongly in W^2_1 to $X_\infty \in W^2_1$, and we have

$$\forall\, Y \in \mathring{W}{}^2_1(M, g, d\mu_g) \qquad \int_M (D^i X^j_\infty + D^j X^i_\infty - \frac{2}{n} D_k X^k_\infty g^{ij}) D_j Y_i \, d\mu_g = 0 . \tag{6.2.27}$$

Elliptic regularity (*cf. e.g.* [**39**]) and $g \in C_2(\bar{M})$ give $X_\infty \in C_{1+\tilde{\alpha}}(\bar{M})$ for some $\tilde{\alpha} \in (0,1)$, while $g \in M^\alpha_3$ implies that $X_\infty \in W^{p,\mathrm{loc}}_3(M, g, d\mu_g)$ for all $p \in (1, \infty)$. Setting $Y = X_\infty$ in (6.2.27) it follows that X_∞ is a conformal Killing vector. Proposition 6.2.2 implies $X_\infty = 0$, which contradicts $\|X_\infty\|_{L^1} = 1$, and proves (6.2.24).

(6.2.25) is proved in a similar (and simpler) way, using

$$\|X\|_{C_{k+2+\lambda}(\bar{M})} \leq C \left(\|\Delta_{L,g} X\|_{C_{k+\lambda}(\bar{M})} + \|X\|_{L^1(M, d\mu_g)} \right)$$

(cf. [**2**, Theorem 9.2]) and the Arzela-Arcoli theorem. □

THEOREM 6.2.7. Let g be a $C_{\ell+\lambda}(\bar{M}) \cap M^\alpha_3$ metric on a compact manifold $\bar{M}$ (with boundary), $\ell \geq 2$, $\lambda \in [0,1]$, $\alpha > 0$.

(i) Suppose that $\lambda \in (0,1)$. For $0 \leq k \leq \ell - 2$ the map

$$\Delta_{L,g} : \{X \in C_{k+2+\lambda}(\bar{M}) : X(p) = 0 \text{ for } p \in \partial M\} \to C_{k+\lambda}(\bar{M}) \tag{6.2.28}$$

is an isomorphism.

(ii) For $p \in [2,\infty)$, $k=0$, or $p \in (1,\infty)$, $1 \le k \le \ell-2$ the map

$$\Delta_{L,g} : \mathring{W}{}^p_1 \cap W^p_{k+2}(M,g,d\mu_g) \to W^p_k(M,g,d\mu_g) \tag{6.2.29}$$

is an isomorphism.

REMARK: If $\ell \ge 3$ then $g \in C_\ell(\bar M)$ implies $g \in M_3^\alpha$ for all $\alpha > 0$, *cf.* Definition 6.2.1.

PROOF: (6.2.24) with $p=2$, $k=0$, shows that the problem

$$\Delta_{L,g} X = Y\,, \qquad Y \in L^2, \quad X \in \mathring{W}{}^2_1 \tag{6.2.30}$$

satisfies the coerciveness condition, existence of a weak solution follows from the Lax–Milgram theorem [**40**, Theorem 5.8], regularity follows from *e.g.* [**39**], uniqueness of solutions follows from (6.2.24). □

THEOREM 6.2.8. Let $\bar M$ be a compact manifold (with boundary), suppose that g is a metric on $M \equiv \bar M \setminus \partial \bar M$ which can be C_∞ extended across ∂M, let x be any smooth defining function for ∂M. Consider the equation

$$\Delta_{L,g} X = x^\alpha Y\,, \qquad Y \in C_\infty(\bar M)\,. \tag{6.2.31}$$

(i) For $\alpha \in \mathbb{N}_0$ and for any $X_0 \in C_\infty(\bar M)$ there exists a solution of (6.2.31) of the form

$$X = X_0 + X_1, \qquad X_1 \in C_\infty(\bar M)\,, \quad X_1 = O(x)\,. \tag{6.2.32}$$

X is uniquely determined by Y and by $\partial M \ni p \to X_0(p)$, in the class of $W^2_1(M,g,d\mu_g)$ solutions of (6.2.31).

(ii) For $\alpha \notin \mathbb{Z}$ there exist $X_{\alpha+2}, X_0 \in C_\infty(\bar M)$ such that the vector field X given by

$$X = x^{\alpha+2} X_{\alpha+2} + X_0 \tag{6.2.33}$$

is a solution of (6.2.31). X is determined by Y, in the class of solutions having the described properties, up to the addition of a $C_\infty(\bar M)$ solution of the homogeneous equation (*cf.* point 1).

(iii) For $-\alpha \in \mathbb{N}$ there exist $X_{\alpha+2}$, X_0, $X_{\log} \in C_\infty(\bar M)$ such that the vector field X given by

$$X = x^{\alpha+2} X_{\alpha+2} + X_0 + \log x \, X_{\log} \tag{6.2.34}$$

is a solution of (6.2.31). If $\alpha = -1$ then there exists $X_{\log,1} \in C_\infty(\bar M)$ such that $X_{\log} = x\, X_{\log,1}$. X is determined by Y, in the class of solutions having the described properties, up to the addition of a $C_\infty(\bar M)$ solution of the homogeneous equation (*cf.* point 1).

REMARKS:

(i) In (6.2.33) one can replace $X_{\alpha+2}$ by $X_{\alpha+2} + \hat X_{\alpha+2}$, with $\hat X_{\alpha+2} \in C^\infty_\infty(M)$ without changing the form of (6.2.33), thus $X_{\alpha+2}$ is not unique. A similar remark applies to (6.2.34).

(ii) For $\alpha+2>1/2$, X in (6.2.33) is unique in the class of solutions which are in $C_2^{1/2+\epsilon}$, $\epsilon>0$, as follows from Proposition 6.2.2.

PROOF: Point 1 folows immediately from Theorem 6.2.7. To prove the remaining claims, let

$$\tilde X_{\alpha+2}=Z_{\alpha+2}+V_{\alpha+2}\,,\qquad Z_{\alpha+2}\perp Dx\,,\quad V_{\alpha+2}\sim Dx\,.$$

A straightforward calculation gives

(6.2.35)

$$2\Delta_{L,g}\,(x^{\alpha+2}\tilde X_{\alpha+2})=(\alpha+1)(\alpha+2)\,x^{\alpha}\,|Dx|_g^2\left\{Z_{\alpha+2}+\frac{2(n-1)}{n}\,V_{\alpha+2}\right\}+W_{\alpha+2}\,,$$

$$W_{\alpha+2}=x^{\alpha+1}\,W'_{\alpha+2}\,,$$

with $W'_{\alpha+2}\in C_\infty(\bar M)$ if $Z_{\alpha+2},V_{\alpha+2}\in C_\infty(\bar M)$. For $\alpha\neq-2,-1$ we can choose $Z_{\alpha+2}$ and $V_{\alpha+2}\in C_\infty(\bar M)$ so that

(6.2.36)

$$(\alpha+1)(\alpha+2)\,\lim_{x\to0}\left\{\left(Z_{\alpha+2}+\frac{2(n-1)}{n}\,V_{\alpha+2}\right)\,|Dx|_g^2\right\}=\lim_{x\to0}\,Y\,,$$

and if we set

$$\begin{aligned}Y_1&=\frac1x\left\{Y-(\alpha+1)(\alpha+2)\,|Dx|_g^2\left(Z_{\alpha+2}+\frac{2(n-1)}{n}\,V_{\alpha+2}\right)\right\}-W'_{\alpha+2}\in C_\infty(\bar M)\\ \hat X_1&=X-x^{\alpha+2}\tilde X_{\alpha+2}\,,\end{aligned}$$

then X will be a solution of (6.2.31) if $\hat X_1$ satisfies

$$2\Delta_{L,g}\,\hat X_1=x^{\alpha+1}\,Y_1\,. \tag{6.2.37}$$

Suppose first that $-\alpha\notin\mathbb N_0$; continuing as above we can find a sequence $\{\tilde X_{\alpha+2+i}\}_{i=0}^\infty$ of vector fields $\tilde X_{\alpha+2+i}\in C_\infty(\bar M)$ such that

$$2\Delta_{L,g}\left(\sum_{i=0}^{N}x^{\alpha+2+i}\,\tilde X_{\alpha+2+i}\right)=x^{\alpha+N}\,Y_N\,, \tag{6.2.38}$$

for some $Y_N\in C_\infty(\bar M)$. By Borel's Lemma 3.1.2 there exists a smooth vector field $X_{\alpha+2}$ such that

$$X_{\alpha+2}\sim\sum_{i=0}^{\infty}x^i\,\tilde X_{\alpha+2+i}\,,$$

and by (6.2.38) we have

$$2\Delta_{L,g}\,(x^{\alpha+2}X_{\alpha+2})-x^\alpha\,Y\in C_\infty^\infty(M)\subset C_\infty(\bar M)\,.$$

By Theorem 6.2.7 there exists a vector field $X_1\in C_\infty(\bar M)$, $X_1(p)=0$ for $p\in\partial M$, such that

$$2\Delta_{L,g}\,X_1=x^\alpha\,Y-\Delta_{L,g}\,(x^{\alpha+2}\,X_{\alpha+2})\,.$$

Setting $X=x^{\alpha+2}\,X_{\alpha+2}+X_1$ one obtains the required solution.

For $-\alpha \in \mathbb{N}$ after a finite number of steps in the procedure leading from (6.2.31) to (6.2.37) one obtains, for some $Y_{-2}, \tilde{X}_i \in C_\infty(\bar{M})$,

$$2\Delta_{L,g}\left(X - \sum_{i=\alpha+2}^{-1} x^i\,\tilde{X}_i\right) = x^{-2}\,Y_{-2}\,, \tag{6.2.39}$$

which can be solved to leading order by setting

$$\tilde{X}_{\log} = (Z_{\log} + V_{\log}) \quad Z_{\log}, V_{\log} \in C_\infty(\bar{M})\,, \quad Z_{\log} \perp Dx\,,\ V_{\log} \sim Dx\,,$$
$$\lim_{x\to 0}\left\{\left(Z_{\log} + \frac{2(n-1)}{n}\,V_{\log}\right)\,|Dx|_g^2\right\} = -\lim_{x\to 0} Y_{-2}\,,$$

and we obtain

$$2\Delta_{L,g}\left(X - \sum_{i=\alpha+2}^{-1} x^i\,\tilde{X}_i - \log x\,\tilde{X}_{\log}\right) = x^{-1}\,Y_{-1} + \log x\,Y_{\log}$$

with some $Y_{-1},\, Y_{\log} \in C_\infty(\bar{M})$. For $i \in \mathbb{N}$ consider

$$\tilde{X}_{\log,i} = (Z_{\log,i} + V_{\log,i})\,, \quad Z_{\log,i}, V_{\log,i} \in C_\infty(\bar{M})\,, \quad Z_{\log,i} \perp Dx\,,\ V_{\log,i} \sim Dx\,.$$

We have

$$2\Delta_{L,g}\,(\tilde{X}_{\log,i}x^i \log x) = x^{i-2}\{i(i-1)\log x + 2i - 1\}\,|Dx|_g^2\,\left\{Z_{\log,i} + \frac{2(n-1)}{n}\,V_{\log,i}\right\}$$
$$+x^{i-1}(W_{\log,i}\log x + W_i)\,,$$

with $W_i, W_{\log,i} \in C_\infty(\bar{M})$, and proceeding as before one shows that there exists $X_{\log} \in C_\infty(\bar{M})$, with $X_{\log} \sim \sum_{i=0}^\infty x^i \tilde{X}_{\log,i}$ such that

$$\Delta_{L,g}\,(X_{\log}\log x) - x^{-1}\,Y_{-1} - \log x\,Y_{\log} \in C_\infty(\bar{M})$$

(note that if $\alpha = -1$ then $X_{\log} = xX_{\log,1}$, for some $X_{\log,1} \in C_\infty(\bar{M})$), and the result follows by point 1. Uniqueness, up to a smooth solution of the homogeneous equation, follows by construction. □

Theorem 6.2.8 shows that a natural space in which solutions of equation (6.2.31) "live" is the space of polyhomogeneous vector fields. This leads us to introduce the following:

Definition 6.2.9. A metric g on M will be said *polyhomogeneous*, and we shall write $g \in M^{\mathrm{phg}}$, if g is strictly positive definite up to ∂M, in the sense of (6.2.4), and if in local coordinates near ∂M as described in Chapter 3 the components g_{ij} of the metric tensor are polyhomogeneous functions. We thus have (*cf.* Chapter 3)

$$g_{ij}(x,v) \sim \sum_{m=0}^{I}\sum_{n=0}^{\infty}\sum_{k=0}^{N_{mn}} g_{ijmnk}(v)\,x^{s_m+n}\,\ln^k x\,,$$

for some functions $g_{ijmnk}(v) \in C_\infty(\partial M)$, and some sequence $\{(s_i, \{N_{ij}\}_{j=0}^\infty)\}_{i=0}^I$, with $s_i \in \mathbb{R}$, $N_{ij} \in \mathbb{N}_0$, $I \in \mathbb{N}_0^\infty$, $s_i > s_j$ for $i > j$.

It should be remarked that (6.2.4) implies that $s_0 = 0$, $\sum_{ij} (g_{ij000}(v))^2 > 0$, and that $N_{00} = 0$. A metric $g \in C_\infty(\bar M)$ is necessarily polyhomogeneous, and we also have $M^{\mathrm{phg}} \subset M^\alpha_\infty \equiv \cap_k M^\alpha_k$ (*cf.* Definition 6.2.1), for some $\alpha > 0$ (we have *e.g.* $\alpha = s_1$ if $s_1 < 1$ and $N_{10} = 0$, etc.).

The proof of the following result is a straightforward repetition of the proof of Theorem 6.2.8, the details are left to the reader:

THEOREM 6.2.10. Let g be a polyhomogeneous metric on a compact manifold $\bar M$ with boundary, which can be C_2 extended across ∂M, consider the equation

$$\Delta_{L,g} X = Y\,, \qquad Y \in \mathcal{A}^{\mathrm{phg}}\,. \tag{6.2.40}$$

(i) For any $Y = o(x^{-2})$ and $X_0 \in C_\infty(\bar M)$ there exists a solution of (6.2.40) of the form

$$X = X_0 + X_*, \qquad X_* \in \mathcal{A}^{\mathrm{phg}}\,, \quad X_*(p) = 0 \ \text{ for } \ p \in \partial M\,. \tag{6.2.41}$$

X is uniquely determined by Y and by $\partial M \ni p \to X_0(p)$, in the class of polyhomogeneous solutions of (6.2.40).

(ii) There exists a solution $X \in \mathcal{A}^{\mathrm{phg}}$ of (6.2.40). Any two polyhomogeneous solutions of (6.2.40) differ by a solution of the homogeneous equation, as described in point (i) above.

6.3. Weighted Sobolev spaces

In this section we shall establish existence of solutions of the equation

$$\Delta_{\tilde L,\tilde g} X = Y\,, \tag{6.3.1}$$

with $\tilde g_{ij} = x^{-2} g_{ij}$, g_{ij} — Riemannian metric on a compact manifold M with boundary ∂M, x — a defining function for ∂M, under various conditions on Y. Let us set

$$\tilde L(X)^{ij} = \frac{1}{2}(\tilde D^i X^j + \tilde D^j X^i) - \frac{1}{n}\, \tilde D_k X^k\, \tilde g^{ij}\,, \tag{6.3.2}$$

where $\tilde D_i$ is the Riemannian connection of the metric $\tilde g_{ij}$, let $\tilde L^*$ be the formal L^2 adjoint of $\tilde L$, we have

$$\Delta_{\tilde L,\tilde g} X^j = \tilde D_i\, \tilde L(X)^{ij} = -\tilde L^*\, \tilde L(X)^j\,. \tag{6.3.3}$$

The operator $\Delta_{\tilde L,\tilde g}$ is of the form considered in Chapter 4, in fact we have

$$\begin{aligned} \Delta_{\tilde L,\tilde g} X^j &= x^{n+2} D_i (x^{-n} L^{ij}(X)) \\ &= x^2\, \Delta_{L,g} X^j - nx\, D_i x\, L^{ij}(X)\,, \end{aligned}$$

with $\Delta_{L,g}$, L^{ij} — defined in (6.2.1)–(6.2.3). In local coordinates as described at the beginning of Chapter 3 this can be written as

$$\Delta_{\tilde L,\tilde g} X = |Dx|^2_g \left[\left(x^2\, \partial_x^2 - nx\partial_x\right)\, \phi + \tilde L \phi \right]\,, \tag{6.3.4}$$

where $\phi^1 = \left(1 - \frac{1}{n}\right) X^1$, $\phi^A = X^A/2$, with $\tilde{L}$ of the form (4.2.3), and the theory of Chapter 4 will apply if we can show the existence of some regularity interval, either in weighted Hölder or in weighted Sobolev spaces. Let us note that the indicial roots for $\Delta_{\tilde{L},\tilde{g}}$ are

$$(\mu_-, \mu_+) = (0, n+1)\,. \tag{6.3.5}$$

In this section we shall make appeal to some results of [**3**], let us shortly discuss the notational correspondence between [**3**] and this paper. We have $\tilde{g}$ here $\leftrightarrow g$ in [**3**], x here $\leftrightarrow \rho$ in [**3**], g here $\leftrightarrow h$ in [**3**], thus $\tilde{g} = x^{-2} g$ here while $g = \rho^{-2} h$ in [**3**]. Let X be an s-covariant r-contravariant tensor field, we have

$$\begin{aligned} X \in C^\alpha_{k+\lambda} \quad \text{here} \quad &\leftrightarrow \quad X \in C^{k,\lambda}_{-\alpha+s-r} \quad \text{in } [\mathbf{3}] \\ X \in H^\alpha_k(M,\tilde{g}) \quad \text{here} \quad &\leftrightarrow \quad X \in H^{2,k}_{-\alpha}(M,g) \quad \text{in } [\mathbf{3}] \\ &\leftrightarrow \quad X \in W^{\alpha-s+r;2}_k(M,g) \quad \text{as defined in Chapter 3}\,. \end{aligned}$$

We will also use spaces

$$H^{\alpha,\beta}_k(M,\tilde{g}) \leftrightarrow W^{\alpha-s+r,\beta;2}_k(M,g) \quad \text{as defined in Chapter 3.} \tag{6.3.6}$$

Let us start with a McKean–type inequality (*cf.* [**61**]):

PROPOSITION 6.3.1. Let g be a M^σ_1, $\sigma > 0$, metric on M and suppose that f is a C_1 tensor field with $\mathrm{supp} f \subset M_\epsilon$, with some $\epsilon > 0$. For $1 \le p$ and $\alpha > -(n-1)/p$ there exists $\epsilon_0 > 0$ such that for $0 < \epsilon \le \epsilon_0$ we have

$$\left\{\left[\frac{n-1}{p}\right] + \alpha - o(1)\right\} \|\, |Dx|_g\, |f|_{\tilde{g}} \|_{x^\alpha L^p(M_\epsilon, d\mu_{\tilde{g}})} \le \|\, |\tilde{D} f|_{\tilde{g}} \|_{x^\alpha L^p(M_\epsilon, d\mu_{\tilde{g}})}\,, \tag{6.3.7}$$

with $o(1) \to 0$ as $\epsilon \to 0$.

PROOF: Suppose first that f is a function. The identity

$$0 = \int_{M_\epsilon} \tilde{D}_i \left(|f|^p x^{-p\alpha-1} |Dx|^{p-2}_g \tilde{D}^i x\right) d\mu_{\tilde{g}}$$

written out in detail gives, after some rearrangements,

$$\begin{aligned} &\left(\frac{n-1}{p} + \alpha - o(1)\right) \int_{M_\epsilon} |Dx|^p_g |f|^p x^{-p\alpha} d\mu_{\tilde{g}} \\ &\qquad \le \int_{M_\epsilon} |Dx|^{p-1}_g |\tilde{D} f|_{\tilde{g}} |f|^{p-1} x^{-p\alpha} d\mu_{\tilde{g}}\,. \end{aligned} \tag{6.3.8}$$

From Hölder's inequality we obtain

$$\begin{aligned} &\int_{M_\epsilon} |Dx|^{p-1}_g |\tilde{D} f|_{\tilde{g}} |f|^{p-1} x^{-p\alpha} d\mu_{\tilde{g}} \\ &\le \left(\int_{M_\epsilon} |\tilde{D} f|^p_{\tilde{g}} x^{-p\alpha} d\mu_{\tilde{g}}\right)^{1/p} \left(\int_{M_\epsilon} |Dx|^p_g |f|^p x^{-p\alpha} d\mu_{\tilde{g}}\right)^{(p-1)/p} \end{aligned}$$

which together with (6.3.8) gives (6.3.7).

If f is a tensor, the result follows by applying (6.3.7) to the function $\phi(x)\,\left(\delta+|f|^2_{\tilde g}\right)^{\frac12}$ and passing to the limit $\delta\to 0$, where δ is a positive constant and ϕ is any function satisfying $\phi\in \mathring{C}_\infty(M_\epsilon)$, $\phi|_{\mathrm{supp} f}=1$. □

We now improve this for $p=2$ and for r–forms (we shall need the result for vectors, which are of course naturally identified with 1–forms). The following computational Lemma is useful:

LEMMA 6.3.2. Let g be a $M^\sigma_{2+\lambda}$ metric on M with $0<\sigma\le 1$ and let $\tilde H_{\log(x)}$ denote the Hessian of $\log(x)$ with respect to the metric $\tilde g$, where x is the defining function of ∂M as above and let $s\in\mathbb{R}$. Then the eigenvalues of $\tilde H_{\log(x)}$ satisfy

$$\lambda_i(\tilde H_{\log(x^s)})\ge -s|Dx|^2_g+O(x^\sigma)\,.$$

Moreover, the Laplacian $\Delta_{\tilde g}x^s$, with $\Delta_{\tilde g}=\tilde D_i\tilde D^i$, satisfies

$$\Delta_{\tilde g}x^s=-x^s\left[s(n-1-s)|Dx|^2_g+O(x^\sigma)\right]\,.$$

Further, if we let $d^{\tilde *}$ denote the exterior co–derivative w.r.t. $\tilde g$, then

$$d^{\tilde *}(\tilde Dx/|\tilde Dx|_{\tilde g})=-\tilde D_i(\tilde D^i x/|\tilde Dx|_{\tilde g})=(n-1)|Dx|_g+O(x^\sigma)\,.$$

In regular coordinates near ∂M the Riemann tensor $\tilde R_{ijkl}$ of $\tilde g$ satisfies

$$\tilde R_{ijkl}=x^{-4}[|Dx|^2_g(g_{ij}g_{kl}-g_{il}g_{jk})+O(x^\sigma)]\,,$$

and the Ricci tensor $\tilde R_{ij}$ of $\tilde g$ satisfies

$$\tilde R_{ij}=x^{-2}[-(n-1)|Dx|^2_g g_{ij}+O(x^\sigma)]\,.$$

The following result was originally proved for manifolds close to hyperbolic space by Donnelly and Xavier [**29**]. The following argument, due to Lee [**52**], gives the result in the general conformally compact case, *cf.* also [**3**] for a proof based on the argument of [**29**]. We give the proof here for completeness; by definition the norm $||\cdot||_{H_k(M_\epsilon,\tilde g)}$ here is the norm in $H^{0,0}_k(M_\epsilon,\tilde g)$, as defined in Section 3.1.3, page 17.

LEMMA 6.3.3. Let ξ be a C^∞ r–form with $\mathrm{supp}|\xi|_{\tilde g}\subset M_\epsilon$, let g be a $M^\sigma_1\cap C^{\mathrm{loc}}_2$ metric on M, $\sigma>0$. Then, for $r\le (n-1)/2$ or $r\ge(n+1)/2$,

(6.3.9)
$$\left[\frac{(n-1-2r)^2}{4}+o(1)\right]|||Dx|_g\xi||^2_{H_0(M_\epsilon,\tilde g)}\le ||d\xi||^2_{H_0(M_\epsilon,\tilde g)}+2||d^{\tilde *}\xi||^2_{H_0(M_\epsilon,\tilde g)}$$

PROOF: To avoid a proliferation of tilde's, we momentarily suspend the convention that geometric quantities refering to the metric $\tilde g$ are decorated with tildes. Thus, until specified otherwise all geometric quantities refer to the $\tilde g$ metric. Let $\{e_j\}$ denote any local orthonormal frame for TM and $\{e^j\}$ the corresponding dual coframe. Let D denote the covariant derivative of the metric acting on tensors, with $D_j \equiv D_{e_j}$. Recall the identities

$$\begin{aligned}
d\xi &= \sum e^j \wedge D_j \xi \,, \\
d^*\xi &= \sum -e^j \vee D_j \xi \,, \\
d(e^{-u}\xi) &= e^{-u}(d\xi - du \wedge \xi) \,, \\
d^*(e^u \xi) &= e^u(d^*\xi - du \vee \xi) \,, \\
d(du \vee \xi) &= -du \vee d\xi + H_u \xi + D_{Du}\xi \,, \\
|du|^2|\xi|^2 - (\Delta_H u)|\xi|^2 &= -e^{-u}\Delta_H(e^u)|\xi|^2 \,.
\end{aligned}$$

Here $\vee$ is the contraction operator[1], $H_u = Ddu$ denotes the Hessian of u and $H_u\xi$ denotes the induced operator on Λ^r,

$$H_u\xi = D_i du \wedge (e^i \vee \xi) \,.$$

In the calculation below, $\Delta_H = dd^* + d^*d$.

$$\begin{aligned}
& \int_M |e^u d(e^{-u}\xi)|^2 + |e^{-u} d^*(e^u\xi)|^2 \\
= & \int_M |d\xi - du \wedge \xi|^2 + |d^*\xi - du \vee \xi|^2 \\
= & \int_M |d\xi|^2 - 2\langle d\xi, du \wedge \xi\rangle + |du \vee \xi|^2 + |d^*\xi|^2 - 2\langle d^*\xi, du \vee \xi\rangle + |du \wedge \xi|^2 \\
= & \int_M \langle \xi, \Delta_H \xi\rangle - 2\langle du \vee d\xi, \xi\rangle + \langle du \vee (du \wedge \xi), \xi\rangle \\
& \qquad -2\langle \xi, d(du \vee \xi)\rangle + \langle du \wedge (du \vee \xi), \xi\rangle \\
= & \int_M \langle \xi, \Delta_H \xi\rangle + |du|^2|\xi|^2 - 2\langle \xi, H_u\xi\rangle - 2\langle \xi, D_{\mathrm{grad}_u}\xi\rangle \\
= & \int_M \langle \xi, \Delta_H \xi\rangle + |du|^2|\xi|^2 - 2\langle \xi, H_u\xi\rangle - (\Delta_H u)|\xi|^2 \\
= & \int_M \langle \xi, \Delta_H \xi\rangle - e^{-u}\Delta_H(e^u)|\xi|^2 - 2\langle \xi, H_u\xi\rangle
\end{aligned}$$

We can state this in the form

$$\begin{aligned}
\int_M \langle \xi, \Delta_H \xi\rangle &= \int_M \phi^{-1}\Delta_H\phi|\xi|^2 + 2\langle \xi, H_{\log\phi}\xi\rangle + |\phi d(\phi^{-1}\xi)|^2 + |\phi^{-1}d^*(\phi\xi)|^2 \\
&\geq \int_M \phi^{-1}\Delta_H\phi|\xi|^2 + 2\langle \xi, H_{\log\phi}\xi\rangle \,.
\end{aligned}$$

Let us now return to our convention, that quantities referring to the $\tilde{g}$ metric are decorated with tildes. Choosing $\phi = x^s$ with $s \neq 0, n-1$, a straightforward

[1] If α is a q–form and β a $p+q$–form, in local coordinates we have $(\alpha \vee \beta)_{i_1\dots i_p} = \alpha^{j_1\dots j_q}\beta_{j_1\dots j_q i_1\dots i_p}$, where the indices on $\alpha_{j_1\dots j_q}$ have been raised with the appropriate metric.

computation gives (recall that $\Delta_H = -\Delta_{\tilde g}$ when acting on functions)

$$\begin{aligned}\phi^{-1}\Delta_{\tilde g}\phi &= -s(n-1-s)|Dx|^2_g + o(1)\,,\\ \tilde H_{\log\phi} \equiv \tilde D_i \tilde D_j \log\phi &= sx^{-2}|Dx|^2_g(n_i n_j - g_{ij} + o(1))\,,\end{aligned}$$

with $n_i = D_i x/|Dx|_g$. Note that if $A \in \mathrm{End}(TM)$ is symmetric with eigenvalues bounded from below by λ then $\langle A\xi,\xi\rangle \geq \lambda r|\xi|^2$ when A is considered in $\mathrm{End}(\Lambda^r)$. This gives the inequality

(6.3.10)
$$[s(n-1-s-2r)+o(1)]\int_M |\xi|^2_{\tilde g}|Dx|^2_g d\mu_{\tilde g} \leq \int_M (|d\xi|^2_{\tilde g} + |d^{\tilde *}\xi|^2_{\tilde g})d\mu_{\tilde g}\,.$$

Here s is a free parameter and we maximize the left hand side of (6.3.10) w.r.t. s. This gives $s = (n-1-2r)/2$ which inserted into (6.3.10) gives the result. □

REMARK: Note that in the case $r = (n\pm 1)/2$ the above result does not give any information better that $\lambda_r \geq 0$ and this is in fact sharp, as can be shown for the case of constant sectional curvature.

PROPOSITION 6.3.4. Let ξ be a C^∞ r–form with $\mathrm{supp}|\xi|_{\tilde g} \subset M_\epsilon$, let g be a M^σ_2 metric on M with $\sigma > 0$. Then

$$\left[\frac{(n-1)^2}{4} + r' - o(1)\right] \|\, |Dx|_g\,\xi\|^2_{H_0(M_\epsilon,\tilde g)} \leq \|\, \tilde D\xi\|^2_{H_0(M_\epsilon,\tilde g)}\,. \tag{6.3.11}$$

where $r' = r$ if $r < n/2$ and $r' = n-r$ if $r > n/2$.

PROOF: This is immediate from the Weitzenbock formula (*cf., e.g.*, [**38**]),

$$< (d\tilde{d^*} + \tilde{d^*}d)\xi,\xi >_{\tilde g} = |\tilde D\xi|^2_{\tilde g} + < \tilde{\mathcal R}_r\xi,\xi >_{\tilde g} + \text{pure divergence}, \tag{6.3.12}$$

and the asymptotic behaviour of the Riemann tensor, *cf.* Lemma 6.3.2. *cf.* [**11**] or [**38**] for the precise form of $\tilde{\mathcal R}_r$. □

PROPOSITION 6.3.5. Let $g \in M^\sigma_2$, $\sigma > 0$, and suppose that

$$|\alpha| < \frac{n+1}{2}\sqrt{\frac{n}{2(n-1)}}\,.$$

For every $\beta \in \mathbb R$ there exists $\epsilon > 0$ and a constant C such that for every $X \in \mathring{C}_2(M_\epsilon)$ we have

(6.3.13)
$$\|\, |X|_{\tilde g}\|_{x^\alpha(1+|\ln x|)^\beta L^2(M_\epsilon,d\mu_{\tilde g})} \leq C\|\, |\Delta_{\tilde L,\tilde g} X|_{\tilde g}\|_{x^\alpha(1+|\ln x|)^\beta L^2(M_\epsilon,d\mu_{\tilde g})}\,.$$

REMARK: We believe that the above bound on α is not sharp, and we expect the inequality to hold for $|\alpha| < (n+1)/2$.

To prove Proposition 6.3.5 we need still another Lemma:

LEMMA 6.3.6. Let $\phi \in C_2(M)$, $X \in \mathring{C}_2(M_{x_0})$, set $Y = \phi X$. Under the hypotheses of Proposition 6.3.5 we have

$$
\begin{aligned}
* \quad \equiv \quad & -\int_{M_{x_0}} Y_i\,\Delta_{\tilde L,\tilde g}\,Y^i\,d\mu_{\tilde g} \le \frac{(n-1)}{n}\int_{M_{x_0}} |\tilde D\,\phi|^2_{\tilde g}\,|X|^2_{\tilde g}\,d\mu_{\tilde g} \\
& -\int_{M_{x_0}} \phi^2\,X_i\,\Delta_{\tilde L,\tilde g}\,X^i\,d\mu_{\tilde g}\,. \qquad (6.3.14)
\end{aligned}
$$

PROOF: From the Ricci identity we have

$$
2\Delta_{\tilde L,\tilde g}\,Y^i = \Delta_{\tilde g}\,Y^i + \tilde R^i{}_j\,Y^j + \frac{n-2}{n}\,\tilde D^i\,\tilde\delta Y\,, \qquad (6.3.15)
$$

where $\Delta_{\tilde g} = \tilde D_i\,\tilde D^i$ is the standard Laplacian, $\tilde R_{ij}$ is the Ricci tensor of $\tilde g_{ij}$, and $\tilde\delta\,Y = \tilde D_i\,Y^i$. This implies

$$
\begin{aligned}
* \quad &= \quad \frac{1}{2}(A+B+C)\,, && (6.3.16)\\
A \quad &= \quad \int |\tilde D\,Y|^2_{\tilde g}\,, && (6.3.17)\\
B \quad &= \quad -\int \tilde R_{ij}\,Y^i\,Y^j\,, && (6.3.18)\\
C \quad &= \quad \frac{n-2}{n}\int \left(\tilde D_i\,Y^i\right)^2\,. && (6.3.19)
\end{aligned}
$$

Setting $Y = \phi\,X$ the identity

$$
0 = \int \tilde D^i\left(\phi^2\,X^j\,\tilde D_i\,X_j\right)
$$

written out in detail and inserted in A gives

$$
A = \int |\tilde D\,\phi|^2_{\tilde g}\,|X|^2_{\tilde g} - \int \phi^2\,X_j\,\Delta_{\tilde g}\,X^j\,. \qquad (6.3.20)
$$

Similarly the identity

$$
0 = \int \tilde D_i\left(\phi^2\,X^i\,\tilde D_k\,X^k\right)
$$

yields

$$
C = \frac{n-2}{n}\int \left[\left(X^i\,\tilde D_i\,\phi\right)^2 - \phi^2\,X^i\,\tilde D_i\,\tilde D_k\,X^k\right]\,. \qquad (6.3.21)
$$

Substituting (6.3.20)–(6.3.21) in (6.3.16) one gets, using (6.3.15),

$$
* = \int\left\{\frac{1}{2}|\tilde D\,\phi|^2_{\tilde g}|X|^2_{\tilde g} + \frac{n-2}{2n}\left(X^i\,\tilde D_i\,\phi\right)^2 - \phi^2\,X_i\,\Delta_{L,\tilde g}\,X^i\right\}\,,
$$

which implies (6.3.14). □

PROOF OF PROPOSITION 6.3.5. Consider again the equation (6.3.16). We have

$$
-\tilde R_{ij}\,Y^i\,Y^j = (n-1)\,|Dx|^2_g\,(1+o(1))\,\phi^2\,|X|^2_{\tilde g}\,,
$$

so that using neglecting C in (6.3.16) and using (6.3.11) to estimate A one has

$$\left[\frac{(n+1)^2}{8} - o(1)\right] \int |Dx|_g^2\, \phi^2\, |X|_{\tilde g}^2\, d\mu_{\tilde g} \le * \,,$$

which together with (6.3.14) yields

$$\int \left\{ \left(\frac{(n+1)^2}{8} - o(1)\right) \phi^2\, |Dx|_g^2 - \frac{(n-1)}{n} |\tilde D\, \phi|_{\tilde g}^2 \right\} |X|_{\tilde g}^2 \le - \int \phi^2\, X_i\, \Delta_{\tilde L, \tilde g}\, X^i \,.$$

Let $\epsilon < 1$, set $\phi = x^{-\alpha}(1 + |\ln x|)^{-\beta}$, for $\alpha \neq 0$ we have

$$|\tilde D\, \phi|_{\tilde g} = |\alpha|\, x^{-\alpha}(1 + |\ln x|)^{-\beta})\, |Dx|_g \left(1 + O\left(\frac{1}{|\ln \epsilon|}\right)\right) \,,$$

while if $\alpha = 0$

$$|\tilde D\, \phi|_{\tilde g} = \beta\, |\ln x|^{\beta - 1}\, |Dx|_g \,,$$

so that in either case

$$|\tilde D\, \phi|_{\tilde g} = \phi\, |Dx|_g (|\alpha| + o(1)) \,,$$

thus

$$\begin{aligned} \left\{ \frac{(n+1)^2}{8} - \frac{(n-1)}{n}\, \alpha^2 - o(1) \right\} \|\phi |Dx|_g\, |X|_{\tilde g}\|_{L^2}^2 &\le - \int \phi^2\, X_i\, \Delta_{L, \tilde g}\, X^i\, d\mu_{\tilde g} \\ &\le \|\phi |X|_{\tilde g}\|_{L^2}\, \|\phi |\Delta_{L, \tilde g}\, X|_{\tilde g}\|_{L^2} \end{aligned}$$

which implies (6.3.13). □

Proposition 6.3.5 shows that Proposition 2.6 of [**3**] applies, and since $\Delta_{\tilde L, \tilde g}$ is formally self-adjoint, Corollary 2.7 of [**3**] shows:

PROPOSITION 6.3.7. *Let M be a manifold with boundary with $\bar M$ — compact. Suppose that $g \in M^\sigma_{k+2+\lambda}$, and $k \ge 0$. For*

$$|\alpha| < \frac{n+1}{2} \sqrt{\frac{n}{2(n-1)}} \,, \qquad \beta \in \mathbb{R} \,,$$

the operator

$$\Delta_{\tilde L, \tilde g} : H^{\alpha,\beta}_{k+2}(M, \tilde g) \to H^{\alpha,\beta}_k(M, \tilde g)$$

is Fredholm.

REMARK: The spaces $H^{\alpha,\beta}_k(M, \tilde g)$ are defined in (6.3.6).

THEOREM 6.3.8. *Let M be a manifold with boundary with $\bar M$ — compact. Let $g \in M^\sigma_{k+2+\lambda}$, $k \ge 1$, $\lambda \in [0,1)$, $\sigma > 0$, $k + \lambda > 1$ (cf. (6.2.5)) be a Riemannian metric on M, let $\tilde g = x^{-2}\, g$ where x is a defining function for ∂M. For*

$$|\alpha| < \frac{n+1}{2} \sqrt{\frac{n}{2(n-1)}} \,, \qquad \beta \in \mathbb{R} \,, \qquad 0 \le \ell \le k \,,$$

the maps

$$\begin{aligned} \Delta_{\tilde L, \tilde g} &: H^{\alpha,\beta}_{\ell+2}(M, \tilde g) \to H^{\alpha,\beta}_\ell(M, \tilde g) \,, \\ \Delta_{\tilde L, \tilde g} &: \mathring{H}{}^{\alpha,\beta}_1 \cap H^{\alpha,\beta}_{\ell+2}(M_{x_0}, \tilde g) \to H^{\alpha,\beta}_\ell(M_{x_0}, \tilde g) \,, \end{aligned} \tag{6.3.22}$$

are isomorphisms.

REMARKS:

(i) As noted at the beginning of this section, $H_m^{\alpha,\beta}(M,\tilde g) = W_m^{\alpha+1,\beta;2}(M,g)$ for vector fields.

(ii) In the physically interesting case $n=3$ we get $\alpha \in \left(-\sqrt{3},\sqrt{3}\right)$. In local coordinates near ∂M this corresponds, roughly speaking, to vector fields the components of which behave as x^β, $\beta \in (2-\sqrt{3}, 2+\sqrt{3})$.

PROOF: By Proposition 6.3.7 $\Delta_{\tilde L,\tilde g}$ is Fredholm, and formal self-adjointness of $\Delta_{\tilde L,\tilde g}$ together with duality arguments imply (*cf. e.g.* [**10, 3**]) that it is sufficient to show that the kernel of $\Delta_{\tilde L,\tilde g}$ on $H_2^{\alpha,\beta}(M,\tilde g)$ is zero for α's in the desired interval. Let thus

$$\Delta_{\tilde L,\tilde g} X = 0\,, \qquad X \in H_2^{\alpha,\beta}(M,\tilde g)\,.$$

The proof of Proposition 3.10 of [**3**] shows that we have $X \in H_2^{\alpha',\beta}(M,\tilde g)$ for any $\alpha' \in (-\omega_+,\omega_+)$ and β in $\mathbb{R}$, where $\omega_+ = \frac{n+1}{2}\sqrt{\frac{n}{2(n-1)}}$. In particular we have $X \in H_2^{0,0}(M,\tilde g)$, the standard H_2 Sobolev space on M equipped with the metric $\tilde g$. For all $Y \in H_2^{0,0}(M,\tilde g)$ which are compactly supported we have the integration by parts identity

$$\int_M Y_i\, \Delta_{\tilde L,\tilde g} Y^i = \int_M |\tilde L(Y)|^2_{\tilde g}\, d\mu_{\tilde g}\,, \tag{6.3.23}$$

where $\tilde L$ has been defined in (6.3.2), and a standard density argument shows that (6.3.23) holds for all (not necessarily compactly supported) $Y \in H_2^{0,0}(M,\tilde g)$. It follows that

$$0 = \int_M X_i\, \Delta_{\tilde L,\tilde g} X^i = \int_M |\tilde L(X)|^2_{\tilde g}\, d\mu_{\tilde g}\,, \tag{6.3.24}$$

and we conclude that

$$\tilde D_i X_j + \tilde D_j X_i - \frac{2}{n}\tilde D_k X^k \tilde g_{ij} = 0\,. \tag{6.3.25}$$

Elliptic regularity implies $X \in H_{k+2}^{\alpha,\beta}(M,\tilde g)$, and from Proposition 3.10 of [**3**] it follows that $X \in C^{1+\alpha'}_{0+\sigma}$, for any

$$|\alpha'| \le \frac{n+1}{2}\sqrt{\frac{n}{2(n-1)}}\,, \tag{6.3.26}$$

and for some $\sigma > 0$. If $\lambda > 0$ the scaling estimates (4.1.22) imply $X \in C^{1+\alpha'}_{k+\min(\lambda,\sigma)}$, while if $\lambda = 0$ then by hypothesis $k > 1$ and $X \in C^{1+\alpha'}_{k-1+\sigma}$, thus in either case $X \in C^{1+\alpha'}_1$. Conformal invariance of (6.3.25) implies that (6.2.6) holds as well, and the vanishing of X follows by Proposition 6.2.2. □

The two Theorems that follow can be proved by arguments similar to those used in the proof of Theorem 6.2.8, using Theorem 6.3.8 (*cf.* also [**6**] or Section 4.2.2 here), the details will be omitted. Let us start with an equivalent of Theorem 6.2.8:

THEOREM 6.3.9. Let $\bar{M}$ be a compact manifold (with boundary), suppose that g is a metric on $M \equiv \bar{M} \setminus \partial\bar{M}$ which can be C_∞ extended across ∂M, let x be any smooth defining function for ∂M. Consider the equation

$$\Delta_{\tilde{L},\tilde{g}} X = x^\alpha Y \,, \qquad Y \in C_\infty(\bar{M}) \,. \tag{6.3.27}$$

(i) For any α and Y there exists a solution of (6.3.27) of the form

$$X = x^\alpha X_\alpha + \log x X_{0,1} + X_0 + x^{n+1} \log^2 x X_{n+1,2} \,,$$

$$X_\alpha, X_{0,1}, X_0, X_{n+1,2} \in C_\infty(\bar{M}) \,.$$

(ii) If moreover $-\alpha \notin \mathbb{N}_0$ or if $X_{0,1}|_{\partial M} = 0$, then there exists a solution of (6.3.27) of the form

$$X = x^\alpha X_\alpha + X_0 + x^{n+1} \log x X_{n+1,1} \,,$$

$$X_\alpha, X_0, X_{n+1,1} \in C_\infty(\bar{M}) \,.$$

If $X_{n+1,1}|_{\partial M} = 0$, then $X_{n+1,1} \equiv 0$.

(iii) Any solution $X \in W_1^{\rm loc}$ of the homogeneous equation satisfying either 1) $X \in H_0^\alpha(M,\tilde{g})$ with $\alpha > -\frac{n+1}{2}\sqrt{\frac{n}{2(n-1)}}$, or 2) $X = O(x^\alpha)$ for some $\alpha > \frac{n+1}{2}(1 - \sqrt{\frac{n}{2(n-1)}})$, vanishes.

We also have the following equivalent of Theorem 6.2.10 (*cf.* Definition 6.2.9 for the definition of a polymogeneous metric). The reader should note that the result below holds for any polyhomogeneous metric g — this should be contrasted with Theorem 6.2.10 in which the metric g is assumed to be polyhomogeneous *and* uniformly C_2 up to boundary.

THEOREM 6.3.10. Let g be a polyhomogeneous metric on a compact manifold $\bar{M}$ with boundary, ∂M, consider the equation

$$\Delta_{\tilde{L},\tilde{g}} X = Y \,, \qquad Y \in \mathcal{A}^{\rm phg} \,. \tag{6.3.28}$$

There exists a solution $X \in \mathcal{A}^{\rm phg}$ of (6.3.28). Any solution of the homogeneous equation (6.3.28) satisfying either 1) $X \in H_0^\alpha(M,\tilde{g})$ with $\alpha > -\frac{n+1}{2}\sqrt{\frac{n}{2(n-1)}}$, or 2) $X = O(x^\alpha)$ for some $\alpha > \frac{n+1}{2}(1 - \sqrt{\frac{n}{2(n-1)}})$, vanishes.

As far as classical regularity of solutions of eq. (6.3.1) is concerned, we have the following Corollary of Theorems 6.3.8 and 4.2.11 (here the case $\dim M = 3$ only will be considered, other cases can be obtained in a similar way):

THEOREM 6.3.11. Suppose that $k \geq 2$, $\lambda \in (0,1]$, and let $g \in C_{k+1+\lambda}(\bar{M})$ be a Riemannian metric on a three dimensional manifold with boundary M, with $\bar{M}$ – compact. Define

$$(\alpha_-, \alpha_+) = \left(2 - \sqrt{3}, 2 + \sqrt{3}\right) \,.$$

Consider the equation

$$\Delta_{\tilde L,\tilde g} X = Y\,, \qquad Y \in x^\alpha\, C_{\hat k+\lambda}(\bar M)\,,$$
$$\alpha > \alpha_-\,, \qquad 1 \le \hat k \le k\,.$$

Suppose that either

$$u = O(x^{\alpha_-+\epsilon})\,, \quad \epsilon > 0\,,$$

or

$$u \in W_0^{\omega;2}(M_{x_1}, x^{-n} d\mu_g)\,, \quad \omega > 1+\sqrt{3}\,.$$

Then there exists $\sigma > 0$ such that the following hold:

(i) Let $\alpha < 2+\sqrt{3}$ and let $\ell_0 \in \mathbb{N}_0$ be the largest number such that $\alpha+\ell_0 < 4$.
  (a) $\alpha + \hat k < 5 \quad\Longrightarrow\quad X \in \cap_{i=0}^3 x^{\alpha-i}\, C_{\hat k-1+i+\sigma}(\bar M)$.
  (b) $\hat k = \ell_0 + k_0 + 1 \quad\Longrightarrow\quad X \in \cap_{i=0}^3 x^{\alpha-i}\, C_{\ell_0+i+\sigma|k_0}(\bar M)$.

(ii) For $\alpha \in [2+\sqrt{3}, 4)$ the conclusions of point (i) hold with k there replaced by $k-1$.

(iii) Let $\alpha \in \mathbb{N}_0 \cap [1,4]$. If $\hat k \ge 6-\alpha$, then there exists

$$X_{\log} \in \cap_{i=-(4-\alpha)}^3 x^{\alpha-i} \log x\, C_{\hat k-1+i+\lambda}(\bar M)$$

such that

$$X - X_{\log} \log x \in \cap_{i=0}^3 x^{\alpha-i} C_{\hat k-1+i+\sigma}(\bar M)\,.$$

If $X_{\log}|_{\partial M} = 0$, then $X_{\log} \equiv 0$.

(iv) Let $\alpha > 4$, suppose that $\hat k \ge 3$, and let $\ell_0 \in \mathbb{N}_0$ be the largest integer such that $4+\ell_0 \le \alpha$.
  (a) If $2+\hat k < \alpha$ or if $\alpha \in \mathbb{N}$ (or both) $\quad\Longrightarrow\quad X \in \cap_{i=0}^4 x^{4-i} C_{\hat k-2+i+\sigma}(\bar M)\,.$
  (b) $\hat k > \ell_0 + 2 \quad\Longrightarrow\quad X \in \cap_{i=0}^4 x^{4-i} C_{\ell_0+i+\sigma|\hat k-2-\ell_0}(\bar M)\,.$

CHAPTER 7

The Lichnerowicz equation.

7.1. Introductory remarks

In this chapter we shall examine the boundary behaviour of the solutions of the scalar constraint equation, under various hypotheses on the differentiability of the metric: this is obtained using the methods of Chapter 4 together with a fairly standard "bootstrap" procedure. We shall construct solutions of the Lichnerowicz–type equation,

$$\frac{4(n-1)}{n-2}\Delta_{\tilde g}\,\phi - \tilde R\phi + \zeta\phi^{-\kappa} - n(n-1)\,\phi^{\frac{n+2}{n-2}} = 0\,, \qquad \kappa > 0\,, \tag{7.1.1}$$

with the boundary condition

$$\phi \underset{x\to 0}{\longrightarrow} |Dx|_g^{\frac{n-2}{2}}\,. \tag{7.1.2}$$

Let us point out that under suitable conditions on the metric any solution of (7.1.1) which is uniformly bounded above and uniformly bounded away from zero has to satisfy (7.1.2), *cf.* Theorem 7.3.2. Here κ is a positive constant[1], ζ is a function satisfying $\zeta \geq 0$. $\Delta_{\tilde g}$ is the Laplace operator of the metric $\tilde g_{ij} = x^{-2} g_{ij}$, $\Delta_{\tilde g} = \tilde D_i \tilde D^i$, and $\tilde R$ is the scalar curvature of $\tilde g$, $n = \dim M \geq 3$. If $n = 3$ and $\kappa = 7$ the equation (7.1.1) is of the form (2.1.23). Note also that if $\zeta = 0$ then (7.1.1) is the Yamabe equation (with the scalar curvature of the metric $\phi^{4/(n-2)}\tilde g_{ij}$ equal to $-n(n-1)$).

We have the equality

$$\tilde R = -n(n-1)\,|Dx|_g^2 + 2(n-1)x\Delta_g x + x^2 R\,, \tag{7.1.3}$$

where $\Delta_g = D_i D^i$ is the Laplace operator of the metric g, and R is the Ricci scalar of g. If we assume that $\zeta = o(1)$, $\Delta_g x = o(x^{-1})$ and $R = o(x^{-2})$ (which holds *e.g.* if $g \in C_{k+2}(\bar M)$ or if[2] $g \in M^{\epsilon}_{k+2+\lambda}$, with $\epsilon > 0, k \in \mathbb{N}_0, \lambda \in [0,1]$), then from (7.1.1)–(7.1.3) it follows that $\Delta_{\tilde g}\,\phi \longrightarrow_{x\to 0} 0$ if and only if (7.1.2)

[1] Most of the results of this chapter remain true when $\kappa < -1$ and $\zeta \leq 0$. When $\kappa \in (-1, 0]$ some further restrictions than non–positivity of ζ are needed for the existence theorem, *cf. e.g.* section 2.2 for a discussion of an example with $\kappa = -1$.

[2] Recall that the space $M^{\epsilon}_{k+2+\lambda}$ has been defined in Definition 6.2.1, page 66.

holds, except perhaps when $\phi \to 0$. In that last case one does not expect $(M, \phi^{4/(n-2)}\tilde{g})$ to be complete; it certainly will not if $\phi \in C_1(\bar{M})$. This shows that the boundary condition (7.1.2) is necessary if the solution ϕ has to have some degree of regularity at ∂M (*e.g.*, if we look for solutions $\phi \in C_2(\bar{M})$), and if we require completeness of $(M, \phi^{4/(n-2)}\tilde{g}))$.

Note that by carrying out a conformal transformation $g \to \psi^2 g$, where ψ is any strictly positive function of the same differentiability as the metric and which coincides with $|Dx|_g$ near ∂M, we get

$$|Dx|_g \equiv 1 \tag{7.1.4}$$

in a neighbourhood of ∂M. Assuming (7.1.4), the linearization of (7.1.1) at $\phi = 1$ gives an operator which in a neighbourhood of ∂M takes the form

$$L = \Delta_{\tilde{g}} + \xi \,, \tag{7.1.5}$$

If we make the same hypotheses about the metric and ζ as in the discussion following eq. (7.1.3), we obtain

$$\xi = -\frac{(n-2)}{4(n-1)}\left[\tilde{R} + \frac{(n-1)n(n+2)}{(n-2)} + \kappa\zeta\right] \underset{x\to 0}{\longrightarrow} -n\,. \tag{7.1.6}$$

We can now rewrite (7.1.1) in the form

$$Lu = F(y,u) + S\,, \qquad F(y,0) = \frac{\partial F(y,u)}{\partial u}\Big|_{u=0} = 0\,, \tag{7.1.7}$$

with

$$u \equiv \phi - 1 \,,$$

L as in (7.1.5) and, assuming (7.1.4),

$$S = \frac{(n-2)}{4(n-1)}(\tilde{R} + n(n-1) - \zeta)\,, \tag{7.1.8}$$

$$\begin{aligned} F(y,u) &= \frac{n(n-2)}{4}\left\{(1+u)^{(n+2)/(n-2)} - 1 - \frac{n+2}{n-2}u\right\} \\ &\quad -\frac{(n-2)}{4(n-1)}\zeta\Big\{(1+u)^{-\kappa} - 1 + \kappa u\Big\}\,. \end{aligned} \tag{7.1.9}$$

Let us point out that the form (7.1.7) of the Lichnerowicz equation is sufficiently general to cover many other matter models than the vacuum case considered above, *cf.* the discussion in Section 2.2. For this reason, and with future applications in mind, we will in the rest of this chapter discuss equations of this general form.

7.2. The linearized equation

Before passing to the non-linear equation (7.1.1) it will be necessary to give a detailed discussion of the properties of the linear operator (7.1.5). In order to be able to apply the results of Chapter 4 to that equation we have to prove existence of a regularity interval for L. It has been shown in [**3**] that $(\alpha_-,\alpha_+) = (-(n+1)/2, (n+1)/2)$ is a strong regularity interval for L for $H_0^\alpha(M,\tilde g) \equiv W_k^{0,0;2}(M,g)$, with $W_k^{0,0;2}(M,g)$ defined in Section 3.1.3. This and Theorem 4.2.3 show that $(\alpha_-,\alpha_+) = (-1,n)$ is a regularity interval for $C^\alpha_{k+\lambda,\ell}$, with $(n-1)/2 < \ell \le k$. We shall show that a direct analysis in the framework of weighted Hölder spaces gives $(\alpha_-,\alpha_+) = (-1,n)$ as a *strong* regularity interval for $C^\alpha_{k+\lambda}$.

Let us start with the observation, that for $g \in C_{k+2+\lambda}(\bar M)$, $k\in\mathbb{N}_0$, $\lambda\in(0,1)$, we can write

$$\begin{aligned} L &= x^2\left(\Delta_g - \frac{(n-2)}{x}\, D^i x\,\partial_i\right) + \xi \\ &= |Dx|^2_g\{\,(x^2\partial_x^2 - x\,(n-2)\,\partial_x - n) + \tilde L\}\,. \end{aligned} \tag{7.2.1}$$

It follows that after a trivial rescaling L can be written in the form (4.2.1)–(4.2.2), with $\tilde L$ of the form (4.2.3). The indicial roots for (7.2.1) are (*cf.* (4.2.7))

$$\mu_- = -1\,, \qquad \mu_+ = n\,.$$

To motivate the hypotheses of the theorem to follow, let us note that if $g \in M^\epsilon_{k+2+\lambda}$, $k\in\mathbb{N}_0$, $\lambda\in(0,1)$ for some $\epsilon>0$, then we can still formally define $\tilde L$ using eq. (7.2.1). Doing so and assuming (7.1.4) we obtain

$$\forall\alpha\in\mathbb{R} \qquad \tilde L x^\alpha = O(x^{\alpha+\epsilon})\,. \tag{7.2.2}$$

In the result that follows, a *scalar* operator is defined as an operator which maps functions on M to functions on M.

THEOREM 7.2.1. Let $L \in OP^2_{C^0_{k+\lambda}(M)}$, $k\in\mathbb{N}_0$, $\lambda\in(0,1)$, be a scalar geometric elliptic operator satisfying (4.2.1)–(4.2.2) and (4.2.4), with $\tilde L$ satisfying (7.2.2) and with indicial exponents $\mu_- < \mu_+$ (*cf.* eq. (4.2.7)). Consider the equation

$$Lu = f\,. \tag{7.2.3}$$

Then

(i) For every $\beta\in\mathbb{R}$ and $\alpha\in(\mu_-,\mu_+)$ there exists $0 < x_1 \le x_0$ such that for every $f \in C^{\alpha,\beta}_{k+\lambda}(M_{x_1})$, $\psi \in C_{k+2+\lambda}(\tilde\partial M_{x_1})$, there exists a solution $u \in C^{\alpha,\beta}_{k+2+\lambda}(M_{x_1})$ of (7.2.3) satisfying $u|_{\tilde\partial M_{x_1}} = \psi$. Moreover L has a strong regularity interval (α_-,α_+) for $C^{\alpha,\beta}_{k+\lambda}(M)$ with $\alpha_\pm = \mu_\pm$, and with no restrictions on β.

(ii) Assume further that $\bar M$ is compact, with $\mu_+ > 0$ and with L – elliptic in the standard sense on CM_{x_0}, with coefficients in $C^{k+\lambda}(CM_{x_0})$ and with $a_0 \le 0$, where a_0 is the 0–order coefficient of L in (4.1.3) (this condition on a_0 is supposed to hold throughout M, not only near ∂M). Then $L : C^{\alpha,\beta}_{k+2+\lambda}(M) \to C^{\alpha,\beta}_{k+\lambda}(M)$ is an isomorphism.

REMARKS:

(i) Assume that g is a $M^{\epsilon}_{k+1+\lambda}$ metric on a compact manifold $\bar{M}$ (with boundary), $k \geq 0$, $\lambda \in (0,1)$, $\epsilon > 0$, and $\tilde{g}_{ij} = x^{-2} g_{ij}$, where x is a defining function for ∂M. Then Theorem 7.2.1 applies to equations of the form

$$Lu \equiv \Delta_{\tilde{g}} u + x^{-1} \xi_1 \, \tilde{g}(dx, du) + \xi_2 u = f \,, \tag{7.2.4}$$

provided that $\xi_1 = p + \psi_1$, $\xi_2 = q|Dx|^2_g + \psi_2$, with $\psi_a \in C^{\epsilon}_{k+\lambda}$, a=1,2, and with $\delta \equiv \left(\frac{n-1-p}{2}\right)^2 - q > 0$. In that case the indicial exponents are $\mu_\pm = \alpha_\pm = \frac{n-1-p}{2} \pm \sqrt{\delta}$.

(ii) Graham and Lee [**42**, Theorem 3.10] have proved the following related result (we use here the notation of Remark (i) above): If $(M, \tilde{g})$ is the hyperbolic space ($M \approx \mathbb{R}^n$, with $\tilde{g}$ — the maximally symmetric metric of sectional curvature equal to -1) and $\xi_1 = \psi_2 = 0$, then for $\alpha \in (\mu_-, \mu_+)$ the operator L defined in (7.2.4) is an isomorphism between $C^{\alpha}_{k+2+\lambda}(M)$ and $C^{\alpha}_{k+\lambda}(M)$.

PROOF: For $\sigma \in \mathbb{R}$, let L_σ be the operator defined by $L_\sigma u = x^\sigma L(x^{-\sigma} u)$. Then L_σ is of the above form with the indicial roots $\tilde{\mu}_\pm = \mu_\pm + \sigma$ (*cf.* eqs. (4.2.7)–(4.2.4)). Let $\tilde{\alpha} = \alpha + \sigma$ and $f_\sigma = x^\sigma f$. By choosing $\sigma > -\mu_-$ and considering the equivalent equation $L_\sigma u = f_\sigma$ we may without loss of generality assume that $\tilde{\alpha} > \tilde{\mu}_- > 0$ in the following. Now $\tilde{L}_\sigma$ can be written in the form

$$\tilde{L}_\sigma = x^2 a^{xx} \frac{\partial^2}{\partial x^2} + x^2 a^{xA} \frac{\partial}{\partial x} \frac{\partial}{\partial v^A} + x^2 a^{AB} \frac{\partial}{\partial v^A} \frac{\partial}{\partial v^B} + x a^A \frac{\partial}{\partial v^A} \quad + x a^x \frac{\partial}{\partial x} + a_0 \,.$$

Setting $\alpha = 0$ in (7.2.2) we obtain $a_0 = O(x^\epsilon)$. Note moreover that the constant b of eq. (4.2.2) for L_σ so defined must be negative as $\tilde{\mu}_- > 0$. It follows that the comparison principle applies for L_σ in M_{x_0}, provided that x_0 is chosen small enough.

Consider (7.2.3) with $f \in C^{\alpha,\beta}_{k+\lambda}(M)$. A straightforward calculation shows that there exist constants C, $0 < x_1 = x_1(\alpha, \beta, L) \leq x_0$ such that for $x \leq x_1$ the functions

$$\phi_\pm = \pm C \, \|f_\sigma\|_{C_0^{\tilde{\alpha},\beta}} \, x^{\tilde{\alpha}} \left(1 + \ln \frac{1}{x}\right)^\beta$$

are super– and sub–solutions of (7.2.3):

$$(L_\sigma \phi_+ - f_\sigma)\big|_{M_{x_1}} \leq 0 \,, \qquad (L_\sigma \phi_- - f_\sigma)\big|_{M_{x_1}} \geq 0 \,.$$

Choose a decreasing sequence $\{x_k\}_{k=0}^\infty \subset \mathbb{R}$ so that $x_k \to 0$ as $k \to \infty$. As the 0–th order coefficient of L_σ is negative on M_{x_0}, there is a unique solution u_k of the equation $L_\sigma u_k = f_\sigma$ in CM_{x_k} satisfying $u_k\Big|_{\tilde{\partial} M_{x_k}} = 0$ and $u_k\Big|_{\tilde{\partial} M_{x_1}} = x_1^{-\sigma}\psi$. By a standard diagonalization procedure we can find a subsequence of $\{u_k\}$

which converges to some $u_\sigma \in C_0^{\tilde\alpha,\beta}(M_{x_1}) \cap C_2^{\mathrm{loc}}(M_{x_1})$. The scaling estimates of Lemma 4.1.1 near the boundary give $u_\sigma \in C_{k+2+\lambda}^{\tilde\alpha,\beta}(M_{x_1})$. This proves existence of a solution in the appropriate space, uniqueness follows from the maximum principle. It follows that L_σ is an isomorphism between $\{u \in C_{k+2+\lambda}^{\tilde\alpha,\beta}(M_{x_1}) : u|_{\tilde\partial M_{x_1}} = \psi\}$ and $C_{k+\lambda}^{\tilde\alpha,\beta}(M_{x_1})$. Consequently L is an isomorphism between $\{u \in C_{k+2+\lambda}^{\alpha,\beta}(M_{x_1}) : u|_{\tilde\partial M_{x_1}} = \psi\}$ and $C_{k+\lambda}^{\alpha,\beta}(M_{x_1})$. Now the existence of a regularity interval with $\alpha_\pm = \mu_\pm$ follows from point 1 of Proposition 4.1.2. The fact that (μ_-, μ_+) is a *strong* regularity interval follows by construction from the form of the barriers $\phi_\pm$.

To prove point 2, note that if $Lu = 0$ in M and $u = O(x^{\mu_-+\epsilon})$ for some $\epsilon > 0$, then it follows by the definition of a regularity interval and by point 1 of this Theorem that $u = O(x^{\mu_+-\epsilon})$ for any $\epsilon > 0$. As $\mu_+ > 0$ by hypothesis, it follows that u goes to zero at ∂M, and then $u \equiv 0$ by the maximum principle. Also note that there exists a constant C such that the functions

$$\tilde\phi_\pm = \pm C \min\left(1, Cx^\alpha(1+\ln\frac{1}{x})^\beta\right) \|f\|_{C_0^{\alpha,\beta}}$$

are (weak) super– and sub–solutions of (7.2.3). The result follows by similar arguments as in the proof of point 1. □

What has been said so far can be summarized in the following:

Proposition 7.2.2. Let $g \in C_{k+2+\lambda}(\bar M)$ and let L be as in (7.2.1). Then L is a geometric elliptic operator in the sense of Chapter 4. Moreover L is of the form (4.2.1)–(4.2.4) with $(\mu_-, \mu_+) = (-1, n)$, and L has a strong regularity interval (α_-, α_+) for $C_{0+\lambda}^{\alpha,\beta}$, with $\alpha_\pm = \mu_\pm$. The same holds for $g \in M_{k+2+\lambda}^\epsilon$, $\epsilon > 0$.

Proposition 7.2.2 shows that we can use Theorems 4.1.3 and 4.2.1 to obtain tangential regularity of solutions u of the linearized counterpart of eq. (7.1.1) whenever the metric and the source f are tangentially regular. Similarly it follows that Theorems 4.2.6, 4.2.9 and 4.2.10 can be applied to give information about the classical regularity of the solutions to equation (7.2.3):

Corollary 7.2.3. Let $\bar M$ be compact and assume that the hypotheses of Theorem 7.2.1 hold. Assume further that $L \in OP^2_{C_{k+\lambda}(\bar M)}$ is elliptic in a standard sense in CM_{x_0}, and that $\tilde L$ is of the form (4.2.3). Consider the equation (7.2.3) with

$$f \in x^\alpha C_{k+\lambda}(\bar M), \qquad \alpha > \mu_-, \qquad u = O(x^{\mu_-+\epsilon}), \qquad \epsilon > 0\,.$$

Then there exists $\sigma \in (0,1)$ such that the following results hold:

(i) Let $\alpha < \mu_+$ and let $\ell_0 \in \mathbb{N}_0$ be the largest integer such that $\alpha + \ell_0 < \mu_+$.
  (a) $\alpha + k < \mu_+ \implies u \in \cap_{i=0}^2 x^{\alpha-i}\, C_{k+i+\sigma}(\bar M)$.
  (b) $k = \ell_0 + k_0 \implies u \in \cap_{i=0}^2 x^{\alpha-i}\, C_{\ell_0+i+\sigma|k_0}(\bar M)$.

(ii) Let $\alpha \le \mu_+$, $\mu_+ - \alpha \in \mathbb{N}_0$. If $k \ge \mu_+ - \alpha + 1$, then there exists $u_{\log} \in \cap_{i=0}^{\mu_+-\alpha+2} x^{\mu_+-i}\, C_{k-(\mu_+-\alpha)+i+\lambda}(\bar M)$ such that

$$u - u_{\log}\log x \in \cap_{i=0}^2 x^{\alpha-i} C_{k+i+\sigma}(\bar M)\,.$$

If $u_{\log}|_{\partial M} = 0$, then $u_{\log} \equiv 0$.

(iii) Let $\alpha > \mu_+$, suppose that $k \geq 1$, and let $\ell_0 \in \mathbb{N}_0$ be the largest integer such that $\mu_+ + \ell_0 < \alpha$.

(a) If $\mu_+ + k < \alpha$ or if $\alpha - \mu_+ \in \mathbb{N}$ (or both)then

$$u \in \cap_{i=0}^{2} x^{\mu_+ - i} C_{k+i+\sigma}(\bar{M}) .$$

(b) $k > \ell_0 \quad \Longrightarrow \quad u \in \cap_{i=0}^{2} x^{\mu_+ - i} C_{\ell_0+i+\sigma|k-\ell_0}(\bar{M})$.

(c) If $u = o(x^{\mu_+})$ and if $k \geq \tilde{\ell}$, where $\tilde{\ell}$ is the smallest integer such that $\tilde{\ell} \geq \alpha - \mu_+$, then

$$u \in \cap_{i=0}^{\tilde{\ell}+2} x^{\alpha - i} C_{k-\tilde{\ell}+i+\sigma}(\bar{M}) .$$

The above results provide rather detailed information about the linear counterpart of eq. (7.1.1).

7.3. Existence of solutions of the non–linear problem

In order to prove existence of solutions to equation (7.1.1) we will first consider a somewhat more general class of equations.

PROPOSITION 7.3.1. Consider the equation for a scalar function u

$$\Delta_{\tilde{g}} u + X(u) + F(y, u) = 0 , \tag{7.3.1}$$

where $\tilde{g} = x^{-2} g$ with x a defining function for ∂M, g a (locally) C_1 metric on M, X a continuous vector field on M, $F(y, u)$ a continuous function in both arguments. Suppose that

(i) There exists a function $u_0 \in C_0(\bar{M}) \cap C_2^{\mathrm{loc}}(M)$ and constants $x_0, C_1, \alpha > 0$ such that

$$\forall\ 0 < x < x_0 \qquad |\Delta_{\tilde{g}} u_0 + X(u_0)| \leq C_1 x^{\alpha} .$$

We shall moreover require that $du_0 \in C_0^{\alpha-1}(M_{x_0})$.

(ii) There exist constants $C_\pm, L$ satisfying $C_- \leq u_0|_{M_{x_0}} \leq C_+$ such that

$$\forall y \in M, \qquad F(y, C_-) \geq 0,\ F(y, C_+) \leq 0 ,$$
$$\forall z_1, z_2 \in [C_-, C_+],\ y \in M \qquad |F(y, z_1) - F(y, z_2)| \leq L|z_1 - z_2| .$$

(iii) There exist constants $C_2 > 0$, $x_1 > 0$ and β satisfying $0 < \beta \leq \alpha$ such that for all $0 \leq x \leq x_1$ we have

$$\Delta_{\tilde{g}} x^{\beta} + X(x^{\beta}) \leq -C_2 x^{\beta} .$$

(iv) There exists a constant $C_3 > 0$ such that for $0 < x < x_0$ we have

$$u_0 \leq u \leq C_+ \quad \Longrightarrow \quad F(y, u) \leq C_3 x^{\alpha} ,$$
$$C_- \leq u \leq u_0 \quad \Longrightarrow \quad F(y, u) \geq -C_3 x^{\alpha} .$$

Then there exists a solution $u \in W_2^{\mathrm{loc};p}(M)$ for any $1 < p \le \infty$ of equation (7.3.1) satisfying

$$C_- \le u \le C_+, \qquad |u - u_0|\Big|_{M_{x_0}} \le C_4 x^\beta \tag{7.3.2}$$

for some constant C_4.

REMARKS: 1. The solution u needs not be unique.

2. Note that point (iv) above will be satisfied if $|F(y, u_0)| \le C_3 x^\alpha$ and if $F(y, u)$ is monotonously decreasing in u for $0 < x < x_0$ and $C_- < u < C_+$.

PROOF: We shall show that there exist weak barriers $u_\pm \in W_1^{\mathrm{loc};\infty}(M)$, the result then follows by a well known method using *e.g.* the monotone iteration scheme (*cf. e.g.* [**66**, Theorem 2.3.1]). Let thus

$$u_+ = \min(u_0 + Bx^\beta, C_+),$$

$$u_- = \max(u_0 - Bx^\beta, C_-),$$

with some constant B which shall be specified later. Let $x_+ = \min(x_0, x_1)$. For those $x \le x_+$ for which $u_+ = u_0 + Bx^\beta$ we have

$$\begin{aligned} \Delta_{\tilde g} u_+ + X(u_+) + F(y, u_+) &= \Delta_{\tilde g} u_0 + X(u_0) \\ &\quad + B(\Delta_{\tilde g} x^\beta + X(x^\beta)) + F(y, u_+) \\ &\le (C_1 + C_3)x^\alpha - BC_2 x^\beta \end{aligned}$$

Let us set $B_+ = \max\left(\frac{(C_1+C_3)x_+^{\alpha-\beta}}{C_2}, (C_+ - \min_{\bar M} u_0)x_+^{-\beta}\right)$; for all $B \ge B_+$ we then have

$$\Delta_{\tilde g} u_+ + X(u_+) + F(y, u_+) \le 0.$$

Define Γ to be that connected component of the set $\{u_0 + Bx^\beta = C_+\}$ which intersects $\{0 < x < x_+\}$. By the definition of B_+, if $B \ge B_+$ we have $\Gamma \subset \{0 \le x \le x_+\}$. Increasing B if necessary, we can assume that $D(u_0 + Bx^\beta)$ does not vanish anywhere in $\{0 \le x \le x_+\}$ so that Γ is a C_2 submanifold of M. An integration by parts calculation shows that u_+ is a weak supersolution of equation (7.3.1). Similarly one shows that increasing B if necessary u_- will be a weak subsolution of equation (7.3.1) and the result follows. □

Existence of bounded solutions of (7.1.1) (with $\zeta = 0$) has been proved under various conditions on $(M, \tilde g)$ in [**8, 9**] and [**6**]. As far as existence of solutions of (7.1.1) satisfying (7.1.2) is concerned, we have the following theorem:

THEOREM 7.3.2. Let $g \in M^\epsilon_{k+2+\lambda}$ be a metric on a compact manifold $\bar M$ with boundary, $0 \le \zeta \in C^\epsilon_{k+\lambda}(M)$, $0 < \epsilon \le 1$, $\lambda \in (0,1)$, $k \ge 0$. There exists a solution ϕ of (7.1.1) such that

$$\phi - |Dx|_g^{\frac{n-2}{2}} \in C^\epsilon_{k+2+\lambda}(M)\,. \tag{7.3.3}$$

ϕ is unique in the class of uniformly bounded, uniformly bounded away from zero, locally C_2 solutions of (7.1.1).

REMARK: For $\zeta = 0$, it has been proved in [**6**, § 3] that, for $g \in C_\infty(\bar M)$, the condition $x^{-(n-2)/2}\phi \to_{x\to\partial M} \infty$ implies uniqueness. (Although the uniqueness question is not discussed in [**62**] in the same context as in [**6**], it should be pointed out that the *a–priori* estimate which is the key for the uniqueness proof of [**6**] has also been obtained in [**62**].) It is not hard to see that this argument holds also when $\zeta \in xC_\infty(\bar M)$, $\zeta \geq 0$. For $\zeta = 0$ and $g \in C_\infty(\bar M)$ it has been announced without proof in [**59**] that u is unique provided that the manifold $\bar M$ equipped with the topology induced by the metric $\phi^{4(n-2)}g$ is complete and is *not* a compact manifold with boundary.

PROOF: We may assume without loss of generality that $|Dx|^2_g \equiv 1$ in a neighbourhood of ∂M. The argument of the proof of theorem 3.4 of [**6**] shows that there exists a function $\phi_1 \in C^0_{k-2+\lambda}(M)$ satisfying $0 < C_1 \leq \phi_1 \leq C_2$ for some constants $C_1, C_2 > 0$, and satisfying $\phi_1 - 1 \in C^\epsilon_{k+2+\lambda}(M)$, such that the metric $\phi_1^{\frac{4}{n-2}}\tilde g$ has constant scalar curvature equal to $-n(n-1)$. Replacing $\tilde g$ by $\phi_1^{\frac{4}{n-2}}\tilde g$ we may without loss of generality assume that $\tilde R = R(\tilde g) = -n(n-1)$. Replacing ϕ by $\phi\phi_1^{-1}$ one obtains an equation of the form (7.1.1) with possibly a different $\zeta \geq 0$. One readily checks that the hypotheses of Proposition 7.3.1 are satisfied with $u_0 = 1$, $\alpha = \beta = \epsilon$, $C_- = 1$ and C_+ – a sufficiently large positive constant. Further regularity than that asserted by Proposition 7.3.1 follows by elliptic regularity and a scaling estimate (Lemma 4.1.1).

To prove uniqueness, let ϕ_2 be the solution constructed above and let ϕ_1 be any other solution bounded from above and bounded away from zero. Conformally rescaling the metric we can choose a "conformal gauge" in which $\phi_2 = 1$. In this gauge eq. (7.1.1) reads

$$\tilde R = -n(n-1) + \zeta\,.$$

Here $\zeta \geq 0$ is an appropriately rescaled counterpart of the function ζ of eq. (7.1.1). The eq. satisfied by $\phi \equiv \phi_1/\phi_2$ is eq. (7.1.1) with $\tilde R$ as above, so that we have

$$\frac{4(n-1)}{n-2}\,\Delta_{\tilde g}\,\phi + \zeta(\phi^{-\kappa} - 1) + n(n-1)(\phi - \phi^{\frac{n+2}{n-2}}) = 0\,. \tag{7.3.4}$$

Set

$$\alpha \equiv \inf_M \phi\,.$$

Suppose first that α is attained at some point $p \in M$, in that case the maximum principle shows that α cannot be smaller than one. Suppose next that α is not attained on M, then by [**42**, Theorem 3.5] there exists a sequence of points $p_k \in M$, $p_k \to p \in \partial M$, such that [3]

$$\phi(p_k) \to \alpha\,, \quad \liminf_{k\to\infty} \Delta_{\tilde g}\,\phi(p_k) \geq 0\,.$$

[3] Here lim inf is taken in $\mathbb{R} \cup \{\pm\infty\}$.

Evaluating eq. (7.3.4) at p_k and passing to the limit $k \to \infty$ one obtains

$$\zeta(\alpha^{-\kappa} - 1) + n(n-1)(\alpha - \alpha^{\frac{n+2}{n-2}}) \le 0 \,.$$

It follows again that α cannot be smaller than one, so that we must have

$$\inf_M \phi \ge 1 \,.$$

One similarly shows that

$$\sup_M \phi \le 1 \,,$$

so that $\phi_1 = \phi_2$ follows. □

7.4. Regularity at the boundary of the solutions

7.4.1. Polyhomogeneous or smooth backgrounds. The result that follows establishes polyhomogeneity for a class of semilinear equations when the metric is polyhomogeneous and when a polyhomogeneous approximate solution can be found.

THEOREM 7.4.1. Let g be a polyhomogeneous metric on M (*cf.* Definition 6.2.9) and suppose that $u \in C_2^{\rm loc}(M)$ is a solution to the scalar equation

$$\Delta_{\tilde g} u + xX(u) + F(x,u) = 0, \tag{7.4.1}$$

where X is a vector field on M satisfying $X - pDx \in \mathcal{A}^{\rm phg} \cap C_0^\beta(M)$ for some $\beta > 0$ and $p \in \mathbb{R}$. Let $u_0 \in \mathcal{A}^{\rm phg} \cap C_0(\bar M)$ and assume that there exists $q \in \mathbb{R}$ such that

$$\frac{\partial F}{\partial u}(x,u_0) - q|Dx|^2_g \in C_0^\beta(M),$$
$$q < \left(\tfrac{n-1-p}{2}\right)^2 \,, \quad \mu_+ > 0 \,,$$

where

$$\mu_\pm = \frac{n-1-p}{2} \pm \sqrt{\left(\frac{n-1-p}{2}\right)^2 - q} \,. \tag{7.4.2}$$

Further assume that for all i we have $(\partial_u^i F)(x,u_0) \in C_0(\bar M) \cap \mathcal{A}^{\rm phg}$, and that for all i,j and all multiindices μ the functions $(x\partial x)^i \partial_v^\mu \partial_u^j F$ are continuous in a neighbourhood of the graph of u_0 in $\bar M_{x_0} \times \mathbb{R}$. If

$$\Delta_{\tilde g} u_0 + xX(u_0) + F(x,u_0) \in C_0^{\alpha_0}(M) \,, \qquad \alpha_0 > \max(0,\mu_-) \,,$$
$$u - u_0 \in C_0^\alpha(M) \,, \qquad \alpha > \max(0,\mu_-) \,,$$

then

$$u \in \mathcal{A}^{\rm phg} \cap C_0(\bar M_{x_0}) \,.$$

REMARKS:

(i) The condition on the function F will be satisfied if *e.g.* F is a rational function of u with coefficients which are polyhomogeneous, appropriately bounded functions of x.

(ii) Let L and M satisfy the conditions of Corollary 7.2.3 and consider the equation

$$Lu = F(y, u)\,, \tag{7.4.3}$$

with F as considered above. Assumer further that the coefficients a_α of L as defined in eq. (4.1.3) are bounded polyhomogeneous functions. Then we can define a metric g and a vector field X on M so that equation (7.4.3) takes the form (7.4.1), with all the hypotheses of Theorem 7.4.1 being satisfied.

PROOF: Decreasing α if necessary without loss of generality we may assume $\alpha < \mu_+$. By a scaling argument one easily shows that $u - u_0 \in C_\infty^{\min(\alpha,\alpha_0)}(M)$. By Theorem 7.2.1 the linearized equation has a regularity interval (α_-, α_+) with $\alpha_\pm = \mu_\pm$ given by (7.4.2), and the result follows from Proposition 5.1.4 and Theorem 5.1.1. □

COROLLARY 7.4.2. Let g be a polyhomogeneous metric on a compact manifold with boundary M, let $0 \le \zeta \in \mathcal{A}^{\rm phg} \cap C_0^\epsilon(M)$, $\epsilon > 0$. There exists a solution ϕ of (7.1.1) such that

$$\phi - |Dx|_g^{\frac{n-1}{2}} \in \mathcal{A}^{\rm phg} \cap C_0^\epsilon(M)\ .$$

The solution ϕ is unique in the class of uniformly bounded, uniformly bounded away from zero, locally C_2 solutions of (7.1.1).

PROOF: Existence and uniqueness follows from Theorem 7.3.2; regularity from Theorem 7.4.1 – the appropriate approximate solution is $\phi_0 = |Dx|_g^{\frac{n-1}{2}}$. □

When $g \in C_\infty(\bar M)$, the solution ϕ of (7.1.1) given by Corollary 7.4.2 will be polyhomogeneous rather than smooth–up–to–boundary in general, *cf.* Appendix A and also [**5, 4**].

7.4.2. Background metrics with finite differentiability at the boundary. In the remainder of this chapter we shall prove various results concerning the regularity at the boundary of the solution of the Lichnerowicz equation, when finite differentiability only of the metric is assumed. As intermediate steps in the proof of the final results we shall need to prove various weighted regularity results. Our next result shows tangential regularity of the solutions of (7.1.1), under hypotheses weaker than those of Theorem 7.4.1. It is convenient to introduce first the following space of metrics:

DEFINITION 7.4.3. We shall say that a metric g is of $M^\epsilon_{k+\lambda,m+\mu}$ class, $0 \le \epsilon \le 1$, $k \ge \max(1, m)$, if $g \in M^\epsilon_{k+\lambda}$ (*cf.* Definition 6.2.1), $g_{ij} \in C^0_{k+\lambda,m+\mu}$, and if for $|\gamma| = 1$ we have $x\partial_y^\gamma g \in C^\epsilon_{k-1+\lambda,\min(k-1,m)+\mu}$ (*cf.* (3.2.4)).

It can be noted that for $g \in M^\epsilon_{k+1+\lambda,m+\mu}$ we have $\Delta_{\tilde g} \in OP^2_{C^0_{k+\lambda,\min(k,m)+\mu}}$. If $g \in M^\epsilon_{k+2+\lambda,m+\mu}$, then the functions ξ of eq. (7.1.5) and S of eq. (7.1.7) satisfy

$$S \in C^\epsilon_{k+\lambda,m+\mu}\,, \quad \xi + n|Dx|^2_g \in C^\epsilon_{k+\lambda,m+\mu}\,.$$

THEOREM 7.4.4 (TANGENTIAL REGULARITY). Let $g \in M^{\epsilon}_{k+2+\lambda,m+\mu}$, $0 \leq \zeta \in C^{\epsilon}_{k+\lambda,m+\mu}$, $\epsilon \in (0,1]$, $\lambda \in (0,1)$, $\mu \in [0,1]$, $1 \leq m \leq k+2$. Then the solution ϕ of (7.1.1) given by Theorem 7.3.2 satisfies

$$\phi - |Dx|_g^{(n-2)/2} \in C^{\epsilon}_{k+2+\lambda,m+\mu'}(M)\,,$$

with $\mu' = 0$ if $\mu = 0$, while if $\mu > 0$ then μ' is an arbitrary number in $[0,\mu)$.

PROOF: The proof is a bootstrap-commutation argument similar to that of Theorem 4.1.3. For $1 \leq i \leq m$ let $X_i = X_i^A(v)\,\frac{\partial}{\partial v^A}$, $X_i^A \in C_\infty(\partial M)$ be smooth vector fields on M_{x_0}. (7.1.7) implies that $X_i(u)$ satisfies the equation

$$\begin{aligned} \bar{L}X_i(u) &\equiv \left[\Delta_{\tilde{g}} + \xi - \frac{\partial F}{\partial u}\right] X_i(u) = \rho_i\,, \\ \rho_i &\equiv X_i(S) - X_i(\xi)u + [\Delta_{\tilde{g}}, X_i]u + X_i^A\,\frac{\partial F}{\partial v^A}\,. \end{aligned}$$

From $u \in C^{\epsilon}_{k+2+\lambda}$ it follows that

$$\xi - \frac{\partial F}{\partial u} + n|Dx|_g^2 \in C^{\epsilon}_{k+\lambda}\,, \qquad \rho_i \in C^{\epsilon}_{k-1+\lambda}\,.$$

By Theorem 7.2.1 $(-1,n)$ is a regularity interval for $C^{\alpha}_{0+\lambda}$ for $\bar{L}$, and from $X_i(u) \in C^{\epsilon-1}_{k+1+\lambda}$ and from scaling estimates one obtains $X_i(u) \in C^{\epsilon}_{k+1+\lambda}$, thus $u \in C^{\epsilon}_{k+2+\lambda,1}$. For $0 \leq \mu \leq 1$ the inclusion $C^{\epsilon}_{k+2+\lambda,1} \subset C^{\epsilon}_{k+2+\lambda,0+\mu}$ implies $u \in C^{\epsilon}_{k+2+\lambda,0+\mu}$, which in turn yields

$$\xi - \frac{\partial F}{\partial u} + n|Dx|_g^2 \in C^{\epsilon}_{k+\lambda,0+\mu}\,, \quad \rho_i \in C^{\epsilon}_{k-1+\lambda,0+\mu}\,.$$

The claim that $u \in C^{\epsilon}_{k+2+\lambda,1+\mu}$ follows by a difference quotient argument as in the proof of Theorem 4.1.3, because by Proposition 7.2.2 $(-1,n)$ is a strong regularity interval for $\bar{L}$. For $m \geq 1$ the result follows by induction as in Theorem 4.1.3 from the equation

$$\bar{L}X_1 \ldots X_i(u) = [\bar{L}, X_1 \ldots X_{i-1}]X_i(u) + X_1 \ldots X_{i-1}\rho_i\,.$$

□

Our next result proves classical regularity at the boundary of solutions of semilinear equations under appropriate conditions – this is the main result of this chapter or, indeed, of this paper. To avoid a tedious but otherwise straightforward discussion of various possibilities we only consider $\alpha, \mu_+ \in \mathbb{N}$, $\alpha \leq \mu_+$ and $\mu_+ = \alpha_+$ — these conditions hold in our applications to the general relativistic constraint equations. Similar results can be obtained without those restrictions using the same arguments. Let us emphasize that the following result holds for semilinear systems, and we are *not* assuming that (7.4.4) is a scalar equation.

THEOREM 7.4.5 (CLASSICAL BOUNDARY REGULARITY FOR SEMILINEAR SYSTEMS). Let (α_-, α_+), $\alpha_+ > 0$, be a strong regularity interval for $C^{\alpha}_{0+\lambda}(M)$ for a geometric elliptic operator $L \in OP^2_{C_{k+\lambda}(\bar{M})}$, $\lambda \in (0,1)$. Let L be of the form (4.2.1)–(4.2.4) and assume that $\mu_+ = \alpha_+$. Suppose that $u \in C^2_{\rm loc}(M)$, let $S \in x^\alpha C_{k+\lambda}(\bar M)$ for some $\alpha > \alpha_-$, and assume further that

$$\alpha,\ \mu_+ \in \mathbb{N}, \qquad k_1 := \mu_+ - \alpha \geq 0\,.$$

Let $u \in C_2^{\rm loc}(M)$, $u = O(x^{\alpha_- + \epsilon})$, $\alpha_- + \epsilon \geq 0$, $\epsilon > 0$, solve the semilinear system of equations

$$Lu = F(y,u) + S\,, \qquad F(y,0) = \partial_s F(y,s)\Big|_{s=0} = 0\,. \tag{7.4.4}$$

Let N be the smallest integer such that $N\alpha > k$. Assume that $F(y,s) \in C_{k+\lambda}(\bar M \times \mathbb{R})$ and that for all $0 \leq i \leq N + k + 2$ and for $s \in \mathbb{R}$ satisfying $|s| \leq K \equiv \sup_{M_{x_o}} |u^a|$ there are constants C_i such that

$$||\partial^i_s F(\cdot, s)||_{C_{k+\lambda}(\bar M)} \leq C_i. \tag{7.4.5}$$

Then there exists $\sigma > 0$ such that

(i) If $\alpha + k < \mu_+$ then $u \in \cap_{i=0}^2 x^{\alpha - i} C_{k+i+\sigma}(\bar M_{x_0})$.

(ii) If $\alpha + k \geq \mu_+$ then

$$u \in \cap_{i=0}^2 x^{\alpha - i} C_{\mu_+ - \alpha - 1 + i + \sigma | k - (\mu_+ - \alpha - 1)}(\bar M_{x_0})\,.$$

(If $\alpha = \mu_+$, the case $i = 0$ should be excluded in the equation above.) Moreover for $\alpha + k \geq \mu_+ + 1$ there exist functions u_i, $i = 1, \dots, \hat N$, such that $u = \sum_{j=0}^{\hat N} u_j \log^j(x)$, with

$$u_0 \in \cap_{i=0}^2 x^{\alpha - i} C_{k+i+\sigma}(\bar M_{x_0})\,, \tag{7.4.6}$$

and for $j \geq 1$,

$$u_j \in \cap_{i=0}^{k_1 + 2} x^{j\mu_+ - i} C_{k - k_1 + i + \lambda}(\bar M_{x_0})\,. \tag{7.4.7}$$

Here $\hat N$ is the smallest integer such that $\hat N > (k+2)/\mu_+$. If, finally, $u_1\big|_{\partial M} = 0$, then $u \in \cap_{i=0}^2 x^{\alpha - i} C_{k+i+\sigma}(\bar M_{x_0})$.

Remarks:

(i) We have imposed the somewhat unnatural condition (7.4.5) on F because it is satisfied by our problem at hand, namely the general relativistic constraint equations. A similar but somewhat worse result as far as differentiability of the functions u_i is concerned can be established by the same methods assuming only that $F(y,s) \in C_{k+\lambda}(\bar M \times \mathbb{R})$.

(ii) For $\mathbb{N} \ni \alpha \geq \mu_+$ and $k \geq 1$, and assuming that the remaining hypotheses of Theorem 7.4.5 remain unchanged, we obtain $u \in \cap_{i=0}^2 x^{\mu_+ - i} C_{k+i+\sigma}(\bar M_{x_0})$. Similarly, for $\mathbb{N} \not\ni \alpha \geq \mu_+$ and $k \geq 1$ the regularity of the solution of the non–linear problem will be the same as the for the solution of the linear problem with the corresponding value of α, *cf.* Theorem 4.2.9 and the remarks following that theorem. Clearly similar results can be obtained under the hypotheses of Theorem 4.2.11.

PROOF: In this proof, $\sigma \in (0,1)$ will be a constant which may vary from line to line. Let $F_m(x) = \partial_s^m F(x,s)\Big|_{s=0}$. From Taylor's Theorem we obtain

$$F(y,s) = \sum_{m=2}^{N} \frac{F_m(y)}{m!} s^m + s^{N+1} F_R(y,s)\,, \tag{7.4.8}$$

with $F_m \in C_{k+\lambda}(\bar M)$ and $F_R \in C_{k+\lambda}(\bar M \times \mathbb{R})$. Clearly, if $v \in C^{\beta}_{k+\lambda,\ell+\sigma}(M)$ for $\ell+\sigma \le k+\lambda$, then $v^{N+1}F_R(x,v) \in C^{(N+1)\beta}_{k+\lambda,\ell+\sigma}$.

Let v be a function of the form given in point (ii) of the Theorem and satisfying (7.4.6)–(7.4.7) with $\sigma = \lambda$. Then from (7.4.8) we get

$$F(y,v) = \sum_{j=0}^{N} \tilde F_j(y) \log^j(x) + \xi\,.$$

From (7.4.6) with $i = 0$ and (7.4.7) with $i = k_1$ we obtain $\xi \in C^{k+\alpha+\lambda}_{k+\lambda,k+\lambda}(M)$ and $\tilde F_0 \in x^{2\alpha} C_{k+\lambda}(\bar M)$. Similarly the choice $i = 0$ in (7.4.7) gives $\tilde F_j \in x^{j\mu_+} C_{k-k_1+\lambda}(\bar M)$ for $j \ge 1$. Expressions of the form $F(y, v_1+v_2) - F(y,v_1)$ can be analyzed in a similar way. Finally we note that if v_1 is of the form given in point (ii) and $v_2 \in C^{\beta}_{k+\lambda,\ell+\sigma}$ with $\ell+\sigma \le k+\lambda$, then

$$F(y, v_1+v_2) - F(y,v_1) \in C^{\alpha+\beta}_{k+\lambda,\ell+\sigma}\,, \tag{7.4.9}$$

which is easily seen by the mean value Theorem.

After those preliminary remarks, let us by induction construct an approximate solution $\hat u$ of eq. (7.4.4). Set $w_{-1} = 0$ and let $w_0 \in \cap_{i\in\mathbb{N}_0} x^{\alpha-i} C_{k+i+\lambda}(\bar M)$ be given by Lemma 4.2.5 so that $Lw_0 - S \in C^{k+\alpha,\beta_0}_{k+\lambda,0+\lambda} \cap C^{k+\alpha+\lambda,\beta_0}_{k+\alpha+\lambda}$, $\beta_0 = 0$ or $\beta_0 = 1$. By interpolation for $\beta \in \mathbb{N}_0$ we have $C^{k+\alpha,\beta}_{k+\lambda,0+\lambda}(M) \cap C^{k+\alpha+\lambda,\beta}_{k+\lambda}(M) \hookrightarrow C^{k+\alpha+\sigma}_{k+\sigma,0+\sigma}(M)$ for some $\sigma > 0$.

Suppose then that for some $m \le N$ we have found functions w_i, $i = -1,\dots,m$, of the form

$$w_i = \sum_{j=0}^{\hat N} w_{i,j} \log^j(x)\,, \tag{7.4.10}$$

with coefficients $w_{i,j}$ satisfying (7.4.6)–(7.4.7). We shall moreover require that $w_i - w_{i-1} \in C^{(i+1)\alpha}_{k+2+\lambda}$. Then a function w_{m+1} of the form (7.4.10) with $w_{m+1} - w_m \in C^{(m+2)\alpha}_{k+2+\lambda}$ can be constructed as a solution of the equation

$$L(w_{m+1} - w_m) - (F(y,w_m) - F(y,w_{m-1})) \in C^{k+\alpha+\sigma}_{k+\sigma,0+\sigma}(M)\,,$$

by using the argument of the proof of Lemma 4.2.5. This completes the induction. Note that for $\alpha + k < \mu_+$ we will have $w_{i,j} \equiv 0$ for $j > 1$.

Letting $\hat u \equiv w_N$, we have

$$\xi := F(y,\hat u) + S - L\hat u \in C^{k+\alpha+\sigma}_{k+\sigma,0+\sigma}\,. \tag{7.4.11}$$

Setting $\tilde{u} = u - \hat{u}$ we obtain

$$L(\tilde{u}) = F(y, \tilde{u} + \hat{u}) - F(y, \hat{u}) + \xi\,, \qquad \xi \in C^{k+\alpha+\sigma}_{k+\sigma,0+\sigma}\,. \tag{7.4.12}$$

We have $u \in C^{\alpha_-+\epsilon}_0$, so that decreasing ϵ if necessary we obtain $\tilde{u} \in C^{\alpha_-+\epsilon}_0$. Equations (7.4.9) and (7.4.12) give $L\tilde{u} \in C^{\alpha}_0$. By point (i) of Lemma 4.1.1 and the weighted Sobolev embedding we obtain $\tilde{u} \in C^{\alpha_-+\epsilon}_{1+\lambda}$. Iterating this argument using point (ii) of Lemma 4.1.1 and (7.4.9) gives $\tilde{u} \in C^{\alpha_-+\epsilon}_{k+2+\lambda}$. We have:

LEMMA 7.4.6. For $0 \le \ell \le k$ and $\gamma < \min(k - \ell + \alpha + \sigma, \mu_+)$ we have

$$\tilde{u} \in C^{\gamma}_{k+2+\lambda,\ell+\sigma}\,.$$

Proof: The argument below is rather similar to that of Lemma 5.1.3. Some care, however, must be taken because of the finite degree of differentiability of the functions here. Let $\ell \ge 0$, choose β so that $0 < \beta \le \alpha_- + \epsilon$, $\beta < \mu_+$ and $2\beta \le \alpha$, and let $\sigma \in (0,1)$ satisfy $\ell + \sigma < k + \lambda$. Let $\theta \in (0, \beta)$ and $K \in \mathbb{N}$ be chosen so that $K\theta = \ell + \sigma$. Let ϵ_κ, $\kappa = 0, \ldots, K$ be any increasing sequence of numbers such that $\epsilon_0 = 0$, and $\epsilon_\kappa < \sigma$. Note that for $0 \le \ell \le k$ eq. (7.4.11) implies that

$$\xi \in C^{k-\ell+\alpha+\sigma}_{k+\lambda,\ell+\sigma}\,.$$

We will argue by induction. Assume that $\tilde{u} \in C^{\beta}_{k+\lambda,\kappa\theta-\epsilon_\kappa}$; we have shown that this holds with $\kappa = 0$. Then (7.4.9) and (7.4.12) imply

$$L\tilde{u} \in C^{2\beta}_{k+\lambda,\kappa\theta-\epsilon_\kappa} \cap C^{2\beta-1}_{k+\lambda,\kappa\theta+1-\epsilon_\kappa}\,.$$

Lemma 4.2.2 gives

$$L\tilde{u} \in C^{2\beta-(1-\tau)}_{k+\lambda,\kappa\theta+1-\tau-\epsilon_{\kappa+1}}\,,$$

for any $\tau \in (0,1)$. Choosing $\tau = 1 - \theta$ we get

$$L\tilde{u} \in C^{\beta}_{k+\lambda,(\kappa+1)\theta-\epsilon_{\kappa+1}}\,,$$

and, increasing slightly ϵ_{k+1}, Theorem 4.1.3 gives

$$\tilde{u} \in C^{\beta}_{k+2+\lambda,(\kappa+1)\theta-\epsilon_{\kappa+1}}\,.$$

This completes the induction step.

We have thus shown that $\tilde{u} \in C^{\beta}_{k+2+\lambda,\ell+\sigma}$ with some $\beta > 0$. An iterative argument based on (7.4.9), (7.4.12) and Theorem 4.1.3 completes the proof. $\square$

Returning to the proof of Theorem 7.4.5, suppose first that $\alpha + k < \mu_+$. In this case Lemma 7.4.6 with $\ell = 0$ gives $\tilde{u} \in C^{k+\alpha+\sigma}_{k+2+\sigma,0+\sigma}(M_{x_0})$, and the result follows from Proposition 3.2.2.

In the case $\alpha + k \ge \mu_+$ some more work is needed. If $k = \mu_+ - \alpha$ our conclusion follows again from Lemma 7.4.6 and Proposition 3.2.2. (Note that in this case our result is weaker than for $k > \mu_+ - \alpha$, *cf.* point (ii).) For $k > 0$ set $\tilde{f} = L(\tilde{u})$ and rewrite equation (7.4.12) in the form (4.2.46). Then arguing as in the proof of Theorem 4.2.9 we find, using Lemma 4.2.7, a function $\hat{\psi} \in \cap_{i\in\mathbb{N}_0} x^{\mu_+-i} C_{k+i+\lambda}(\bar{M})$ with $L\hat{\psi} \in C^{k+\mu_++\sigma}_{k+\sigma,0+\sigma}(M)$ and $\tilde{u} \equiv u - \hat{u} -$

$\hat{\psi} = o(x^{\mu_+})$. Now we can change w_0 as defined above to $w_0 + \hat{\psi}$ and repeat the induction argument described previously to obtain a new approximate solution, still denoted $\hat{u}$, of the form (7.4.10) and satisfying (7.4.11). A direct analysis of the identity (4.2.47) of the proof of Theorem 4.2.9 gives $\tilde{u} \equiv u - \hat{u} \in C^{\mu_+ + \sigma}_{k+2+\sigma, k-(\mu_+ - \alpha)+\sigma}(M_{x_0})$. We wish to show that in fact we must have $\tilde{u} \in C^{k+\alpha+\sigma}_{k+2+\sigma,0+\sigma}(M_{x_0})$. For suppose first that $k \le \mu_+$. In this case $L\tilde{u} \in C^{k+\alpha+\sigma}_{k+\sigma,0+\sigma}(M)$ by (7.4.9), and Lemma 4.2.8 with $\beta = \alpha$, $m = \ell = k$ yields $\tilde{u} \in C^{k+\alpha+\sigma}_{k+2+\sigma,0+\sigma}(M_{x_0})$. To handle the case $k > \mu_+$, suppose that $\tilde{u} \in C^{\beta+\sigma}_{k+2+\sigma,\ell+\sigma}(M_{x_0})$ with some $\beta \in \mathbb{N}$ satisfying $\beta < k + \alpha$ and some $\ell \ge m := \min(k+\alpha-\beta, \alpha)$. Then (7.4.9) and Lemma 4.2.8 yield $\tilde{u} \in C^{\beta+m+\sigma}_{k+2+\sigma,\ell-m+\sigma}(M_{x_0})$. Applying this argument repeatedly leads indeed after a finite number of steps to $\tilde{u} \in C^{k+\alpha+\sigma}_{k+2+\sigma,0+\sigma}(M_{x_0})$. The proof of point (ii) is completed by Proposition 3.2.2. $\square$

The classical regularity up–to–boundary of solutions of the Lichnerowicz equation up to the threshold $n-1 = \dim M - 1$ is a straightforward corollary of the above:

THEOREM 7.4.7 (BOUNDARY REGULARITY). Let M be an n–dimensional manifold with boundary with $\bar{M}$ – compact. Let $g \in C_{k+2+\lambda}(\bar{M})$, suppose that $\zeta \ge 0$ and suppose moreover that $\zeta \in C_{k+2+\lambda}(\bar{M}) \cap C^1_0(M)$, or $\zeta \in x\, C_{k+1+\lambda}(\bar{M})$, or $\zeta \in x^2 C_{k+\lambda}(\bar{M})$, $\lambda \in (0,1)$, $k \in \mathbb{N}_0$. There exists $\sigma \in (0,1)$ such that the solution ϕ of (7.1.1) given by Theorem 7.3.2 satisfies

$$\begin{aligned} k+2 \le n-1 \quad &\Longrightarrow \quad \phi \in C_{k+2+\sigma}(\bar{M})\,, \\ k+2 = n+k_1,\ k_1 \in \mathbb{N}_0 \quad &\Longrightarrow \quad \phi \in C_{n-1+\sigma|k_1+1}(\bar{M})\,. \end{aligned}$$

Moreover in the case $k+2 > n$ there exist functions $\phi_j \in \cap_{i=0}^n x^{jn-i} C_{k+2-n+i+\lambda}(\bar{M})$, $j = 1, \ldots, N$, such that

$$\phi - \sum_{i=1}^N \phi_i \log^i x \in C_{k+2+\sigma}(\bar{M})\,.$$

Here N is the smallest integer such that $N > (k+2)/n$. If finally $\phi_1|_{\partial M} = 0$, then $\phi \in C_{k+2+\sigma}(\bar{M})$.

REMARK: The results of Theorem 7.4.7 are optimal (except perhaps for the modulus of Hölder continuity σ which we leave unspecified), as the $x^n \log x$ term generically arises in the solution of (7.1.1), *cf.* Appendix A. The reader is also referred to [**6, 5**] for an analysis under which conditions on the geometry it holds that $\phi_1|_{\partial M} = 0$; in these references $\dim M = 3$ is assumed.

APPENDIX A

Genericity of log-terms.

In this section we shall show that for generic "background fields", in a sense to be made precise in the statements of Proposition A.1.1 and Theorems A.2.2, A.2.3, the Cauchy data constructed by the conformal method will "pick-up" log-terms, even though the background fields are smooth up to boundary. The results of this section are somewhat related to those of [**5**]. In [**5**] rather more geometric results can be found; on the other hand, the proofs and the discussion here are much simpler.

To avoid a tedious differentiability–chase we have stated all the results in a C_∞ setting. However, the calculations presented here carry immediately over to the finite differentiability case. Some of the equations below can be somewhat simplified if one makes use of the "almost Gaussian coordinates" of Appendix B.

Throughout this section M is a manifold with boundary with $\bar{M}$ – compact, and with $\dim M \geq 3$. In some results we shall specialize to the physically relevant case $n = 3$, but this condition is not assumed unless specified otherwise. The reader is referred to Chapter 3 for any notation which is not defined here.

Let us start with the vector constraint equation. As explained at the beginning of Chapter 6, there are at least two methods of constructing solutions to this equation. Here we shall analyze in detail the method described in section 6.2; assuming smoothness of the background fields the method of section 6.3 gives equivalent results. Note, however, that the discussion of decay rates of various fields of section 2 has been tailored to fit the construction of section 6.3. This explains the difference between the decay rates of section 2 and those of section A.1 below.

A.1. The vector constraint equation

PROPOSITION A.1.1. Let $g \in C_\infty(\bar{M})$, denote by $i_{\partial M}$ the embedding $i_{\partial M} : \partial M \to \bar{M}$, and by $\Gamma_{C_\infty(\partial M)}(i^*_{\partial M}T^2\bar{M})$ the space of smooth sections of the bundle $i^*_{\partial M}T^2\bar{M}$ over ∂M obtained by pull-back by $i_{\partial M}$ of the bundle $T^2\bar{M}$ of two-contravariant tensors over $\bar{M}$, suppose that $\alpha = -2$ or $\alpha = -1$. For

$A_{ij} \in C_\infty(\bar{M})$ let X be any solution of the equation

$$D_i\, L^{ij} \equiv \frac{1}{2} D_i \left(D^i X^j + D^i X^i - \frac{2}{n}\, D_k X^k\, g^{ij} + x^\alpha\, A^{ij} \right) = 0 \tag{A.1.1}$$

given by Theorem 6.2.8, point 3 thus

$$L^{ij} = x^\alpha\, L^{ij}_\alpha + \log x\, L^{ij}_{\log}\,, \quad L^{ij}_\alpha\,,\, L^{ij}_{\log} \in C_\infty(\bar{M})\ . \tag{A.1.2}$$

(i) Let $\alpha = -1$. There exists a closed subspace $\mathcal{A} \subset \Gamma_{C_\infty(\partial M)}(i^*_{\partial M} T\bar{M})$ of finite dimension not larger than $(\dim \partial M + 1)(\dim \partial M + 2)/2 = n(n+1)/2$ such that for all A^{ij} satisfying $D_i x\Big(|Dx|^2_g A^{ij} - Dx_k\, Dx^j\, A^{ik}\Big)(0, v) \notin \mathcal{A}$ we have

$$L^{ij}_{\log}(0, v) \not\equiv 0\,.$$

(ii) Let $\alpha = -2$. For any $\mathring{A} \in \Gamma_{C_\infty(\partial \bar{M})}(i^*_{\partial M} T^2\, M)$ and $\psi \in C_\infty(\partial \bar{M})$ there exists a closed subspace $\mathcal{A}_{\mathring{A}\,,\psi} \subset \Gamma_{C_\infty(\partial M)}(i^*_{\partial M} T\bar{M})$, of finite dimension not larger than $(\dim \partial M + 1)(\dim \partial M + 2)/2$, such that for A^{ij} satisfying $A^{ij}(0, v) = \mathring{A}^{\,ij}(v)$, $D_i x\, D_j x \frac{\partial A^{ij}}{\partial x}(0, v) = \psi$ and $D_i x\Big(|Dx|^2_g \frac{\partial A^{ij}}{\partial x} - Dx_k\, Dx^j\, \frac{\partial A^{ik}}{\partial x}\Big)(0, v) \notin \mathcal{A}_{\mathring{A}\,,\psi}$ we have

$$L^{ij}_{\log}(0, v) \not\equiv 0\,.$$

Moreover for all A_{ij} for which a solution with $L^{ij}_{\log}(0, v) \not\equiv 0$ exists, there exists no solution of (A.1.1) satisfying

$$X \in x^{\alpha+1}\left(C_\infty(\bar{M}) + x \log x\, C_\infty(\bar{M}) \right)\,,$$

such that $x^{-\alpha}\, L^{ij} \in C_{-\alpha}(\bar{M})$.

PROOF: Uniqueness up to the addition of a smooth (up to boundary) solution of the homogeneous equation follows from Theorem 6.2.8, thus the only thing to show is the non-vanishing of $L^{ij}_{\log}$. Let thus

$$\alpha = -1 \qquad X = X_0 + \log x X_{\log}\,, \tag{A.1.3}$$

$$\alpha = -2 \qquad X = x^{-1}\, X_{-1} + \log x X_{\log}\,, \tag{A.1.4}$$

$X_0, X_{-1}, X_{\log} \in C_\infty(\bar{M})$. X solves the equation

$$\begin{aligned} \Delta_{L,g}\, X^j &\equiv \frac{1}{2} D_i \left(D^i X^j + D^j X^i - \frac{2}{n}\, D_k X^k\, g^{ij} \right) \\ &= x^{\alpha-1}\, Z^j\,, \end{aligned} \tag{A.1.5}$$

with

$$Z^j = -x^{-\alpha+1}\, D_i(x^\alpha\, A^{ij}) = -\left(\alpha A^{xj} + x D_i\, A^{ij} \right)\ . \tag{A.1.6}$$

Inserting (A.1.3) in (A.1.5) one finds, with $m = |Dx|_g^{-1}\, Dx$ (thus m is the *inward* pointing unit normal to ∂M),

$$\begin{aligned}
\alpha = -1 \qquad \Delta_{L,g} X^j &= -\frac{|Dx|_g^2}{2x^2}\left(X^j_{\log} + \frac{n-2}{n}\, g(m, X_{\log})\, m^j\right) + O\left(x^{-1}\right) ,\\
\alpha = -2 \qquad \Delta_{L,g} X^j &= \frac{|Dx|_g^2}{x^3}\left\{X^j_{-1} + \frac{n-2}{n}\, g(m, X_{-1})\, m^j\right\}(0,v)\\
&\quad -\frac{|Dx|_g^2}{2x^2}\left\{X^j_{\log} + \frac{n-2}{n}\, g(m, X_{\log})\, m^j + L_1(X_{-1})\right\}\\
&\quad +O(x^{-1}) ,
\end{aligned}$$

where L_1 is a homogeneous linear first order differential operator which has the property that $L_1(X_{-1})(0,v)$ does *not* depend upon $\partial_x X_{-1}(0,v)$. For $\alpha = -1$ we thus must have (*cf.* the beginning of Chapter 3 for conventions on coordinate systems)

$$\left(X^j_{\log} + \frac{n-2}{n}\, g(m, X_{\log})\, m^j\right)(0,v) = -\left(|Dx|_g^{-2}\, A^{xj}\right)(0,v) ,$$

which can be algebraically solved for $X^j_{\log}(0,v)$ in terms of $A^{xj}(0,v)$, and which shows that $X^j_{\log}(0,v)$ runs over $\Gamma_{C_\infty(\partial M)}(i^*_{\partial M} T\bar{M})$ as $A^{xj}(0,v)$ does. For $\alpha = -2$ one obtains

$$\left(X^j_{-1} + \frac{n-2}{n}\, g(m, X_{-1})\, m^j\right)(0,v) = \left(|Dx|_g^{-2}\, A^{xj}\right)(0,v) , \tag{A.1.7}$$

$$\left(X^j_{\log} + \frac{n-2}{n}\, g(m, X)\, m^j\right)(0,v) = -\left(|Dx|_g^{-2}\, \frac{\partial A^{xj}}{\partial x}\right)(0,v) + F , \tag{A.1.8}$$

$$F = \left(|Dx|_g^{-2}\left[D_i\, A^{ij} - \partial_x\, A^{xj}\right]\right)(0,v) - L_1(X_{-1})(0,v) . \tag{A.1.9}$$

From (A.1.7) one can algebraically determine $X^j_{-1}(0,v)$ in terms of $A^{xj}(0,v)$ and since $\left(D_i\, A^{ij} - \frac{\partial A^{ij}}{\partial x}\right)(0,v)$ depends upon $A^{ij}(0,v)$ and its tangential derivatives only, one finds that F in (A.1.8) is uniquely determined by $A^{ij}(0,v)$; (A.1.8) implies then that for any fixed $A^{ij}(0,v)$ the vector field $X^j_{\log}(0,v)$ runs over $\Gamma_{C_\infty(\partial M)}(i^*_{\partial M} T\bar{M})$ as $\frac{\partial A^{xj}}{\partial x}(0,v)$ does. Now a simple calculation shows that

$$2L^{ij}_{\log} \equiv D^i\, X^j_{\log} + D^j\, X^i_{\log} - \frac{2}{n}\, D_k\, X^k_{\log}\, g^{ij} ,$$

and to finish the proof we have to show that the vanishing of $L^{ij}_{\log}(0,v)$ implies that $X^j_{\log}$ is of rather special form. Let h_{ij} be the induced metric on ∂M,

$$h_{ij} = g_{ij} - m_i m_j , \tag{A.1.10}$$

let $\mathcal{D}_i$ denote the covariant Riemannian derivative operator of the metric h_{ij}, set $\beta_i = m^j\, D_j\, m_i$. Setting

$$\phi = X^i_{\log}\, m_i\,, \qquad Y^i = X^i_{\log} - \phi m^i\,, \tag{A.1.11}$$

one finds

$$2m_i m_j\, L^{ij}_{\log} = \frac{2}{n}\,\left[(n-1)\left(m^k D_k\phi - Y^k\beta_k\right) - \mathcal{D}_k Y^k + \phi\lambda\right]\,, \tag{A.1.12}$$

$$2h^i_k m_j\, L^{kj}_{\log} = \mathcal{D}_m Y^i + \mathcal{D}^i\phi + \lambda^i{}_k\, Y^k + \phi\beta^i\,, \tag{A.1.13}$$

$$\begin{aligned} 2(h^i_k h^j_\ell L^{k\ell}_{\log} + \frac{1}{n-1} m_k m_\ell\, L^{k\ell}_{\log} h^{ij}) &= \mathcal{D}^i Y^j + \mathcal{D}^j Y^i - \frac{2}{n-1}\,\mathcal{D}_k Y^k h^{ij} \\ &\quad -2\phi\left(\lambda^{ij} - \frac{1}{n-1}\,\lambda h^{ij}\right)\,, \end{aligned} \tag{A.1.14}$$

where λ_{ij} is the extrinsic curvature of ∂M ($\lambda_{ij} = -h^k_i h^\ell_j D_k m_\ell$), $\lambda = h^{ij}\,\lambda_{ij}$, and

$$\mathcal{D}_m Y^i = h^i_j\, m^k\, D_k\, Y^j\,.$$

For $\dim\partial M \geq 3$ eq. (A.1.14) shows that the vanishing of $L^{k\ell}_{\log} h^i_k h^j_\ell$ leads to an overdetermined system of equations for $X^i_{\log} - m_k X^k_{\log} m^i$, and a straightforward repetition of the argument which shows that the space of conformal Killing vectors cannot exceed $(\dim\partial M + 1)(\dim\partial M + 2)/2$ yields the same upper bound for the dimension of the space of Y's for which we have $L^{ij}_{\log} = 0$. In the physically relevant case $\dim\partial M = 2$, the operator

$$Y^i \longrightarrow \mathcal{D}^i Y^j + \mathcal{D}^j Y^i - \frac{2}{n-1}\,\mathcal{D}_k Y^k h^{ij}$$

(which maps vectors to symmetric traceless tensors) is elliptic, with (as is well known)

(i) trivial kernel if $g(\partial M) > 1$, where $g(\partial M)$ is the genus of ∂M;
(ii) two dimensional kernel if $\partial M \approx S^1 \times S^1$;
(iii) six dimensional kernel if $\partial M \approx S^2$.

It follows again that the condition $L^{k\ell}_{\log} = 0$ will hold for at most a $(\dim\partial M + 1)(\dim\partial M + 2)/2$ dimensional space of Y's, and the result follows. □

For the proof of Theorem A.2.2 we shall need the following proposition:

PROPOSITION A.1.2. Let m_i be the unit normal to ∂M, define

$$\mathcal{T} \subset \Gamma_{C_\infty(\partial M)}\left(i^*_{\partial M} T^2 \bar M\right)$$

as the space of tensors $A^{ij}(v)$ over ∂M satisfying $m_i A^{ij} = g_{ij} A^{ij} = 0$, let P be the projection on $\mathcal{T}$: $(PA)^{ij} = h^i{}_k h^j{}_\ell\left(A^{k\ell} - h^{mn} A_{mn} h^{k\ell}/(n-1)\right)$. (Here h is the metric induced by g on ∂M, *cf.* eq. (A.1.10).) For any $\mathring{A}$, $\hat{A} \in \mathcal{T}$ there exists a transverse traceless tensor field $L^{ij} \in x^{-2}C_\infty(\bar{M}) + \ln x\, C_\infty(\bar{M})$ such that

$$
\begin{aligned}
&\text{(A.1.15)} \qquad & L^{ij}(x,v) &= \mathring{A}(v)x^{-2} + L^{ij}_{-1}(v)x^{-1} + O(\ln x)\,, \\
&\text{(A.1.16)} \qquad & (PL_{-1})^{ij} &= \hat{A}^{ij}\,.
\end{aligned}
$$

REMARKS:

(i) Note that for a transverse traceless tensor of the form (A.1.15) the condition $\mathring{A}(v) \in \mathcal{T}$ is necessary.

(ii) The question of existence of transverse traceless tensors $L^{ij} \in x^{-\alpha}\, C_\infty(\bar{M})$, $\alpha = 1, 2$, such that $x^\alpha L^{ij}|_{\partial M} \not\equiv 0$ is considered in [**5**], and is shown to be related to the vanishing of the space–time Weyl tensor at ∂M.

PROOF: Let $A_{ij} \in C_\infty(\bar{M})$, we have

(A.1.17)
$$
D_i(x^{-2}A^{ij})(x,v) = -2x^{-3}A^{xj}(0,v) + x^{-2}\left[D_iA^{ij}(x,v) - 2x^{-1}\left(A^{xj}(x,v) - A^{xj}(0,v)\right)\right]\,,
$$

where $A^{xj} = D_i x A^{ij}$. Let $A^{ij}(0,v) = \mathring{A}^{ij}(v)$, so that $A^{xj}(0,v) = 0$. Taylor expanding there exist $R^j \in C_\infty(\bar{M})$, $S^j \in C_\infty(\bar{M})$ such that

$$
\begin{aligned}
A^{xj}(x,v) &= \partial_x A^{xj}(0,v)x + \frac{1}{2!}\,\partial_x^2 A^{xj}(0,v)x^2 + x^3R^j\,, \\
\partial_x A^{xj}(x,v) &= \partial_x A^{xj}(0,v) + \partial_x^2 A^{xj}(0,v)x + x^2S^j\,.
\end{aligned}
$$

Define

$$
\text{“}\Gamma A\text{”} = D_x A^{xj} - \partial_x A^{xj}\,;
$$

(A.1.17) can be rewritten as

$$
D_i(x^{-2}A^{ij})(x,v) = x^{-2}\left[-\partial_x A^{xj}(0,v) + \left(\text{“}\Gamma A\text{”} + D_A A^{Aj}\right)(x,v)\right] + S^j - 2R^j\,.
$$

If we choose $A^{ij}(x,v)$ so that

$$
\begin{aligned}
\partial_x A^{xj}(0,v) &= \left(\text{“}\Gamma\, \mathring{A}\text{”} + D_A \mathring{A}^{Aj}\right)(0,v)\,, \\
P\partial_x A^{ij}(0,v) &= \hat{A}^{ij}(v)\,,
\end{aligned}
$$

we obtain

$$
D_i(x^{-2}A^{ij}) \in x^{-1}C_\infty(\bar{M})\,,
$$

therefore by Theorem 6.2.8 there exists a solution $X \in C_\infty(\bar{M}) + x\ln x C_\infty(\bar{M})$ of equation (A.1.1). $\square$

A.2. The coupled system

In the remainder of this section we shall restrict our considerations to $n = \dim M = 3$. Before analysing the presence or absence of log-terms in the solutions of the Lichnerowicz equation, recall that we are mainly interested in conformal classes of "background fields" $[(g, L)]$, where L is a TT-tensor, in which the pair (g, L) is identified with the pair $(\psi^2 g, \psi^{-5} L)$ for any positive function $\psi \in C_\infty(\bar{M})$: (g, L) and $(\psi^2 g, \psi^{-5} L)$ lead to the same solution of the constraint equations. We shall assume that $L \in x^{-2} C_\infty(\bar{M}) + \log x C_\infty(\bar{M})$, as forced upon us by Proposition A.1.1 in generic situations, it should be stressed that *the log terms in L will not affect the conclusion of Theorem* A.2.2, because they do not affect the $x^3 \log x$ terms in the expansion of solutions of the Lichnerowicz equation.

LEMMA A.2.1. Let $x, x_1 \in C_\infty(\bar{M})$ be two defining functions for ∂M and let L^{ij} be a transverse traceless tensor for the metric g. Suppose that $\left(g_{ij}, L^{ij}\right) = \left(\psi^4 g^1_{kj}, \psi^{-10} L^{ij}_1\right)$, $g \in C_\infty(\bar{M})$, $L \in x^{-2} C_\infty(\bar{M}) + \ln x \, C_\infty(\bar{M})$, with some bounded positive function $\psi \in C_3(\bar{M})$, ψ uniformly bounded away from zero. Set

$$\tilde{g}_{ij} = x^{-2} g_{ij} \,, \quad \tilde{g}^1_{ij} = x_1^{-2} g^1_{ij} \,, \quad \tilde{L}^{ij} = x^3 L^{ij} \,, \quad \tilde{L}^{ij}_1 = x_1^3 L^{ij}_1 \,.$$

Let ϕ, ϕ_1 be uniformly bounded, locally C_2 solutions of

$$\begin{aligned} 8\Delta_{\tilde{g}} \phi - \tilde{R}\phi + |\tilde{L}|^2_{\tilde{g}} \phi^{-7} - 6\phi^5 &= 0 \\ 8\Delta_{\tilde{g}_1} \phi_1 - \tilde{R}_1 \phi_1 + |\tilde{L}_1|^2_{\tilde{g}_1} \phi_1^{-7} - 6\phi_1^5 &= 0 \end{aligned}$$

ϕ has a $x^3 \log x$ term in the asymptotic expansion at ∂M if and only if ϕ_1 has one.

PROOF: By the uniqueness part of Theorem 2.1.1 we have

$$\phi_1 = \psi \left(\frac{x}{x_1}\right)^{\frac{1}{2}} \phi \,. \tag{A.2.1}$$

Since $\frac{x}{x_1}$ is in $C_\infty(\bar{M})$, the result follows by comparing coefficients in an asymptotic expansion of both sides of (A.2.1). $\square$

THEOREM A.2.2. Let X be the collection of pairs (g, L), where $g \in C_\infty(\bar{M})$ is a Riemannian metric and $L^{ij} \in x^{-2} C_\infty(\bar{M}) + \log x \, C_\infty(\bar{M})$ is transverse traceless.

(i) The subset $\mathring{X}$ of X consisting of pairs for which the log terms in the expansion of the solutions of the Lichnerowicz equation do not vanish is open and dense in the $C_\infty(\bar{M})$ topology on X.
(ii) The set $\hat{X} = X \backslash \mathring{X}$ is an infinite dimensional closed subspace of X.

PROOF: Choose some defining function x for ∂M, by Lemma A.2.1 the vanishing or not of the log terms does not depend upon this choice. Let $\bar{g}_{ij} = |Dx|^2_g g_{ij}$, in the metric $\bar{g}_{ij}$ we have $\bar{g}^{ij} x_{,i} x_{,j} = 1$, set $\tilde{g}_{ij} = x^{-2} \bar{g}_{ij}$. Let ϕ be the unique solution of

$$8\Delta_{\tilde g}\phi - \tilde R\phi + |\tilde L|^2_{\tilde g}\phi^{-7} - 6\phi^5 = 0 \tag{A.2.2}$$

satisfying $\phi|_{\partial M} = 1$. By the results of Chapter 7 there exist ϕ_1, ϕ_2, $\phi_{\log} \in C_\infty(\partial M)$ such that

$$\phi - 1 - \phi_1 x - \phi_2 x^2 - \phi_{\log} x^3 \log x \in C_3^{3+\epsilon}\,, \quad \epsilon > 0\,. \tag{A.2.3}$$

Let $\bar m^i = \bar g^{ij}x_{,j}$, $\bar h_{ij} = \bar g_{ij} - \bar m_i \bar m_j$, let $\bar\lambda_{ij}$ be the extrinsic curvature (in the metric $\bar g_{ij}$) of the sets $x = \text{const.}$, $\bar\lambda = \bar h^{ij}\bar\lambda_{ij}$, define the expansion functions $\bar\lambda_0$, etc., by

$$\begin{aligned}
\bar\lambda(x,v) &= \bar\lambda_0(v) + x\bar\lambda_1(v) + x^2\bar\lambda_2(v) + O(x^3) && \text{(A.2.4)}\\
|Dx|_g^{-5}\, L^{ij}(x,v) &= \frac{\mathring{A}^{ij}(v)}{x^2} + \frac{\hat A^{ij}(v)}{x} + O(\ln x) && \text{(A.2.5)}\\
R(\bar g)(x,v) &= \bar R_0(v) + x\bar R_1(v) + O(x^2)\,, && \text{(A.2.6)}\\
\bar g^{xA}(x,v) &= \bar g^{xA}(0,v) + x\dot{\bar g}^{xA}(v) + O(x^2)\,, && \text{(A.2.7)}
\end{aligned}$$

where $R(\bar g)$ is the scalar curvature of the metric $\bar g_{ij}$. Inserting (A.2.3)–(A.2.7) in (A.2.2) one finds with the help of a simple REDUCE code

$$\begin{aligned}
\phi_1 &= \bar\lambda_0/8\,, && \text{(A.2.8)}\\
\phi_2(v) &= \frac{1}{24}(\bar g_{ij}\bar g_{k\ell})(0,v)\mathring{A}^{ij}(v)\mathring{A}^{j\ell}(v)\\
&\quad + \frac{1}{6}\left[\bar\lambda_1(v) - \bar R_0(v) + (\bar D_i x \bar D^i \bar\lambda_0)(0,v)\right]\\
&\quad - \frac{23}{384}\bar\lambda_0^2(v)\,, && \text{(A.2.9)}\\
\phi_{\log}(v) &= -\frac{1}{16}(\bar g_{ij}\bar g_{k\ell})(0,v)\left[\mathring{A}^{ik}(v)\hat A^{j\ell}(v) + \mathring{A}^{ik}(v)\mathring{A}^{j\ell}(v)\bar\lambda_0(v)\right]\\
&\quad + \psi(v)\,, && \text{(A.2.10)}\\
\psi(v) &= \left\{\frac{1}{32}\left[\bar R_1 - 4\bar\lambda_2 + 5\bar\lambda_0\bar\lambda_1 - \bar\lambda_0\bar R_0 - \Delta_g\bar\lambda_0 + \dot{\bar g}^{xA}\partial_A\bar\lambda_0\right]\right.\\
&\quad \left. - \frac{207}{4096}\bar\lambda_0^3 + \frac{9}{256}\bar\lambda_0\bar D_i x\bar D^i\bar\lambda_0 - \frac{3}{4}\bar D_i x \bar D^i\phi_2\right\}(0,v)\,. && \text{(A.2.11)}
\end{aligned}$$

It follows from (A.2.10)–(A.2.11) that the condition $\phi_{\log} = 0$ defines a closed subspace of the set of all (g, L), thus $\mathring{X}$ is an open subset.

Suppose that $(g, L) \in \hat X$, thus $\phi_{\log} = 0$. By Proposition A.1.2 $\mathring{A}^{ij}(v)$ and $B^{ij} = \bar h^i_k \bar h^j_\ell\ (\hat A^{k\ell} - \bar h^{mn}\hat A_{mn}\bar h^{k\ell})$ can assume arbitrary values satisfying $\mathring{A}^{ij}m_j = B^{ij}m_j = \bar h_{ij}\mathring{A}^{ij} = \bar h_{ij}B^{ij} = 0$, and one can thus find a sequence $\left(g, L^{ij}_k\right) \longrightarrow_{k\to\infty} (g, L^{ij})$ such that $\phi_{\log}(v) \not\equiv 0$ by choosing $\mathring{A}_k{}^{ij}$ and B^{ij}_k suitably, and the theorem follows. □

In Theorem A.2.2 it was essential that the space of TT-tensors considered included those for which $(x^2L^{ij})|_{\partial M} \not\equiv 0$. When $x^2L^{ij}|_{\partial M} = 0$ we have the following:

THEOREM A.2.3. Let $g \in C_\infty(\bar M)$ be a Riemannian metric, and let L be a transverse traceless tensor with respect to g. Consider the sets X_a, $a = 1, 2, 3$ of pairs (g, L) such that

$$\begin{array}{llll} X_1: & L & \in & x^{-1}C_\infty(\bar M) + \log x\, C_\infty(\bar M) \\ X_2: & L & \in & C_\infty(\bar M) \\ X_3: & L & \equiv & 0\,. \end{array}$$

(i) The set $\mathring{X}_a$, $a = 1, 2, 3$, of pairs (g, L) for which the solution of the Lichnerowicz equation has a $x^3 \log x$ term is open and dense in X_a.
(ii) In each case the sets $\hat X_a = X_a \backslash \mathring{X}_a$ are closed infinite dimensional subsets of X_a.

REMARK: Note that for $(g, L) \in \hat X_a$, $a = 2, 3$, the solution ϕ of the Lichnerowicz equation is smooth. On the other hand for generic $(g, L) \in \hat X_1$, ϕ is expected to be $C_4(\bar M)$ but not $C_5(\bar M)$ because of $x^5 \log x$ terms coming from $|\tilde L|^2_{\tilde g}$. The blow up of the fifth derivatives is in this case mild enough so that one gets $\phi \in H_5(M, g, d\mu_g)$, $x\phi^{-1/2} \in H_6(M, g, d\mu_g)$.

PROOF: In the notation of the proof of Theorem A.2.2, let $\bar{\mathcal{D}}_i$ be the Riemannian connection of the metric $\bar h_{ij} = \bar g_{ij} - \bar m_i \bar m_j$. A 2+1 decomposition calculation gives (note that $\bar m^i \bar D_i \bar m^j = 0$ because $\bar m_i dx^i = dx$)

$$\bar R = 2\bar m^k \bar D_k \bar\lambda - \bar\lambda^{ij}\bar\lambda_{ij} - \bar\lambda^2 + \bar{\mathcal{R}}\,, \tag{A.2.12}$$

where $\bar{\mathcal{R}}$ is the curvature scalar of $\bar h_{AB}$, so that if the coordinates v^A are chosen to satisfy $\bar g^{xA} \equiv 0$ one finds

$$\begin{aligned} \bar R_1 \ &\equiv\ \frac{\partial \bar R}{\partial x}\Big|_{\partial M} = \bar m^k \bar D_k R\Big|_{\partial M} \\ &=\ \left(2\bar m^i \bar m^j \bar D_i \bar D_j \bar\lambda - 2\bar\lambda^{ij}\bar m^k \bar D_k \bar\lambda_{ij} - 2\bar\lambda \bar m^k \bar D_k \bar\lambda + \bar m^k \bar D_k \bar{\mathcal{R}}\right) + 2\lambda^{ik}\lambda_i^j\lambda_{jk}\Big|_{\partial M} \end{aligned}$$

and (A.2.10) gives

$$\phi_{\log}(v) \ = \ \frac{1}{32}\left[-2\bar\lambda^{ij}\bar m^k \bar D_k \bar\lambda_{ij} + \bar\lambda_0 \bar m^k \bar D_k \bar\lambda\right](0, v) + \bar\psi(v)\,, \tag{A.2.13}$$

Here $\bar\psi$ depends only upon $g_{ij}\Big|_{\partial M}$, $\frac{\partial g_{ij}}{\partial x}\Big|_{\partial M}$ and their derivatives in directions tangent to ∂M. Suppose that (g, L) is such that $\phi_{\log} = 0$. If $\bar\lambda_{ij}\Big|_{\partial M} = 0$, in an arbitrary neighbourhood of g one can find a metric g' such that $\bar\lambda'_{ij}\Big|_{\partial M} \neq 0$. An appropriate small perturbation of $\frac{\partial^2 g'_{ij}}{\partial x^2}$ which does not change $g'_{ij}\Big|_{\partial M}$ and $\frac{\partial g'_{ij}}{\partial x}\Big|_{\partial M}$ will then make $\phi_{\log}$ non-zero, which shows density of $\mathring{X}_a$. □

APPENDIX B

B.1. "Almost Gaussian" coordinates.

Consider a metric $g \in C^{k+\lambda}(\bar{M})$; if $k \geq 2$ Gauss coordinates near ∂M can be introduced. Namely, 1) there exists a finite cover $\{\mathcal{U}_i\}_{i=0}^I$ of ∂M and $x_0 > 0$ such that $\cup_{i=1}^I [0, x_0) \times \mathcal{U}_i$ is a neighbourhood of ∂M and 2) in local coordinates $y = (x, v^A)$, $x \in [0, x_0)$, $v^A \in \mathcal{U}_i$ we have

$$g^{ij} = g(dy^i, dy^j) \in C^{k-2+\lambda}(\overline{(0, x_0) \times \mathcal{U}_i}), \tag{B.1.1}$$

$$g^{xx} \equiv 1, \quad g^{xA} \equiv 0. \tag{B.1.2}$$

The problem is, that the coordinates (x, v) above are obtained through a solution of the geodesic equation, which leads to the threshold $k \geq 2$, and which for $k < \infty$ leads to the loss of differentiability of g in the new coordinate system, as emphasized in eq. (B.1.1). This loss of differentiability is quite annoying, as then the use of *exact* Gauss coordinates in various applications leads to stronger restrictions on the degree of differentiability of the metric. On the other hand, for many applications it is sufficient for (B.1.2) to hold only approximately near ∂M. The aim of this Appendix is to show that this can be achieved *without losing differentiability* of g^{ij}. More precisely, we have the following:

Proposition B.1.1 ("Almost Gaussian Coordinates at ∂M"). Let $0 \leq k \leq \infty$, $\lambda \in [0, 1]$, consider a Riemannian metric $g \in C^{k+\lambda}(\bar{M})$ on $\bar{M}$, with ∂M — compact. There exists $x_0 > 0$ and a finite cover $\{\mathcal{U}_i\}_{i=1}^I$ of ∂M together with coordinates $(y^i) = (x, v^A) \in [0, x_0] \times \mathcal{U}_i$, such that $\cup_{i=1}^I [o, x_0] \times \mathcal{U}_i$ is a neighbourhood of ∂M, and such that

$$g(dy^i, dy^j) = g^{ij} \in C^{k+\lambda}(\overline{(0, x_0) \times \mathcal{U}_i}), \tag{B.1.3}$$

$$g(dx, dx) - 1 = o(x^k), \tag{B.1.4}$$

$$g(dx, dv^A) = o(x^k). \tag{B.1.5}$$

Remark: If $\lambda > 0$, (B.1.4) – (B.1.5) can be strengthened to $g^{xx} - 1 = O(x^{k+\lambda})$, $g^{xA} = O(x^{k+\lambda})$.

PROOF: If $k = \infty$ we can use Gauss coordinates near ∂M, and the result follows. Suppose thus that $k < \infty$, let $p \in \partial M$, let $\mathcal{O}$ be any conditionally compact coordinate neighbourhood of ∂M in which $\mathcal{O} \cap \partial M = \{x = 0\}$, with $g^{ij} \equiv g(dy^i, dy^j) \in C^{k+\lambda}(\bar{\mathcal{O}})$. Passing to a subset of $\mathcal{O}$ if necessary without loss of generality we may assume $\mathcal{O} \approx [0, x_0) \times \mathcal{U}$, $\mathcal{U} \subset \partial M$. Coordinate systems of this form will be called cylindrical. By Lemma 3.3.1 there exists $\bar{x} \in C^{k+1+\lambda}(\bar{\mathcal{U}})$ satisfying

$$\frac{\partial \bar{x}}{\partial x}\bigg|_{x=0} = (g^{xx})^{-1/2}, \quad \bar{x}\bigg|_{x=0} = 0.$$

The implicit function theorem implies that there exists $\mathcal{O}_1 \subset \mathcal{O}$ such that $(\bar{x}, v)$ are coordinates on $\mathcal{O}_1$, $\mathcal{O}_1$ being cylindrical in the coordinates $(\bar{x}, v)$. We also have $\bar{g}^{xx}|_{\bar{x}=0} = g(d\bar{x}, d\bar{x})|_{x=0} = g^{xx}(\frac{\partial \bar{x}}{\partial x})^2|_{x=0} = 1$. Passing to the coordinates $(\bar{x}, v)$, dropping the "1" on $\mathcal{O}_1$ and dropping bars on $\bar{x}$ and on $\bar{g}^{xx}$ we thus have

$$g^{xx}\bigg|_{x=0} = 1. \tag{B.1.6}$$

Let $\phi, f^A \in C^{k+1+\lambda}(\bar{\mathcal{O}})$, consider

$$\bar{x} = x + \phi(x, v), \tag{B.1.7}$$

$$\bar{v}^A = v^A + f^A(x, v), \tag{B.1.8}$$

hence

$$\bar{g}^{xx} = g^{xx} + 2g^{xi}\frac{\partial \phi}{\partial x^i} + g^{ij}\frac{\partial \phi}{\partial x^i}\frac{\partial \phi}{\partial x^j}, \tag{B.1.9}$$

$$\bar{g}^{xA} \equiv g(d\bar{x}, d\bar{v}^A) = g^{xA} + g^{xi}\frac{\partial f^A}{\partial x^i} + g^{Ai}\frac{\partial \phi}{\partial x^i} + g^{ij}\frac{\partial \phi}{\partial x^i}\frac{\partial f^A}{\partial x^j}. \tag{B.1.10}$$

Suppose that for some $\ell \geq 0$ we have

$$g^{xx} - 1 = o(x^\ell), \quad g^{xA} = o(x^{\ell-1}). \tag{B.1.11}$$

It follows from (B.1.6) that (B.1.11) holds with $\ell = 0$. By Lemma 3.3.1 there exists $f^A \in C^{k+1+\lambda}(\bar{\mathcal{O}})$ satisfying

$$\frac{\partial^{\ell+1} f^A}{\partial x^{\ell+1}}\bigg|_{x=0} = -\frac{\partial^\ell g^{xA}}{\partial x^\ell}\bigg|_{x=0},$$

while all the lower order x–derivatives of f^A vanish at $x = 0$. Passing to coordinates $(x, \bar{v})$ on a (possibly smaller) cylindrical neighbourhood $\mathcal{O}$, and dropping bars, one finds from (B.1.9)–(B.1.10) that (B.1.10) still holds with, moreover

$$g^{xA} = o(x^\ell), \tag{B.1.12}$$

so that for $\ell = k = 0$ the proof is completed. If $0 < \ell \leq k$ by Lemma 3.3.1 there exists $\phi \in C^{k+1+\lambda}(\bar{\Omega})$ such that

$$\frac{\partial^{\ell+1}\phi}{\partial x}\bigg|_{x=0} = -\frac{\partial^\ell g^{xA}}{\partial x}\bigg|_{x=0},$$

with all the lower order x–derivatives of ϕ vanishing at $x = 0$. Passing to the coordinates $(\bar{x}, v)$ and dropping the bar on $\bar{x}$ one finds that (B.1.11) holds with ℓ replaced by $\ell + 1$, and the induction step is completed. $\square$

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