ASYMPTOTICALLY SIMPLE SOLUTIONS OF THE VACUUM EINSTEIN EQUATIONS IN EVEN DIMENSIONS

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Abstract. We show that a set of conformally invariant equations derived from the Fefferman-Graham tensor can be used to construct global solutions of vacuum Einstein equations, in all even dimensions. This gives, in particular, a new, simple proof of Friedrich's result on the future hyperboloidal stability of Minkowski space-time, and extends its validity to even dimensions.

1. Introduction

Consider the class of vacuum solutions to the Einstein equations $(\mathscr{M}, g)$ in $n+1$ dimensions, which are future asymptotically simple, i.e. conformally compact, in the sense of Penrose, to the future of a complete Cauchy surface $(\mathscr{S}, \gamma)$. A natural method to try to construct such space-times is to solve a Cauchy problem for the compactified, unphysical space-time $(\mathscr{M}, \bar{g})$, and then recover the associated physical space-time via a conformal transformation. However, a direct approach along these lines leads to severe difficulties, since the conformally transformed vacuum Einstein equations form, at best, a degenerate system of hyperbolic evolution equations, for which it is very difficult to prove existence and uniqueness of solutions.

Friedrich [20, 23] has developed a method to overcome this difficulty in $3+1$ dimensions, by introducing a system of "conformal Einstein equations" whose solutions include the vacuum Einstein metrics and which transforms naturally under conformal changes. A variation upon Friedrich's approach, again in $3+1$ dimensions, has been presented in [6].

In this paper, we develop a different approach to this issue which, besides its simplicity, has the advantage of working in all *even* dimensions.

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The method, carried out for vacuum space-times with $\Lambda > 0$ in [2], is based on use of the Fefferman-Graham (ambient obstruction) tensor $\mathcal{H}$, introduced in [16]. The tensor $\mathcal{H}$ is a symmetric bilinear form, depending on a metric g and its derivatives up to order $n+1$, cf. Section 2 for further discussion. It is conformally covariant, (of weight $n-1$) and metrics conformal to Einstein metrics satisfy the system

$$\mathcal{H} = 0 \; . \tag{1.1}$$

When $n = 3$, i.e. in space-time dimension 4, the Fefferman-Graham tensor is the well-known Bach tensor.

The main result of the paper, Theorem 4.1, is the proof of the well-posedness of the Cauchy problem for the equation (1.1), for Lorentz metrics. This leads to a new proof of Friedrich's result on the future "hyperboloidal" stability of Minkowski space-time [21] (see Theorem 6.1), and extends the validity of this result to all even dimensions.[1] As a corollary we additionally obtain existence of a large class of non-trivial, vacuum, even dimensional space-times which are *asymptotically simple* in the sense of Penrose, see Theorem 6.2.

We further note that in [2] existence of solutions of the Cauchy problem for (1.1) is obtained by pseudo-differential techniques. Here we show that the Anderson-Fefferman-Graham (AFG) equations (1.1) can be solved using an auxiliary, first order, symmetrisable hyperbolic system of equations. This shows that (1.1) is a well posed evolutionary system, directly amenable to numerical treatment. Thus, in space-time dimension four we provide an alternative to Friedrich's conformal equations for the numerical construction of global space-times [18, 19, 28].

Our methods do not apply in odd space-time dimensions, where the situation is rather different in any case, as one generically expects polyhomogeneous expansions with half-integer powers of $1/r$, where r is, say, the luminosity distance, compare [11, 26, 27, 30].

2. The Anderson-Fefferman-Graham equations

Let, as before, $n+1$ denote space-time dimension, with n odd. The Fefferman-Graham tensor $\mathcal{H}$ is a conformally covariant tensor, built out

[1]Once most of the work on this paper was completed we have been informed of the work by Schimming [32], who has done a local analysis, related to ours, of the Cauchy problem for the Bach equations in dimension four. The application of his work to global issues, for instance concerning the vacuum Einstein equations, seems not to have been addressed. We are grateful to R. Beig for pointing out that reference to us.

of the metric g and its derivatives up to order $n+1$, of the form

$$\mathcal{H} = (\nabla^*\nabla)^{\frac{n+1}{2}-2}[\nabla^*\nabla(P) + \nabla^2(trP)] + \mathcal{F}^n \ , \tag{2.1}$$

where

$$P = \mathrm{Ric}_g - \frac{R_g}{2n} g \ , \tag{2.2}$$

and where $\mathcal{F}^n$ is a tensor built out of lower order derivatives of the metric (see, e.g., [25], where the notation $\mathcal{O}$ is used in place of $\mathcal{H}$). It turns out that $\mathcal{F}^n$ involves only derivatives of the metric up to order $n-1$: this is an easy consequence of Equation (2.4) in [25], using the fact that odd-power coefficients of the expansion of the metric g_x in [25, Equation (2.3)] vanish. (For $n = 3, 5$ this can also be verified by inspection of the explicit formulae for $\mathcal{F}^3$ and $\mathcal{F}^5$ given in [25].)

The system of equations

$$\mathcal{H} = 0 \tag{2.3}$$

will be called the Anderson-Fefferman-Graham (AFG) equations. It has the following properties [25]:

(1) The system (2.3) is conformally invariant: if g is a solution, so is $\varphi^2 g$, for any positive function φ.
(2) If g is conformal to an Einstein metric, then (2.3) holds.
(3) $\mathcal{H}$ is trace-free.
(4) $\mathcal{H}$ is divergence-free.

Recall that $\mathcal{H}$ was originally discovered by Fefferman and Graham [16] as an obstruction to the existence of a formal power series expansion for conformally compactifiable Einstein metrics, with conformal boundary equipped with the conformal equivalence class $[g]$ of g. This geometric interpretation is irrelevant from our point of view, as here we are interested in (2.3) as an equation on its own.

3. Reduction to a symmetrisable hyperbolic system

Let g be a Lorentzian metric, let ∇ be a connection (not necessarily g-compatible), and let $\Box$ denote an operator with principal part $g^{\mu\nu}\nabla_\mu\nabla_\nu$ (acting perhaps on tensors). Let u be a tensor field, and let $\nabla^{(k)}u$ denote the tensor formed from the k-th order covariant derivatives of u. For $k \geq 0$ consider the system of equations

$$\Box^{k+1} u = F(x, u, \nabla u, \nabla^{(2)} u, \ldots, \nabla^{(2k+1)} u) \ , \tag{3.1}$$

for some smooth F. Here we allow the coefficients of $\Box$ as well as the connection coefficients to depend smoothly upon x as well as upon the collection of fields $(u, \nabla u, \nabla^{(2)} u, \ldots, \nabla^{(2k+1)} u)$, in particular the metric g is allowed to depend (smoothly) upon those fields. We will assume that (3.1) is invariant under diffeomorphisms, although this is not necessary for some of our results below, such as local existence and local uniqueness of solutions.

We want to show that solutions of (3.1) can be found by solving a first order symmetric hyperbolic system of PDEs. The idea of the proof can be illustrated by the following example. Consider the equation

$$\Box^2 u = 0 \; . \tag{3.2}$$

Introducing

$$\psi^{(0)} = u \; , \qquad \psi^{(1)} = \Box u \; ,$$

it is easily seen that solutions of (3.2) are in one-to-one correspondence with solutions of the system

$$\Box \begin{pmatrix} \psi^{(0)} \\ \psi^{(1)} \end{pmatrix} = \begin{pmatrix} \psi^{(1)} \\ 0 \end{pmatrix} \; . \tag{3.3}$$

It is then standard to write a symmetrisable-hyperbolic first order system so that solutions of (3.3) are in one-to-one correspondence with solutions of the first order system with appropriate initial data (compare the calculations in the proof below).

Some work is needed when we want to allow lower order derivatives as in the right-hand-side of (3.1):

PROPOSITION 3.1. *There exists a symmetrisable hyperbolic first order system*

$$P\Phi = H(\Phi) \; , \tag{3.4}$$

where P is a linear first order operator and H does not involve derivatives of Φ, such that every solution of (3.1)*, with $(\mathscr{M}, g)$ time orientable, satisfies* (3.4)*.*

Proof. Let $\{e_a\}_{a=0,\ldots,n} = \{e_0, e_i\}_{i=1,\ldots,n}$ be an orthonormal frame for g, with e_0 a globally defined unit timelike vector; (such vector fields always exist on time orientable manifolds). We set

$$\varphi^{(j)} = \{\varphi^{(j)}_{a_1\ldots a_i}\}_{1\le i\le 2(k-j)} \; , \text{ where } \quad \varphi^{(j)}_{a_1\ldots a_i} = e_{a_1}\cdots e_{a_i}\Box^j u \; , \tag{3.5}$$

$$\varphi = \{\varphi^{(j)}\}_{0\le j\le k} \; , \tag{3.6}$$

$$\psi = \{\psi^{(j)}\}_{0\le j\le k} \; , \text{ where } \qquad \psi^{(j)} = \Box^j u \; . \tag{3.7}$$

Let us derive a convenient system of equations for φ. First,

$$\text{for } 1 \le i \le 2(k-j)-1 \qquad e_0\varphi^{(j)}_{a_1\ldots a_i} = \varphi^{(j)}_{0a_1\ldots a_i} = L(\varphi^{(j)}) \; , \tag{3.8}$$

where we use a generic symbol L to denote a linear map which may change from line to line. This gives evolution equations for those

$\varphi^{(j)}_{a_1\ldots a_i}$'s which have a number of indices strictly smaller than the maximum number allowed.

We continue by noting that, again for $1 \le i + 2j \le 2k-1$, we have on the one hand

$$\Box\varphi^{(j)}_{a_1\ldots a_i} = -e_0\varphi^{(j)}_{0a_1\ldots a_i} + \sum_{\ell=1}^{n} e_\ell\varphi^{(j)}_{\ell a_1\ldots a_i} + L(\varphi^{(j)}) \; , \tag{3.9}$$

and on the other

$$\begin{aligned} \Box\varphi^{(j)}_{a_1\ldots a_i} &= e_{a_1}\cdots e_{a_i}\Box\psi^{(j)} + [\Box, e_{a_1}\cdots e_{a_i}]\psi^{(j)} \\ &= \varphi^{(j+1)}_{a_1\ldots a_i} + L(\varphi^{(j)}) \; . \end{aligned} \tag{3.10}$$

Combining those two equations we obtain

$$e_0\varphi^{(j)}_{0a_1\ldots a_i} = \sum_{\ell=1}^{n} e_\ell\varphi^{(j)}_{\ell a_1\ldots a_i} + L(\varphi^{(j)}, \varphi^{(j+1)}) \; . \tag{3.11}$$

Note that the condition $i+2j \le 2k-1$ implies $j < k$ so that (3.11) can also be rewritten as

$$e_0\varphi^{(j)}_{0a_1\ldots a_i} = \sum_{\ell=1}^{n} e_\ell\varphi^{(j)}_{\ell a_1\ldots a_i} + L(\varphi) \; . \tag{3.12}$$

Next, for $i + 2j = 2k-1$ and for ℓ running from 1 to n we write

$$\begin{aligned} e_0\varphi^{(j)}_{\ell a_1\ldots a_i} &= e_\ell\varphi^{(j)}_{0a_1\ldots a_i} + [e_0, e_\ell]\varphi^{(j)}_{a_1\ldots a_i} \\ &= e_\ell\varphi^{(j)}_{0a_1\ldots a_i} + L(\varphi^{(j)}) \; . \end{aligned} \tag{3.13}$$

The rewriting of (3.12)-(3.13) in the form

$$\begin{pmatrix} e_0\varphi^{(j)}_{0a_1\ldots a_i} & -e_1\varphi^{(j)}_{1a_1\ldots a_i} & \cdots & -e_n\varphi^{(j)}_{na_1\ldots a_i} \\ -e_1\varphi^{(j)}_{0a_1\ldots a_i} & +e_0\varphi^{(j)}_{1a_1\ldots a_i} & +0 & +0 \\ \vdots & +0 & \ddots & \vdots \\ -e_n\varphi^{(j)}_{0a_1\ldots a_i} & +0 & \cdots & +e_0\varphi^{(j)}_{na_1\ldots a_i} \end{pmatrix} = L(\varphi) \tag{3.14}$$

makes explicit the symmetric character of (3.12)-(3.13). It is well known that this system is symmetrisable hyperbolic in the sense of [34, Volume III] when e^0 is a nowhere vanishing vector field.[2] This provides the desired system of evolution equations for those $\varphi^{(j)}_{a_0a_1\ldots a_i}$'s which have

[2]In fact, (3.14) is symmetric hyperbolic in a coordinate system with $e_0 = \partial_t$ and $e_i t = 0$. However, when g depends upon u and its derivatives it is not useful to use such coordinates, as the construction of the Gauss coordinate system leads to differentiability loss. In any case Gauss coordinates are not well adapted to the proof of existence of solutions when g depends upon u.

the maximum number of indices. (One could also use (3.14) for any number of indices, but (3.8) is obviously simpler.)

If we write (3.14) as

$$\mathring{P}\varphi = 0 \ ,$$

where $\mathring{P}$ is a linear first order operator, then the derivatives $e_a\varphi$ satisfy a first order symmetrisable-hyperbolic system of equations

$$\mathring{P}e_a\varphi = L(\varphi, \nabla\varphi) \ , \qquad L(\varphi, \nabla\varphi) := [\mathring{P}, e_a]\varphi \ . \tag{3.15}$$

The evolution equations for ψ are simply

$$\Box\psi^{(i)} = \psi^{(i+1)} \ , \qquad 0 \le i \le k-1 \ , \tag{3.16}$$

$$\Box\psi^{(k)} = F(x, \psi^{(0)}, \varphi^{(0)}, \nabla\varphi^{(0)}) \ , \tag{3.17}$$

where in (3.17) we have expressed the derivatives of u appearing in (3.1) in terms of $\varphi^{(0)}$ and $\nabla\varphi^{(0)}$ using (3.5). By obvious modifications of the calculation starting at (3.9) and ending at (3.14) one can rewrite the left-hand-side of (3.16)-(3.17) as a first order symmetrisable hyperbolic operator acting on the collection of fields $(\psi, \nabla\psi) := \{(\psi^{(i)}, \nabla\psi^{(i)})\}_{0\le i\le k}$. Setting

$$\Phi = (\psi, \nabla\psi, \varphi, \nabla\varphi) \ , \tag{3.18}$$

and letting P be the linear part of the system of equations just described, the proposition follows. □

The interest of Proposition 3.1 relies in the fact, that it is standard to prove existence and uniqueness of solutions of (3.4) when the initial data for Φ are in H^s, $s \in \mathbb{N}$, for $s > n/2 + 1$, provided that $(\mathscr{M}, g)$ is globally hyperbolic. If g does *not* depend on $\nabla^{2k+1}u$, then the threshold can be lowered[3] to $s > n/2$.

Now, not every solution of (3.4) will be a solution of (3.1). Let us show that appropriate initial data for (3.4) will provide the desired solutions. When the space-time metric g is independent of u, let $\mathscr{S}$ be a spacelike hypersurface in the space-time $(\mathscr{M}, g)$. We choose e_0 to be a unit time-like vector field normal to $\mathscr{S}$, so that the e_i's are tangential at $\mathscr{S}$, and we extend e_0 off $\mathscr{S}$ in some convenient way, which might vary according to the context. Since (3.1) is an equation of order $2k+2$, the associated Cauchy data consist of a set of tensor fields $\{f_{(i)}\}_{i=0,\dots,2k+1}$

[3]For $s > n/2 + 1$ the result follows from [34, Volume III, Theorem 2.3, p. 375]. However, when the symmetric hyperbolic system has the structure considered here, with g *not* depending upon $\nabla^{2k+1}u$, the proof in [34] applies for $s > n/2$.

defined on $\mathscr{S}$ which provide initial data for $(e_0)^i u$ on $\mathscr{S}$:

$$(e_0)^i u|_{\mathscr{S}} := \underbrace{e_0 \cdots e_0}_{i \text{ times}} u|_{\mathscr{S}} = f_{(i)} \ , \qquad 0 \le i \le 2k+1 \ . \tag{3.19}$$

For any $i \ge 0$ and $\ell \ge 0$ we can use (3.19) to calculate formally $\psi^{(i)}|_{\mathscr{S}}$ and $\psi^{(i)}_0|_{\mathscr{S}}$ by replacing each occurrence of $(e_0)^j u$ by $f_{(j)}$, e.g., $\psi^{(0)}|_{\mathscr{S}} = f_{(0)}$, $e_0\psi^{(0)}|_{\mathscr{S}} = f_{(1)}$,

$$\begin{aligned} \psi^{(1)}|_{\mathscr{S}} &= (\Box u)|_{\mathscr{S}} = \Big(- e_0 e_0 u + \sum_{i=1}^n e_i e_i u + \Gamma^\alpha e_\alpha u + \Gamma u \Big)\Big|_{\mathscr{S}} \\ &= -f_{(2)} + \sum_{i=1}^n e_i e_i f_{(0)} + \Gamma^0 f_{(1)} + \sum_{i=1}^n \Gamma^i e_i f_{(0)} + \Gamma f_{(0)} \ , \end{aligned}$$

for some linear maps Γ^α, Γ arising from the detailed structure of $\Box$, and so on. We will write $g^{(i)}_{(0)}$ for the resulting functions $\psi^{(i)}|_{\mathscr{S}}$ and $g^{(i)}_{(1)}$ for the resulting functions $e_0\psi^{(i)}|_{\mathscr{S}}$, so that

$$\psi^{(i)}|_{\mathscr{S}} = g^{(i)}_{(0)} \ , \quad e_0\psi^{(i)}|_{\mathscr{S}} = g^{(i)}_{(1)} \ .$$

Similarly we can calculate

$$\varphi^{(i)}_{0_1\ldots 0_\ell}|_{\mathscr{S}} := \varphi^{(i)}_{\underbrace{0\ldots 0}_{\ell \text{ factors}}}|_{\mathscr{S}}$$

where we replace each occurrence of $(e_0)^j u$ by $f_{(j)}$, e.g. $\varphi^{(0)}_{0_1\ldots 0_\ell}|_{\mathscr{S}} = f_{(\ell)}$. We will write $h^{(i)}_{(\ell)}$ for the resulting functions, so that

$$\varphi^{(i)}|_{\mathscr{S}} = h^{(i)}_{(0)} = g^{(i)}_{(0)} \ , \quad \varphi^{(i)}_{(1)}|_{\mathscr{S}} = h^{(i)}_{(1)} = g^{(i)}_{(1)} \ , \quad \varphi^{(i)}_{0_1\ldots 0_\ell}|_{\mathscr{S}} = h^{(i)}_{(\ell)} \ .$$

When g *does* depend upon u, then the space-time will be built in the process of solving the equations. In the simplest case of g depending only upon u, the procedure just described should be understood as follows: the initial metric $g|_{\mathscr{S}}$ is determined by the initial data f_0. We choose an orthonormal basis $\{e_i\}_{i=0,\ldots,n}$ for $g|_{\mathscr{S}}$, and interpret e_0 as the unit normal to $\mathscr{S}$ in the space-time that will arise out of the initial data. Thus, $f_{(1)}$ will be interpreted as the value of the normal derivative of u at $\mathscr{S}$, and so on, and the above considerations remain unchanged when this interpretation is used.

PROPOSITION 3.2. *Let $\Phi \in C(I, H^s(\mathscr{O}))$, $s > n/2 + 2k + 2$, $s \in \mathbb{N}$, be a solution of* (3.4) *on a globally hyperbolic region $I \times \mathscr{O}$ with initial data constructed as described above. Then $u := \psi^{(0)}$ is a solution of* (3.1) *and* (3.19).

Proof. From (3.16) one has $\psi^{(i)} = \Box^i \psi^{(0)}$ for $0 \le i \le k$. It remains to show that if $\varphi^{(0)}_{a_1\ldots a_i} = e_{a_1}\cdots e_{a_i}\psi^{(0)}$, then (3.17) will coincide with (3.1). This can be proved by a standard calculation. One sets

$$\chi^{(j)}_{a_1\ldots a_i} = \varphi^{(j)}_{a_1\ldots a_i} - e_{a_1}\cdots e_{a_i}\psi^{(j)} ,$$

and using (3.4) one derives a system of equations which show that $\chi^{(j)}_{a_1\ldots a_i} = 0$ for the initial data under consideration.

However, the computations involved are avoided by the following argument. Suppose, first, that g, F and ∇ are analytic functions of all their variables. Let us denote by $f = \{f_{(j)}\}_{0\le j\le 2k+1}$ the initial data for (3.1); by an abuse of notation we will write $f \in H^s$ if $f_{(j)} \in H^{s-j}$ for $0 \le j \le 2k+1$. We note, first, that by using an exhaustion of $I \times \mathscr{O}$ by compact subsets thereof it suffices to prove the result when $I \times \mathscr{O}$ is a conditionally compact subset of the domain of definition of the solution. Let f_n be any sequence of analytic initial data which converges in $H^s(\mathscr{O})$ to f. Let u_n be the corresponding solution of (3.4); by stability all u_n's will be defined on $I \times \mathscr{O}$ for n large enough. Similarly, the stability estimates[4] for symmetric hyperbolic systems [29] prove that u_n is Cauchy in $C(I, H^s(\mathscr{O})) \cap C^1(I, H^{s-1}(\mathscr{O}))$. The results in [1] show that u_n is analytic throughout $I \times \mathscr{O}$.

Let $\widehat{u}_n$ be a solution of (3.1) on an open neighborhood $\mathscr{U}_n$ of $\mathscr{S}$ in $I \times \mathscr{O}$ obtained by the Cauchy-Kowalevska theorem. (Note that $\mathscr{U}_n$ could in principle shrink as n tends to infinity, but it is nevertheless open and nonempty for each n.) Passing to a subset of $\mathscr{U}_n$ if necessary we can without loss of generality assume that $\mathscr{U}_n$ is globally hyperbolic.

Now, uniqueness of the solutions of the Cauchy problem for (3.4) shows that u_n coincides with $\widehat{u}_n$ on $\mathscr{U}_n$. Thus u_n satisfies (3.1) there and thus, by analyticity, everywhere. This shows that maximal globally hyperbolic solutions of (3.1) with analytic initial data are in one-to-one correspondence with maximal globally hyperbolic solutions of (3.4) with the initial data constructed as above. Then, for H^s initial data, Proposition 3.2 follows from continuity of solutions upon initial data for (3.4).

Finally, if the fields g, ∇ and F are smooth functions of their arguments, they can be approximated by a sequence of fields $g(n)$, $\nabla(n)$ and $F(n)$ which are analytic in their arguments. The estimates for (3.4) just described can similarly be used to show that solutions of the approximate problem converge to solutions of the problem at hand both for equation (3.4) and (3.1), which finishes the proof. $\square$

[4]Note that $I \times \partial\mathscr{O}$ is non-timelike by global hyperbolicity, so that integration by parts gives harmless contributions as far as energy estimates are concerned.

From what has been said so far we obtain

THEOREM 3.3. *Let $s > n/2 + 2k + 2$, $s \in \mathbb{N}$. For any fields*

$$f_{(i)} \in H^{s-i}_{\mathrm{loc}}(\mathscr{S}) , \qquad i = 0, \dots, 2k+1 ,$$

there exists a unique solution of (3.1) *satisfying* (3.19). *If the metric g does* not *depend upon $\nabla^{2k+1}u$, then $s > n/2 + 2k + 1$ suffices.*

We note that in local coordinate systems (t, x^i) on an open neighborhood $\mathscr{U}$ of $\mathscr{O} \subset \mathscr{S}$ of the form $\mathscr{U} = I \times \mathscr{O}$, with $\mathscr{S} \cap \mathscr{U} = \{t = 0\}$ and $\overline{\mathscr{O}}$-compact, the solutions are in

$$u \in \cap_{i=0}^{2k+1} C^i(I, H^{s-i}(\mathscr{O})) .$$

As usual, the lower bound for the local time of existence of the solution does not depend upon the differentiability class s, so in particular smooth initial data provide smooth solutions. In addition the Cauchy problem for (3.1) is well-posed, in that given a pair of initial data $f^1_{(i)}$, $f^2_{(i)}$ which are close in $H^{s-i}_{\mathrm{loc}}(\mathscr{S})$, then the solutions u^1, u^2 are also close in $\cap_{i=0}^{2k+1} C^i(I, H^{s-i}_{\mathrm{loc}}(\mathscr{S}))$.

When g does not depend upon u, there exists a unique maximal globally hyperbolic subset $\mathscr{O}$ of $\mathscr{M}$, with $\mathscr{S}$ being Cauchy for $\mathscr{O}$, on which the solution exists. This is proved by the usual methods.

In the quasi-linear case one also has the existence of a maximal globally hyperbolic development of the Cauchy data, giving a space-time $(\mathscr{M}, g)$. This follows from the fact that the domains of dependence for the system constructed above are determined by the light-cones of the metric g, so that a proof along the lines of [5], (compare [7,8]), applies.

4. The Cauchy problem for the AFG equations

The Cauchy problem for (2.3) has a similar form to that for the Einstein equations. Since the system (2.3) is of order $n+1$, the initial data consist of an n-dimensional Riemannian manifold $(\mathscr{S}, \gamma)$, $n = 2k+1 \geq 3$, with n symmetric two-tensors $K^{(i)}$ prescribed on $\mathscr{S}$. The tensor fields $K^{(i)}$ represent the i-th time derivative of the metric g in a Gauss coordinate system around $\mathscr{S}$. Thus, in a neighborhood of $\mathscr{S}$, (or more precisely a neighborhood of a bounded domain in $\mathscr{S}$), one may write

$$g = -dt^2 + \gamma(t), \tag{4.1}$$

where $\gamma(t)$ is a curve of metrics on $\mathscr{S}$. Setting $e_0 = -\nabla t$, one has

$$K^{(i)} = \frac{1}{2}\mathcal{L}^i_{e_0} g|_{t=0} = \frac{1}{2}\partial^i_t \gamma(t)|_{t=0}. \tag{4.2}$$

In particular $K = K^{(1)}$ is the extrinsic curvature tensor of $\mathscr{S}$ in the final space-time $(\mathscr{M}, g)$.

The set $(\gamma, K^{(1)}, \ldots, K^{(n)})$ is not arbitrary, since the equations

$$\mathcal{H}(e_0, \cdot) = 0, \tag{4.3}$$

only involve t-derivatives of g up to order n, and so induce $(n+1)$ equations on $(\gamma, K^{(1)}, \ldots, K^{(n)})$. Because (4.3) is diffeomorphism invariant, this is most easily seen in the coordinates (4.1) where $g_{0\alpha} = -\delta_{0\alpha}$, so that (4.3) only involves t-derivatives of g_{ab}, $a, b \geq 1$, up to order n. The fact that the constraint equations (4.3) are preserved under the evolution follows in the usual way from the equation $\delta\mathcal{H} = 0$, where δ denotes the divergence of a tensor.

To describe the system of $n+1$ constraint equations (4.3) in more detail[5], the Gauss-Codazzi equations for the embedding $\mathscr{S} \subset (\mathscr{M}, g)$ are:

$$R_\gamma - |K|^2 + H^2 = R + 2\,\mathrm{Ric}(e_0, e_0), \tag{4.4}$$

$$\delta K - dH = \mathrm{Ric}(e_0, \cdot), \tag{4.5}$$

where $H = trK$. In addition, in a Gauss coordinate system (t, x^i) near $\mathscr{S}$, the Raychaudhuri equation gives

$$\partial_t H + |K|^2 = -\,\mathrm{Ric}(e_0, e_0). \tag{4.6}$$

We first point out that the curvature scalar $R = R_g$ is determined directly by the initial data; this is in constrast to the situation with the Einstein equations, where R is determined by the evolution equations for the metric. Namely, the left side of (4.4) is determined by the initial data, as is $\mathrm{Ric}(e_0, e_0)$, by (4.6). Thus R is determined by γ and $K^{(i)}$, for $i = 1, 2$.

To describe the form of the "scalar constraint equation" $\mathcal{H}_{\mu\nu} n^\mu n^\nu = 0$, $n = e_0$, note that from (2.2) one has $\mathrm{tr}P = (n-1)R_g/2n$. Together with (4.4) and (4.6), this leads to

$$\begin{aligned} \Box P_{00} - \nabla_0 \nabla_0 \mathrm{tr}P &= \left(\frac{1-n}{n}\right) \left(\Box + \nabla_0 \nabla_0\right) \partial_t H \\ &\quad + \frac{1}{2n} \left(\Box + (1-n)\nabla_0 \nabla_0\right) R_\gamma + \ldots\,, \end{aligned} \tag{4.7}$$

where "..." stands for terms which contain less derivatives of *the space-time metric.* The t-derivatives of the metric of order 4 cancel out, as

[5] An explicit form of (4.3) in space dimension 3 can be found in [32]. The parameterisation of the initial data there is rather different from ours.

expected, and one easily finds that the equation $\mathcal{H}_{\mu\nu}n^\mu n^\nu = 0$ takes the form

$$\Delta^{\frac{n-1}{2}} R_\gamma = \rho_n \ , \tag{4.8}$$

where, as before, R_γ is the curvature scalar of γ, with $\Delta = D^k D_k$ the Laplace operator of γ. Finally, ρ_n is a functional of $(\gamma, K^{(1)}, \ldots, K^{(n)})$ which does *not* involve derivatives of *initial data* of order $n+1$, while the left-hand-side does. This shows in particular that (4.3) is a non-trivial restriction on the initial data.

It also follows from (4.7) that ρ_n will contain terms of the form

$$\frac{1}{n}\Delta(\mathrm{tr}K^{(n-1)}) - D^k D^l K^{(n-1)}_{kl} \ , \tag{4.9}$$

with other occurrences of $K^{(n-1)}$ there, if any, being also linear with at most one space-derivative. Thus, the scalar constraint equation $\mathcal{H}_{\mu\nu}n^\mu n^\nu = 0$ can be viewed either as a non-linear equation of order $n+1$ which puts restrictions on γ in terms of the remaining data, or as a second order linear PDE for the trace of $K^{(n-1)}$.

One can similarly check that the equation $\mathcal{H}_{0i} = 0$ takes the form of a linear first order PDE for $K^{(n)}$, with principal part

$$D^i(K^{(n)}_{ij} - \frac{1}{n}\mathrm{tr}K^{(n)} g_{ij}) \ , \tag{4.10}$$

where D denotes the Levi-Civita connection of γ.

A set $(\gamma, K^{(1)}, \ldots, K^{(n)})$ will be called an *initial data set* if the fields $(\gamma, K^{(1)}, \ldots, K^{(n)})$ satisfy the constraint equations (4.3). The collection of initial data sets is not empty, as every solution of the general relativistic constraint equations solves (4.3).

The equations (4.3) are preserved by the following family of transformations, related to the conformal invariance of (2.3). Suppose that the data set $(\gamma, K^{(1)}, \ldots, K^{(n)})$ arises from a space-time $(\mathscr{M}, g)$ satisfying (2.3). Then $(\gamma, K^{(1)}, \ldots, K^{(n)})$ satisfies (4.3), and if Ω is any strictly positive function on $\mathscr{M}$, then the set $(\tilde\gamma, \tilde K^{(1)}, \ldots, \tilde K^{(n)})$ obtained on $\mathscr{S}$ from the metric $\Omega^2 g$ also satisfies (4.3). For example, if we set

$$\omega := \Omega|_{\mathscr{S}} \ , \quad \omega^{(j)} := \underbrace{e_0 \cdots e_0}_{j \text{ factors}}(\Omega)|_{\mathscr{S}} \ , \tag{4.11}$$

where e_0 is, e.g., a geodesic extension of e_0 off $\mathscr{S}$, then it holds that

$$\tilde\gamma = \omega^2\gamma \ , \qquad \tilde K^{(1)} = \omega K^{(1)} + \omega^{(1)}\gamma \ . \tag{4.12}$$

Similar but more complicated transformation formulae hold for $\tilde K^{(i)}$, $i \geq 2$, see Appendix A. This leads to a family of transformations preserving

(4.3) which are parameterized by $n+1$ functions ω, $\omega^{(j)}$, $j=1,\ldots,n$, on $\mathscr{S}$, which are arbitrary except for the requirement that $\omega > 0$.

Recall that under a conformal transformation of the space-time metric we have

$$\tilde g_{ij} = \Omega^{\frac{4}{n-1}} g_{ij} \quad \Longrightarrow \quad \tilde R = \Omega^{-\frac{4}{n-1}} \left(R - \frac{4n}{(n-1)\Omega} \Box_g \Omega \right) . \tag{4.13}$$

It follows from this formula that for any given $\omega = \Omega|_{\mathscr{S}} > 0$ and $\omega^{(1)} = e_0(\Omega)|_{\mathscr{S}}$ one can choose $\omega^{(2)}$ so that Ω solves the linear wave equation

$$\frac{4n}{(n-1)} \Box_g \Omega = R\Omega \tag{4.14}$$

when restricted to $\mathscr{S}$. (This equation is globally solvable in globally hyperbolic space-times, but this is irrelevant for the current discussion. Note that solutions of (4.14) might sometimes develop zeros; these are essential in the analysis of the vacuum Einstein equations). As remarked above that the curvature scalar R is determined by the order 2 part of the initial data set. Taking further e_0-derivatives shows that the remaining $\omega^{(j)}$'s may be chosen so that the conformally transformed curvature scalar $\tilde R$, together with its normal derivatives up to order $n-2$, vanish at $\mathscr{S}$.

Now the conformal and diffeomorphism invariance of (2.3) requires a suitable choice of gauge in order to obtain a well-posed system. As in [2], we use constant scalar curvature for the conformal gauge and harmonic coordinates for the diffeomorphism gauge; the treatment of the conformal gauge is somewhat different here than in [2].

Thus, we require first that

$$R = 0 . \tag{4.15}$$

If (4.15) holds, then (2.3) takes the form

$$(\nabla^*\nabla)^{\frac{n-1}{2}} \operatorname{Ric} = -\mathcal{F}^n , \tag{4.16}$$

In harmonic coordinates $\{y^\alpha\} = \{(\tau, y^i)\}$ with respect to the conformal gauge (4.15), one has

$$\operatorname{Ric}_{ab} = -\frac{1}{2} g^{\mu\nu}\partial_\mu\partial_\nu g_{ab} + Q_{ab}(g,\partial g) ,$$

where Q is quadratic in g and ∂g. Applying $(\nabla^*\nabla)^{(n-1)/2}$ to this and commuting $(\nabla^*\nabla)^{(n-1)/2}$ with the ∂g terms in Q shows that (4.16) has the form

$$\Box_g^{\frac{n+1}{2}} g_{\alpha\beta} = -\widehat{F}^n_{\alpha\beta}, \tag{4.17}$$

where $\widehat{F}^n_{\alpha\beta}$ still has the form (3.1) and $\Box_g = g^{\mu\nu}\partial_\mu\partial_\nu$ acts on scalars.

Finally, as initial data for the (gauge-dependent) variables $g_{0\alpha}$, $0 \le \alpha \le n$, we choose

$$g_{0\alpha} = -\delta_{0\alpha}, \quad \text{on} \quad \mathscr{S}.$$

The data g_{ab} and $\partial_\tau g_{ab}$, $a, b \ge 1$, are determined by the initial data γ and $K^{(1)}$. The derivatives $\partial_\tau g_{0\alpha}$ are then fixed by the requirement that the coordinates $\{y^\beta\}$ are harmonic when restricted to $\mathscr{S}$, i.e.

$$\Box y^\beta = \partial_\alpha g^{\alpha\beta} + \frac{1}{2} g^{\alpha\beta} g^{\mu\nu} \partial_\alpha g_{\mu\nu} = 0 \;\; \text{on} \;\; \mathscr{S} \,. \tag{4.18}$$

The higher derivatives $\partial^i_\tau g_{\alpha\beta}$, $i \le n$, on $\mathscr{S}$ are then determined by the given initial data $(\gamma, K^{(1)}, \ldots, K^{(n)})$ and by setting to zero the τ-derivatives of (4.18) up to order $n-1$.

These choices lead to the following:

THEOREM 4.1. *Consider any class* $(\mathscr{S}, [\gamma, K^{(1)}, \ldots, K^{(n)}])$ *satisfying the constraint equations* (4.3)*, with*

$$(\gamma, K^{(1)}, \ldots, K^{(n)}) \in H^s_{\rm loc}(\mathscr{S}) \times H^{s-1}_{\rm loc}(\mathscr{S}) \times \cdots \times H^{s-n}_{\rm loc}(\mathscr{S}) \,,$$

$s > n/2+n+1$*,* $s \in \mathbb{N}$*, where* $(\mathscr{S}, \gamma)$ *is a Riemannian metric and where the* $K^{(i)}$*'s are symmetric two-covariant tensors; the equivalence class is taken with respect to the transformations of the data discussed above.*

Then there exists a unique maximal globally-hyperbolic conformal space-time $(\mathscr{M}, [g])$ *satisfying* (2.3)*, where* $[\cdot]$ *denotes the conformal class, and an embedding*

$$i : \mathscr{S} \to \mathscr{M} \,,$$

for which γ *is the metric induced on* $\mathscr{S}$ *by* g*, with* $K^{(i)}$ *given by* (4.2)*. One can always choose local representatives of* $[g]$ *by imposing* (4.15)*. Moreover, the Cauchy problem with such initial data is well-posed in* $H^s_{\rm loc}(\mathscr{S}) \times H^{s-1}_{\rm loc}(\mathscr{S}) \times \cdots \times H^{s-n}_{\rm loc}(\mathscr{S})$*.*

REMARK 4.2. The inequality $s > n/2+n$ suffices for existence of unique solutions in local coordinate patches in Theorem 4.1. One expects that the use of local foliations with prescribed mean curvature and space-harmonic coordinates as in [4] should allow one to lower the threshold $s > n/2+n+1$ to $s > n/2+n$ in this result.

REMARK 4.3. The conformal space-time $(\mathscr{M}, [g])$ is smooth if the initial data are. Similarly, real-analytic initial data lead to real-analytic solutions.

Proof. Given any initial data set $[(\gamma, K^{(1)}, \ldots, K^{(n)})]$ satisfying the constraint equations, by using the functions $\omega^{(j)}$, $j = 2, \ldots, n$ from (4.11), one can adjust the tensor fields $K^{(i)}$, $i = 2, \ldots, n$, so that R_g, together with its transverse derivatives up to order $n-2$, vanish on $\mathscr{S}$, (see the discussion following (4.14)). Note that this holds for any $\gamma \in [\gamma]$, so that $\omega > 0$ and $\omega^{(1)}$ are otherwise freely specifiable. Solving (4.17) with this initial data, as described in Section 3, one obtains a collection of space-time coordinate patches with a solution of (4.17) there when $s > n/2 + n$. We recall again that the Cauchy problem for (4.17) is well-posed in the H^s_{loc} spaces above.

The argument that these gauge choices are preserved, so that (4.15) holds and the coordinates y^α remain harmonic in these local space-time coordinate patches generated by (4.17), is rather similar to the one for the Einstein equations [17]; we give details because of some differences in the analytical tools used.

First recall that (4.17) can be written in the form

$$\Box_g^{\frac{n-1}{2}}(R_{\mu\nu} - \nabla_\mu\lambda_\nu - \nabla_\nu\lambda_\mu) = -\mathcal{F}^n_{\mu\nu}\ , \tag{4.19}$$

where

$$\lambda^\mu := -\frac{1}{2}\Box_g y^\mu\ .$$

As g solves (4.19), its obstruction tensor equals

(4.20)

$$\mathcal{H}_{\mu\nu} = (-1)^{\frac{n-1}{2}}\left\{\Box_g^{\frac{n-1}{2}}(\nabla_\mu\lambda_\nu + \nabla_\nu\lambda_\mu - \frac{1}{2n}R_g g_{\mu\nu}) - \frac{n-1}{2n}\Box_g^{\frac{n-3}{2}}\nabla_\mu\nabla_\nu R_g\right\}\ .$$

The divergence identity $\nabla_\mu\mathcal{H}^{\mu\nu} = 0$ gives then an equation involving λ and R_g:

(4.21)

$$\Box_g^{\frac{n-1}{2}}\left\{\Box\lambda_\nu + \nabla_\nu(\nabla^\mu\lambda_\mu - \frac{1}{2}R_g)\right\} = \text{lower order commutator terms}\ .$$

Since $\mathcal{H}$ is trace-free, we further have from (4.20)

$$\Box_g^{\frac{n-1}{2}}(\nabla^\mu\lambda_\mu - \frac{1}{2}R_g) = 0\ . \tag{4.22}$$

Because R_g vanishes to order $n-2$ at $\mathscr{S}$, and λ_j vanishes to order $n-1$ at $\mathscr{S}$ by the discussion following (4.18), the initial data for this equation vanish. It follows, for instance from the work in Section 3, that

$$R_g = 2\nabla^\mu\lambda_\mu\ . \tag{4.23}$$

This can be used to rewrite (4.21) as

$$\Box_g^{\frac{n+1}{2}}\lambda_\nu = \text{lower order commutator terms}\ . \tag{4.24}$$

In this last equation all commutator terms in (4.21) that involve gradients of R_g have been replaced by derivatives of λ using (4.23).

Since g satisfies the constraint equation (4.3) at $\mathscr{S}$, (4.23) and (4.20) imply that λ_j vanishes to order n at $\mathscr{S}$, so that (4.24) has vanishing initial data. This system has the form considered in Section 3, so we conclude that λ vanishes. Hence, the coordinates y^α remain harmonic, $R_g = 0$ by (4.23), and so $\mathcal{H} = 0$ as well, as desired.

The usual procedure, as used in the context of the Cauchy problem for Einstein's equations, then allows one to patch the solutions together provided $s > n/2 + n + 1$. An argument as in [5] leads to a unique, (up to diffeomorphism) maximal globally hyperbolic manifold $(\mathscr{M}, [g])$, with $[g]$ satisfying (4.16), with an embedding $i : \mathscr{S} \to \mathscr{M}$, with the desired initial data on $i(\mathscr{S})$.

The well-posedness statement follows immediately from the discussion following Theorem 3.3. $\square$

Remark 4.4. In general, there will not be a global smooth gauge for $(\mathscr{M}, [g])$ in which $R_g = 0$. The local coordinate patches where $R_g = 0$ need not patch together smoothly, preserving $R_g = 0$, since the initial data $\omega > 0$, $\omega^{(1)}$ on local space-like slices $\mathscr{S}$ are freely chosen, and so not uniquely determined.

To see this in more detail, consider for example the de Sitter space-time $\mathscr{M} = \mathbb{R} \times S^n$, with metric

$$g_{dS} = -dt^2 + \cosh^2 t \, g_{S^n(1)}.$$

This is a geodesically complete solution of the Einstein equations with $R_g = n(n+1)$, and so satisfies (2.3). The linear wave equation (4.14) with initial data $\Omega = c_1$, $\partial_t \Omega = c_2$ on $\mathscr{S} = \{t = 0\}$ has a global solution on $\mathscr{M}$ given by

$$\Omega = -\frac{n-1}{4} \sinh t + d_2 \int_0 \frac{1}{\cosh^n t} dt + d_1,$$

for suitable d_1, d_2. One sees that there are no values of c_1, c_2 for which $\Omega > 0$ everywhere on $\mathscr{M}$, so that there is no natural global $R = 0$ gauge for $(\mathscr{M}, [g_{dS}])$ with such initial data.

Consider for example the solution $\Omega = -\frac{n-1}{4} \sinh t$, giving an $R = 0$ gauge in the region $t < 0$, which does not extend to $\{t = 0\}$. For $t = -\varepsilon$ small, the induced metric on $\mathscr{S}_{-\varepsilon} = \{t = -\varepsilon\}$ is the round metric γ_δ on S^n of small radius $\delta = \delta(\varepsilon)$. To obtain an $R = 0$ gauge starting at $\mathscr{S}_{-\varepsilon}$ which extends up to and beyond $\{t = 0\}$, one must choose ω to be a large constant. This causes a discontinity in the choice of gauge for the metric, but not in the structure of the conformal class.

Remark 4.5. One may also construct the maximal solution $(\mathscr{M}, [g])$ by means of local gauges satisfying

$$R_g = c_0 \tag{4.25}$$

in place of the scalar-flat gauge (4.15), for any $c_0 \in \mathbb{R}$. The proof of this is the same as before, noting from the form of (4.13) that given ω and $\omega^{(1)}$ on $\mathscr{S}$, one can find $\omega^{(2)}$ on $\mathscr{S}$ such that $\tilde{R} = c_0$ on $\mathscr{S}$.

For example, the de Sitter space-time is a geodesically complete solution in the gauge $R_g = n(n+1)$. However, it is well-known that $(\mathscr{M}, g_{dS})$ conformally compactifies to the bounded domain in the Einstein static cylinder $g_E = -dT^2 + g_{S^n(1)}$ where $T \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The metric g_E is of course also a solution of (2.3) with $R_{g_E} = n(n-1)$, which is thus a globally hyperbolic extension of $(\mathscr{M}, g_{dS})$, since $T \in (-\infty, \infty)$; it is easily seen that this is the maximal solution.

Thus, the choice $R_{g_E} = n(n-1)$ gives a global gauge for $[g_{dS}]$. The de Sitter metric itself, with gauge $R_g = n(n+1)$, has conformal factor Ω relative to g_E blowing up to ∞ as $T \to \pm\frac{\pi}{2}$. To obtain an extension of $[g_{dS}]$ past the range $(-\frac{\pi}{2}, \frac{\pi}{2})$ of g_{dS} requires a rescaling of the large Ω factor to a factor of unit size.

5. AFG equations vs. Einstein equations

Consider an initial data set $(\mathscr{S}, \gamma, K)$ for the vacuum Einstein equations in $n+1$ dimensions, n odd. Thus $(\mathscr{S}, \gamma)$ is a Riemannian manifold, and the pair (γ, K) satisfies the vacuum constraint equations with cosmological constant $\Lambda \in \mathbb{R}$. Using Einstein's equations one can formally calculate the derivatives

$$K^{(i)}_{kl} := \frac{1}{2}\partial^i_t \gamma_{kl}|_{t=0}\ , \qquad 1 \le i \le n \tag{5.1}$$

in a hypothetical Gauss coordinate system near $\mathscr{S}$ in which the space-time metric g takes the form $-dt^2 + \gamma(t)$, as in (4.1). This gives

Proposition 5.1. *The initial data set for* (2.3) *so obtained solves the constraints* (4.3)*, and any such globally hyperbolic solution of* (2.3) *given by Theorem 4.1 is conformally Einstein.*

Proof. Let g be the associated maximal globally hyperbolic solution of the vacuum Einstein equations. Then g also solves (2.3), and the result follows from the uniqueness part of Theorem 4.1. □

A space-time with boundary $(\overline{\mathscr{M}}, \bar{g})$ is said to be a *conformal completion at infinity* of a space-time $(\mathscr{M}, g)$ if the usual definition of Penrose is satisfied; that is, there exists a diffeomorphism Φ from $\mathscr{M}$ to the interior of $\overline{\mathscr{M}}$ and a function $\Omega : \overline{\mathscr{M}} \to \mathbb{R}^+ \cup \{0\}$, which is a defining function for the boundary

$$\mathscr{I} := \partial\overline{\mathscr{M}}\ ,$$

such that

$$g = \Phi^*(\Omega^{-2}\bar{g}) \ .$$

A conformal completion $(\overline{\mathscr{M}}, \bar{g})$ is H^s smooth if $\bar{g} \in H^s(\overline{\mathscr{M}})$, (in suitable local coordinates for $\overline{\mathscr{M}}$).

A set $(\overline{\mathscr{S}}, \bar{\gamma}, \bar{K}, \omega, \omega^{(1)}, \ldots, \omega^{(n)})$ is said to be a *smooth conformal completion at infinity* of a general relativistic initial data set $(\mathscr{S}, \gamma, K)$ if $(\overline{\mathscr{S}}, \bar{\gamma})$ is a Riemannian manifold with boundary, with $\overline{\mathscr{S}} = \mathscr{S} \cup \partial\mathscr{S}$, and with

$$\omega, \omega^{(j)} : \overline{\mathscr{S}} \to \mathbb{R}$$

being smooth-up-to-boundary functions such that ω is a defining function for the boundary

$$\dot{\mathscr{S}} := \partial\overline{\mathscr{S}} \ ,$$

with

$$\bar{\gamma} = \omega^2 \gamma \tag{5.2}$$

on $\mathscr{S}$. Finally $\bar{K}$ is a smooth-up-to-boundary symmetric tensor field on $\overline{\mathscr{S}}$ such that the equation

$$\bar{K} = \omega K + \frac{\omega^{(1)}}{\omega^2}\bar{\gamma} \tag{5.3}$$

holds on $\mathscr{S}$ (compare (4.12)). We further assume that the functions $\omega^{(i)}$, $2 \le i \le n$, are such that the fields $\bar{K}^{(i)}$, calculated using the $K^{(i)}$'s as in (5.1), and the functions $\omega^{(i)}$, defined before (4.12), can be extended by continuity to smooth tensor fields on $\overline{\mathscr{S}}$.

A conformal completion will be said to be H^s if $\bar{\gamma} \in H^s(\overline{\mathscr{S}})$ and $\bar{K}^{(i)} \in H^{s-i}(\overline{\mathscr{S}})$, $i = 1, \ldots, n$.

The above conditions are clearly necessary for the existence of a smooth conformal completion *à la Penrose* of the maximal globally hyperbolic development of $(\mathscr{S}, \gamma, K)$; we will see shortly that they are also sufficient. We emphasise, however, that Equations (5.2) and (5.3) alone, together with the requirement of smoothness of $\bar{\gamma}$ and $\bar{K}$ are *not* sufficient for the existence of such space-time completions. Indeed, it follows from the results in [3] that, in dimension $3+1$, the requirement of smoothness up-to-boundary of the $\bar{K}^{(i)}$'s, $i \ge 2$, imposes further constraints on $\bar{\gamma}$ and $\bar{K}$. It would be of interest to work out the explicit form of those last conditions, analogously to [3], in all dimensions.

We have the following, conformal version of Proposition 5.1; it allows one to repeat several constructions of Friedrich (see [19] and references therein) for vacuum space-times with vanishing cosmological constant in all even space-time dimensions:

THEOREM 5.2. *Let $(\mathscr{S},\gamma,K)$ be a general relativistic vacuum initial data set, $\Lambda=0$, which admits an H^s* conformal completion at infinity*, $s>n/2+n+1$, with $s\in\mathbb{N}$ and n odd. Then there exists an H^s space-time with boundary $(\overline{\mathscr{M}},\bar g)$, equal to the conformal completion at infinity of the unique maximal development $(\mathscr{M},g)$ of $(\mathscr{S},\gamma,K)$, so that $g=\Omega^{-2}\bar g$ on $\mathscr{M}$, and*

$$\mathscr{I}\supset\dot{\mathscr{S}}\ .$$

Proof. The proof is essentially identical to that of Proposition 5.1. Consider the initial data $(\gamma,K^{(1)},\dots,K^{(n)})$ as constructed at the beginning of this section. One can conformally transform them to initial data

$$\bar K^{(i)}_{kl}:=\frac12\partial^i_s\bar\gamma_{kl}|_{t=0}\ ,\qquad 1\le i\le n$$

in a hypothetical Gauss coordinate system near $\overline{\mathscr{S}}$ in which the space-time metric $\bar g$ takes the form $\bar g=-ds^2+\bar\gamma(s)$. Here we use a normalisation as in the proof of Theorem 4.1 or Remark 4.5, requiring the vanishing (or constancy) of $R_{\bar g}$. Theorem 4.1 provides a solution of this Cauchy problem, while the fact that this solution is conformally Einstein follows from Proposition 5.1.

The choice of the conformal factor Ω transforming $(\overline{\mathscr{M}},[\bar g])$ to the vacuum Einstein solution $(\mathscr{M},g)$ of course depends on the choice of gauge for $[\bar g]$; we describe here how Ω is determined at least in the natural settings corresponding to (4.25). Suppose that $\bar\gamma=\omega^2\gamma$ is a geodesic compactification of $(\mathscr{S},\gamma)$, so that for x near $\dot{\mathscr{S}}$, $\omega(x)=dist_{\bar\gamma}(\dot{\mathscr{S}},x)$. Such a compactification is uniquely determined by the choice of a boundary metric on the boundary $\dot{\mathscr{S}}$. Now the value of $\omega^{(1)}$, at the zero level set of ω is determined by the initial data, (compare Eq. (3.13) of [3] in dimension $3+1$; an obvious modification of that equation holds in all dimensions). Changing time-orientation if necessary, one will have $\omega^{(1)}=-1$ at $\dot{\mathscr{S}}$ and we extend $\omega^{(1)}$ to a neighborhood of $\dot{\mathscr{S}}$ in $\mathscr{S}$ to have the same value. These data determine the compactification of the initial data set $(\mathscr{S},\gamma,K)$.

Hence, the proof of Theorem 4.1 gives a unique local solution $\bar g$ of (2.3) in the conformal class $[\bar g]$, with $R_{\bar g}=c_0$, for any given c_0, satisfying the initial conditions. Let $\varphi=\Omega^{(n-1)/2}$, and set $g=\Omega^{-2}\bar g=\varphi^{-4/(n-1)}\bar g$. Then φ, (and so Ω), is uniquely determined in a chart for $\overline{\mathscr{M}}$ containing a portion of $\dot{\mathscr{I}}$ by the requirement that φ solves the linear wave equation $\frac{4n}{n-1}\Box_{\bar g}\varphi-R_{\bar g}\varphi=0$, with initial data $\varphi=\omega$, $\partial_s\varphi=\omega^{(1)}=-1$ on $\mathscr{S}$ near $\dot{\mathscr{S}}$, (where s is the Gaussian coordinate).

Given such a solution Ω, let $\mathscr{I}$ be the connected component of the set $\{\Omega=0\ ,\ d\Omega\neq0\}$ intersecting $\dot{\mathscr{S}}$. Since, by construction, $Ric_g=0$, and $\bar{g}$ is smooth up to $\mathscr{I}$, standard formulas for the Ricci curvature under conformal changes show that Ω has the usual structure on $\mathscr{I}$, in that $\nabla\Omega$ is null, and $\nabla^2\Omega$ is pure trace, on $\mathscr{I}$. $\square$

REMARK 5.3. A version of Theorem 5.2, and its proof, also holds for de Sitter-type vacuum solutions of the Einstein equations, where $\Lambda > 0$. In this case, the completion is at future or past space-like infinity $\mathscr{I}^+$ or $\mathscr{I}^-$; the Cauchy data for $(\overline{\mathscr{M}}, \bar{\gamma})$ at $\mathscr{I}^+$, consist of the two undetermined terms $g_{(0)}$, $g_{(n)}$ in the formal Fefferman-Graham expansion for vacuum Einstein solutions with $\Lambda > 0$. This gives an alternate proof of one of the results of [2].

The case $\Lambda < 0$ leads to initial-boundary value problems. While it is clear that a generalization of Friedrich's analysis of this case [22] should exist, precise statements require further investigation.

6. APPLICATIONS TO SEMI-GLOBAL AND GLOBAL STABILITY OF GENERAL RELATIVISTIC INITIAL VALUE PROBLEMS IN ALL EVEN SPACE-TIME DIMENSIONS

Let $(\mathscr{S}, \gamma_0)$ be the Poincaré metric on the $(n+1)$-dimensional ball, with $n \geq 3$ and n odd. Setting

$$K_0 = \gamma_0,$$

the set $(\mathscr{S}, \gamma_0, K_0)$ is an initial data set for the vacuum Einstein equations, denoted *standard hyperboloidal initial data*. The maximal globally hyperbolic development $(\mathscr{M}, g_0)$ of $(\mathscr{S}, \gamma_0, K_0)$ is given by

$$g_0 = -d\tau^2 + \tau^2\gamma_0, \tag{6.1}$$

for $\tau \in \mathbb{R}^+$, with $\mathscr{S} = \{\tau = 1\}$. This space-time is the interior of the future light cone about a point in Minkowski space-time (the "Milne universe"). With respect to the standard smooth conformal compactification of Minkowski space-time as a bounded domain in the static Einstein cylinder,

$$\bar{g}_0 = -dT^2 + dR^2 + \sin^2 R\, g_{S^{n-1}(1)}, \tag{6.2}$$

the data $(\bar{\gamma}_0, \bar{K}_0^{(1)}, ..., \bar{K}_0^{(n)})$ are C^∞ smooth up to the boundary. As noted in Remark 4.5, this choice of gauge is global and satisfies $R_{g_0} = n(n-1)$.

As an example of application of the results of the previous section, one now easily obtains:

THEOREM 6.1. *Let $\mathscr{S}$ be an n-dimensional open ball, n odd, and consider a general relativistic initial data set $(\mathscr{S}, \gamma, K)$ which admits an H^s* conformal completion at infinity*, $s > n/2 + n + 1$, $s \in \mathbb{N}$.*

Then there exists $\varepsilon = \varepsilon(n) > 0$ such that if the associated data $(\mathscr{S}, [(\bar\gamma, \bar K^{(1)}, \dots, \bar K^{(n)})])$ are ε-close in $H^s \times \dots \times H^{s-n}$ to the data $(\mathcal{S}, ([\bar\gamma_0, \bar K_0^1, \dots, \bar K_0^n)])$ corresponding to the standard hyperboloidal initial data, then the maximal globally hyperbolic development $(\mathscr{M}, g)$ of $(\mathscr{S}, \gamma, K)$ is causally geodesically complete to the future.

The H^s conformal compactification $(\overline{\mathscr{M}}, \bar g)$ is H^s close to $(\overline{\mathscr{M}}, \bar g_0)$, and extends to a larger H^s space-time containing a regular future time-like infinity ι^+ for $(\overline{\mathscr{M}}, \bar g)$.

Proof. The standard space-time $(\mathscr{M}, g_0)$ has a conformal compactification to a bounded domain $\mathcal{D}$ in the static Einstein cylinder (6.2), where $\mathcal{D}$ corresponds to the range of parameters $T+R \in [0,\pi]$, $T-R \in [0,\pi]$, $R \geq 0$. Future null infinity $\mathscr{I}^+$ is given by $\mathscr{I}^+ = \{T+R=\pi\}, T \in (\frac{\pi}{2}, \pi)$, with future time-like infinity $\iota^+ = \{T=\pi, R=0\}$. The future development of $(\mathscr{S}, \gamma_0, K_0)$ corresponds to the domain $\mathcal{D}^+ = \mathcal{D} \cap \{T \geq \frac{\pi}{2}\}$.

Clearly, the compactification $(\mathcal{D}, \bar g_0)$ extends smoothly to a neighborhood $\tilde{\mathcal{D}}$ of $\mathcal{D}$ as a globally hyperbolic solution of (2.3). The Cauchy data for such an extension are an extension of the standard Cauchy data $(\mathcal{S}, ([\bar\gamma_0, \bar K_0^1, \dots, \bar K_0^n)])$ past the boundary $\dot{\mathscr{S}}$. Similarly, the initial data $(\mathscr{S}, [(\bar\gamma, \bar K^{(1)}, \dots, \bar K^{(n)})])$ for $(\mathscr{M}, \bar g)$ extend in H^s past a neighbhorhood of the boundary $\dot{\mathscr{S}}$ and generate a maximal globally hyperbolic space-time $(\tilde{\mathscr{M}}, \tilde g)$, satisfying (2.3). By the Cauchy stability associated with Theorem 5.2, for ε small, the solution $(\tilde{\mathscr{M}}, \tilde g)$ is close in H^s to $(\tilde{\mathcal{D}}, \bar g_0)$, and in particular is an H^s extension of $(\mathscr{M}, \bar g)$, where $\mathscr{M} = \{\Omega > 0\}$ in $\tilde{\mathscr{M}}$. This shows that, to the future of $\mathscr{S}$, $(\mathscr{M}, g)$ has an H^s conformal completion, which extends in H^s to a neighborhood of $\mathscr{I}$ and ι^+. This gives the result. □

Note that in dimension $3+1$ the mere requirement

$$[(\bar\gamma, \bar K^{(1)}, \dots, \bar K^{(n)})] \in H^s \times \dots \times H^{s-n}\ , \quad s > n/2 + n + 1\ ,$$

regardless of any smallness condition, forbids solutions which have logarithmic terms with small powers of $1/r$ in polyhomogeneous expansions. Thus, (similarly to the results of Friedrich), the above theorem applies for non-generic initial data sets only.

Using Corvino-Schoen type constructions together with the above stability result, as in [9], one obtains:

Theorem 6.2. *There exists an infinite dimensional space of complete, asymptotically simple globally hyperbolic solutions of the Einstein vacuum equations in all even dimensions $n+1 = 2(k+1)$, $n \geq 3$. Thus,* *such solutions are geodesically complete both to the future and past, and have a smooth conformal completion at infinity.*

Proof. The Corvino-Schoen gluing technique [12, 13] can be used, as in [9, 10], to construct static, parity-symmetric initial data on $\mathbb{R}^n$, for any $n \geq 3$, which are Schwarzschild with $m \neq 0$ outside a compact set, and which are as close to the Minkowskian data as desired. The resulting maximal globally hyperbolic space-time then contains smooth hyperboloids, close to standard hyperbolic initial data, as in Theorem 6.1, both in the future and in the past. In even space-time dimensions the result follows as in the proof of Theorem 6.1. □

We note that all the space-times constructed in the proof of Theorem 6.2 possess a "complete $\mathscr{I}$"; this should be understood as completeness of generators of $\mathscr{I}$ in the zero-shear gauge, compare [24].[6]

One expects the above construction to generalise to initial data which are stationary, asymptotically flat outside of a spatially compact set (rather than exactly Schwarzschild there). This would require proving that the resulting space-times have smooth conformal compactifications near ι^0, (in space-time dimension four this follows from [14,15,33]), and working out "reference families of metrics" needed for the arguments in [10]. Those results are very likely to hold, but need detailed checking; note that one step of [33] requires dimension four, and that the family of asymptotically flat stationary metrics in higher dimensions might be richer than that in dimension $3 + 1$, [31].

Appendix A. The infinitesimal invariance group of the constraint equations.

In this appendix we study the Lie algebra associated to the group of conformal transformations preserving the constraint equations (4.3). This allows one to derive identities which shed light on the structure of those equations.

Suppose that $\mathcal{H} = 0$ and that the space-time metric g is rescaled by Ω^2, where, in a Gauss coordinate system (t, x^i), so that $\mathscr{S} = \{t = 0\}$, we have

$$\Omega = 1 + \varepsilon \frac{\psi(x^i)}{j!} t^j \ ,$$

for some $j \geq 0$, and $\varepsilon > 0$ small. In the new Gauss coordinates $(\bar{t}, \bar{x}^i)$ we thus have

$$\Omega^2(-dt^2 + g_{ij}dx^i dx^j) = -d\bar{t}^2 + \bar{g}_{k\ell}d\bar{x}^k d\bar{x}^\ell \ ,$$

[6]We are grateful to H. Friedrich for useful discussions concerning this point.

leading to

$$(\partial_t\bar t)^2 - \bar g_{k\ell}\partial_t\bar x^k\partial_t\bar x^\ell = \Omega^2\ , \tag{A.1}$$

$$\partial_t\bar t\partial_i\bar t = \bar g_{k\ell}\partial_i\bar x^k\partial_t\bar x^\ell\ , \tag{A.2}$$

$$g_{ij} = \Omega^{-2}\Big(\bar g_{k\ell}\partial_i\bar x^k\partial_j\bar x^\ell - \partial_i\bar t\partial_j\bar t\Big)\ . \tag{A.3}$$

By definition of Gauss coordinates we have $(\bar t,\bar x^i) = (O(t), x^i + O(t))$, and also $(\bar t,\bar x^i) = (t + O(\varepsilon), x^i + O(\varepsilon))$. Inserting this in the equations above, matching powers in Taylor expansions we find

$$\bar t = t+\varepsilon\frac{\psi}{(j+1)!}t^{j+1}+O(\varepsilon^2t^{2j+3})\ ,\quad \bar x^i = x^i+\varepsilon\frac{D^i\psi}{(j+2)!}t^{j+2}+O(\varepsilon^2t^{2j+2})\ .$$

At the right-hand-side of (A.3) we have variations related to the fact that all the quantities there are evaluated at the point $\bar x = x+\varepsilon\times(\cdot)+\ldots$; to first order, this produces a Lie derivative-type contribution. Next, there are variations related to the fact that $\bar t = t + \varepsilon\times(\cdot) + \ldots$ Each term $\bar t^i\bar K^{(i)}/i!$ in (twice) the Taylor expansion of $\bar g_{ij}$ at $t = 0$ gives then a contribution to the right-hand-side of (A.3) equal to

$$\Omega^{-2}(t+\varepsilon\frac{\psi}{(j+1)!}t^{j+1})^i\frac{\bar K^{(i)}}{i!}+O(\varepsilon^2) = \Big(t^i + \frac{i-2j-2}{(j+1)!}\varepsilon\psi t^{i+j}\Big)\frac{\bar K^{(i)}}{i!}+O(\varepsilon^2)\ .$$

From this we can calculate the coefficients of an expansion in powers of t of the right-hand-side of (A.3); inverting those relations, for $j = 0$ this leads to

$$\bar K^{(i)} = \begin{cases} (1-\varepsilon(i-2)\psi)K^{(i)} + O(\varepsilon^2), & i \neq 2;\\ K^{(i)} - \frac{\varepsilon}{4}\mathscr{L}_{D\psi}g + O(\varepsilon^2), & i = 2,\end{cases} \tag{A.4}$$

where $\mathscr{L}$ denotes a Lie derivative; note that $\mathscr{L}_{D\psi}g$ is twice the Hessian of ψ.

For $j > 0$ we obtain instead

$$\bar K^{(i)} = K^{(i)}\ ,\quad 0 \le i < j\ , \tag{A.5}$$

$$\bar K^{(j)} = K^{(j)} + \varepsilon\psi g\ , \tag{A.6}$$

$$\begin{aligned}\bar K^{(j+i)} &= K^{(j+i)} - \frac{(i-2j-2)(j+i)!}{2(j+1)!i!}\varepsilon\psi K^{(i)}\\ &\quad+\begin{cases} O(\varepsilon^2), & 1\le i\neq 2;\\ -\frac{\varepsilon}{4}\mathscr{L}_{D\psi}g + O(\varepsilon^2), & i = 2.\end{cases}\end{aligned} \tag{A.7}$$

The terms proportional to ε in those equations describe the desired infinitesimal action.

We can view $\mathcal{H}_{0\mu}$ as functions of the metric, its derivatives, and of the symmetric derivatives $D_{(\ell_1\ldots\ell_s)}K^{(i)}_{k\ell}$, with $i = 1,\ldots,n$. As $\mathcal{H}_{\mu\nu}$ is a differential operator in the space-time metric of order $n+1$, the possibly non-trivial contributions arise from those $D_{(\ell_1\dots\ell_s)}K^{(i)}_{k\ell}$ for which $s+i\le n+1$. Differentiating the equations $0=\mathcal{H}_{0\mu}$ with respect to ε, at $\varepsilon=t=0$ for $j>0$ one obtains

(A.8)
$$\begin{aligned}0 \;=\; & \sum_{s=0}^{n+1-j}\frac{\partial\mathcal{H}_{0\mu}}{\partial D_{(\ell_1\dots\ell_s)}K^{(j)}_{k\ell}}g_{k\ell}D_{\ell_1\dots\ell_s}\psi\\ & -\frac{1}{2}\sum_{i=1}^{n-j}\frac{(i-2j-2)(j+i)!}{(j+1)!i!}\sum_{s=0}^{n+1-i-j}\frac{\partial\mathcal{H}_{0\mu}}{\partial D_{(\ell_1\dots\ell_s)}K^{(j+i)}_{k\ell}}D_{\ell_1\dots\ell_s}(\psi K^{(i)}_{k\ell})\\ & -\frac{1}{2}\sum_{s=0}^{n-1-j}\frac{\partial\mathcal{H}_{0\mu}}{\partial D_{(\ell_1\dots\ell_s)}K^{(j+2)}_{k\ell}}D_{\ell_1\dots\ell_s k\ell}\psi\;.\end{aligned}$$

At any point x the derivatives $D_{\ell_1\dots\ell_s}\varphi(x)=D_{(\ell_1\dots\ell_s)}\varphi(x)$ can be chosen independently, which leads to various identities. The simplest one is obtained for $j=n$, then the last two lines give a vanishing contribution. We parameterise the $K^{(i)}$'s by their trace-free part and by trace, obtaining

$$0=\frac{\partial\mathcal{H}_{0\mu}}{\partial D_{\ell_1\dots\ell_s}\mathrm{tr}K^{(n)}}\;,\quad s\ge 0\;; \tag{A.9}$$

this gives a check of the general form of (4.10).

When $j=n-1$ the last line in (A.8) is zero again, and we obtain

$$\begin{aligned}0 \;=\; & \sum_{s=0}^{2}\frac{\partial\mathcal{H}_{0\mu}}{\partial D_{(\ell_1\dots\ell_s)}K^{(n-1)}_{k\ell}}g_{k\ell}D_{\ell_1\dots\ell_s}\psi\\ & +\frac{1}{2}(2n-1)\sum_{s=0}^{1}\frac{\partial\mathcal{H}_{0\mu}}{\partial D_{\ell_1\dots\ell_s}K^{(n)}_{k\ell}}D_{\ell_1\dots\ell_s}(\psi K^{(1)}_{k\ell})\\ =\; & \sum_{s=0}^{2}\frac{\partial\mathcal{H}_{0\mu}}{\partial D_{\ell_1\dots\ell_s}\mathrm{tr}K^{(n-1)}}D_{\ell_1\dots\ell_s}\psi\\ & +\frac{1}{2}(2n-1)\Big(\frac{\partial\mathcal{H}_{0\mu}}{\partial K^{(n)}_{k\ell}}\psi K^{(1)}_{k\ell}+\frac{\partial\mathcal{H}_{0\mu}}{\partial D_\ell K^{(n)}_{k\ell}}(K^{(1)}_{k\ell}D_\ell\psi+\psi D_\ell K^{(1)}_{k\ell})\Big)\;.\end{aligned}$$

The vanishing of the coefficients in front of the second derivatives of ψ leads to the identity

$$0=\frac{\partial\mathcal{H}_{0\mu}}{\partial D_{\ell_1\ell_2}\mathrm{tr}K^{(n-1)}}\;, \tag{A.10}$$

consistently with (4.9).

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