Boundary value problems for Dirac–type equations

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Abstract

We prove regularity for a class of boundary value problems for first order elliptic systems, with boundary conditions determined by spectral decompositions, under coefficient differentiability conditions weaker than previously known. We establish Fredholm properties for Dirac-type equations with these boundary conditions. Our results include sharp solvability criteria, over both compact and non-compact manifolds; weighted Poincaré and Schrödinger-Lichnerowicz inequalities provide asymptotic control in the non-compact case.

Contents

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1 Introduction

Elliptic systems based on the Dirac equation arise frequently in problems in geometry and analysis. Applications to positive mass and related conjectures in general relativity motivate this paper, and involve boundary value problems on compact and non-compact domains [10, 13, 14, 30].

Previous existence and regularity results [2, 5, 6, 16, 23] are insufficient for these applications, for various reasons. The Agmon-Douglas-Nirenberg approach based on freezing coefficients and explicit kernels for the constant coefficient inverse operator, leads only to boundary conditions of Lopatinski-Shapiro type [17]. The pseudo-differential operator approach [16, 26] handles non-local boundary conditions such as the spectral projection condition of Atiyah-Patodi-Singer [2], but the assumptions of smooth coefficients and product-type boundary metric [2, 5, 6, 16] are unnatural and, as we shall show, unnecessary.

In this paper we provide an essentially elementary proof of existence and regularity for first order elliptic systems with "Dirac-type" boundary value conditions. These encompass both pointwise (Lopatinski-Shapiro) and non-local (spectral) boundary conditions, and do not require product metric structures on the boundary. We obtain explicit necessary and sufficient conditions which ensure the solvability of natural inhomogeneous boundary value problems, over both compact and non-compact manifolds with compact boundary.

The coefficient regularity conditions, for both the elliptic system and the boundary conditions, are rather general. For example, they are weaker than those in the pseudo-differential operator approach of Marschall [20]. It seems likely that the boundary conditions will admit some generalizations; the boundary data is $H^{1/2}$ whereas there are recent results for a certain constant coefficient Dirac equation with L^2 boundary values on a Lipschitz hypersurface [3].

Note that there is an extensive literature on applications of Dirac operators to index problems on compact and non-compact manifolds [5, 7] which we do not address, although many aspects of our results are no doubt relevant to such applications; the results here are focussed on applications to energy theorems in general relativity, which will be described elsewhere.

The motivating example of the Dirac (Atiyah-Singer) operator is described in some detail in §2, where the Schrödinger-Lichnerowicz identity with suitable boundary conditions combines with a Lax-Milgram argument to reduce the existence question to that of showing that a weak (L^2) solution of an adjoint problem is in fact a strong (H^1) solution. This *weak-strong* regularity property turns out to be the key technical step, and the focus of much of the paper. The difficult case is regularity at the boundary; interior regularity is established in §3 using standard Fourier techniques, for general first order elliptic systems.

§4 reviews conditions under which a symmetric operator has a complete set of eigenfunctions; these are used to to control the boundary operator in later sections. In §5 we prove regularity results at the boundary, for a class of operators much broader than Dirac equations, with weak assumptions on the continuity/regularity of the operator coefficients. The main technical tools are the H^1 identity (5.15), and some basic spectral theory. The boundary conditions of §5 follow from the requirements of the arguments of the regularity theorem, and some additional work is required to apply them to first order systems. This is carried out in §6, for equations of Dirac-type near the boundary, for which the boundary operator is self-adjoint. The resulting boundary conditions are naturally presented in terms of graphs over the space of negative eigenfunctions of the boundary operator.

The boundary value problems considered have a Fredholm property, and admit an explicit solvability criteria involving solutions of the homogeneous adjoint problem. These properties are established for compact manifolds with boundary in §7, and for a large class of non-compact manifolds with boundary in §8. The analysis of the non-compact case relies on two *a priori* inequalities: a weighted Poincaré inequality, and a Schrödinger-Lichnerowicz inequality. These inequalities imply the manifold is *non-parabolic at infinity* in the sense of [7]. The weighted Poincaré inequality is established in §9 in a number of cases, including the important cases of manifolds with asymptotically flat or hyperbolic ends. The Schrödinger-Lichnerowicz inequality follows in applications from an H^1 estimate derived from an identity of Schrödinger-Lichnerowicz type.

2 The model problem

In this section we use the Riemannian Dirac equation to illustrate and motivate the existence and regularity results of the following sections.

Consider an oriented manifold M with Riemannian metric g and a representation $c : \mathrm{C}\ell(TM) \to \mathrm{End}(S)$ of the Clifford algebra $\mathrm{C}\ell(TM)$ on some bundle S; with our conventions,

$$c(v)c(w) + c(w)c(v) = -2g(v, w) .$$

Clifford representations are discussed in detail in [1, 18]. S carries an invariant inner product, $\langle c(v)\psi, c(v)\psi\rangle = |v|^2 \langle \psi, \psi\rangle = |v|^2 |\psi|^2$, with respect to which $c(v)$ is skew-symmetric, for all vectors v.

A *Dirac* connection [5, 18] is a connection on the space of sections of S which satisfies the compatibility relation

$$d\langle\phi, c(v)\psi\rangle = \langle\nabla\phi, c(v)\psi\rangle + \langle\phi, c(v)\nabla\psi\rangle + \langle\phi, c(\nabla v)\psi\rangle , \qquad (2.1)$$

where ∇ also denotes the Levi-Civita connection on vector fields.

Spin manifolds provide the fundamental example, with S a bundle of spinors associated with a Spin principal bundle which double covers the Riemannian orthonormal frame bundle. In this case there is a covariant derivative ∇ defined in terms of a local orthonormal frame e_k, $k = 1, \ldots, n$, with Riemannian connection matrix $\omega_{ij}(e_k) = g(e_i, \nabla_{e_k} e_j)$, by

$$\nabla_{e_k}\psi = D_{e_k}\psi^I\,\phi_I - \tfrac{1}{4}\psi^I\,\omega_{ij}(e_k)c(e^ie^j)\phi_I\ , \tag{2.2}$$

where $\psi = \psi^I\phi_I$ and ϕ_I, $I = 1,\ldots,\dim S$, is a choice of spin frame associated with the orthonormal frame e_k. The expression (2.2) may be abbreviated to $\nabla = d - \frac{1}{4}\omega_{ij}e^ie^j$. Note that there are other examples of Dirac bundles and connections, eg. [18, example II.5.8].

The *Dirac operator* of a Dirac connection ∇ is

$$\mathcal{D}\psi = c(e^i)\nabla_{e_i}\psi\ ; \tag{2.3}$$

in the spin case this is sometimes called the *Atiyah-Singer operator*. When the spinor representation is irreducible[1] a classical and very important computation [25] shows that

$$\mathcal{D}^2\psi = \nabla^*\nabla\psi + \tfrac{1}{4}R(g)\psi\ , \tag{2.4}$$

where $R(g)$ is the (Ricci) scalar curvature of g. This leads to the Schrödinger-Lichnerowicz identity [19, 25]

$$\left(|\nabla\psi|^2 + \tfrac{1}{4}R(g)|\psi|^2 - |\mathcal{D}\psi|^2\right) *1 = d\left(\langle\psi, (c(e_ie_j) + g_{ij})\nabla^j\psi\rangle *e^i\right)\ , \tag{2.5}$$

which when integrated over the compact manifold M with boundary[2] Y becomes

$$\int_M \left(|\nabla\psi|^2 + \tfrac{1}{4}R(g)|\psi|^2 - |\mathcal{D}\psi|^2\right) = \oint_Y \langle\psi, c(ne^A)\nabla_A\psi\rangle\ . \tag{2.6}$$

Here n is the outer normal vector at $Y = \partial M$ and $\{e_A\}$ is a compatible orthonormal frame on Y. The boundary term may be simplified by introducing the boundary covariant derivative

$$\bar{\nabla} = d - \tfrac{1}{4}\omega_{AB}c(e^Ae^B)\ ,$$

and the *boundary* Dirac operator[3]

$$\mathcal{D}_Y\psi = c(ne^A)\bar{\nabla}_A\psi\ . \tag{2.7}$$

Denoting the mean curvature by $H = H_Y = g(n, \nabla_{e_A}e^A)$ gives

$$\oint_Y \langle\psi, c(ne^A)\nabla_A\psi\rangle = \oint_Y \langle\psi, \mathcal{D}_Y\psi + \tfrac{1}{2}H\psi\rangle\ . \tag{2.8}$$

We use conventions which give $H = 2/r > 0$ for $M = \mathbb{R}^3 - B(0, r)$, the exterior of a ball of radius r, with the outer normal $n = -\partial_r$. If x is a Gaussian boundary coordinate ($x \geq 0$ in M, $x = 0$ on Y and $\partial_x = -n$), then near the boundary we have

$$\mathcal{D}\psi = -c(n)(\partial_x + \mathcal{D}_Y + \tfrac{1}{2}H)\psi\ . \tag{2.9}$$

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[1] Reducible representations lead to interesting formulas with $\frac{1}{4}R(g)$ replaced by more complicated curvature endomorphisms.

[2] Throughout this paper we use the geometer's convention, that a manifold with boundary contains its boundary as a point set.

[3] Both $e^A \to c(e^A)$ and $e^A \to c(ne^A)$ give representations of the Clifford algebra of the boundary tangent space; the choice of $c(ne^A)$ is made here for convenience [13].

We now seek boundary conditions for which the equation $\mathcal{D}\psi = f$ is solvable, following a well-known argument [11, 13, 24]. Suppose M is a compact manifold with non-negative scalar curvature, $R(g) \geq 0$, and $\mathcal{K} : H^{1/2}(Y) \to H^{1/2}(Y)$ is a bounded linear operator such that

$$\oint_Y \langle \psi, \mathcal{D}_Y \psi + \tfrac{1}{2} H \psi \rangle \leq 0 \qquad \text{whenever} \quad \mathcal{K}\psi = 0 \,. \tag{2.10}$$

Suppose further that M admits no parallel spinors. Define the space $H^1_{\mathcal{K}}(M)$ as the completion of the smooth spinor fields with compact support (in $M \cup Y$) which satisfy the boundary condition $\mathcal{K}\psi = 0$, in the norm

$$\|\psi\|^2_{H^1_{\mathcal{K}}(M)} := \int_M (|\nabla\psi|^2 + \tfrac{1}{4} R(g) |\psi|^2) \,. \tag{2.11}$$

The boundary condition (2.10) combined with the Lichnerowicz identity (2.6) and the curvature condition $R(g) \geq 0$ now ensures that the bilinear form

$$a(\psi, \phi) = \int_M \langle \mathcal{D}\psi, \mathcal{D}\phi \rangle, \quad \phi, \psi \in H^1_{\mathcal{K}}(M) \,,$$

is strictly coercive, $a(\psi, \psi) \geq \|\psi\|^2_{H^1_{\mathcal{K}}(M)}$. For any spinor field $f \in L^2(M)$, the linear functional $\phi \mapsto \int_M \langle f, \mathcal{D}\phi \rangle$ is bounded on $H^1_{\mathcal{K}}(M)$. Coercivity and the Lax-Milgram lemma show there is a unique $\psi \in H^1_{\mathcal{K}}(M)$ such that

$$\int_M \langle \mathcal{D}\psi - f, \mathcal{D}\phi \rangle = 0 \qquad \forall \phi \in H^1_{\mathcal{K}}(M) \,,$$

and we would like to deduce that $\mathcal{D}\psi = f$. Now $\Psi := \mathcal{D}\psi - f \in L^2(M)$ is a *weak solution* of the Dirac equation; that is,

$$\int_M \langle \Psi, \mathcal{D}\phi \rangle = 0 \qquad \forall\ \phi \in H^1_{\mathcal{K}}(M) \,. \tag{2.12}$$

If we could show that Ψ is in fact a strong solution, that is, $\Psi \in H^1(M)$, then we could integrate by parts to conclude

$$\int_M \langle \mathcal{D}\Psi, \phi \rangle + \oint_Y \langle \Psi, c(n)\phi \rangle = 0 \qquad \forall \phi \in H^1_{\mathcal{K}}(M) \,,$$

and thus $\mathcal{D}\Psi = 0$ and $\oint_Y \langle \Psi, c(n)\phi \rangle = 0$ for all $\phi \in H^{1/2}(Y)$ such that $\mathcal{K}\phi = 0$. This would give the boundary condition $\Psi|_Y \in c(n)(\ker \mathcal{K})^{\perp}$, which we suppose may be re-expressed as $\widetilde{\mathcal{K}}\Psi = 0$, for some "adjoint" boundary operator $\widetilde{\mathcal{K}}$. This would give $\Psi \in H^1_{\widetilde{\mathcal{K}}}$, so if finally we suppose that $\widetilde{\mathcal{K}}$ also satisfies the boundary positivity condition (2.10), then we could conclude from $a(\Psi, \Psi) = 0$ and the coercivity of $a(\cdot, \cdot)$ with respect to the norm $\| \cdot \|_{H^1_{\widetilde{\mathcal{K}}}}$, that $\Psi = 0$ as desired.

The key technical difficulty in this classical argument lies in establishing the "Weak-Strong" property, that weak (L^2) solutions lie in H^1. In the following sections we will prove this property for a large class of elliptic systems, under rather general boundary conditions; see §5 and §6.

Two model boundary operators illustrate the possibilities for achieving the required conditions. The *APS* (or *spectral projection* [2]) condition arose in Herzlich's work [13]:

$$\mathcal{K} = P_+ \ , \tag{2.13}$$

where P_+ is the $L^2(M)$-orthogonal projection onto the positive spectrum eigenspace of the boundary Dirac operator $\mathcal{D}_Y$. Using the relation $c(n)\mathcal{D}_Y = -\mathcal{D}_Y c(n)$, which shows that the spectrum of $\mathcal{D}_Y$ is symmetric about $0 \in \mathbb{R}$, we find that $\widetilde{\mathcal{K}} = \mathcal{K}$, provided there are no zero eigenvalues.

The eigenvalue estimate for $Y \simeq S^2$ of Hijazi and Bär [4, 15]

$$|\lambda(\mathcal{D}_Y)| \geq \sqrt{4\pi/\text{area}(Y)}, \tag{2.14}$$

shows that in this case there are no zero eigenvalues. In addition, if we have the mean curvature condition

$$H_Y \leq \sqrt{16\pi/\text{area}(Y)}, \tag{2.15}$$

then $\mathcal{K}$ (and $\widetilde{\mathcal{K}}$) will satisfy the boundary positivity condition (2.10). In conclusion, if $Y = \partial M \simeq S^2$ satisfies (2.15), then (assuming the Weak-Strong property can be established) the above argument shows $\Psi = 0$ and thus the equation $\mathcal{D}\psi = f$ with boundary condition $P_+\psi = 0$ is uniquely solvable, for any $f \in L^2(M)$.

The *chirality* condition was used in [9, 10]. For a slightly simplified version of [10], suppose M is a totally geodesic hypersurface in a Lorentz spacetime, with future unit normal vector e_0, and consider the connection on spacetime spinors, restricted to M. Along $Y = \partial M$ with outer normal n we define

$$\epsilon = c(e_0 n), \tag{2.16}$$

which satisfies the *chiral conditions*

$$\begin{array}{rclcrcl} \epsilon^2 &=& 1\ , & \qquad & \epsilon c(n) + c(n)\epsilon &=& 0\ , \\ \langle \phi, \epsilon\psi \rangle &=& \langle \epsilon\phi, \psi \rangle\ , & & \epsilon\mathcal{D}_Y + \mathcal{D}_Y\epsilon &=& 0\ , \end{array} \tag{2.17}$$

and then define the boundary operators

$$\mathcal{K}_\pm = \tfrac{1}{2}(1 \pm \epsilon)\ . \tag{2.18}$$

Assuming either of the two conditions $\mathcal{K}_\pm\psi = 0$ gives $\epsilon\psi = \mp\psi$ which implies

$$\begin{aligned} \langle \psi, \mathcal{D}_Y\psi \rangle &= \mp\langle \psi, \mathcal{D}_Y\epsilon\psi \rangle &&= \pm\langle \psi, \epsilon\mathcal{D}_Y\psi \rangle \\ &= \pm\langle \epsilon\psi, \mathcal{D}_Y\psi \rangle &&= -\langle \psi, \mathcal{D}_Y\psi \rangle = 0\ . \end{aligned}$$

If we further assume that $H_Y \leq 0$ then (2.10) follows directly. In general relativity the condition $H_Y \leq 0$ is the defining property for Y to be a *trapped surface.*

Since the $\mathcal{K}_\pm$'s are complementary orthogonal projections, we have $(\ker \mathcal{K}_\pm)^\perp = \ker \mathcal{K}_\mp$, so $\psi \in c(n)(\ker \mathcal{K}_\pm)^\perp$ exactly when $c(n)\psi \in \ker \mathcal{K}_\mp$, which gives $\psi \in \ker \mathcal{K}_\pm$, and $\widetilde{\mathcal{K}}_\pm = \mathcal{K}_\pm$. In this case we conclude (still assuming the Weak-Strong property can be established) that if $H_Y \leq 0$ then $\mathcal{D}\psi = f$, with either of the boundary conditions $\epsilon\psi = \pm\psi$, is uniquely solvable.

3 Interior Regularity

In this section we establish regularity away from the boundary for weak (L^2) solutions of first order elliptic systems. We consider equations locally of the form

$$\mathcal{L}u := a^j \partial_j u + bu = f \ , \tag{3.1}$$

where u, f are sections respectively of N-dimensional real vector bundles E, F, both over an n-dimensional manifold M without boundary[4], and a^j, b, $j = 1, \ldots, n$, are sections of the bundle of endomorphisms of E to F. We assume that E, F are equipped with fixed smooth inner products, denoted by $\langle \cdot, \cdot \rangle$. The length determined by $\langle \cdot, \cdot \rangle$ will be denoted invariably by $|u|^2 = \langle u, u \rangle$. To simplify notation, the respective bundles usually will be understood, and thus $L^2(M)$ will generally mean $L^2\Gamma(E)$, the space of L^2 sections of E, or $L^2\Gamma(F)$, depending on context.

Remark 3.1 There is no loss of generality in considering real bundles, since complex and quaternionic bundles may be viewed simply as real bundles with additional algebraic structure. For example, a Hermitean vector space of dimension n is equivalent to a real vector space of dimension $2n$ with a skew endomorphism J satisfying $J^2 = -1$, with the Hermitean inner product $(\ , \ )$ and real inner product $\langle \ , \ \rangle$ related by $(u, v) = \langle u, v \rangle - i \langle u, Jv \rangle$.

Define the indices $\hat{2} = \hat{2}(n)$, $n^* = n^*(n)$ by

$$\begin{array}{lll} \hat{2} = \frac{2n}{n-2}, & n^* = n & \text{for } \ n \geq 3 \ , \\ \hat{2} = 10^6, & n^* = \frac{2}{1-2/\hat{2}} & \text{for } \ n = 2 \ , \\ \hat{2} = \infty \ , & n^* = 2 & \text{for } \ n = 1 \ , \end{array} \tag{3.2}$$

where 10^6 represents any large constant. Note that if M admits a Sobolev inequality with constant C_S

$$\|u\|_{L^{\hat{2}}} \leq C_S \|u\|_{H^1},$$

then we also have

$$\|fu\|_{L^2} \leq C_S \|f\|_{L^{n^*}} \|u\|_{H^1} \ . \tag{3.3}$$

Another basic fact is the inequality

$$\|fg\|_{W^{1,p}} \leq C\Big(\|f\|_{L^\infty} \|g\|_{W^{1,p}} + \|g\|_{L^\infty} \|f\|_{W^{1,p}} \Big),$$

which shows that $W^{1,n^*} \cap C^0$ (in particular) forms a ring under addition and multiplication of functions. For $n = 1, 2$ the C^0 is superfluous here, of course.

It suffices to assume throughout that the underlying manifold has a C^∞ differentiable structure. We will assume that a^j, b satisfy the regularity conditions

$$\begin{array}{rcl} a^j & \in & W^{1,n^*}_{\mathrm{loc}}(M) \cap C^0(M) \ , \\ b & \in & L^{n^*}_{\mathrm{loc}}(M) \ , \end{array} \tag{3.4}$$

[4]If M has boundary $\partial M \neq \emptyset$, then the interior $\mathring{M} = M - \partial M$ is a (noncompact) manifold without boundary, to which the results of this section will apply.

by which we mean that M can be covered by open neighbourhoods $\mathscr{O}_\alpha$ with $W^{1,n^*} \cap C^0$ bundle transition functions, with respect to which the local coefficients a^j, b satisfy (3.4).

The conditions (3.4) are preserved by bundle frame changes in $W^{1,n^*} \cap C^0$, by the above ring property. In particular, even if the bundle metrics $\langle \cdot, \cdot \rangle$ on E, F are only in $W^{1,n^*} \cap C^0$, by the Gram-Schmidt process we may construct $W^{1,n^*} \cap C^0$ frame changes which make the metric coefficients locally constant. Since this changes the operator coefficients a^j, b respectively by $W^{1,n^*} \cap C^0$, W^{n^*} affine linear transformations, there is no loss of generality in assuming the metrics on E, F to be locally constant.

We require that a^j satisfy the ellipticity condition, that for each $p \in M$ there is a coordinate neighbourhood $p \in U \subset M$ and a constant $\eta > 0$ such that

$$\eta^2 |\xi|^2 |V|^2 \le |\xi_j a^j(x) V|^2 \le \eta^{-2} |\xi|^2 |V|^2 \,, \tag{3.5}$$

for all $x \in U$, $\xi \in T^*_x M$ and $V \in E_x$, where $|\xi|^2$ is measured by a fixed background metric $\mathring{g}$, which we may assume to be C^∞. Note that (3.5) implies the fibres of E, F must be of the same dimension.

A *weak solution* of (3.1) is $u \in L^2_{\rm loc}(E)$ such that

$$\int_M \langle \mathcal{L}^\dagger \phi, u \rangle \, dv_M = \int_M \langle \phi, f \rangle \, dv_M \,, \tag{3.6}$$

for all $\phi \in C^\infty_c(M)$, where $dv_M = \gamma dx$, $\gamma > 0$, is a coordinate-invariant volume measure on M with $\gamma \in W^{1,n^*}(U) \cap C^0(U)$ and dx is coordinate Lebesgue measure, in any local coordinate neighbourhood U. Here the formal adjoint $\mathcal{L}^\dagger$ is defined with respect to dv_M and the inner products on E, F. Thus in local coordinates,

$$\mathcal{L}^\dagger \phi = -{}^t a^j \partial_j \phi + ({}^t b - \gamma^{-1} \partial_j ({}^t a^j \gamma)) \phi \,, \tag{3.7}$$

where the transposes ${}^t a^j$ are defined with respect to the local framing forms of the inner products of E, F.

The proof proceeds by establishing various special cases, starting with a constant coefficient operator acting on sections of a trivial bundle E over the torus $\mathbb{T}^n$. This type of argument is very standard.

Proposition 3.2 *Suppose $u \in L^2(\mathbb{T}^n)$ is a weak solution of $\mathcal{L}_0 u = f$ where $f \in L^2(\mathbb{T}^n)$, $\mathcal{L}_0 = a_0^j \partial_j$ with a_0^j constant and satisfying the ellipticity condition (3.5). Then $u \in H^1(\mathbb{T}^n)$.*

Proof: We regard $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$. Fix a mollifier $\phi_\epsilon = \epsilon^{-n} \phi((x-y)/\epsilon) \in C^\infty(\mathbb{R}^n)$ with $\phi(-x) = \phi(x)$, and set $u_\epsilon = \phi_\epsilon * u \in C^\infty(\mathbb{T}^n)$. Then

$$\mathcal{L}_0 u_\epsilon = \int_{\mathbb{T}^n} a_0^j \frac{\partial}{\partial x^j} \phi_\epsilon(x-y) u(y) \, dy$$

and thus the definition of weak solution gives

$$\int_{\mathbb{T}^n} \langle \psi, \mathcal{L}_0 u_\epsilon \rangle dx \;\; = \;\; \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \phi_\epsilon(x-y) \langle \mathcal{L}_0^* \psi(x), u(y) \rangle \, dy \, dx$$

$$
\begin{aligned}
&= \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \langle -{}^t a_0^j \frac{\partial}{\partial y^j}\phi_\epsilon(y-x)\psi(x), u(y)\rangle\, dy\, dx \\
&= \int_{\mathbb{T}^n} \langle \mathcal{L}_0^*(\phi_\epsilon * \psi)(y), u(y)\rangle\, dy \\
&= \int_{\mathbb{T}^n} \langle \phi_\epsilon * \psi(y), f(y)\rangle\, dy \\
&= \int_{\mathbb{T}^n} \langle \psi(y), \phi_\epsilon * f(y)\rangle\, dy\ .
\end{aligned}
$$

Thus $\mathcal{L}_0 u_\epsilon = f_\epsilon = \phi_\epsilon * f$, and we note that $f_\epsilon \to f$ strongly in L^2. Now the ellipticity condition (3.5) and the Plancherel theorem ensure that for all $v \in H^1(\mathbb{T}^n)$,

$$
\begin{aligned}
\int_{\mathbb{T}^n} |\partial v|^2\, dx &= \int_{\mathbb{T}^n} |\xi|^2 |\hat{v}|^2\, d\xi \\
&\leq \eta^{-1} \int_{\mathbb{T}^n} |a_0^j \xi_j \hat{v}|^2\, d\xi \\
&= \eta^{-1} \int_{\mathbb{T}^n} |\mathcal{L}_0 v|^2\, dx\ ,
\end{aligned}
$$

and thus

$$
\int_{\mathbb{T}^n} |\partial u_\epsilon|^2\, dx \leq \eta^{-1} \int_{\mathbb{T}^n} |f_\epsilon|^2\, dx\ .
$$

Since $u, f \in L^2$, it follows that $u_\epsilon \to u$ strongly in H^1. ■

Proposition 3.3 *Under the conditions of Proposition 3.2, the map $\mathcal{L}_0 + \lambda : H^1(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ where $\lambda = \pi\eta$, is uniquely invertible, and for all $u \in H^1(\mathbb{T}^n)$,*

$$
\|u\|_{H^1} \leq \sqrt{5}/\eta \|(\mathcal{L}_0 + \lambda)u\|_{L^2}\ . \tag{3.8}
$$

Proof: Write $u = \sum_{k\in\mathbb{Z}^n} u_k e^{2\pi i k\cdot x}$, where the coefficients $u_k = \int_{\mathbb{T}^n} u(x) e^{-2\pi i k\cdot x} dx$ are valued in $\mathbb{C}^N$, the complexification of the real vector space modelling the fibres of E. We then have

$$
\int_{\mathbb{T}^n} |(\mathcal{L}_0 + \lambda)u|^2\, dx = \sum_{k\in\mathbb{Z}^n} |(2\pi i k_j a_0^j + \lambda)u_k|^2,
$$

and using the vector length inequality $|a+b|^2 \geq \chi|a|^2 - \chi/(1-\chi)|b|^2$ with $\chi = \frac{1}{2}$, we find

$$
|(2\pi i k_j a_0^j + \lambda)u_k|^2 \geq \tfrac{1}{2}|2\pi i k_j a_0^j u_k|^2 - \lambda^2 |u_k|^2.
$$

For $k \neq 0$ this is greater than $\pi^2\eta^2|u_k|^2$, whilst for $k = 0$ we have $|(2\pi i k_j a_0^j + \lambda)u_k|^2 = \lambda^2|u_k|^2 = \pi^2\eta^2|u_k|^2$, hence

$$
\int_{\mathbb{T}^n} |(\mathcal{L}_0 + \lambda)u|^2\, dx \geq \sum_{k\in\mathbb{Z}^n} \pi^2\eta^2|u_k|^2 = \pi^2\eta^2 \int_{\mathbb{T}^n} |u|^2\, dx\ ,
$$

which shows $\mathcal{L}_0 + \lambda$ has trivial kernel. Choosing $\chi = 1 - 1/(2|k|)$ shows in fact that

$$|(2\pi i k_j a_0^j + \lambda)u_k|^2 \geq \pi^2\eta^2(2|k|-1)^2|u_k|^2,$$

for all $k \in \mathbb{Z}^n$ and all $u_k \in \mathbb{C}^N$. Since $(2|k|-1)^2 \geq 1$ for all $k \in \mathbb{Z}^n$, we obtain (3.8). Moreover, this shows also that the $N \times N$ complex matrices $2\pi i k_j a_0^j + \lambda$ are invertible for any $k \in \mathbb{Z}^n$, which gives a direct construction of the inverse of the operator $\mathcal{L}_0 + \lambda$. ∎

THEOREM 3.4 *Suppose $u \in L^2(\mathbb{T}^n)$ is a weak solution of*

$$\mathcal{L}_0 u + B_0 u + B_1 u = f \tag{3.9}$$

where $f \in L^2$ and $\mathcal{L}_0 = a_0^j \partial_j$ is a constant coefficient first order operator satisfying the conditions of Proposition 3.3 with ellipticity constant η, where $B_1 : L^2 \to L^2$ is bounded, and where $B_0 : H^1 \to L^2$ is a linear map satisfying

$$\|B_0\|_{H^1 \to L^2} \leq \eta/3, \quad \|B_0^\dagger\|_{H^1 \to L^2} \leq \eta/3\,, \tag{3.10}$$

where $B_0^\dagger$ is the $L^2(\mathbb{T}^n)$-adjoint of B_0. Then $u \in H^1(\mathbb{T}^n)$ is a strong solution of (3.9), and there is a constant C, depending only on η and $\|B_1\|_{L^2 \to L^2}$, such that

$$\|u\|_{H^1} \leq C\left(\|f\|_{L^2} + \|u\|_{L^2}\right). \tag{3.11}$$

PROOF: Construct the iteration sequence $w^{(k)} \in H^1$, $k = 0, 1, \ldots$ by defining $w^{(k+1)}$ to be the solution of

$$(\mathcal{L}_0 + \lambda)w^{(k+1)} = -B_0 w^{(k)} + \tilde{f}\,, \tag{3.12}$$

with $w^{(0)} = 0$, where $\tilde{f} = f + \lambda u - B_1 u \in L^2$ by the assumptions. This equation with $\lambda = \pi\eta$ is uniquely solvable by Proposition 3.3. The difference $v^{(k+1)} = w^{(k+1)} - w^{(k)}$ satisfies $(\mathcal{L}_0 + \lambda)v^{(k+1)} = -B_0 v^{(k)}$ and the estimate (3.8) shows that

$$\frac{\eta}{\sqrt{5}}\|v^{(k+1)}\|_{H^1} \leq \|B_0 v^{(k)}\|_{L^2} \leq \frac{\eta}{3}\|v^{(k)}\|_{H^1}.$$

The iteration is thus a contraction and converges in H^1, to $w \in H^1$ satisfying $(\mathcal{L}_0 + B_0 + \lambda)w = \tilde{f}$, and then $v = u - w \in L^2$ is a weak solution of $(\mathcal{L}_0 + B_0 + \lambda)v = 0$. Now $\mathcal{L}_0^\dagger$ is also elliptic with the same ellipticity constant η, so there is $z \in H^1$ satisfying $(\mathcal{L}_0^\dagger + B_0^\dagger + \lambda)z = v$. Since v is a weak solution,

$$\int_{\mathbb{T}^n} \langle (\mathcal{L}_0^\dagger + B_0^\dagger + \lambda)\phi, v\rangle\, dx = 0 \quad \forall \phi \in H^1(\mathbb{T}^n),$$

we may test with $\phi = z$ to see that $\int |v|^2 = 0$ and $v = 0$. Thus $u = v + w = w \in H^1$ as required. By Proposition 3.3 and (3.10), we have

$$\begin{aligned}
\frac{\eta}{\sqrt{5}}\|u\|_{H^1} &\leq \|(\mathcal{L}_0 + B_0 + B_1)u\|_{L^2} + \|B_0 u\|_{L^2} + \|B_1 u\|_{L^2} + \|\lambda u\|_{L^2} \\
&\leq \|f\|_{L^2} + \frac{\eta}{3}\|u\|_{H^1} + (\|B_1\|_{L^2 \to L^2} + \eta\pi)\|u\|_{L^2}\,.
\end{aligned}$$

Since $\sqrt{5} < 3$, the estimate (3.11) follows. ■

Next we consider operators with non-constant coefficients. Let C_S be the $\mathbb{T}^n$ Sobolev constant

$$\|u\|_{L^{\hat{2}}(\mathbb{T}^n)} \le C_S \|u\|_{H^1(\mathbb{T}^n)} , \tag{3.13}$$

where $\hat{2}$ is defined in (3.2).

PROPOSITION 3.5 *Suppose $u \in L^2(\mathbb{T}^n)$ is a weak solution of the system of equations*

$$\mathcal{L}u := a^j \partial_j u + bu = f \tag{3.14}$$

where $f \in L^2$ and the coefficients $a^j \in W^{1,n^*} \cap C^0$, $b \in L^{n^*}$ satisfy

$$\begin{aligned} \|a^j - a^j_0\|_{L^\infty} &\le \frac{\eta}{10} && (3.15)\\ \|\partial_j a^j\|_{L^{n^*}} &\le \frac{\eta}{10 C_S} && (3.16) \end{aligned}$$

where a^j_0, $j = 1, \ldots, n$, are constant matrices with ellipticity constant η. Then $u \in H^1(\mathbb{T}^n)$ is a strong solution of (3.14) .

PROOF: It will suffice to show that $\mathcal{L}$ admits a decomposition satisfying the conditions of Theorem 3.4. Since L^∞ is dense in L^{n^*}, for any $\epsilon > 0$ we may find $b_0 \in L^{n^*}$, $b_1 \in L^\infty$, such that $b = b_0 + b_1$ and $\|b_0\|_{L^{n^*}} < \epsilon$. We choose $\epsilon = \eta/(10C_S)$. Then $B_0 u := b_0 u + (a^j - a^j_0)\partial_j u$ satisfies

$$\|B_0 u\|_{L^2} \le \|b_0\|_{L^{n^*}} \|u\|_{L^{\hat{2}}} + \|a^j - a^j_0\|_{L^\infty} \|\partial u\|_{L^2}.$$

Using (3.16) and the Sobolev inequality (3.13) gives

$$\|B_0 u\|_{L^2} \le \frac{\eta}{10}(\|u\|_{H^1} + \|\partial u\|_{L^2}) \le \frac{\eta}{3}\|u\|_{H^1},$$

so $\|B_0\|_{H^1 \to L^2} \le \eta/3$. Clearly $B_1 u := b_\infty u$ is bounded on L^2, and it remains to verify the $H^1 \to L^2$ bound on the adjoint operator

$$B_0^\dagger w := {}^t b_0 w - ({}^t a^j - {}^t a^j_0)\partial_j w - \partial_j({}^t a^j - {}^t a^j_0) w.$$

Again using the Sobolev inequality and the conditions (3.15),(3.16) we find

$$\begin{aligned} \|B_0^\dagger w\|_{L^2} &\le C_S \|b_0\|_{L^{n^*}} \|w\|_{H^1} + \|{}^t a^j - {}^t a^j_0\|_{L^\infty} \|\partial w\|_{L^2} + C_S \|\partial_j a^j\|_{L^{n^*}} \|w\|_{H^1} \\ &\le \frac{\eta}{3}\|w\|_{H^1} , \end{aligned}$$

so the conditions of Theorem 3.4 are met and the result follows. ■

On a general compact manifold we define the Sobolev space $H^1(M)$ by the norm

$$\|u\|^2_{H^1(M)} = \int_M (|\nabla u|^2 + |u|^2)\, dv_M , \tag{3.17}$$

where the lengths $|u|^2$, $|\nabla u|^2$ are measured using the metric $\langle\ ,\ \rangle$ on sections of E and a fixed smooth background metric $\mathring{g}$ on TM, and where ∇ is a (covariant) derivative defined in local coordinates on M and a local framing on E by

$$\nabla_i = \partial_i - \Gamma_i \ . \tag{3.18}$$

We assume the charts on E, M are such that

$$\Gamma_i \in L^{n^*}_{\mathrm{loc}} \ . \tag{3.19}$$

Note we do not require that ∇ be compatible with the metric on E. If M is compact then the space $H^1(M)$ is independent of the choice of covariant derivative:

Lemma 3.6 *Suppose M is compact and $\nabla, \hat{\nabla}$ are covariant derivatives satisfying (3.19). Then there is $C > 0$ such that for all $u \in H^1(M)$,*

$$C^{-1}\int_M (|\nabla u|^2 + |u|^2) dv_M \le \int_M (|\hat{\nabla} u|^2 + |u|^2) dv_M \le C\int_M (|\nabla u|^2 + |u|^2) dv_M \ . \tag{3.20}$$

Moreover, there is a constant C_S, depending on M, ∇, such that

$$\left(\int_M |u|^{\hat{2}} dv_M\right)^{2/\hat{2}} \le C_S \int_M (|\nabla u|^2 + |u|^2) dv_M \ . \tag{3.21}$$

Proof: There is a finite covering of M by charts U_α with a corresponding partition of unity ϕ_α. Using the Sobolev inequality for $U_\alpha \subset \mathbb{R}^n$, in each chart we may estimate the localisation $u_\alpha = \phi_\alpha u$ by

$$\int_{U_\alpha} |\partial u_\alpha|^2 \, dx \le C \int_M (|\nabla u_\alpha|^2 + |u_\alpha|^2) \, dv_M \ ,$$

where C depends also on the decomposition $\Gamma = \Gamma^\infty + \Gamma^{n^*} \in L^\infty + L^{n^*}$, with Γ^{n^*} small. Again using the $\mathbb{R}^n$ Sobolev inequality and $\Gamma, \hat{\Gamma} \in L^{n^*}$ we have

$$\begin{aligned}
\int_M (|\hat{\nabla} u|^2 + |u|^2) \, dv_M \ &\le\ C\sum_\alpha \int_{U_\alpha} (|\hat{\nabla} u_\alpha|^2 + |u_\alpha|^2) \, dx \\
&\le\ C\sum_\alpha \int_{U_\alpha} (|\partial u_\alpha|^2 + |u_\alpha|^2) \, dx \\
&\le\ C\sum_\alpha \int_{U_\alpha} (|\nabla u_\alpha|^2 + |u_\alpha|^2) \, dv_M \ ,
\end{aligned}$$

from which the equivalence of the norms follows easily. The Sobolev inequality follows from very similar arguments. ■

We may now complete the proof of interior regularity.

Theorem 3.7 *Suppose M is a C^∞ n-dimensional manifold without boundary, and E, F are real vector bundles over M, each with fibres modelled on $\mathbb{R}^N$. Suppose $u \in L^2_{\mathrm{loc}}(M)$ is a weak solution of $\mathcal{L}u = f$, where $\mathcal{L}$ is a first order* *operator satisfying the conditions (3.4,3.5). Then $u \in H^1_{\rm loc}(M)$ and u is a strong solution of $\mathcal{L}u = f$. Moreover, if M is compact there is a constant $C > 0$, depending on a^j, b and Γ, such that for all $u \in H^1(M)$,*

$$\|u\|_{H^1(M)} \leq C(\|\mathcal{L}u\|_{L^2(M)} + \|u\|_{L^2(M)}) \ . \tag{3.22}$$

Proof: Since $\mathcal{L}$ is locally of the form $\mathcal{L}u = a^j\partial_j u + bu$ with $a^j \in W^{1,n^*}_{\rm loc} \cap C^0$, $b \in L^{n^*}_{\rm loc}$, for each $p \in M$ there is a coordinate neighbourhood U and a constant $\eta > 0$ such that η is the ellipticity constant of $a^j_0 = a^j(p)$, and with respect to the local trivialisation of $E|_U \simeq U \times \mathbb{R}^N$ we have the bounds

$$\begin{aligned} \|a^j - a^j_0\|_{L^\infty(U)} &\leq \frac{\eta}{10}, \\ \|\partial_j a^j\|_{L^{n^*}(U)} &\leq \frac{\eta}{10C_S} \ , \end{aligned}$$

where we assume without loss of generality that $U = Q_R = (0,R)^n$ is a cube of side length $R \leq 1$. By paracompactness there is a locally finite countable covering $\{p_\alpha, U_\alpha\}_{\alpha\in\mathbb{Z}}$ of M by such charts, with a subordinate C^∞ partition of unity $\{\phi_\alpha\}_{\alpha\in\mathbb{Z}}$. Noting that $\operatorname{supp}(\phi_\alpha u) \Subset Q_R$ and that $\phi_\alpha u$ satisfies

$$\mathcal{L}(\phi_\alpha u) = \phi_\alpha f + a^j\partial_j\phi_\alpha u$$

weakly, we see that it suffices to consider the case where $\operatorname{supp} u \Subset Q_R$. Assuming this, rescaling by $y = x/R, x \in Q_R$ and defining $\tilde{u}(y) = u(x)$, $\tilde{f}(y) = Rf(x)$, $\tilde{a}^j(y) = a^j(x)$ and $\tilde{b}(y) = Rb(x)$, it follows that $\tilde{u} \in L^2(\mathbb{T}^n)$ is a weak solution of

$$\tilde{a}^j\frac{\partial}{\partial y^j}\tilde{u}(y) + \tilde{b}(y)\tilde{u}(y) = \tilde{f}(y).$$

In particular we have $\tilde{b} \in L^{n^*}(\mathbb{T}^n)$ and

$$\begin{aligned} \|\tilde{a}^j - a^j_0\|_{L^\infty(\mathbb{T}^n)} &= \|a^j - a^j_0\|_{L^\infty(Q_R)} \leq \eta/10, \\ \|\partial_{y^j}\tilde{a}^j\|_{L^{n^*}(\mathbb{T}^n)} &\leq \|\partial_j a^j\|_{L^n(Q_R)} \leq \eta/10 \ . \end{aligned}$$

The conditions of Proposition 3.5 are satisfied, so $\tilde{u} \in H^1(\mathbb{T}^n)$ and thus $u \in H^1_{\rm loc}(M)$.

When M is compact there is a finite covering by charts $\{p_\alpha, U_\alpha\}$, and by Theorem 3.4, in each chart we may estimate the localisation $u_\alpha = \phi_\alpha u$ by

$$\|u_\alpha\|_{H^1(U_\alpha)} \leq C_\alpha(\|\mathcal{L}u_\alpha\|_{L^2(M)} + \|u_\alpha\|_{L^2(M)}),$$

where the $H^1(U_\alpha)$ norm uses the coordinate partial derivatives ∂_i in U_α. To estimate $\int_{U_\alpha}|\nabla u_\alpha|^2$, note that the Sobolev inequality (3.3) in U_α gives

$$\int_{U_\alpha}|\Gamma u_\alpha|^2\,dv_M \leq C\|\Gamma\|^2_{L^{n^*}(U_\alpha)}\int_{U_\alpha}(|\partial u_\alpha|^2 + |u_\alpha|^2)\,dv_M \ ,$$

so by the $H^1(U_\alpha)$ estimate we have

$$\|u_\alpha\|_{H^1(M)} \leq C_\alpha(\|\mathcal{L}u_\alpha\|_{L^2(M)} + \|u_\alpha\|_{L^2(M)}),$$

for some constant C_α depending also on $\|\Gamma\|_{L^{n^*}(U_\alpha)}$. Since $u = \sum u_\alpha$ and $\mathcal{L}u_\alpha = \phi_\alpha \mathcal{L}u + \partial_j(\phi_\alpha) a^j u$, with $|\partial\phi_\alpha| \le c$ and $|\phi_\alpha| \le 1$, the estimate (3.22) follows easily. ■

The constant C of (3.22) can be controlled by $\|a^j\|_{W^{1,p}}$, $\|b\|_{L^p}$ for any $p > n^*$, or by otherwise controlling the decompositions $\partial_j a^j, b \in L^\infty + L^{n^*}$.

Higher regularity follows easily from Theorem 3.7 by a standard bootstrap argument:

THEOREM 3.8 *Suppose $u \in L^2_{\text{loc}}$ is a weak solution of $\mathcal{L}u = f$ in the situation of Theorem 3.7, where the coefficients of $\mathcal{L}u = f$ in local charts satisfy the regularity conditions*

$$a^j \in W^{k,n^*}_{\text{loc}} \cap C^0, \quad b \in W^{k,n^*}_{\text{loc}}, \quad \textit{and} \quad f \in H^k_{\text{loc}} , \tag{3.23}$$

for some integer $k \ge 1$. Then $u \in H^{k+1}_{\text{loc}}$. If M is a compact manifold without boundary then there is a constant $C = C(k, \mathcal{L})$, depending on k and $\|a^j\|_{W^{k,n^*}}$, $\|b^j\|_{W^{k,n^*}}$, such that

$$\|u\|_{H^{k+1}(M)} \le C(\|f\|_{H^k(M)} + \|u\|_{H^k(M)}) . \tag{3.24}$$

Thus for any $u \in L^2(M)$ such that $\mathcal{L}u$ (defined weakly) satisfies $\mathcal{L}u \in H^k(M)$, we have

$$\|u\|_{H^{k+1}(M)} \le C(\|\mathcal{L}u\|_{H^k(M)} + \|u\|_{L^2(M)}) . \tag{3.25}$$

PROOF: For simplicity we first treat the case $k = 1$. Theorem 3.7 shows $u \in H^1_{\text{loc}}$, so the vector of first derivatives $\partial u \in L^2_{\text{loc}}$ itself is a weak solution of the system of equations

$$\mathcal{L}\partial u + \partial(a^j)\,\partial_j u = \partial f - \partial(b)\,u . \tag{3.26}$$

Since $\partial a^j \in L^{n^*}$ and

$$\|\partial(b)\,u\|_{L^2} \le C\|b\|_{W^{1,n^*}}\|u\|_{H^1},$$

so $\|\partial(b)\,u\|_{L^2}$ is bounded, this system satisfies the conditions of Theorem 3.7, hence $u \in H^2_{\text{loc}}$. The general induction step applies a similar argument: if the result is established $\forall\ k \le K-1$, and if $\mathcal{L}u = f$ with coefficient conditions (3.23) with $k = K$, then ∂u satisfies an elliptic system (3.26) of the same form with coefficient conditions (3.23) with $k = K-1$, so by induction $\partial u \in H^K(M)$ and thus $u \in H^{K+1}(M)$ as required. The estimates (3.24), (3.25) follow easily by a similar argument and Theorem 3.7. ■

The coefficient conditions in Theorem 3.8 are not optimal in most cases. For example, if $n = 3$ then $b \in W^{1,2}$ suffices to show $u \in H^2$ (rather than $b \in W^{1,3}$). This follows by interpolation,

$$\begin{aligned} \|\partial b\, u\|_{L^2} &\le \|\partial b\|_{L^2}\|u\|_{L^\infty} \\ &\le \epsilon\|u\|_{H^2} + C(\epsilon, \|\partial b\|_{L^2})\|u\|_{L^2} , \end{aligned}$$

which shows that $\partial b\, u$ may be thought of as the sum of a small second order operator, and a large bounded operator on L^2. The small operator term may be absorbed as a perturbation of $\mathcal{L}$, and the remainder contributes to the right hand side source term.

4 Spectral Condition

In this section we review conditions under which an operator will have a complete set of eigenfunctions. These conditions will be used in §5 to analyse boundary conditions, and thus the case of most interest concerns operators on a compact manifold without boundary, and in particular the first order elliptic systems considered in §3. However, the main result, Theorem 4.1, is stated in slightly more generality, which could be used to extend the eigenfunction representation to operators on manifolds with boundary.

Let H be a closed subspace of $W^{1,2}(Y)$, with the induced norm, where Y is a compact manifold perhaps with boundary, and as in §3, it is understood that these spaces refer to sections of a (real) vector bundle E over Y.

The abstract spectral theorem for the map $A : H \to L^2(Y)$ uses the following conditions:

($\mathcal{C}0$) $A : H \to L^2(Y)$ is linear and bounded in the $W^{1,2}$ topology on H.

($\mathcal{C}1$) The Gårding inequality holds: there exists a constant C such that for all $\psi \in H$ we have

$$\|\psi\|_H^2 \quad \leq \quad C \int_Y \left(\langle A\psi, A\psi \rangle + \langle \psi, \psi \rangle \right) \, dv_Y \ . \tag{4.1}$$

($\mathcal{C}2$) Weak solutions are strong solutions ("elliptic regularity"): If $\phi \in L^2(Y)$ satisfies

$$\int_Y \langle A\psi, \phi \rangle \, dv_Y = 0 \ , \qquad \forall\, \psi \in H, \tag{4.2}$$

then $\phi \in H$.

($\mathcal{C}3$) A is symmetric:

$$\forall\ \phi, \psi \in H \qquad \int_Y \langle A\phi, \psi \rangle \, dv_Y = \int_Y \langle \phi, A\psi \rangle \, dv_Y \ . \tag{4.3}$$

($\mathcal{C}4$) density:

$$H \text{ is dense in } L^2(Y). \tag{4.4}$$

Note that in the case $\partial Y \neq \emptyset$, the space H must incorporate boundary conditions, and these will play an important role in verifying ($\mathcal{C}2$), as will be seen in §5.

The main result of this section is the following:

Theorem 4.1 *Under the conditions ($\mathcal{C}0$)–($\mathcal{C}4$), there exists a countable orthonormal basis of L^2 consisting of eigenfunctions of A, with eigenvalues all real and having no accumulation point in $\mathbb{R}$.*

Proof: Let $\mathrm{Ker}(A) \subset H$ be the kernel of A; it is a standard fact that $\mathrm{Ker}(A)$ is finite dimensional when the Gårding inequality holds — we give the proof for completeness. Let $\{\psi_i\}_{i=1}^I$, $I \leq \infty$, be an L^2–orthonormal basis of $\mathrm{Ker}(A)$, the equation $A\psi_i = 0$ together with (4.1) shows that $\{\psi_i\}_{i=1}^I$ is bounded in

$W^{1,2}$. The Rellich theorem [12, Theorem 7.22] implies that from the sequence ψ_i we can extract a subsequence ψ_{i_j} converging strongly in L^2, weakly in $W^{1,2}$. The Gårding inequality (4.1) with ψ replaced by $\psi_{i_j} - \psi_{i_k}$ shows that ψ_{i_j} is Cauchy in $W^{1,2}$, hence converges in norm to some $\psi \in W^{1,2}$. By continuity of A, condition ($\mathcal{C}0$), we have $A\psi = 0$, by continuity of L^2 norm on $W^{1,2}$ it holds that $\|\psi\|_{L^2} = 1$, and it easily follows that $\psi \in \{\psi_i\}_{i=1}^I$. We have thus shown that $\{\psi_i\}_{i=1}^I$ is compact, which yields $I < \infty$, as desired.

Let now

$$\hat{H} = \{\psi \in H : \forall\, \phi \in \mathrm{Ker}(A) \quad \int_Y \langle \phi, \psi \rangle \, dv_Y = 0\} \ .$$

For $\phi \in L^2$ the map $H \ni \psi \to \int_Y \langle \phi, \psi \rangle \, dv_Y \in \mathbb{R}$ is continuous in the L^2 topology (and therefore also in the $W^{1,2}$ topology), thus $\hat{H}$ is closed (being an intersection of closed spaces), and hence a Banach space. We note the following:

Lemma 4.2 *There exists a constant C such that*

$$\forall\, \psi \in \hat{H} \qquad \|\psi\|_{L^2} \leq C\|A\psi\|_{L^2} \ . \tag{4.5}$$

Proof: Suppose that this is not the case, then there exists a sequence $\psi_n \in \hat{H}$ such that

$$\|\psi_n\|_{L^2} \geq n\|A\psi_n\|_{L^2} \quad . \tag{4.6}$$

Rescaling ψ_n if necessary we can without loss of generality assume that $\|\psi_n\|_{L^2} = 1$. The inequality (4.1) shows that ψ_n is bounded in $W^{1,2}$ norm. By the Rellich theorem [12, Theorem 7.22] we can extract a subsequence, still denoted ψ_n, converging to a $\psi_\infty \in \hat{H}$, weakly in $W^{1,2}$ and strongly in L^2. Equation (4.6) shows that the sequence $A\psi_n$ converges to zero in L^2, and (4.1) with ψ replaced with $\psi_n - \psi_m$ shows that ψ_n is Cauchy in the $W^{1,2}$ norm. Continuity of A and Equation (4.6) imply that $A\psi_\infty = 0$, and since A has no kernel on $\hat{H}$ we obtain $\psi_\infty = 0$, which contradicts $\|\psi_\infty\|_{L^2} = 1$, and the lemma follows. ■

Returning to the proof of Theorem 4.1, define $\mathrm{Im}(A)$ to be the image of $\hat{H}$ under A. Then $\mathrm{Im}(A)$ is a closed subspace of L^2, which can be seen as follows: Let ψ_i be any sequence in $\hat{H}$ such that the sequence $\chi_i \equiv A\psi_i$ converges in L^2 to $\chi_\infty \in L^2$. The inequality (4.5) shows that ψ_i is Cauchy in L^2, which together with the Gårding inequality shows that ψ_i is Cauchy in the $W^{1,2}$ norm. As $\hat{H}$ is closed, it follows that there exists $\psi_\infty \in \hat{H}$ such that ψ_i converges to ψ_∞ in the $W^{1,2}$ norm, and the equality $\chi_\infty = A\psi_\infty$ follows from continuity of A.

Let $\phi \in L^2$ be any element of $\mathrm{Im}(A)^\perp$, the L^2 orthogonal of $\mathrm{Im}(A)$; by definition we have

$$\forall\, \psi \in H \qquad \int_Y \langle \phi, A\psi \rangle \, dv_Y = 0 \ .$$

The hypothesis ($\mathcal{C}2$) of elliptic regularity implies that $\phi \in H$, so we can use the symmetry of A to conclude

$$\forall\, \psi \in H \qquad \int_Y \langle A\phi, \psi \rangle \, dv_Y = 0 \ .$$

Density of H in L^2 implies $A\phi = 0$, thus

$$\mathrm{Im}(A)^{\perp} = \mathrm{Ker}(A) \ . \tag{4.7}$$

Define $\hat{A} : \hat{H} \to \mathrm{Im}(A)$ by $\hat{A}\psi = A\psi$. By the definition of all the objects involved the map $\hat{A}$ is continuous, surjective and injective, hence bijective. Let $\hat{K} : \mathrm{Im}(A) \to \hat{H}$ denote its inverse, then $\hat{K}$ is continuous by the open mapping theorem. Let i be the embedding of $W^{1,2}(Y)$ into $L^2(Y)$; we have $i(\hat{H}) \subset \mathrm{Ker}(A)^{\perp}$ which coincides with $\mathrm{Im}(A)$ by (4.7). It follows that for all $\chi \in \mathrm{Im}(A)$ we have $i \circ \hat{K}(\chi) \in \mathrm{Im}(A)$, so that $i \circ \hat{K}$ defines a map of $\mathrm{Im}(A)$ into $\mathrm{Im}(A)$, which we will denote by K. Now $\hat{K}$ is continuous and i compact, which implies compactness of K.

We note that $\mathrm{Im}(A)$ is a closed subset of the Hilbert space L^2, hence a Hilbert space with respect to the induced scalar product. The operator K is self–adjoint with respect to this scalar product, which can be seen as follows: let $\psi_a = K\phi_a$, $\phi_a \in \mathrm{Im}(A)$, $a = 1, 2$, thus $\psi_a \in H$ and $A\psi_a = \phi_a$. We then have

$$\int_Y \langle \phi_1, K\phi_2 \rangle \, dv_Y = \int_Y \langle A\psi_1, \psi_2 \rangle \, dv_Y = \int_Y \langle \psi_1, A\psi_2 \rangle \, dv_Y = \int_Y \langle K\phi_1, \phi_2 \rangle \, dv_Y \ ,$$

as desired. By the spectral theorem for compact self adjoint operators [29] there exists a countable L^2–orthonormal basis of $\mathrm{Im}(A)$ consisting of eigenfunctions of K:

$$K\phi_\alpha = \mu_\alpha \phi_\alpha \ ,$$

with eigenvalues μ_α accumulating only at 0. Since K is invertible we have $\mu_\alpha \neq 0$, hence

$$A\phi_\alpha = \lambda_\alpha \phi_\alpha \ , \qquad \lambda_\alpha = \mu_\alpha^{-1} \ .$$

The required basis of L^2 is obtained by completing $\{\phi_\alpha\}$ with any L^2–orthonormal basis of the finite dimensional kernel of A. ■

Definition 4.3 *A is said to satisfy the* spectral condition *if A is an operator on C^∞ sections of E over Y which is symmetric with respect to the L^2 integration pairing with measure dv_Y and inner product $\langle \cdot, \cdot \rangle$, and there is a countable orthonormal basis $\{\phi_\alpha\}_{\alpha \in \Lambda}$ of $L^2\Gamma(E)$ consisting of eigenfunctions,*

$$A\phi_\alpha = \lambda_\alpha \phi_\alpha, \quad \alpha \in \Lambda \ , \tag{4.8}$$

such that the eigenvalues $\lambda_\alpha \in \mathbb{R}$, counted as always with multiplicity, have no accumulation point in $\mathbb{R}$.

Corollary 4.4 *Suppose Y is a compact manifold without boundary and $A : H^1(Y) \to L^2(Y)$ is an elliptic system between sections of the bundles E, F, which satisfies the conditions (3.4,3.5) of Theorem 3.7. If $A = A^\dagger$ is formally self-adjoint (see (3.7)), then A satisfies the spectral condition, Definition 4.3.*

Proof: Take $H = H^1(Y)$. Condition $(\mathcal{C}0)$ follows from the coefficient bounds (3.4) and the inequality (3.3), and condition $(\mathcal{C}1)$ is conclusion (3.22) of Theorem 3.7, which also provides condition $(\mathcal{C}2)$. Finally, $(\mathcal{C}3)$ follows from the definition (3.7) of the L^2-adjoint $A^\dagger$, since integration by parts is permitted in H, and $(\mathcal{C}4)$ is standard. The conclusions now follow from Theorem 4.1. ■

Corollary 4.5 *Suppose Y is a compact manifold without boundary and $A : H^1(Y) \to L^2(Y)$ is an elliptic system between sections of the bundles E, F, which satisfies the conditions of Theorem 3.7. There are bases $\phi_\alpha \in L^2(E)$, $\psi_\alpha \in L^2(F)$, $\alpha \in \Lambda$, with real numbers λ_α having no accumulation point in $\mathbb{R}$, which satisfy*

$$A\phi_\alpha = \lambda_\alpha \psi_\alpha, \quad A^\dagger \psi_\alpha = \lambda_\alpha \phi_\alpha \ . \tag{4.9}$$

The fields ϕ_α, ψ_α are all $H^1(Y)$.

Proof: This follows directly by applying Corollary 4.4 to the formally self-adjoint operator

$$\mathbb{A} = \begin{bmatrix} 0 & A^\dagger \\ A & 0 \end{bmatrix} , \tag{4.10}$$

which acts between sections of the bundle $E \oplus F$. ■

5 Boundary Regularity

In this section we introduce a broad class of boundary conditions which are elliptic in the sense that the Weak-Strong property can be established, at least for solutions supported near the boundary. When combined with the interior regularity results of §3, this will give the Weak-Strong property for compact manifolds with boundary (§6), and for a large class of noncompact manifolds with compact boundary (§8). The main result is the boundary regularity Theorem 5.11, and the primary ingredient in the arguments is the energy identity (5.15) cf. [2] and (2.6).

We consider operators which may be written abstractly in the form

$$L = L_0 + B = \partial_x + A + B \ , \tag{5.1}$$

acting on sections of a (real) vector bundle E over $Y \times I$, where Y is a compact manifold without boundary and for some constant $\delta > 0$,

$$I = [0, \delta] \ ,$$

with ∂_x tangent to I. Let $E|_Y = i^*E$ be the pullback bundle over Y, where $i : Y \to Y \times I$, $y \mapsto (y, 0)$. We assume that A is an operator on sections of $E|_Y$ which is formally self-adjoint with respect to the pairing defined by integration over Y with the measure dv_Y and the real inner product $\langle \cdot, \cdot \rangle$ on the fibres of $E|_Y$. The operator A and the inner product extend naturally to act on sections of E over $Y \times I$, and we likewise extend the definition of the integration pairing by using the product measure $dv_Y dx$ on $Y \times I$. Thus, A is x–independent, but we allow B to depend upon x.

We assume also that A satisfies the spectral condition, Definition 4.3, so there is a countable index set Λ and an orthonormal basis $\{\phi_\alpha\}_{\alpha \in \Lambda}$ of $L^2(E|_Y)$ consisting of eigenfunctions

$$A\phi_\alpha = \lambda_\alpha \phi_\alpha \ , \qquad \alpha \in \Lambda \ , \tag{5.2}$$

with eigenvalues $\lambda_\alpha \in \mathbb{R}$ having no accumulation points in $\mathbb{R}$. A formally self-adjoint first order elliptic operator with the coefficient conditions of Theorem 3.7, will satisfy these conditions, by Corollary 4.4.

Although we have in mind primarily the case where A, B are first order differential (Dirac-type) operators, the results here will be presented in an abstract form, because they could be applied more widely. For example, $A = -\Delta_Y$ will also satisfy the spectral conditions, so the boundary regularity result Theorem 5.11 may also be applied to the heat equation.

We fix an eigenvalue cutoff parameter $\kappa > 0$, which is used to partition the index set Λ into

$$\begin{aligned} \Lambda^+ &= \{\alpha \in \Lambda, \lambda_\alpha \geq \kappa\} , & (5.3)\\ \Lambda^- &= \{\alpha \in \Lambda, \lambda_\alpha \leq -\kappa\} , & (5.4)\\ \Lambda^0 &= \{\alpha \in \Lambda, |\lambda_\alpha| < \kappa\} , & (5.5) \end{aligned}$$

and we set $\Lambda' = \Lambda^+ \cup \Lambda^-$. It will be useful also to introduce a scale parameter

$$\theta_0 = \kappa^{-1} \max_{\alpha \in \Lambda^0} |\lambda_\alpha| < 1 , \tag{5.6}$$

which measures the size of the "small" eigenvalues. For example, we could choose κ to be the smallest nonzero eigenvalue, $\kappa = \inf_{\lambda_\alpha \neq 0} |\lambda_\alpha|$, in which case $\theta_0 = 0$. Choosing κ appropriately will lead to estimates for L which are uniform under perturbations of A which create or destroy small and zero eigenvalues.

The eigenfunction expansion $u = \sum_{\alpha \in \Lambda} u_\alpha \phi_\alpha$ of $u \in L^2(Y)$, where

$$u_\alpha := \oint_Y \langle u, \phi_\alpha \rangle \, dv_Y,$$

leads to projection operators P_+, P_-, P_0, P', defined by

$$\begin{aligned} P_\pm u &= \sum_{\alpha \in \Lambda^\pm} u_\alpha \phi_\alpha , & (5.7)\\ P_0 u &= \sum_{\alpha \in \Lambda^0} u_\alpha \phi_\alpha , & (5.8) \end{aligned}$$

and $P' = 1 - P_0 = P_+ + P_-$.

For $s \geq 0$, the Sobolev-type space $H^s_*(Y)$ is defined as the completion of the space of smooth sections $C^\infty(Y)$, with respect to the norm

$$\|u\|^2_{H^s_*(Y)} = \sum_{\alpha \in \Lambda'} |\lambda_\alpha|^{2s} |u_\alpha|^2 + \kappa^{2s} \sum_{\alpha \in \Lambda^0} |u_\alpha|^2 . \tag{5.9}$$

The space $H^1_*(Y \times I)$ is likewise the completion of $C^\infty(Y \times I)$ with respect to the norm

$$\begin{aligned} \|u\|^2_{H^1_*(Y\times I)} &= \int_0^\delta \left(\sum_{\alpha \in \Lambda} |u'_\alpha|^2 + \sum_{\alpha \in \Lambda'} |\lambda_\alpha|^2 |u_\alpha|^2 + \kappa^2 \sum_{\alpha \in \Lambda^0} |u_\alpha|^2 \right) dx \\ &\leq \int_0^\delta \oint_Y \left(|\partial_x u|^2 + |Au|^2 + \kappa^2 |P_0 u|^2 \right) dv_Y dx , & (5.10) \end{aligned}$$

where $' = \frac{d}{dx}$. Of course in the typical case where A is a first order uniformly elliptic operator, these norms will be equivalent to the usual Sobolev norms, defined using the Fourier transform. Note that the normalization (5.9) ensures that the L^2 norm is controlled by the Sobolev norm:

$$\|u\|_{L^2(Y)} \leq \kappa^{-s}\|u\|_{H^s_*(Y)}, \quad s \geq 0 .$$

This formulation leads simply to a useful trace lemma in terms of the parameter ℓ,

$$\ell = \kappa\delta , \tag{5.11}$$

which measures the thickness of the boundary layer $Y \times [0, \delta]$ in units of κ^{-1}.

LEMMA 5.1 *The restriction map $r_Y : u \mapsto r_Y u = u(\cdot, 0)$ from $C^\infty(Y \times I)$ to $C^\infty(Y)$, $I = [0, \delta]$, extends to a bounded linear map $r_Y : H^1_*(Y \times I) \to H^{1/2}_*(Y)$ satisfying*

$$\|r_Y u\|^2_{H^{1/2}_*} \leq c_1 \|u\|^2_{H^1_*(Y\times I)} , \tag{5.12}$$

where $c_1 = c_1(\ell) = \ell^{-1}(1 + \sqrt{1+\ell^2})$. The map $x \mapsto r_{Y,x}u$ (where $r_{Y,x}u = u(x, \cdot)$ is the restriction to $Y \times \{x\}$), is likewise bounded and continuous in x from I to $H^{1/2}_*(Y)$. Moreover, r_Y is surjective: there is an extension map $e_Y : H^{1/2}_*(Y) \to H^1_*(Y \times I)$ such that $r_Y e_Y u = u$ for all $u \in H^{1/2}_*(Y)$ and e_Y satisfies $r_{Y,\delta} e_Y(u) = 0$ and

$$\|e_Y u\|^2_{H^1_*(Y\times I)} \leq \frac{2}{\sqrt{3}} \|u\|^2_{H^{1/2}_*} . \tag{5.13}$$

PROOF: For $x \in [0, \infty)$ set $\chi(x) = \max(0, 1 - x)$. For any $u \in C^\infty(Y \times [0, \delta])$ with $u_\alpha(x)$, $\alpha \in \Lambda$ denoting the spectral coefficients, and with $\tilde{\chi}(x) = \chi(x/\delta)$, we find that

$$\begin{aligned} |u_\alpha(0)|^2 &= -2\int_0^\delta \langle \tilde{\chi} u_\alpha, \tfrac{d}{dx}(\tilde{\chi} u_\alpha)\rangle \, dx \\ &\leq \int_0^\delta \left((2\tilde{\chi}|\tilde{\chi}'| + \eta\tilde{\chi}^2)\,|u_\alpha|^2 + \eta^{-1}\tilde{\chi}^2|u'_\alpha|^2\right) dx \end{aligned}$$

for any $\eta > 0$. For $\alpha \in \Lambda'$ we take $\eta = a|\lambda_\alpha|$ and find

$$|\lambda_\alpha||u_\alpha(0)|^2 \leq \int_0^\delta \left(a^{-1}|u'_\alpha|^2 + |\lambda_\alpha|^2(a + 2/\ell)|u_\alpha|^2\right) dx .$$

Likewise for $\alpha \in \Lambda^0$, setting $\eta = a\kappa$ gives

$$\kappa|u_\alpha(0)|^2 \leq \int_0^\delta \left(a^{-1}|u'_\alpha|^2 + \kappa^2(a + 2/\ell)|u_\alpha|^2\right) dx .$$

Choosing $a = \ell^{-1}(\sqrt{1+\ell^2} - 1)$ and combining the two estimates gives (5.12) for all $u \in C^\infty(Y \times [0, \delta])$. But this space is dense in $H^1_*(Y \times I)$ by definition, and it follows easily that $r_Y u$ is defined and (5.12) is valid for all $u \in H^1_*(Y \times I)$.

A very similar argument shows that for $x \in [0, \delta]$,

$$\|r_{Y,x}u\|^2_{H^{1/2}_*} \le \ell^{-1}(2 + \sqrt{4 + \ell^2})\|u\|^2_{H^1_*(Y\times I)}.$$

To establish continuity of $x \mapsto r_{Y,x}u$ as a map $[0, \delta] \to H^{1/2}_*(Y)$, note first that for any $v \in H^1_*(Y \times [x_0, x_1])$, $x_0 < x_1$, the spectral coefficients v_α lie in $H^1([x_0, x_1])$ and we may compute:

$$\begin{aligned}
\big||v_\alpha(x_1)|^2 - |v_\alpha(x_0)|^2\big| &\le 2\int_{x_0}^{x_1} \big|\langle v_\alpha, v'_\alpha\rangle\big|\, dx \\
&\le \eta_\alpha^{-1}\int_{x_0}^{x_1} \left(|v'_\alpha|^2 + \eta_\alpha^2|v_\alpha|^2\right) dx\ .
\end{aligned}$$

Choosing $\eta_\alpha = |\lambda_\alpha|$ for $\alpha \in \Lambda'$ and $\eta_\alpha = \kappa$ for $\alpha \in \Lambda^0$ and summing gives

$$\begin{aligned}
\Big|\|r_{Y,x_1}v\|^2_{H^{1/2}_*} - \|r_{Y,x_0}v\|^2_{H^{1/2}_*}\Big| &\le \int_{x_0}^{x_1}\Big(\sum_{\alpha\in\Lambda}|u'_\alpha|^2 + \sum_{\alpha\in\Lambda'}|\lambda_\alpha|^2|u_\alpha|^2 + \sum_{\alpha\in\Lambda^0}\kappa^2|u_\alpha|^2\Big)\, dx \\
&\le \|v\|_{H^1_*(Y\times[x_0,x_1])}\ . \qquad (5.14)
\end{aligned}$$

Given $u \in H^1_*(Y \times I)$ and $\bar{x} \in [0, \delta)$, $\epsilon \in (0, (\delta - \bar{x})/2)$, we set

$$v(x) = u(\bar{x} + \epsilon + x) - u(\bar{x} + \epsilon - x),$$

where $x \in [-\epsilon, \epsilon]$ and the Y-dependence of u, v is understood. Applying (5.14) with $x_0 = 0$, $x_1 = \epsilon$, gives $v(x_0) = 0$, $v(x_1) = u(\bar{x} + 2\epsilon) - u(\bar{x})$ and

$$\begin{aligned}
\|r_{Y,\bar{x}+2\epsilon}u - r_{Y,\bar{x}}u\|^2_{H^{1/2}_*} &\le \|v\|^2_{H^1_*(Y\times[0,\epsilon])} \\
&\le 2\|u\|^2_{H^1_*(Y\times[\bar{x},\bar{x}+2\epsilon])} \\
&= o(1) \quad \text{as} \quad \epsilon \searrow 0\ ,
\end{aligned}$$

since the measure of $[\bar{x}, \bar{x} + 2\epsilon] \to 0$. This establishes continuity from the right, and left continuity follows similarly.

To see that r_Y is surjective, we construct an extension map $e_Y : H^{1/2}_*(Y) \to H^1_*(Y \times I)$, such that $r_Y \circ e_Y = Id$. For any $u \in H^{1/2}_*(Y)$ let $\{u^k\}$ be an approximating Cauchy sequence of smooth fields with spectral coefficients u^k_α and consider the sequence $\{\tilde{u}^k\}$ defined by

$$\tilde{u}^k(x, y) = \sum_{\alpha\in\Lambda} u^k_\alpha \phi_\alpha(y)\chi(x/\eta_\alpha),$$

where $\eta_\alpha = \sqrt{3}/|\lambda_\alpha|$ for $\alpha \in \Lambda'$ and $\sqrt{3}/\kappa$ for $\alpha \in \Lambda^0$, so

$$\begin{aligned}
\|\tilde{u}^k\|^2_{H^1_*(Y\times I)} &\le \int_0^\delta \sum_{\alpha\in\Lambda'}|\lambda_\alpha|^2|u^k_\alpha|^2\Big(\chi^2(x/\eta_\alpha) + \tfrac{1}{3}\chi'^2(x/\eta_\alpha)\Big)\, dx \\
&\quad + \int_0^\delta \sum_{\alpha\in\Lambda^0}\kappa^2|u^k_\alpha|^2\Big(\chi^2(x/\eta_\alpha) + \tfrac{1}{3}\chi'^2(x/\eta_\alpha)\Big)\, dx\ .
\end{aligned}$$

Using the bounds $\int_0^\infty \chi^2(x)dx \le 1/3$, $\int_0^\infty \chi'^2(x)dx \le 1$, and noting that

$$\int_0^\delta \psi^2(x/\eta)dx \le \eta \int_0^\infty \psi^2(x)dx$$

for any ψ, we have

$$\begin{aligned} \|\tilde{u}^k\|^2_{H^1_*(Y\times I)} &\le \frac{2}{\sqrt{3}}\Big\{\sum_{\alpha\in\Lambda'}|\lambda_\alpha||u^k_\alpha|^2 + \sum_{\alpha\in\Lambda_0}\kappa|u^k_\alpha|^2\Big\} \\ &\le \frac{2}{\sqrt{3}}\|\tilde{u}^k\|^2_{H^{1/2}_*} . \end{aligned}$$

Hence the sequence $\{\tilde{u}^k\}$ is uniformly bounded, and a similar argument shows that it is also Cauchy, with limit $\tilde{u} = e_Y u \in H^1_*(Y\times I)$. It follows easily that the sequence has boundary values converging to $u \in H^{1/2}_*(Y)$, and $u, e_Y u$ satisfy the bound (5.13). ∎

The next result relates H^1 estimates to boundary conditions, and is the key to understanding the nature of ellipticity for boundary data. It may be considered as a generalization either of the integration by parts formula for the Dirac operator (2.6), or of the estimate underlying the analysis in [2]. We use $u(0), u(\delta)$ to denote the restrictions $r_Y u = u|_{Y\times\{0\}}$, $r_{Y,\delta}u = u|_{Y\times\{\delta\}}$ respectively.

Lemma 5.2 *Suppose $f \in L^2(Y\times I)$ and $u \in H^1_*(Y\times I)$ satisfies $L_0 u = f$, then*

$$\begin{aligned} \|u\|^2_{H^1_*(Y\times I)} &\le \|(1-P_0)f\|^2_{L^2(Y\times I)} + (1+\theta_0^2)\|P_0 f\|^2_{L^2(Y\times I)} \\ &\quad + 3\kappa^2\|P_0 u\|^2_{L^2(Y\times I)} \\ &\quad + \sum_{\alpha\in\Lambda^+}\lambda_\alpha\left(|u_\alpha(0)|^2 - |u_\alpha(\delta)|^2\right) \\ &\quad + \sum_{\alpha\in\Lambda^-}|\lambda_\alpha|\left(|u_\alpha(\delta)|^2 - |u_\alpha(0)|^2\right) . \end{aligned} \tag{5.15}$$

Proof: The coefficient functions $u_\alpha(x)$ are measurable and, by Fubini's theorem, square-integrable over $[0,\delta]$. Testing the weak formulation with $\phi(x,y) = \chi(x)\phi_\alpha(y)$ where $\chi \in C_c^\infty((0,\delta))$ shows that u_α satisfies

$$\int_0^\delta \left(-u_\alpha\chi' + u_\alpha\lambda_\alpha\chi - f_\alpha\chi\right) dx = 0 , \tag{5.16}$$

for all $\chi \in C_c^\infty((0,\delta))$. Because $u \in H^1_*(Y\times I)$, the spectral coefficient $u_\alpha(x)$ is differentiable for *a.e.* $x \in [0,\delta]$, with u'_α square-integrable, and (5.16) shows that it satisfies the ordinary differential equation $u'_\alpha(x) + \lambda_\alpha u_\alpha(x) = f_\alpha(x)$. The trace lemma also shows that the restrictions $u_\alpha(0)$, $u_\alpha(\delta)$ are well defined. From the ODE we derive the fundamental identity

$$\begin{aligned} \int_0^\delta |f_\alpha|^2\, dx &= \int_0^\delta |u'_\alpha + \lambda_\alpha u_\alpha|^2\, dx \\ &= \int_0^\delta \left(|u'_\alpha|^2 + \lambda_\alpha^2|u_\alpha|^2\right) dx \\ &\quad + \lambda_\alpha(|u_\alpha(\delta)|^2 - |u_\alpha(0)|^2) . \end{aligned} \tag{5.17}$$

Summing over $\alpha \in \Lambda^+ \cup \Lambda^-$ and noting that the boundary restrictions $u(0)$, $u(\delta)$ are in $H_*^{1/2}(Y)$ by Lemma 5.1 since $u \in H_*^1(Y \times I)$ by assumption, we find

$$\begin{aligned}
\|P'u\|^2_{H^1_*(Y\times I)} &= \int_0^\delta \sum_{\alpha\in\Lambda^+\cup\Lambda^-} (|u_a'|^2 + |\lambda_\alpha|^2|u_\alpha|^2)\,dx \\
&= \int_0^\delta \oint_Y |P'f|^2 dv_Y dx \\
&\quad + \sum_{\alpha\in\Lambda^+\cup\Lambda^-} \lambda_\alpha \left(|u_\alpha(0)|^2 - |u_\alpha(\delta)|^2\right) . \qquad (5.18)
\end{aligned}$$

For $\alpha \in \Lambda^0$ we use $u_\alpha' = f_\alpha - \lambda_\alpha u_\alpha$ to estimate

$$\begin{aligned}
\|P_0 u\|^2_{H^1_*(Y\times I)} &= \int_0^\delta \sum_{\alpha\in\Lambda^0} (|u_a'|^2 + \kappa^2|u_\alpha|^2)\,dx \\
&\le \int_0^\delta \sum_{\alpha\in\Lambda^0} \left((1+\varepsilon)|f_\alpha|^2 + (\kappa^2 + (1+\varepsilon^{-1})\kappa_0^2)|u_\alpha|^2\right) \\
&\le 3\kappa^2\|P_0u\|^2_{L^2(Y\times I)} + (1+\theta_0^2)\|P_0 f\|^2_{L^2(Y\times I)} , \qquad (5.19)
\end{aligned}$$

having chosen $\varepsilon = \theta_0^2$. Combining (5.18) and (5.19) gives (5.15). ■

REMARK 5.3 The above proof could be generalized to allow $u \in L^2$ and to show then that u is in $H^1_{\rm loc}$, but this refinement is unnecessary as we soon will show a more general regularity theorem. Working with $u \in H^1$ allows us to use the boundary terms with impunity—a freedom that is not possible with weak solutions at this stage.

The fundamental estimate (5.15) shows that in order to obtain a useful *a priori* elliptic estimate for a general solution of $L_0 u = f$, it is necessary to impose boundary conditions which control $P_+u(0)$ (and $P_-u(\delta)$). Motivated by the examples of the spectral and pointwise boundary conditions for the Dirac equation (see §2), we introduce a class of boundary conditions which allow us to exploit the "good" terms in $P_-u(0)$ in (5.15) to provide the required control. The effect of the parameterization below is to describe the class of admissible boundary data as graphs over the complementary subspace of "good" data $(1-P_+)H_*^{1/2}(Y)$. The first justification of this approach is the following existence result and its corresponding elliptic estimate (5.24).

LEMMA 5.4 *Let $P = P_{\Lambda^+\cup\hat\Lambda}$ be the spectral projection determined by Λ^+ and some subset $\hat\Lambda \subset \Lambda^0$ of the set of small eigenvalues. Let $\sigma \in PH_*^{1/2}(Y)$ and $f \in L^2(Y \times I)$ be given, and suppose $K : (1-P)H_*^{1/2}(Y) \to PH_*^{1/2}(Y)$ is a continuous linear operator, so there is a constant $k \ge 0$ such that for all $w \in H_*^{1/2}(Y)$,*

$$\|K(1-P)w\|_{H_*^{1/2}} \le k\|(1-P)w\|_{H_*^{1/2}} . \qquad (5.20)$$

Then there exists a solution $u \in H^1_*(Y \times I)$ to the boundary value problem

$$\begin{aligned} L_0 u &= f & (5.21)\\ Pu(0) &= \sigma + K(1-P)u(0) & (5.22)\\ (1-P)u(\delta) &= 0. & (5.23) \end{aligned}$$

Moreover, the solution u satisfies the estimate

$$\|u\|^2_{H^1_*(Y\times I)} \le c_4(\|f\|^2_{L^2(Y\times I)} + \|\sigma\|^2_{H^{1/2}_*}) \,, \tag{5.24}$$

where the constant c_4 depends on ℓ, θ_0 and k.

PROOF: The solution to an ordinary differential equation $u'(x) + \lambda u = f$ may be written in either of the two forms

$$u(x) = \begin{cases} e^{-\lambda x}u(0) + \displaystyle\int_0^x e^{\lambda(s-x)} f(s)ds \\ e^{\lambda(\delta-x)}u(\delta) - \displaystyle\int_x^\delta e^{\lambda(s-x)} f(s)ds \,. \end{cases} \tag{5.25}$$

Consider first the spectral coefficients $u_\alpha(x)$ for $\alpha \in \Lambda^- \cup \hat{\Lambda}'$, where $\hat{\Lambda}' = \Lambda^0 \backslash \hat{\Lambda}$. The boundary condition (5.23) is achieved for u_α by letting $u_\alpha(\delta) = 0$, so we define

$$u_\alpha(x) = -\int_x^\delta e^{\lambda_\alpha(s-x)} f_\alpha(s)\, ds, \qquad \alpha \in \Lambda^- \cup \hat{\Lambda}' \,, \tag{5.26}$$

where $f_\alpha(x)$ is the spectral coefficient of f. Note that $f_\alpha \in L^2(I)$, so the integral in (5.26) is well-defined. The identity (5.17) and $u_\alpha(\delta) = 0$ shows that

$$\int_0^\delta (|u'_\alpha|^2 + \lambda_\alpha^2 |u_\alpha|^2)\, dx = \lambda_\alpha |u_\alpha(0)|^2 + \int_0^\delta |f_\alpha|^2\, dx \,, \quad \forall \alpha \in \Lambda^- \cup \hat{\Lambda}' \,. \tag{5.27}$$

It follows that for all $\alpha \in \Lambda^-$,

$$|\lambda_\alpha||u_\alpha(0)|^2 \le \int_0^\delta |f_\alpha|^2 dx - \int_0^\delta (|u'_\alpha|^2 + \lambda_\alpha^2|u_\alpha|^2)\, dx \,. \tag{5.28}$$

To control the small eigenvalues $\alpha \in \hat{\Lambda}'$ we use Hölder's inequality to show

LEMMA 5.5 *For any $f \in L^2([0,\delta])$ and $\lambda \in \mathbb{R}$,*

$$\int_0^\delta \left(\int_x^\delta e^{\lambda(s-x)} f(s) ds \right)^2 dx \le \begin{cases} \frac{1}{2}\delta^2 e^{2\lambda\delta} \int_0^\delta f^2(x)dx & \lambda > 0 \\ \frac{1}{2}\delta^2 \int_0^\delta f^2(x)dx & \lambda \le 0 \,. \end{cases} \tag{5.29}$$

From (5.26) and Lemma 5.5 it follows that for $\alpha \in \hat{\Lambda}'$,

$$\int_0^\delta \kappa^2 |u_\alpha|^2 dx \le \tfrac{1}{2}\ell^2 e^{2\ell\theta_0} \int_0^\delta |f_\alpha|^2\, dx \,.$$

Using $u'_\alpha = f_\alpha - \lambda_\alpha u_\alpha$ as in (5.19) we obtain

$$\int_0^\delta (|u'_\alpha|^2 + \kappa^2 |u_\alpha|^2) dx \le (1 + c_2) \int_0^\delta |f_\alpha|^2 dx \,, \tag{5.30}$$

where

$$c_2 = c_2(\ell, \theta_0) = \theta_0^2 + \tfrac{3}{2}\ell^2 e^{2\ell\theta_0} \ . \tag{5.31}$$

Combining (5.27) and (5.30) shows that

$$u^{(-)}(x,y) := \sum_{\alpha \in \Lambda^- \cup \hat{\Lambda}'} u_\alpha(x)\phi_\alpha(y)$$

is a sum converging in $H^1_*(Y \times I)$, and $u^{(-)}$ satisfies

$$\begin{aligned} \|u^{(-)}\|^2_{H^1_*(Y\times I)} &\leq \|(1-P)f\|^2_{L^2(Y\times I)} + c_2\|P_{\hat{\Lambda}'}f\|^2_{L^2(Y\times I)} \\ &\quad - \sum_{\alpha\in\Lambda^-} |\lambda_\alpha|\,|u_\alpha(0)|^2 \ , \end{aligned} \tag{5.32}$$

where $1 - P = P_- + P_{\hat{\Lambda}'}$. For $\alpha \in \hat{\Lambda}'$ such that $\lambda_\alpha \neq 0$, using (5.26) we find that

$$\begin{aligned} |u_\alpha(0)|^2 &\leq \left(\int_0^\delta e^{\lambda_\alpha s} f_\alpha(s) ds\right)^2 \\ &\leq \frac{1}{2\lambda_\alpha}(e^{2\lambda_\alpha\delta} - 1)\int_0^\delta |f_\alpha|^2 dx \ , \end{aligned}$$

while if $\lambda_\alpha = 0$ then $|u_\alpha(0)|^2 \leq \delta \int_0^\delta |f_\alpha|^2 dx$. Defining

$$c_3 \ = \ \ell e^{2\ell\theta_0}, \tag{5.33}$$

it follows that

$$\kappa|u_\alpha(0)|^2 \leq c_3 \int_0^\delta |f_\alpha|^2 dx \ , \quad \forall \ \alpha \in \hat{\Lambda}' \ .$$

Thus, combining with (5.32) we obtain good interior and boundary control on $u^{(-)}$,

$$\begin{aligned} &\|u^{(-)}\|^2_{H^1_*(Y\times I)} + \|u^{(-)}(0)\|^2_{H^{1/2}_*} \\ &\leq \ \|(1-P)f\|^2_{L^2(Y\times I)} + (c_2 + c_3)\|P_{\hat{\Lambda}'}f\|^2_{L^2(Y\times I)} \ . \end{aligned} \tag{5.34}$$

For $\alpha \in \Lambda^+ \cup \hat{\Lambda}$ we use (5.25) to define u_α by

$$u_\alpha(x) = e^{-\lambda_\alpha x}(\sigma_\alpha + K_\alpha u^{(-)}(0)) + \int_0^x e^{\lambda_\alpha(s-x)} f_\alpha(s)\, ds, \tag{5.35}$$

where σ_α, $K_\alpha u^{(-)}(0)$ denote the ϕ_α coefficients of σ and $Ku^{(-)}(0)$ respectively. Note in particular that (5.34) shows that $u^{(-)}(0) \in H^{1/2}_*(Y)$, so $Ku^{(-)}(0) \in H^{1/2}_*(Y)$ by the hypothesis (5.20), hence the coefficients $K_\alpha u^{(-)}(0)$ are well defined.

For $\alpha \in \Lambda^+$ we estimate using (5.17) and (5.22):

$$\begin{aligned} \int_0^\delta (|u'_\alpha|^2 + \lambda_\alpha^2) dx &\leq \lambda_\alpha|\sigma_\alpha + K_\alpha u^{(-)}(0)|^2 + \int_0^\delta |f_\alpha|^2 dx \\ &\leq 2\lambda_\alpha(|\sigma_\alpha|^2 + |K_\alpha u^{(-)}(0)|^2) + \int_0^\delta |f_\alpha|^2 dx \ . \end{aligned} \tag{5.36}$$

For $\alpha \in \hat{\Lambda}$ we estimate directly from (5.35):

$$\begin{aligned}
\kappa^2 \int_0^\delta |u_\alpha|^2 dx &\leq 3\kappa^2(|\sigma_\alpha|^2 + |K_\alpha u^{(-)}(0)|^2) \int_0^\delta e^{-2\lambda_\alpha x} dx \\
&\quad + 3\kappa^2 \int_0^\delta \left(\int_0^x e^{\lambda_\alpha (s-x)} f_\alpha(s) ds \right)^2 dx \\
&\leq 3c_3\kappa(|\sigma_\alpha|^2 + |K_\alpha u^{(-)}(0)|^2) + \tfrac{3}{2}\ell c_3 \int_0^\delta |f_\alpha|^2 dx \ , \quad (5.37)
\end{aligned}$$

where Lemma 5.5 has been used to control the final term. Using $u'_\alpha = f_\alpha - \lambda_\alpha u_\alpha$ to estimate $|u'_\alpha|^2 \leq (1+\varepsilon)|f_\alpha|^2 + (1+\varepsilon^{-1})\lambda_\alpha^2 |u_\alpha|^2$ with $\varepsilon = \theta_0^2$, (5.37) gives for $\alpha \in \hat{\Lambda}$,

$$\begin{aligned}
\int_0^\delta \left(|u'_\alpha|^2 + \kappa^2 |u_\alpha|^2 \right) dx &\leq 9c_3\kappa(|\sigma_\alpha|^2 + |K_\alpha u^{(-)}(0)|^2) \\
&\quad + (1+3c_2) \int_0^\delta |f_\alpha|^2 dx \ . \qquad (5.38)
\end{aligned}$$

Combining (5.36) and (5.38) we have (setting $u^{(+)} = \sum_{\Lambda^+ \cup \hat{\Lambda}} u_\alpha \phi_\alpha$)

$$\begin{aligned}
\|u^{(+)}\|^2_{H^1_*(Y\times I)} &\leq \|Pf\|^2_{L^2(Y\times I)} + 3c_2 \|P_{\hat{\Lambda}} f\|^2_{L^2(Y\times I)} \\
&\quad + 2(\|P_+\sigma\|^2_{H^{1/2}_*} + \|P_+ K u^{(-)}(0)\|^2_{H^{1/2}_*}) \\
&\quad + 9c_3(\|P_{\hat{\Lambda}}\sigma\|^2_{H^{1/2}_*} + \|P_{\hat{\Lambda}} K u^{(-)}(0)\|^2_{H^{1/2}_*}) \ , \quad (5.39)
\end{aligned}$$

where $P_+ = P_{\Lambda^+}$, $P = P_+ + P_{\hat{\Lambda}}$. Since we have already shown that $K_\alpha u^{(-)}(0) \in H^{1/2}_*(Y)$, all terms on the right hand side of (5.39) are bounded, which shows that $u^{(+)}$ is well-defined in $H^1_*(Y \times I)$.

With $u = u^{(+)} + u^{(-)}$, we add an appropriate multiple of (5.34) to (5.39) to control the bad terms in $Ku^{(-)}(0)$ with the good term $u^{(-)}(0)$ of (5.34). This gives the elliptic estimate (5.24):

$$\begin{aligned}
\|u\|^2_{H^1_*(Y\times I)} &\leq \|u^{(+)}\|^2_{H^1_*(Y\times I)} + \max(1, 2k^2\ , 9c_3k^2)\|u^{(-)}\|^2_{H^1_*(Y\times I)} \\
&\leq c_4^2(\|f\|^2_{L^2(Y\times I)} + \|\sigma\|^2_{H^{1/2}_*}) \ ,
\end{aligned}$$

where $c_4 = c_4(\ell, \theta_0, k)$ as required. The definitions (5.26), (5.35) ensure u is a solution satisfying the boundary conditions (5.22), (5.23). ■

Explicitly, we may take $c_4^2 = 3c_2 + (k^2+1)(2+9c_3)$ in general, and $c_4^2 = 2\max(1, k^2)$ if $\theta_0 = 0$.

The next result is the key to handling operators with coefficients depending on x. Recall that the operator norm $\|B\|_{op}$ of a linear map $B : X_1 \to X_2$ between Banach spaces is the smallest constant such that

$$\|Bu\|_{X_2} \leq \|B\|_{op} \|u\|_{X_1}, \quad \forall u \in X_1 \ . \qquad (5.40)$$

LEMMA 5.6 *Suppose L_0, A, f, σ, K are as in Lemma 5.4, and suppose $B : H^1_*(Y \times I) \to L^2(Y \times I)$ is a linear map satisfying*

$$c_4 \, \|B\|_{op} < 1 \ . \tag{5.41}$$

Then there exists $u \in H^1_*(Y \times I)$ satisfying

$$(L_0 + B)u = f$$

and the boundary conditions (5.22),(5.23), such that

$$\|u\|^2_{H^1_*(Y\times I)} \le \frac{c_4}{1 - c_4\|B\|_{op}} (\|f\|_{L^2(Y\times I)} + \|\sigma\|_{H^{1/2}_*}) \ . \tag{5.42}$$

PROOF: Let $u^{(0)} \in H^1_*(Y \times I)$ be any function satisfying the boundary conditions (5.22),(5.23); the trace lemma 5.1 ensures the existence of a suitable $u^{(0)}$. Construct a sequence $\{u^{(k)}\} \subset H^1_*(Y \times I)$ by solving the problems

$$\begin{aligned} L_0 u^{(k)} &= f - Bu^{(k-1)} && (5.43)\\ Pu^{(k)}(0) &= \sigma + K(1-P)u^{(k)}(0) && (5.44)\\ (1-P)u^{(k)}(\delta) &= 0, \quad n = 1, 2, \dots \ . && (5.45) \end{aligned}$$

Lemma 5.4 ensures this problem has a solution for every $n \ge 1$, and the difference $w^{(k)} = u^{(k)} - u^{(k-1)}$ satisfies

$$\begin{aligned} L_0 w^{(k)} &= -Bw^{(k-1)}\\ Pw^{(k)}(0) &= K(1-P)w^{(k)}(0)\\ (1-P)w^{(k)}(\delta) &= 0 \ . \end{aligned}$$

The elliptic estimate (5.24) gives

$$\|w^{(k)}\|_{H^1_*(Y\times I)} \le c_4 \|Bw^{(k-1)}\|_{L^2(Y\times I)} \ .$$

If $\|B\|_{op} < 1/c_4$ then the iteration is a contraction and thus the sequence $u^{(k)}$ is Cauchy, converging to $u = \lim_{n\to\infty} u^{(k)}$ strongly in $H^1_*(Y \times I)$. Taking the limit of (5.43) shows that $(L_0+B)u = f$, and boundedness of the trace operator r_Y shows that u satisfies the boundary conditions (5.22)–(5.23). The elliptic estimates (5.24) satisfied by $u^{(k)}$ are preserved in the limit, so u satisfies

$$\|u\|_{H^1_*(Y\times I)} \le c_4(\|f - Bu\|_{L^2(Y\times I)} + \|\sigma\|_{H^{1/2}_*}),$$

from which (5.42) follows easily. ■

Observe that the proof of Lemma 5.6 relies on just two properties of the operator L_0; namely, the solvability of the problem (5.43) with boundary conditions (5.44), (5.45), and the elliptic estimate (5.24), which provides the size bound (5.41) for the perturbation B. This suggest that it should be possible to extend this existence result to more general operators $L = L_0 + B$, for which a strictly coercive estimate such as (5.24) can be established.

Consider, for example, the case where E has a complex structure $J : E \to E$, $J^2 = -1$, and A is a *normal* operator ($[A, A^*] = 0$), so $A = A_0 + JA_1$ where A_0, A_1 are self-adjoint and commuting, A_0 satisfies the spectral condition, and both commute with J. Then A admits an eigenfunction basis

$$A\phi_\alpha = (\lambda_\alpha + \mu_\alpha J)\phi_\alpha, \quad \lambda_\alpha, \mu_\alpha \in \mathbb{R}, \ \forall \alpha \in \Lambda,$$

and the results of this section extend with only minor modifications, provided the eigenvalues $\lambda_\alpha, \mu_\alpha$ satisfy the sectorial condition

$$\sup_{\alpha \in \Lambda} |\mu_\alpha|/(1 + |\lambda_\alpha|) < \infty \ .$$

Before stating the main uniqueness theorem, a definition of weak solution with boundary conditions is required. Note that although the definition is consistent with just L^2 boundary data, the regularity theorem 5.11 will require data in $H_*^{1/2}$.

Definition 5.7 *Suppose $L = \partial_x + A + B$ where A satisfies the spectral conditions (Definition 4.3) and $B : H_*^1(Y \times I) \to L^2(Y \times I)$ is a bounded linear operator for which there exists an L^2-adjoint $B^\dagger : H_*^1(Y \times I) \to L^2(Y \times I)$ such that:*

$$\int_{Y \times I} \langle Bu, v \rangle \, dv_Y dx = \int_{Y \times I} \langle u, B^\dagger v \rangle \, dv_Y dx \ , \quad \forall \ u, v \in H_*^1(Y \times I) \ . \tag{5.46}$$

Suppose further that $P = P_+ + P_{\hat{\Lambda}}$ (as in Lemma 5.4), that $K : (1-P)L^2(Y) \to PL^2(Y)$ is a bounded linear map with L^2-adjoint $K^\dagger : PL^2(Y) \to (1-P)L^2(Y)$, and let $\sigma \in PL^2(Y)$, $f \in L^2(Y \times I)$ be given. A weak solution *of the boundary value problem*

$$\begin{aligned} Lu &= f & (5.47)\\ Pu(0) &= \sigma + K(1-P)u(0) & (5.48)\\ (1-P)u(\delta) &= 0 & (5.49) \end{aligned}$$

is a field $u \in L^2(Y \times I)$ satisfying (with $L^\dagger = -\partial_x + A + B^\dagger$)

$$\int_{Y \times I} \langle u, L^\dagger \phi \rangle \, dv_Y dx = \int_{Y \times I} \langle f, \phi \rangle \, dv_Y dx + \oint_Y \langle \sigma, \phi(0) \rangle \, dv_Y \ , \tag{5.50}$$

for all $\phi \in H_*^1(Y \times I)$ satisfying the adjoint boundary conditions

$$\begin{aligned} (1 - P + K^\dagger P)\phi(0) &= 0 & (5.51)\\ P\phi(\delta) &= 0 \ . & (5.52) \end{aligned}$$

The boundary values $\phi(0), \phi(1)$ both lie in $H_*^{1/2}(Y)$ by the trace lemma, so the adjoint boundary conditions are well-defined on the space of test fields. Since C^∞ fields are dense in $H_*^1(Y \times I)$ and in $H_*^{1/2}(Y)$, to verify the weak equation (5.50) it suffices to test just with C^∞ fields ϕ; however the uniqueness argument of Lemma 5.10 requires the use of an H_*^1 test field.

The structure of the adjoint boundary condition (5.51) is explained by the next lemma, which is applied with $v = u(0) - \sigma$ and $H = L^2(Y)$.

Lemma 5.8 *If H is a Hilbert space, $P : H \to H$ is an orthogonal projection and $K : \ker P \to \operatorname{range} P$ is bounded, and if $v \in H$ satisfies*

$$\langle v, \phi \rangle_H = 0 \quad \forall\, \phi \in \ker(1 - P + K^\dagger P) \tag{5.53}$$

(where $K^\dagger$ is the adjoint of K in H), then $v \in \ker(P - K(1-P))$.

Proof: Since $\ker P = \operatorname{range}(1-P) \perp \operatorname{range} P$, it follows that P is self-adjoint and there is an orthogonal decomposition $H = (1-P)H \oplus PH$. Setting $\phi_1 = (1-P)\phi$, $\phi_2 = P\phi$, the condition $\phi = \phi_1 + \phi_2 \in \ker(1 - P + K^\dagger P)$ is equivalent to $\phi_1 = -K^\dagger \phi_2$, which exhibits $\ker(1 - P + K^\dagger P)$ as a graph over PH. Similarly decomposing $v = v_1 + v_2$, the condition $\langle v, \phi \rangle = 0$ is equivalent to $\langle v_2 - K v_1, \phi_2 \rangle = 0$. Since this holds for all $\phi_2 \in PH$, it follows that $v_2 = K v_1$, or equivalently, $v \in \ker(P - K(1-P))$. ■

In other words, $H = \ker(P - K(1-P)) \oplus \ker(1 - P + K^\dagger P)$ is an orthogonal splitting of H, where $P - K(1-P)$, $1 - P + K^\dagger P$ are projections, which are not orthogonal in general.

Using Lemma 5.8 we next show that an H^1 weak solution of the boundary value problem (5.47)-(5.49), in fact satisfies the equation (5.47) and the boundary conditions (5.48,5.49) in the strong sense:

Lemma 5.9 *If $u \in H^1_*(Y \times I)$ is a weak solution of (5.47) with the boundary conditions (5.48), (5.49), then u satisfies the equation $Lu = f$ in the sense of strong (H^1) derivatives, and the restrictions $u(0) = r_Y(u)$, $u(\delta) = r_{Y,\delta} u$ satisfy the boundary conditions (5.48),(5.49) in $L^2(Y)$. Conversely, if $u \in H^1_*(Y \times I)$ is a strong solution of (5.47,5.48,5.49), then u is also a weak solution.*

Proof: Integration by parts gives

$$\int_{Y \times I} \langle u, L^\dagger \phi \rangle = \int_{Y \times I} \langle Lu, \phi \rangle + \oint_Y \langle u(0), \phi(0) \rangle - \oint_Y \langle u(\delta), \phi(\delta) \rangle$$

for any $u, \phi \in H^1_*(Y \times I)$. Testing u with arbitrary $\phi \in C^\infty_c(Y \times I)$ shows that a H^1_* weak solution satisfies $Lu = f$ in the sense of strong derivatives. Comparing this formula with (5.50) shows also that

$$\oint_Y \langle u(0) - \sigma, \phi(0) \rangle = 0, \qquad \oint_Y \langle u(\delta), \phi(\delta) \rangle = 0,$$

for all $\phi(0), \phi(\delta)$ satisfying the adjoint boundary conditions (5.51), (5.52). Since $H^{1/2}_*(Y)$ is dense in $\ker(1 - P + K^\dagger P) \subset L^2(Y)$, Lemma 5.8 may be applied with $v = u(0) - \sigma$ to show the boundary condition (5.48) holds in $L^2(Y)$, and (5.49) follows similarly.

To show the converse, integration by parts again gives

$$\begin{aligned}
&\int_{Y \times I} \langle u, L^\dagger \phi \rangle - \langle Lu, \phi \rangle \\
&= \oint_Y \langle u(0), \phi(0) \rangle - \langle u(\delta), \phi(\delta) \rangle \\
&= \oint_Y \langle \sigma + (1+K)(1-P)u(0), \phi(0) \rangle - \langle Pu(\delta), \phi(\delta) \rangle \\
&= \oint_Y \langle \sigma, \phi(0) \rangle + \langle u(0), (1 - P + K^\dagger P)\phi(0) \rangle - \langle u(\delta), P\phi(\delta) \rangle \,,
\end{aligned}$$

and the final two terms vanish by the adjoint boundary conditions (5.51,5.52). ■

By solving an adjoint problem, we now show that weak solutions of (5.47)-(5.49) are unique.

LEMMA 5.10 *Let $u \in L^2(Y \times I)$ be a weak solution of the boundary value problem (5.47)-(5.49), with $\sigma \in L^2(Y)$ and $f \in L^2(Y \times I)$. Suppose that the operator $L = \partial_x + A + B$ satisfies the conditions of Lemma 5.6 and Definition 5.7, and the $L^2(Y \times I)$-adjoint $B^\dagger : H^1_*(Y \times I) \to L^2(Y \times I)$ satisfies*

$$c_4 \|B^\dagger\|_{op} < 1 \ . \tag{5.54}$$

Suppose also that the boundary operators $K, K^\dagger$ of Definition 5.7 satisfy

$$\begin{aligned} \|K(1-P)w\|_{H^{1/2}_*} &\leq k\,\|(1-P)w\|_{H^{1/2}_*} \ , && (5.55)\\ \|K^\dagger P w\|_{H^{1/2}_*} &\leq k\,\|Pw\|_{H^{1/2}_*} \ . && (5.56)\end{aligned}$$

for some constant $k \geq 0$ and all $w \in H^{1/2}_*(Y)$. Then u is unique.

PROOF: It will suffice to show that any weak solution $\tilde{u}$ of (5.47)–(5.49) with $\sigma = 0$, $f = 0$, must vanish. Consider the adjoint problem $L^\dagger\phi = \tilde{u}$ with boundary conditions (5.51),(5.52); writing $L^\dagger \phi = \tilde{u}$ as $(\partial_x - A - B^\dagger)\phi = -\tilde{u}$, we see that $L^\dagger$, $K^\dagger$ satisfy the conditions required by Lemma 5.6, since interchanging $A \leftrightarrow -A$ means replacing P by $1-P$, and perhaps changing a finite number of eigenfunctions in $\hat{\Lambda}$ (without modifying Λ_0). The elliptic estimate (5.42) does not depend on $\hat{\Lambda}$. Thus, by Lemma 5.6 there exists a solution $\phi \in H^1_*(Y \times I)$ of this boundary value problem. By construction, ϕ satisfies the boundary conditions required of test functions in (5.50), so testing $\tilde{u}$ in (5.50) with ϕ gives

$$\int_{Y\times I} |\tilde{u}|^2 = \int_{Y \times I} \langle \tilde{u}, L^\dagger \phi\rangle = 0$$

and thus $\tilde{u} = 0$. ■

It is easy to check that (5.56) is equivalent to requiring that $K : (1-P)H^{-1/2}_*(Y) \to PH^{-1/2}_*(Y)$ is bounded, with constant k.

We now obtain the main result on boundary regularity of weak solutions. Note that although the definition of weak solution assumes boundary data $\sigma \in L^2(Y)$ only, and uniqueness of weak solutions holds also in this generality, this condition is incompatible with regularity $u \in H^1_*(Y \times I)$, which would imply (by simple restriction) that $\sigma \in H^{1/2}_*(Y)$. However, some results for L^2 boundary conditions on domains with uniformly Lipschitz boundary are known [3, 21], so it is plausible that the results here could be extended.

THEOREM 5.11 *Suppose $u \in L^2(Y \times I)$ is a weak solution of the boundary value problem (5.47)-(5.49) with operator $L = \partial_x + A + B_0 + B_1$, where A satisfies the spectral conditions (Definition 4.3), B_0 satisfies the size condition (5.41) with L^2-adjoint $B_0^\dagger$ satisfying (5.54), and $B_1 : L^2(Y \times I) \to L^2(Y \times I)$*

is bounded. Further suppose the boundary operators $K, K^\dagger$ satisfy (5.55,5.56), and $\sigma \in PH_*^{1/2}(Y)$. Then $u \in H_*^1(Y \times I)$ (so u is a strong solution) and u satisfies the a priori *estimate*

$$\|u\|_{H^1_*(Y\times I)} \le \frac{c_4}{1 - c_4\|B_0\|_{op}} \left(\|f\|_{L^2(Y\times I)} + \|\sigma\|_{H_*^{1/2}} + \|B_1\|_{L^2\to L^2}\|u\|_{L^2(Y\times I)} \right) . \tag{5.57}$$

Proof: Since $\|B_1 u\|_{L^2(Y\times I)} \le \|B_1\|_{op}\|u\|_{L^2(Y\times I)}$, u satisfies $\tilde{L}u := (\partial_x + A + B_0)u = \tilde{f}$ where $\tilde{f} = f - B_1 u \in L^2(Y \times I)$. Lemma 5.6 constructs a solution $\bar{u} \in H_*^1(Y \times I)$ of $\tilde{L}\bar{u} = \tilde{f}$ satisfying the same boundary conditions, and it follows that $\bar{u}$ is also a weak solution. By the Uniqueness Lemma 5.10 we have $u = \bar{u}$ and thus $u \in H_*^1(Y \times I)$, as required. The estimate (5.42) of Lemma 5.6 (with f replaced by $\tilde{f}$) leads directly to (5.57). ■

6 Boundary regularity for first order systems

In this section we determine conditions under which the boundary regularity results of §5 apply to a first order equation of Dirac type (see (6.7), (6.9)) at the boundary, with suitable boundary operator, to show H^1 regularity of an L^2 weak solution. We assume M is a smooth manifold with compact boundary Y, $E \to M$ and $F \to M$ are real vector bundles over M with scalar products, and $\mathcal{L}$ is a first order elliptic operator on sections of E to sections of F, which in local coordinates x^j, $j = 1, \dots, n$ takes the form

$$\mathcal{L}u = a^j \partial_j u + bu , \tag{6.1}$$

where a^j, b are homomorphisms of E to F as before. Note that we do not assume Y to be connected. To apply the preceding results, the coefficients must satisfy the interior regularity conditions (3.4), and the boundary restrictions must be defined and satisfy the corresponding conditions in dimension $n-1$,

$$\begin{array}{rcl} a^j|_Y & \in & W^{1,(n-1)^*}(Y) \cap C^0, \; j = 1, \dots, n, \\ b|_Y & \in & L^{(n-1)^*}(Y) . \end{array} \tag{6.2}$$

Conditions (3.4,3.5) and (6.2) will be assumed throughout this section.

Remark 6.1 The H^s conditions

$$\begin{array}{rcl} a^j & \in & W^{s,2}_{\mathrm{loc}}(M) \cap C^0(M) , \\ b & \in & W^{s-1,2}_{\mathrm{loc}}(M) , \end{array} \tag{6.3}$$

where

$$\begin{array}{lll} s = n/2 & \text{for} & n \ge 4 , \\ s > 3/2 & \text{for} & n = 3 , \\ s = 3/2 & \text{for} & n = 2 , \end{array} \tag{6.4}$$

imply the interior (3.4) and boundary (6.2) coefficient regularity conditions, through the Sobolev embedding and trace theorems [28]. The C^0 condition in (6.3) is superfluous for $n = 2, 3$.

Let $x = x^n$ be a boundary coordinate, defining a tubular neighbourhood $Y \times [0,1] \subset M$ of Y with local coordinates $(y^i, x) \in Y \times [0,1]$ (where we identify Y with $Y \times \{0\}$). Let dv_M be a volume measure on M, and define $dv_Y = (-1)^{n-1}\partial_x \lrcorner dv_M|_Y$ on $Y \times \{0\}$. The local coordinate integration factor γ is defined in $Y \times [0,1]$ by $dv_M = \gamma\, dy\, dx$, where $dy\, dx$ is coordinate Lebesgue measure. We assume the local coordinate condition[5]

$$\gamma \in (W^{1,n^*} \cap C^0)(Y \times [0,1]) \; . \tag{6.5}$$

In order to directly apply the results of the previous section, we assume that $\gamma = \gamma(0)$ is independent of x in $Y \times [0,1]$, so $dv_M = dv_Y\, dx$. This entails no loss of generality, as the x-dependence of γ in the integral form (6.15) of the equation can be absorbed through a rescaling of the coefficients a^i, b. Since the restrictions to $Y \times \{0\}$ are unchanged, this does not affect the boundary operator A.

To minimize confusion with the outer unit normal $n = -\partial_x$ we set

$$\nu := {}^t a^n : F \to E \; , \tag{6.6}$$

where ${}^t a^n$ is the transpose of a^n with respect to the metrics on E, F, and for simplicity we also assume

$${}^t\nu = \nu^{-1} \; . \tag{6.7}$$

This is satisfied by the Dirac operator. Equation (6.7) can always be achieved locally by replacing $\mathcal{L}$ with $p\mathcal{L}$, for suitable frame change $p : F \to F$, and globally on Y if F is trivial or more generally a Dirac bundle (i.e. carries a representation of the Clifford bundle of Y). It would be interesting to know if these examples cover all first order elliptic operators. Note that such a pre-multiplication by p does not affect the values of $\tilde{a}^i$, $\tilde{b}$, where we define

$$\tilde{a}^i := \nu a^i|_Y, \; i = 1, \ldots, n-1, \quad \tilde{b} := \nu b|_Y \; . \tag{6.8}$$

By extending independent of x we regard $\tilde{a}^i, \tilde{b}$ as defined on $Y \times [0,1]$. We assume the important boundary symmetry condition

$$\tilde{a}^i = -\,{}^t\tilde{a}^i, \quad i = 1, \ldots, n-1 \; , \tag{6.9}$$

where the transpose ${}^t\tilde{a}^i$ is taken with respect to the inner product on E. We encapsulate these conditions into a definition.

Definition 6.2 *A first order system (6.1) is of* boundary Dirac type *if the local coefficients a^j satisfy the conditions (6.7,6.9) in some neighbourhood of the boundary.*

Using (6.9) we define the boundary operator

$$Au := \sum_{i=1}^{n-1} \tilde{a}^i \partial_i u + \tilde{a}_0 u + \tilde{b}_0 u \; , \tag{6.10}$$

[5] As in §3, this means that we can cover Y by a finite number of smooth coordinate charts $\mathscr{O}_\alpha$ so that γ has the stated regularity in the local coordinates on $\mathscr{O}_\alpha \times [0,1]$.

where

$$\tilde{a}_0 = \tfrac{1}{2}\sum_{i=1}^{n-1}\left(\partial_i \tilde{a}^i + \tilde{a}^i \partial_i \log\gamma\right) , \tag{6.11}$$

and $\tilde{b}_0 = b_0|_Y$ for some symmetric endomorphism b_0 of E. We require that $b_0 \in L^{n^*}(Y \times I))$ and $\tilde{b}_0 \in L^{(n-1)^*}(Y)$, compare (3.4), (6.2). Then A is formally self-adjoint ($A^\dagger = A$) on the bundle $E|_Y$ over Y, with respect to the measure dv_Y.

Note that the choice of zero-order term b_0 gives some freedom in the definition of the boundary operator A, which is thus not uniquely determined by $\mathcal{L}$. Near the boundary, $\mathcal{L}u = {}^t\nu(\partial_x + A + B)$, which may be expressed as

$$\tilde{L}u := (\partial_x + A + B)u = \tilde{f} , \tag{6.12}$$

where $\tilde{L}u = \nu\mathcal{L}u$, $\tilde{f} = \nu f$, and B denotes the difference

$$Bu = \sum_{i=1}^{n-1}(\nu a^i - \tilde{a}^i)\partial_i u + (\nu b - \tilde{a}_0 - \tilde{b}_0)u . \tag{6.13}$$

By Corollary 4.4, under the above conditions, A will satisfy the spectral condition 4.3. Denote the eigenvalue index set by Λ and fix a cutoff κ, as in §5. Similarly let $\Lambda^+ = \{\alpha \in \Lambda : \lambda_\alpha \geq \kappa\}$, fix some subset $\hat{\Lambda} \subset \{\alpha \in \Lambda : |\lambda_\alpha| < \kappa\}$ and let $P = P_{\Lambda^+} + P_{\hat{\Lambda}}$ be the associated spectral projection operator. Note that Theorem 3.7 (see (3.22)) shows that the H^s_* norms defined using A will be equivalent to the corresponding H^s norms defined on Y and $Y \times I$, at least for $s = 0, 1$; it seems likely that this will hold, by interpolation, for all $s \in [0, 1]$. If the coefficient regularity allows an H^{k+1} elliptic estimate (*cf.* Theorem 3.8) then this should extend to $s \in [0, k+1]$.

The definition of weak solution on a manifold with boundary which we are about to give is slightly simpler than the tubular neighbourhood Definition 5.7 of §5, since the conditions (5.49), (5.52) may be imposed by localising with a boundary cutoff function; see the proof of Theorem 6.4. Let $H^1_c(M)$ denote the dense subspace of $H^1(M)$ consisting of functions of compact support, where we recall that because M is a manifold with boundary $Y = \partial M$, $H^1_c(M)$ includes functions which are non-zero on Y. As in §5, the boundary condition is expressed using a positive spectrum projection $P : L^2(Y) \to L^2(Y)$ and a bounded linear map $K : (1-P)L^2(Y) \to PL^2(Y)$ and its $L^2(Y)$ adjoint $K^\dagger$.

Definition 6.3 *Let $f \in L^2_{\text{loc}}(M)$ and $\sigma \in PL^2(Y)$ be given. A section $u \in L^2_{\text{loc}}(M)$ is a* weak solution of $\mathcal{L}u = f$ with boundary condition

$$Pu_0 = \sigma + K(1-P)u_0 , \tag{6.14}$$

if

$$\int_M \langle u, \mathcal{L}^\dagger\phi\rangle \, dv_M = \int_M \langle f, \phi\rangle \, dv_M + \oint_Y \langle \sigma, \nu\phi_0\rangle \, dv_Y , \tag{6.15}$$

for all $\phi \in H^1_c(M)$ satisfying the boundary condition

$$(1 - P + K^\dagger P)(\nu\phi_0) = 0 , \tag{6.16}$$

where $\mathcal{L}^\dagger$ is the L^2 adjoint given by (3.7).

Note we are using the notation u_0, ϕ_0, *etc.*, to denote the restriction (trace) on the boundary Y. The additional term ν in (6.15,6.16) (cf. (5.50,5.51)) arises from the relation $\mathcal{L} = {}^t\nu(\partial_x + A + B)$ between $\mathcal{L}$ and the boundary form $\partial_x + A + B$ used in §5.

The boundary condition (6.14) restricts $u_0 = u|_Y$ to lie in the affine subspace of $L^2(Y)$ given by the graph of $x \mapsto \sigma + Kx$ over the negative spectrum subspace $x \in (1-P)L^2(Y)$. It will be useful to re-express (6.14) as

$$\mathcal{K}u_0 = \sigma \ , \tag{6.17}$$

where we have introduced the operator $\mathcal{K}$ on $L^2(Y)$,

$$\mathcal{K} := P - K(1-P) \ , \tag{6.18}$$

and likewise to re-express the "adjoint" boundary condition (6.16) as

$$\mathcal{K}^\dagger \phi_0 = 0 \ , \tag{6.19}$$

where we define

$$\mathcal{K}^\dagger := \nu^{-1}(1 - P + K^\dagger P)\nu \ . \tag{6.20}$$

The next result generalizes the interior weak-strong Theorem 3.7 to boundary value problems.

Theorem 6.4 *Suppose $\mathcal{L}$ and A satisfy the conditions (3.4,3.5,6.2,6.5,6.7,6.9), and suppose $\sigma \in PH^{1/2}_*(Y)$, $f \in L^2_{\mathrm{loc}}(M)$. Further suppose $K : (1-P)L^2(Y) \to PL^2(Y)$ is bounded linear and satisfies (5.55), with L^2 adjoint $K^\dagger$ satisfying (5.56). Assume $u \in L^2_{\mathrm{loc}}(M)$ is a weak solution of $\mathcal{L}u = f$ with the boundary condition $\mathcal{K}u_0 = \sigma$ (6.17). Then $u \in H^1_{\mathrm{loc}}(M)$ and u is a strong solution. Moreover, there are constants $\delta \in (0,1]$, c_5, depending only on κ, k and a^i, b, and intervals $I' = [0, \delta/2]$, $I = [0, \delta]$ such that*

$$\|u\|_{H^1(Y\times I')} \le c_5(\|f\|_{L^2(Y\times I)} + \|\sigma\|_{H^{1/2}_*(Y)} + \|u\|_{L^2(Y\times I)}) \ . \tag{6.21}$$

Proof: Theorem 3.7 ensures $u \in H^1_{\mathrm{loc}}(\mathring{M})$, where $\mathring{M}$ is the interior of M, so it suffices to consider u compactly supported in $Y \times [0, \delta)$, for any choice of $\delta \in (0,1)$. In particular, because u then vanishes near $Y \times \{\delta\}$, it follows from (6.14) that u is a weak solution with boundary conditions, in the sense of Definition 5.7.

It will suffice to show, for a sufficiently small choice of $\delta > 0$, that we may decompose $B = B_0 + B_1$ into pieces satisfying the size conditions of Theorem 5.11. Write $B = \beta^i \partial_i + \beta$ where

$$\beta^i(y,x) = (a^n(y,x))^{-1}a^i(y,x) - (a^n(y,0))^{-1}a^i(y,0), \quad i = 1, \dots, n-1,$$

so $\beta^i \in W^{1,n^*} \cap C^0$ and $\beta^i(y,0) = 0$. Since the constant c_4 of Lemma 5.4 depends only on $\theta_0 < 1$, $\ell = \kappa\delta$ and the constant k of (5.55,5.56), it is bounded uniformly in $\delta \le 1$. Consequently for any $\epsilon > 0$ there is $\delta_0 > 0$ such that $c_4\|\beta^i\|_{L^\infty(Y\times[0,\delta])} < \epsilon$ for all $\delta \le \delta_0$.

Likewise, since $\gamma \in W^{1,n^*}$ (6.5), we have $\beta \in L^{n^*}$ and there is a decomposition $\beta = \beta_0 + \beta_1$ with $c_4\, C_S\, \|\beta_0\|_{L^{n^*}(Y\times[0,1])} \le \epsilon$, where C_S is the Sobolev constant on $Y \times [0,1]$, and $\beta_1 \in L^\infty$. Then $B_0 = \beta^i \partial_i + \beta_0$ satisfies (5.41), as does $B_0^\dagger$ (possibly after decreasing δ), and $B_1 = \beta_1$ is bounded on L^2. Theorem 5.11 now applies and shows $u \in H^1(Y \times [0,\delta])$, since $H^1(Y\times I) = H^1_*(Y \times I)$ as remarked above. The elliptic estimate (6.21) follows by applying (5.57) to $\tilde u = \chi u$, where $\chi = \chi(x)$ is a cutoff function, $\chi(x) = 1$ for $0 \le x \le \delta/2$, $\chi(x) = 0$ for $x \ge \frac{3}{4}\delta$. ■

COROLLARY 6.5 *Suppose $\mathcal{L}, A, K, \mathcal{K}$ satisfy the conditions of Theorem 6.4. Then for all $u \in H^1_{\mathrm{loc}}(M)$ we have the boundary estimate*

$$\|u\|_{H^1(Y\times I')} \le c_5(\|\mathcal{L}u\|_{L^2(Y\times I)} + \|\mathcal{K}u_0\|_{H^{1/2}_*(Y)} + \|u\|_{L^2(Y\times I)}) \, . \tag{6.22}$$

PROOF: If $u \in H^1_{\mathrm{loc}}(M)$ then $f := \mathcal{L}u \in L^2_{\mathrm{loc}}(M)$, $\sigma := \mathcal{K}u_0 \in H^{1/2}_*(Y)$, and u is a strong solution. Since integration by parts may be applied to show u satisfies the weak equation (6.15), Theorem 6.4 applies and gives (6.22). ■

A bootstrap argument, slightly more complicated than that used for the interior bounds Theorem 3.8, leads to higher, H^{1+k}, regularity. Rather than stating complicated conditions for general k, we describe the details only for the case $k = 1$ ($u \in H^2$). The coefficient regularity conditions are most likely not optimal.

THEOREM 6.6 *In the setting of Theorem 6.4 let $u \in H^1(Y \times I)$ be the solution and suppose the following additional regularity conditions are satisfied,*

$$a^j\,, b \in W^{1,\infty}(Y \times I)\,, \tag{6.23}$$

$$\gamma \in W^{2,n^*}(Y \times [0,1])\,, \tag{6.24}$$

$$f \in H^1(Y\times I), \quad \sigma \in H^{3/2}_*(Y)\,, \tag{6.25}$$

$$[A,K](1-P) : (1-P)H^{1/2}_*(Y) \to PH^{1/2}_*(Y) \text{ is bounded.} \tag{6.26}$$

Then there exists $\delta'' \le \delta/2$ such that $u \in H^2(Y \times I'')$, where $I'' = [0,\delta'']$, and there is a constant c_6 depending on the coefficient bounds (6.23)-(6.26), such that

$$\|u\|_{H^2(Y\times I'')} \le c_6(\|f\|_{H^1(Y\times I)} + \|\sigma\|_{H^{3/2}_*(Y)} + \|u\|_{L^2(Y\times I)}) \, . \tag{6.27}$$

REMARK 6.7 For the APS and chiral boundary conditions (2.13), (2.18), A commutes with K and thus (6.26) is trivially satisfied.

PROOF: The idea is to show that Au satisfies a similar boundary value problem. For convenience, let $\tilde H^1_0(Y \times I)$, $I = [0,\delta]$, denote the H^1 completion of the C^∞ functions of compact support in $Y \times [0,\delta)$, where δ is the constant of Theorem

6.4. In particular, functions in $\tilde{H}^1_0(Y\times I)$ have vanishing trace on $Y\times\{\delta\}$. For any $v,\psi\in C^\infty_c(Y\times[0,\delta))$, we have the identity

$$\begin{aligned}\int_{Y\times I}\langle Av,\tilde{L}^\dagger\psi\rangle\,dx\,dv_Y &= \int_{Y\times I}\Big(\langle[\tilde{L},A]v,\psi\rangle+\langle\tilde{L}v,A\psi\rangle\Big)\,dx\,dv_Y\\ &\quad+\oint_Y\langle v_0,A\psi_0\rangle\,dv_Y\ . \end{aligned}\qquad(6.28)$$

Since A is formally self-adjoint with respect to dv_Y, this formula follows by direct calculation. The terms with $\tilde{L}^\dagger\psi,\tilde{L}v$ are well-defined for $v,\psi\in\tilde{H}^1_0(Y\times I)$. Since $v_0,\psi_0\in H^{1/2}_*(Y)$ by Lemma 5.1, the boundary integral extends also by writing it as

$$\oint_Y\langle v_0,A\psi_0\rangle\,dv_Y=\oint_Y\langle Jv_0,J^{-1}A\psi_0\rangle\,dv_Y\ .\qquad(6.29)$$

Here $J=(1+|A|)^{1/2}$ is defined as $Ju=\sum_{\alpha\in\Lambda}u_\alpha(1+|\lambda_\alpha|)^{1/2}\psi_\alpha$, with $u=\sum_{\alpha\in\Lambda}u_\alpha\psi_\alpha$; J^{-1} is defined similarly. This shows that

$$\left|\oint_Y\langle v_0,A\psi_0\rangle\,dv_Y\right|\le c\|v_0\|_{H^{1/2}_*(Y)}\|\psi_0\|_{H^{1/2}_*(Y)},$$

for all $v_0,\psi_0\in H^{1/2}_*(Y)$, where the constant c is determined by A.

More care is required to control the commutator $[\tilde{L},A]$, which takes the form

$$[\tilde{L},A]v=\alpha^{ij}\partial^2_{ij}v+\alpha^i_1\partial_iv+\alpha_2v$$

where

$$\begin{aligned}\alpha^{ij}&=[\nu a^i,\tilde{a}^j],\\ \alpha^i_1&=\nu a^j\partial_j\tilde{a}^i-\tilde{a}^j\partial_j(\nu a^i)+[\nu a^i,\tilde{a}_0],\\ \alpha_2&=\nu a^i\partial_i\tilde{a}_0-\tilde{a}^i\partial_i(\nu b)\ .\end{aligned}$$

Observe that $\alpha^{ij}|_Y=0$. The coefficient conditions (6.23) and $v\in H^1$ ensure that $\alpha^i_1\partial_iv$ is in L^2 and hence may be combined with the source term $A\tilde{f}$. Likewise (6.23) ensures α_2v is bounded in L^2.

Let $\lambda_0\in\mathbb{R}\backslash\Lambda$, so $A-\lambda_0$ has trivial kernel and satisfies elliptic estimates $\|v\|_{H^{s+1}}\le c\|(A-\lambda_0)v\|_{H^s}$ for $s=0,1$ at least. Since A is self-adjoint and elliptic, the cokernel of $A-\lambda_0$ is also trivial, so $A-\lambda_0:H^{s+1}\to H^s$ is invertible. Now decompose $\alpha^{ij}\partial^2_{ij}v=B_2Av+B_3v$ where

$$B_2=\alpha^{ij}\partial^2_{ij}(A-\lambda_0)^{-1}$$

is bounded from $H^1\to L^2$, and

$$B_3=-\lambda_0\alpha^{ij}\partial^2_{ij}(A-\lambda_0)^{-1}$$

is also bounded from $H^1\to L^2$. Note that by perhaps decreasing δ we may ensure that B_2 and $B_2^\dagger$ satisfy a smallness condition similar to (5.41).

Direct calculation (noting that $[A,P]=0$) establishes the boundary formula

$$\begin{aligned}\oint_Y \langle (1+K)(1-P)v_0, A\psi_0\rangle\, dv_Y &= \oint_Y \langle Av_0, (1-P+K^\dagger P)\psi_0\rangle\, dv_Y \\ &\quad + \oint_Y \langle [A,K](1-P)v_0, \psi_0\rangle\, dv_Y \qquad (6.30)\end{aligned}$$

for all $v_0, \psi_0 \in H^{1/2}_*(Y)$, since K satisfies (6.26) and (5.55) by assumption.

Now $u \in \tilde{H}^1_0(Y \times I)$ satisfies $\tilde{L}u = \tilde{f}$ and $u_0 = \sigma + (1+K)(1-P)u_0$. Substituting u for v in (6.28) and using these relations shows that u satisfies

$$\begin{aligned}&\int_{Y\times I} \langle Au, (\tilde{L} - B_2)^\dagger \psi\rangle\, dx\, dv_Y = \\ &\qquad \int_{Y\times I} \langle A\tilde{f} + B_3 u + \alpha_1^i \partial_i u + \alpha_2 u, \psi\rangle\, dx\, dv_Y \\ &\qquad + \oint_Y \langle A\sigma + [A,K](1-P)u_0, \psi_0\rangle\, dv_Y + \oint_Y \langle Au_0, (1-P+K^\dagger P)\psi_0\rangle\, dv_Y\ .\end{aligned}$$

In particular, if $\psi_0 \in \ker(1-P+K^\dagger P) \cap H^{1/2}_*(Y)$ then $w = Au \in L^2(Y \times I)$ satisfies

$$\int_{Y\times I} \langle w, (\tilde{L} - B_2)^\dagger \psi\rangle\, dx\, dv_Y = \int_{Y\times I} \langle \tilde{f}_1, \psi\rangle\, dx\, dv_Y + \oint_Y \langle \sigma_1, \psi_0\rangle\, dv_Y,$$

for all $\psi \in H^1(Y \times I)$ such that $\psi \in \ker(1-P+K^\dagger P)$, where

$$\begin{aligned}\tilde{f}_1 &= A\tilde{f} + B_3 u + \alpha_1^i \partial_i u + \alpha_2 u \in L^2, \\ \sigma_1 &= A\sigma + [A,K](1-P)u_0 \in H^{1/2}_*(Y)\ .\end{aligned}$$

In other words, $w \in L^2(Y \times I)$ is a weak solution of the problem

$$\begin{aligned}(\tilde{L} - B_2)w &= \tilde{f}_1, \\ Pw_0 &= \sigma_1 + K(1-P)w_0\ .\end{aligned}$$

By shrinking the boundary layer we may assume $\|B_2\|_{H^1\to L^2}$ and $\|B_2^\dagger\|_{H^1\to L^2}$ are sufficiently small that the conditions of Theorem 6.4 are met, so $w = Au \in H^1(Y \times [0,\delta])$. The equation now gives $\partial_x u = \tilde{f} - Au - Bu \in H^1$ and thus $u \in H^2(Y \times [0,\delta])$. ■

7 Fredholm properties on compact manifolds

The interior and boundary estimates of § 6 lead to solvability (Fredholm) results, by standard arguments. The main interest lies in identifying the cokernel, and we give a simple necessary and sufficient condition for solvability, in Theorem 7.3. This section treats only compact manifolds, leaving the more difficult case of non-compact manifolds to the following section. Because more detailed descriptions are given in § 8, some of the arguments are only briefly summarised here.

Throughout this section we assume the coefficients a^j , $j = 1, \ldots, n$ and b of $\mathcal{L}$ satisfy the conditions of § 6, namely (3.4),(3.5), (6.2), (6.5), (6.7), (6.9); $\mathcal{L}^\dagger$ is given by (3.7), and the boundary operators $K, K^\dagger$ satisfy (5.55,5.56), where $P = P_\Lambda + P_{\hat{\Lambda}}$ is a positive spectrum projection of A (6.10), and $\mathcal{K}$, $\mathcal{K}^\dagger$ are defined by (6.18,6.20).

Recall the Sobolev space $H^1(M)$ of sections of E over M is defined by the norm (3.17)

$$\|u\|^2_{H^1(M)} = \int_M (|\nabla u|^2 + |u|^2)\, dv_M \;, \tag{7.1}$$

where lengths are measured using the metric $\langle\ ,\ \rangle$ on E and a fixed smooth background metric $\mathring{g}$ on TM, and the connection ∇ satisfies (3.18,3.19). Note again that ∇ need not be compatible with the metric on E, and the space $H^1(M)$ is independent of the choice of ∇.

The following basic elliptic estimate extends (3.22) of Theorem 3.7 to manifolds with boundary, using the boundary neighbourhood estimate (6.21) of Theorem 6.4.

Proposition 7.1 *There is a constant $C > 0$ depending on a^j , b, Γ and $\mathcal{K}$ such that for all $u \in H^1(M)$,*

$$\|u\|_{H^1(M)} \le C(\|\mathcal{L}u\|_{L^2(M)} + \|\mathcal{K}u_0\|_{H^{1/2}_*(Y)} + \|u\|_{L^2(M)}) \;. \tag{7.2}$$

Proof: The argument used in Theorem 3.7 to prove the interior estimate (3.22) may be applied using Theorem 6.4, estimate (6.21), to estimate $\|u_\alpha\|_{H^1(U_\alpha)}$ over boundary neighbourhoods U_α. The remaining details are unchanged. ■

Theorem 7.2 *The linear operator*

$$(\mathcal{L}, \mathcal{K}) : H^1(M) \to L^2(M) \times PH^{1/2}_*(Y) \tag{7.3}$$

is semi-Fredholm *(i.e. has finite dimensional kernel and closed range).*

Proof: Suppose $\{u_k\}_1^\infty$ is a sequence in $\ker(\mathcal{L}, \mathcal{K})$, normalised by $\|u_k\|_{H^1(M)} = 1$. By Rellich's lemma there is a subsequence (which we also denote u_k) which converges strongly in $L^2(M)$, to $\bar{u} \in L^2(M)$ say. The elliptic estimate applied to the differences $u_j - u_k$ shows that the sequence is Cauchy in $H^1(M)$ and thus converges strongly to $\bar{u} \in H^1(M)$. Since (7.3) is bounded, it follows that $\bar{u} \in \ker(\mathcal{L}, \mathcal{K})$, so the unit ball in the kernel is compact and hence the kernel is finite dimensional.

To show the range is closed, let $\mathring{H}^1(M)$ be the finite codimension complement of $\ker(\mathcal{L}, \mathcal{K})$ in $H^1(M)$ defined by the condition

$$\int_M (\langle \nabla u, \nabla\phi\rangle + \langle u, \phi\rangle)\, dv_M = 0 \quad \forall\ \phi \in \ker(\mathcal{L}, \mathcal{K}).$$

A Morrey-type argument by contradiction using (7.2) shows there is a constant $C > 0$ such that for all $u \in \mathring{H}^1(M)$,

$$C^{-1} \int_M |u|^2\, dv_M \le \int_M |\mathcal{L}u|^2\, dv_M + \oint_Y |J\mathcal{K}u_0|^2\, dv_Y \;, \tag{7.4}$$

where $J = (1+|A|)^{1/2}$. Now suppose $\{u_k\}_1^\infty \subset H^1(M)$ is such that $\mathcal{L}u_k = f_k \to f \in L^2(M)$ and $\mathcal{K}(u_k)_0 = s_k \to \sigma \in PH_*^{1/2}(Y)$. (Note that by the definition (6.18) of $\mathcal{K}$, the range of $\mathcal{K}$ is a subspace of $PH_*^{1/2}(Y)$). Since the kernel is finite dimensional we may normalise $u_k \in \mathring{H}^1(M)$, and then (7.4) and (7.2) show that $\{u_k\}_1^\infty$ is bounded in $H^1(M)$. It then follows as above that there is a subsequence converging strongly in $H^1(M)$ to $\bar{u}$, and that $\mathcal{L}\bar{u} = \lim_{k\to\infty} f_k = f$ and $\mathcal{K}\bar{u}_0 = \lim_{k\to\infty} s_k = \sigma$, so the range of $(\mathcal{L}, \mathcal{K})$ is closed. ∎

The general boundary value problem

$$\begin{cases} \mathcal{L}u &= f \quad \text{in } M \\ \mathcal{K}u_0 &= \sigma \quad \text{on } Y \end{cases} \tag{7.5}$$

is solvable for $u \in H^1(M)$ provided (f, σ) satisfies the condition (7.6) of the following main result.

THEOREM 7.3 *$(f,\sigma) \in L^2(Y) \times PH_*^{1/2}(Y)$ lies in the range of $(\mathcal{L},\mathcal{K})$ (that is, (7.5) admits a solution $u \in H^1(M)$), if and only if*

$$\int_M \langle f, \phi\rangle \, dv_M + \oint_Y \langle \sigma, \nu\phi_0\rangle \, dv_Y = 0 \quad \forall\ \phi \in \ker(\mathcal{L}^\dagger, \mathcal{K}^\dagger)\ . \tag{7.6}$$

PROOF: If $u \in H^1(M)$ satisfies $\mathcal{L}u = f$ and $\mathcal{K}u_0 = \sigma$ then u is also a weak solution. Condition (7.6) then follows directly from the definition 6.3 of weak solution, hence (7.6) is a necessary condition for solvability.

To establish the converse, consider first the case $\sigma = 0$. Thus we suppose $f \in L^2(M)$ satisfies $\int_M \langle f, \phi\rangle \, dv_M = 0$ for all $\phi \in \ker(\mathcal{L}^\dagger, \mathcal{K}^\dagger)$, and we must find $u \in H^1(M)$ satisfying $\mathcal{L}u = f$, $\mathcal{K}u_0 = 0$.

By Lemma 5.1 the trace map $r_Y : u \mapsto u_0$ is bounded, hence

$$\begin{aligned} \mathring{H}^1_{\mathcal{K}} \ := \ & \{u \in H^1(M) : \mathcal{K}u_0 = 0, \text{ and} \\ & \int_M (\langle \nabla u, \nabla\phi\rangle + \langle u, \phi\rangle) \, dv_M = 0 \quad \forall\ \phi \in \ker(\mathcal{L},\mathcal{K})\} \end{aligned} \tag{7.7}$$

is a closed subspace of $H^1(M)$. The argument of Theorem 7.2 (ii) shows there is a constant C such that

$$\int_M (|\nabla u|^2 + |u|^2) \, dv_M \le C \int_M |\mathcal{L}u|^2 \, dv_M \tag{7.8}$$

for all $u \in \mathring{H}^1_{\mathcal{K}}(M)$. In particular, $\int_M |\mathcal{L}u|^2 \, dv_M$ is strictly coercive on $\mathring{H}^1_{\mathcal{K}}$, so the Lax-Milgram lemma gives $u \in \mathring{H}^1_{\mathcal{K}}$ satisfying

$$\int_M \langle f, \mathcal{L}\phi\rangle \, dv_M = \int_M \langle \mathcal{L}u, \mathcal{L}\phi\rangle \, dv_M$$

for all $\phi \in \mathring{H}^1_{\mathcal{K}}$. This equality also holds if $\phi \in \ker(\mathcal{L},\mathcal{K})$, so $\Psi = \mathcal{L}u - f$ satisfies

$$\int_M \langle \Psi, \mathcal{L}\phi\rangle \, dv_M = 0 \quad \forall\ \phi \in H^1(M)\ ,\ \mathcal{K}\phi_0 = 0\ . \tag{7.9}$$

Lemma 5.8 and the identity

$$\int_M \langle \Psi, \mathcal{L}\phi\rangle \, dv_M = \int_M \langle \mathcal{L}^\dagger \Psi, \phi\rangle \, dv_M - \oint_Y \langle \nu\Psi_0, \phi_0\rangle \, dv_M \tag{7.10}$$

show that (7.9) is the weak form of the adjoint problem

$$\mathcal{L}^\dagger \Psi = 0, \quad \mathcal{K}^\dagger \Psi_0 = 0 \, . \tag{7.11}$$

By (3.7), $\mathcal{L}^\dagger$ is elliptic with boundary representation

$$\mathcal{L}^\dagger = -\nu(\partial_x + \hat{A} + \hat{B}),$$

where $\hat{A} = -\nu^{-1} A \nu$ since $A^\dagger = A$, and $\hat{B} = -\nu^{-1} B^\dagger \nu$. By (6.8,6.9) the leading terms in $\hat{A}$ are $\tilde{a}^i \partial_i$ so $\hat{A}$ is elliptic on Y, and self-adjoint by (6.7). Since $\hat{A}(\nu^{-1}\phi_\alpha) = -\lambda_\alpha \nu^{-1}\phi_\alpha$ if $A\phi_\alpha = \lambda_\alpha \phi_\alpha$, we see that $\hat{A}$ satisfies the spectral conditions, and $\operatorname{spec} \hat{A} = -\operatorname{spec} A$. (Note that in the usual case of Dirac operators, $\hat{A} = A$ and the spectrum is symmetric). Now $\hat{P} := 1 - \nu^{-1} P \nu$ is a positive eigenspace projector for $\hat{A}$, with eigenvalues $-\lambda_\alpha$ for $\alpha \in \Lambda^- \cup (\Lambda^0 \backslash \hat{\Lambda})$, and the boundary operator satisfies

$$\mathcal{K}^\dagger \Psi_0 = (\hat{P} + \nu^{-1} K^\dagger \nu (1 - \hat{P}))\Psi_0 \, . \tag{7.12}$$

Since $\hat{K} = -\nu^{-1} K^\dagger \nu$ maps negative eigenvectors (of $\hat{A}$) to positive eigenvectors, it follows that $\mathcal{K}^\dagger \Psi_0 = 0$ is an elliptic boundary condition for $\mathcal{L}^\dagger$. The boundedness conditions (5.55,5.56) for $\hat{K}$ follow from the corresponding conditions for K.

Since $(\mathcal{L}^\dagger, \mathcal{K}^\dagger)$ is elliptic and satisfies the conditions for Theorem 6.4, we conclude that $\Psi \in H^1(M)$ and Ψ satisfies the strong form (7.11).

Since $\Psi \in \ker(\mathcal{L}^\dagger, \mathcal{K}^\dagger)$, assumption (7.6) with $\sigma = 0$ gives

$$\int_M \langle f, \Psi\rangle \, dv_M = 0.$$

By construction $\mathcal{K} u_0 = 0$, so we may use u as a test function in the weak form (7.9) of the equation satisfied by Ψ, giving

$$\int_M \langle \mathcal{L} u, \Psi\rangle \, dv_M = 0.$$

It follows from $\Psi = \mathcal{L} u - f$ that $\Psi = 0$ and thus u is the required solution.

Now consider the case $\sigma \neq 0$. By Lemma 5.1 there is an extension $v = e_Y(\sigma) \in H^1(M)$ supported in a neighbourhood of Y such that $v_0 = \sigma$, $\|v\|_{H^1(M)} \le 2\|\sigma\|_{H^{1/2}_*}$. Let $\tilde{f} = f - \mathcal{L} v$ and consider the equation

$$\mathcal{L}\tilde{u} = \tilde{f}, \quad \mathcal{K}\tilde{u}_0 = 0 \, . \tag{7.13}$$

The previous case shows there is a solution provided $\tilde{f}$ satisfies

$$\int_M \langle \tilde{f}, \psi\rangle \, dv_M = 0 \quad \forall \, \psi \in \ker(\mathcal{L}^\dagger, \mathcal{K}^\dagger) \subset H^1(M).$$

Now (7.10) shows that for all $\psi \in \ker(\mathcal{L}^\dagger, \mathcal{K}^\dagger)$,

$$\begin{aligned}
\int_M \langle \tilde{f}, \psi \rangle \, dv_M &= \int_M \langle f, \psi \rangle \, dv_M - \int_M \langle v, \mathcal{L}^\dagger \psi \rangle \, dv_M + \oint_Y \langle v_0, \nu\psi_0 \rangle \, dv_Y \\
&= \int_M \langle f, \psi \rangle \, dv_M + \oint_Y \langle \sigma, \nu\psi_0 \rangle \, dv_Y \; .
\end{aligned}$$

Thus if (7.6) is satisfied then there exists a solution $\tilde{u}$ of (7.13), and then $u = \tilde{u} + v$ is the required full solution. This establishes sufficiency for the condition (7.6). ■

We note two important consequences of Theorems 7.2, 7.3.

Corollary 7.4 *(7.5) admits a solution for all $(f, \sigma) \in L^2(M) \times PH_*^{1/2}(Y)$ if and only if* $\ker(\mathcal{L}^\dagger, \mathcal{K}^\dagger) = \{0\}$.

Corollary 7.5 $(\mathcal{L}, \mathcal{K}) : H^1(M) \to L^2(M) \times PH_*^{1/2}(Y)$ *is Fredholm.*

Proof: The argument of Theorem 7.3 shows that $(\mathcal{L}^\dagger, \mathcal{K}^\dagger)$ is elliptic and thus has finite dimensional kernel by Theorem 7.2. Now (7.6) shows that the range of $(\mathcal{L}, \mathcal{K})$ has finite codimension. ■

8 Fredholm properties on complete noncompact manifolds

In this section we establish conditions under which the Fredholm and existence results of the previous section for the operator $(\mathcal{L}, \mathcal{K}) : H^1(M) \to L^2(M) \times H_*^{1/2}(Y)$, may be extended to non-compact manifolds. This includes in particular, a generalisation of the solvability criterion (7.6) of Theorem 7.3. Results of this type may be applied to establish positive mass results in general relativity, for example.

The non-compactness of M causes some difficulties not found in the compact case. A classical result [18, 31] shows that a Dirac operator $\mathcal{D}$ on a non-compact manifold is essentially self-adjoint on $L^2(M)$. However, this elegant result is useless for our purposes, since it implies only that $\{(\phi, \mathcal{D}\phi) : \phi \in \operatorname{dom} \mathcal{D} \subset L^2(M)\}$ is closed in the graph topology on $L^2(M) \times L^2(M)$. This is weaker than the closed range property, which is necessary for useful solvability criteria. In fact, because $L^2(M)$ often does not encompass natural decay rates of solutions, the self-adjoint closure may not have closed range. In such cases the Dirac operator defined on $L^2(M)$ will not be semi-Fredholm. This is shown explicitly in the following example.

Consider the self-adjoint closure $\overline{\mathcal{D}} : \operatorname{dom} \overline{\mathcal{D}} \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ of the constant coefficient Dirac operator $\mathcal{D} = \gamma^i \partial_i$ and let $f = \mathcal{D}u$, $u = (1-\chi)|x|^{-1}\psi$, where $\chi(r)$ is a smooth compactly supported function identically one around 0 and ψ is a constant spinor on $\mathbb{R}^3$. Clearly $f \in L^2(\mathbb{R}^3)$ but $u \notin L^2(\mathbb{R}^3)$, so in particular, $u \notin \operatorname{dom} \overline{\mathcal{D}}$. However, f still lies in the closure of the range of $\overline{\mathcal{D}}$, since $\mathcal{D}(\chi_R u) = \chi_R f + D\chi_R u \to f$ in $L^2(\mathbb{R}^3)$, where $\chi_R(x) = \chi(x/R)$, but $\chi_R u$

can not converge in $L^2(\mathbb{R}^3)$. Clearly $(u,f)\not\in \mathrm{graph}\,\overline{\mathcal{D}}$ since $u\not\in L^2(\mathbb{R}^3)$, and it can be shown (using the corresponding Schrödinger-Lichnerowicz identity) that there is no $\bar u\in L^2(\mathbb{R}^3)$ satisfying $\mathcal{D}\bar u=f$. Thus the self-adjoint closure $\overline{\mathcal{D}}$ does not have closed range.

In order to obtain an operator with closed range, it is thus necessary to enlarge the domain, which raises the question of determining the appropriate decay rate. We sidestep this problem by using the L^2 size of the covariant derivative as a norm. To obtain sufficient control on the L^2_{loc} behaviour, we then must postulate a *weighted Poincaré inequality* (8.3). The existence of such inequalities can be established for the applications of most interest in general relativity; see Proposition 8.3 and §9.

The elliptic estimate (7.2) plays a central role in the analysis over a compact manifold, but its noncompact analogue cannot be obtained directly by similar localisation arguments. However, in cases of geometric interest an identity of Schrödinger-Lichnerowicz form (generalising (2.5)) is available, and can be used to construct suitable *global* estimates.

The weighted Poincaré and Schrödinger-Lichnerowicz estimates are the two additional ingredients needed for establishing solvability and Fredholm properties on a non-compact manifold.

For ease of further reference, let us summarize the hypotheses which will be made throughout this section:

HYPOTHESES 8.1 *M is a non-compact manifold with compact boundary Y, which is complete with respect to a C^∞ background metric $\mathring{g}$. The case $Y=\emptyset$ is admitted. The operator $\mathcal{L}=a^j\partial_j+b$ satisfies the global uniform ellipticity and boundedness condition*

$$\eta^2|V|^2\le \mathring{g}_{jk}\langle a^j(x)V,a^k(x)V\rangle\le \eta^{-2}|V|^2\ , \tag{8.1}$$

for some $\eta>0$, for all $V\in E_x$ and all $x\in M$. The coefficients of $\mathcal{L}$ satisfy the interior regularity conditions (3.4), and the boundary regularity and structure conditions of §6, namely (6.2), (6.5), (6.7), (6.9). Let A be the boundary operator and P its associated positive spectrum projection, as in §6. The boundary operator $K:(1-P)L^2(Y)\to PL^2(Y)$ satisfies (5.55,5.56), and $\mathcal{K}$,$\mathcal{K}^\dagger$ are defined in (6.18,6.20). The connection

$$\nabla=\partial-\Gamma \tag{8.2}$$

satisfies (3.18,3.19) and we note again that ∇ need not be compatible with the metric on E — this is important in some applications.

We may express $\mathcal{L}$ in terms of ∇ by

$$\mathcal{L}=a^j\nabla_j+(b+a^j\Gamma_j)=a^j\nabla_j+\beta,$$

where $\beta\in L^{n^*}_{\mathrm{loc}}(M)$. Additional, rather weak, decay conditions will be imposed on β (8.8), on the negative part of the curvature endomorphism ρ (8.13), and on $\Gamma^S=\frac12(\Gamma+{}^t\Gamma)$ in §9.

Definition 8.2 *The covariant derivative* ∇ *on* E *over* M *admits a* weighted Poincaré inequality *if there is a weight function* $w \in L^1_{\mathrm{loc}}(M)$ *with* $\operatorname{ess\,inf}_\Omega w > 0$ *for all relatively compact* $\Omega \Subset M$, *such that*

$$\int_M |u|^2 \, w \, dv_M \le \int_M |\nabla u|^2 \, dv_M \quad \forall \; u \in C^1_c(M) \; . \tag{8.3}$$

Here the length $|\nabla u|^2$ is measured by the metric on E and the background Riemannian metric $\mathring{g}$ on M, and dv_M is the volume measure of $\mathring{g}$. It is clear that the weight function w can be chosen to be smooth.

The semi-norm

$$\|u\|^2_{\mathbb{H}} = \int_M |\nabla u|^2 \, dv_M \tag{8.4}$$

on $C^\infty_c(M)$ may be completed to form the space

$$\mathbb{H} := \| \cdot \|_{\mathbb{H}}\text{-completion of } C^\infty_c \Gamma(E) \; , \tag{8.5}$$

which consists of equivalence classes of $\mathbb{H}$-convergent sequences in $C^\infty_c(M)$. Note that if a weighted Poincaré inequality holds, in the sense of Definition 8.2, then (8.3) holds for all $u \in \mathbb{H}$.

The weighted Poincaré inequality (8.3) ensures that an $\mathbb{H}$-convergent sequence converges locally in L^2, so the equivalence classes may be identified with cross-sections in the usual Lebesgue sense: with cross-sections having coefficient functions agreeing dv_M–a.e.

If there is no weighted Poincaré inequality, then it may be that $\mathbb{H}$ can not be identified with a space of Lebesgue-measurable cross-sections in this sense. For example, the trivial spinor bundle over $M = \mathbb{T}^2 \times \mathbb{R}$ with the flat connection $\nabla_i = \partial_i$ admits a global parallel spinor $\nabla_i \psi = 0$ which is approximated in the $\mathbb{H}$ seminorm by $\psi_k = \chi(x/k)\psi$ for $\chi \in C^\infty_c(\mathbb{R})$, $\chi = 1$ on $[-1,1]$. Now $\int_M |\nabla \psi_k|^2 \, dv_M \to 0$, but $\lim_{k\to\infty} \psi_k = \psi \neq 0$, so the $\mathbb{H}$-equivalence class $[0]$ contains $\psi \neq 0$ everywhere. In other words, (8.4) does not define a norm on spinors in this example. This shows, *inter alia*, that (8.3) will not hold in all cases.

More generally, a weighted Poincaré inequality fails for manifolds of the form $N \times \mathbb{R}$, where N is compact and itself admits a parallel spinor. It follows from the proof of Theorem 9.3 below that in such cases the orthogonal complement in $\mathbb{H}$ of the subspace of all parallel spinors will admit a weighted Poincaré inequality. Note also that the presence of a weighted Poincaré inequality (8.3) does not imply there are no global parallel spinors — $\mathbb{R}^3$ provides a simple counterexample.

However, weighted Poincaré inequalities can be demonstrated in many cases of interest. In the next section we will prove:

Proposition 8.3 *A covariant derivative* ∇ *on* E *admits a weighted Poincaré inequality if any one of the following conditions holds:*

1. *there is a relatively compact domain* $\Omega \subset M$ *and a constant* $c > 0$ *such that*

$$\int_\Omega |u|^2 \, dv_M \le c \int_M |\nabla u|^2 \, dv_M \tag{8.6}$$

for all $u \in C_c^\infty(M)$*;*

2. *there are no nontrivial globally parallel sections* ($\nabla u = 0 \Rightarrow u = 0$)*;*

3. *M has a weakly asymptotically flat end $\widetilde{M}$ (see Definition 9.4), with* $\dim M \geq 3$*;*

4. *M has a weakly asymptotically hyperboloidal end (see Definition 9.9), with* $\dim M \geq 2$.

When M is non-compact, the global Gårding inequality (generalizing (7.2)) cannot be constructed from local estimates. Motivated by some classical and fundamental identities, we instead introduce the following definition.

DEFINITION 8.4 *The operator pair* $(\mathcal{L}, \mathcal{K})$ *admits a* Schrödinger-Lichnerowicz estimate *if there is* $C > 0$ *and a non-negative function* ρ *such that for all* $u \in C_c^1(M)$,

$$C^{-1} \int_M |\nabla u|^2 \, dv_M \leq \int_M (|\mathcal{L}u|^2 + \rho |u|^2) \, dv_M + \oint_Y |J\mathcal{K}u_0|^2 \, dv_Y \quad , \tag{8.7}$$

where $J = (1 + |A|)^{1/2}$.

LEMMA 8.5 *Suppose that the Schrödinger-Lichnerowicz estimate (8.7) holds for all* $u \in C_c^1(M)$ *with* ρ *and* $\beta = \mathcal{L} - a^j \nabla_j$ *satisfying* $\rho \in L^{n^*/2}_{\text{loc}}$, $\beta \in L^{n^*}_{\text{loc}}$, *and*

$$\lim_{R \to \infty} \sup_{M \setminus M_R} \frac{\rho + |\beta|^2}{w} < \infty \ , \tag{8.8}$$

where $\{M_R\}_{R \to \infty}$ *is an exhaustion of* M*. Then* $\mathcal{L} : \mathbb{H} \to L^2(M)$ *is bounded and (8.7) holds for all* $u \in \mathbb{H}$.

REMARK 8.6 The condition (8.8) is not particularly restrictive in applications, where typically $\beta = 0$ ($\mathcal{L} = a^j \nabla_j$) and ρ is the negative part of the curvature endomorphism of a Schrödinger-Lichnerowicz identity, and hence $\rho = 0$ when a non-negative curvature condition is assumed. Moreover, the form of w in an asymptotically flat, respectively hyperboloidal, end is known ($w \geq C/r^2$, $\geq C$ respectively — see §9), so (8.8) can follow from simple asymptotic conditions.

PROOF: It will suffice to show that the individual terms of the right-hand-side of (8.7) are bounded by $\|u\|^2_{\mathbb{H}}$. Now

$$\int_M |\mathcal{L}u|^2 \leq C \int_M |\nabla u|^2 + 2 \int_M |\beta|^2 |u|^2 \ , \tag{8.9}$$

and we use (8.8) and (8.3) to estimate

$$\begin{aligned}
\int_{M \setminus M_R} (\rho + |\beta|^2)|u|^2 &\leq \sup_{M \setminus M_R} \frac{\rho + |\beta|^2}{w} \int_{M \setminus M_R} |u|^2 w \\
&\leq \sup_{M \setminus M_R} \frac{\rho + |\beta|^2}{w} \int_{M \setminus M_R} |\nabla u|^2 \\
&\leq C \int_M |\nabla u|^2 \ ,
\end{aligned}$$

for some $R < \infty$. Let $\chi_R \in C_c^\infty(M)$ be a cut-off function with support contained in M_{2R}, $\chi_R = 1$ on M_R. Then

$$\begin{aligned}\int_{M_R} (\rho + |\beta|^2)|u|^2 &\leq \int_{M_{2R}} (\rho + |\beta|^2)|\chi_R u|^2 \\ &\leq \left(\|\rho\|_{L^{n^*/2}(M_{2R})} + \|\beta\|^2_{L^{n^*}(M_{2R})}\right) \|\chi_R u\|^2_{L^{n^*}(M_{2R})} .\end{aligned}$$

Applying the Sobolev inequality for ∇ on the compact set M_{2R} and the weighted Poincaré inequality show that the last term is controlled by $\int_M |\nabla u|^2$. Finally, the K-bound (5.55) and the restriction Lemma 5.1 show that the boundary term is also controlled by $\int_M |\nabla u|^2$. ■

Schrödinger-Lichnerowicz *identities* hold for many common examples, and can easily be adapted to produce estimates of the form (8.7). We will not attempt to give general conditions which imply such inequalities — it is simpler to ask only that (8.7) be established separately in any particular case of interest.

For example, consider the classical Dirac operator $\mathcal{D}$ of the metric g as in §2, on a non-compact spin manifold M. Combining (2.6) and (2.8) gives

$$\int_M |\nabla\psi|^2 \, dv_M = \int_M (|\mathcal{D}\psi|^2 - \tfrac{1}{4}R(g)|\psi|^2) \, dv_M + \oint_Y \langle\psi_0, (\mathcal{D}_Y + \tfrac{1}{2}H_Y)\psi_0\rangle \, dv_Y \ , \tag{8.10}$$

for any C^1_c spinor field on M. Suppose the boundary operator is $\mathcal{K} = P_+$, the orthogonal projection onto the positive spectrum eigenspinors of $\mathcal{D}_Y$. If the boundary mean curvature H_Y satisfies $H_Y \leq \sqrt{16\pi/\mathrm{Area}(Y)}$, then the argument in §2 shows that the H_Y term can be absorbed by the $P_-\psi_0$ contributions, so the boundary term in (8.10) is not greater than

$$\oint_Y \langle P_+\psi_0, \mathcal{D}_Y P_+\psi_0\rangle \, dv_Y \leq \|\mathcal{K}\psi_0\|^2_{H^{1/2}_*(Y)} \ ,$$

and (8.7) follows immediately, with

$$\rho = \max(0, -\tfrac{1}{4}R(g)) \ . \tag{8.11}$$

Since $\beta = 0$ in this example, the inequality holds for all $u \in \mathbb{H}$ provided ρ satisfies (8.8). For general mean curvatures $H_Y \in L^\infty(Y)$, note again that

$$\oint_Y \langle\psi_0, \mathcal{D}_Y\psi_0\rangle \, dv_Y \leq \|P_+\psi_0\|^2_{H^{1/2}_*(Y)} \ .$$

If $H_Y \in L^\infty(Y)$ then $\oint_Y H_Y|\psi_0|^2 \leq \|H_Y\|_{L^\infty(Y)}\|\psi_0\|^2_{H^{1/2}_*(Y)}$. Using a fractional Sobolev inequality, the control on H_Y may be weakened to $H_Y \in L^p(Y)$, $p = n-1$ for $n \geq 3$ and $p > 1$ for $n = 2$. Lemma 5.1 shows that $\|\psi_0\|_{H^{1/2}_*(Y)} \leq c\|\tilde{\psi}_0\|_{H^1(Y\times I')}$, where $\tilde{\psi} = \chi\psi$ and $\chi = \chi(x)$ is a cutoff function supported in $I' = [0, \delta/2]$, as in the proof of Theorem 6.4. Now Corollary 6.5 shows that

$$C^{-1}\|\tilde{\psi}\|^2_{H^1_*(Y\times I')} \leq \int_{Y\times I} (|\mathcal{D}\psi|^2 + |\psi|^2) \, dv_M + \oint_Y |JP_+\psi_0|^2 \, dv_Y,$$

which provides the required Schrödinger-Lichnerowicz estimate (8.7).

In applications, a Schrödinger-Lichnerowicz estimate is usually obtained in the special case of homogeneous boundary data ($\mathcal{K}u_0 = 0$). The above trick shows that the homogeneous estimate implies the general case (8.7):

LEMMA 8.7 *Under the hypotheses of Lemma 8.5, suppose there is $\bar{C} > 0$ such that for all $u \in \mathbb{H}$ with $\mathcal{K}u_0 = 0$ we have*

$$\bar{C}^{-1} \int_M |\nabla u|^2 \, dv_M \le \int_M (|\mathcal{L}u|^2 + \bar{\rho}|u|^2) \, dv_M \ , \tag{8.12}$$

for some $\bar{\rho}$. Then there is $C > 0$ such that (8.7) holds for all $u \in \mathbb{H}$.

PROOF: Suppose $u \in \mathbb{H}$ and let $\tilde{u} = u - \chi u$, where $\chi = \chi(x) \in C^\infty(M)$ is a cutoff function supported in $Y \times I'$ as in the proof of Theorem 6.4. Then $\mathcal{K}\tilde{u}_0 = 0$ so (8.12) applies to $\tilde{u}$, giving

$$\begin{aligned}
\int_M |\nabla u|^2 \, dv_M &\le 2 \int_M (|\nabla \tilde{u}|^2 + |\nabla(\chi u)|^2) \, dv_M \\
&\le C \int_M (|\mathcal{L}\tilde{u}|^2 + \bar{\rho}|\tilde{u}|^2) \, dv_M + 2 \int_{Y \times I'} |\nabla(\chi u)|^2 \, dv_M \\
&\le C \int_M (|\mathcal{L}u|^2 + (\bar{\rho} + |d\chi|^2)|u|^2) \, dv_M + 3 \int_{Y \times I'} |\nabla u|^2 \, dv_M \ .
\end{aligned}$$

Now it follows easily from Corollary 6.5 that

$$C^{-1} \int_{Y \times I'} |\nabla u|^2 \, dv_M \le \int_{Y \times I} (|\mathcal{L}u|^2 + |u|^2) \, dv_M + \oint_Y |J\mathcal{K}u_0|^2 \, dv_Y,$$

which gives the required inequality. ■

THEOREM 8.8 *Under the hypotheses 8.1, suppose $(M, \nabla, \mathcal{L}, \mathcal{K})$ admits a weighted Poincaré inequality (8.3) and a Schrödinger-Lichnerowicz inequality (8.7) with ρ and β satisfying the conditions of Lemma 8.5. If $\rho \in L^p_{\mathrm{loc}}(M)$ for some $p > n^*/2$, and if*

$$\lim_{R \to \infty} \sup_{M \setminus M_R} \frac{\rho}{w} = 0 \ , \tag{8.13}$$

where $\{M_R\}_{R \to \infty}$, is any exhaustion of M, then

$$(\mathcal{L}, \mathcal{K}) : \mathbb{H} \to L^2(M) \times H^{1/2}_*(Y) \tag{8.14}$$

is semi-Fredholm.

PROOF: Lemma 8.5 gives $\mathcal{L}u \in L^2(M)$ for $u \in \mathbb{H}$. We first show the unit ball in the kernel is compact. Let $\{u_k\}_{k=1}^\infty$ be a sequence in the kernel of $(\mathcal{L}, \mathcal{K})$, normalised by $\|u_k\|_{\mathbb{H}} = 1$. Weak compactness of bounded sets in $\mathbb{H}$ shows there is $\bar{u} \in \mathbb{H}$ and a subsequence, which we also denote by u_k, such that $u_k \rightharpoonup \bar{u} \in \mathbb{H}$ and $\|\bar{u}\|_{\mathbb{H}} \le \liminf \|u_k\|_{\mathbb{H}} = 1$.

Since (8.13) is independent of the choice of exhaustion, we may suppose for definiteness that $M_R = \{x \in M : d(x) < R\}$ where $d(x)$ is the smoothed distance function from some fixed base point. Let $\chi \in C_c^\infty(\mathbb{R})$ satisfy $\chi(x) = 1$ for $x \le 1$, $\chi(x) = 0$ for $x \ge 2$ and $0 \le \chi(x) \le 1$, $|\chi'(x)| \le 2$ for all x. Then the functions $\chi_R(x) = \chi(d(x)/R)$ form support functions for the exhaustion M_R which satisfy $\operatorname{supp}\chi_R \subset M_{2R}$, $\chi_R = 1$ on M_R and $|d\chi_R| \le 2$. Using the weighted Poincaré inequality we have

$$\begin{aligned}\int_M |\nabla(\chi_R u_k)|^2\, dv_M &\le 2\int_{M_{2R}\setminus M_R} |d\chi_R|^2 |u_k|^2\, dv_M + 2\int_{M_{2R}} |\nabla u_k|^2\, dv_M \\ &\le 2(1 + 2\sup_{M_{2R}\setminus M_R} w^{-1}) \int_M |\nabla u_k|^2\, dv_M\ ,\end{aligned}$$

which shows that for any $R > 1$ the sequence $\chi_R u_k$ is bounded in $H^1(M_{2R})$. Since $\chi_R u_k \rightharpoonup \chi_R \bar{u}$ in $H^1(M_{2R})$, the Rellich lemma implies $\chi_R u_k \to \chi_R \bar{u}$ strongly in $L^q(M_{2R})$ for any $q < \hat{2} = 2n/(n-2)$ and any $R > 1$.

Applying (8.7) to any difference $u_j - u_k$ gives

$$\begin{aligned}\int_M |\nabla(u_j - u_k)|^2\, dv_M &\le \int_M \rho\, |u_j - u_k|^2\, dv_M \\ &\le \|\rho\|_{L^p(M_R)} \|u_j - u_k\|^2_{L^q(M_R)} \\ &\quad + \sup_{M\setminus M_R} \frac{\rho}{w} \int_M |u_j - u_k|^2\, w\, dv_M \qquad (8.15)\end{aligned}$$

where, since $p > n^*/2$, we have $q = 2p/(p-1) < \hat{2}$. Now (8.3) and $\|u_k\|_{\mathbb{H}} = 1$ combine to show that

$$\int_M |u_j - u_k|^2\, w\, dv_M \le 4,$$

so by (8.13), for any $\epsilon > 0$ there is $R = R(\epsilon)$ such that the second term of (8.15) is less than $\epsilon/2$ for all j, k. Since u_k converges in $L^q(M_R)$ there is $N = N(\epsilon, R)$ such that the first term is less than $\epsilon/2$ for all $j, k \ge N$. This shows u_k is a Cauchy sequence, hence strongly convergent to $\bar{u}$, in $\mathbb{H}$.

As noted above, $\|\mathcal{L}u\|_{L^2(M)} \le C\|u\|_{\mathbb{H}}$ and thus

$$\begin{aligned}\int_M |\mathcal{L}\bar{u}|^2\, dv_M &= \int_M |\mathcal{L}(\bar{u} - u_k)|^2\, dv_M \\ &\le C\int_M |\nabla(\bar{u} - u_k)|^2\, dv_M \\ &\to 0 \ \text{ as } k \to \infty\ ,\end{aligned}$$

which shows that $\mathcal{L}\bar{u} = 0$. Similarly, since $\mathcal{K} : H_*^{1/2}(Y) \to H_*^{1/2}(Y)$ is bounded, for any $u \in \mathbb{H}$ we have

$$\begin{aligned}\oint_Y |J\mathcal{K}u_0|^2\, dv_Y &\le c\|\mathcal{K}u_0\|_{H_*^{1/2}(Y)} \le ck\|u_0\|_{H_*^{1/2}(Y)} \\ &\le C\int_M |\nabla u|^2\, dv_M\ ,\end{aligned}$$

by (5.55) and the trace lemma 5.1. Choosing $u = \bar{u} - u_k$ gives

$$\oint_Y |J\mathcal{K}\bar{u}_0|^2\, dv_Y \le c\int_M |\nabla(\bar{u} - u_k)|^2\, dv_M = o(1),$$

which shows also that $\mathcal{K}\bar{u}_0 = 0$. Thus $\bar{u} \in \ker(\mathcal{L}, \mathcal{K})$ and the kernel is finite dimensional.

To show the closed range property, observe that by (8.13) and (8.3), the elliptic estimate (8.7) may be strengthened to

$$C^{-1} \int_M \left(|\nabla u|^2 + |u|^2 w \right) dv_M \leq \int_M |\mathcal{L}u|^2 dv_M + \int_\Omega \rho |u|^2 \, dv_M + \oint_Y |J\mathcal{K}u_0|^2 \, dv_Y \ , \tag{8.16}$$

for some relatively compact domain $\Omega \Subset M$. Now we claim there is a constant $C > 0$ such that

$$\int_\Omega \rho |u|^2 \, dv_M \leq C \left(\int_M |\mathcal{L}u|^2 dv_M + \oint_Y |J\mathcal{K}u_0|^2 \, dv_Y \right) \ , \tag{8.17}$$

for all $u \in \mathbb{H}$ such that

$$\int_M \langle \nabla u, \nabla \phi \rangle \, dv_M = 0 \quad \forall \ \phi \in \ker(\mathcal{L}, \mathcal{K}) \ . \tag{8.18}$$

Suppose (8.17) fails, so there is a sequence $u_k \in \mathbb{H}$, $k = 1, 2, \ldots$, such that (8.18) holds for each u_k, and

$$\int_\Omega \rho |u_k|^2 \, dv_M = 1, \qquad \int_M |\mathcal{L}u_k|^2 dv_M + \oint_Y |J\mathcal{K}(u_k)_0|^2 \, dv_Y \leq 1/k.$$

The sequence is bounded in $\mathbb{H}$ by (8.16), so by passing to a subsequence we may assume u_k converges weakly to $\bar{u} \in \mathbb{H}$ and strongly in $L^q(\Omega)$, $q = 2p/(p-1) < \hat{2}$ as before. Applying (8.16) to $u_j - u_k$ shows the sequence is Cauchy and thus converges strongly in $\mathbb{H}$. It follows that

$$\int_M |\mathcal{L}\bar{u}|^2 dv_M + \oint_Y |J\mathcal{K}\bar{u}_0|^2 \, dv_Y = 0,$$

so $\bar{u} \in \ker(\mathcal{L}, \mathcal{K})$. Strong convergence shows that (8.18) is also satisfied by $\bar{u}$, so testing (8.18) for $\bar{u}$ with $\phi = \bar{u}$ shows that $\bar{u} = 0$. However, strong convergence in $L^q(\Omega)$ shows that $\int_\Omega \rho |\bar{u}|^2 dv_M = 1$, which is a contradiction and establishes the claim (8.17).

Combining (8.17) with (8.16) gives

$$\int_M \left(|\nabla u|^2 + |u|^2 w \right) dv_M \leq C \left(\int_M |\mathcal{L}u|^2 dv_M + \oint_Y |J\mathcal{K}u_0|^2 \, dv_Y \right) \tag{8.19}$$

for all $u \in \mathbb{H}$ satisfying (8.18). Now suppose $u_k \in \mathbb{H}$ is a sequence such that $\mathcal{L}u_k = f_k \to f \in L^2(M)$ and $\mathcal{K}(u_k)_0 = s_k \to s \in H_*^{1/2}(Y)$. These convergence properties are retained if we replace u_k by $u_k + y_k$ for any convergent sequence $y_k \in \ker(\mathcal{L}, \mathcal{K})$, so we may assume the u_k all satisfy (8.18). In particular, applying (8.19) to $u_j - u_k$ shows that u_k is Cauchy in $\mathbb{H}$ and converges to $\bar{u}$ satisfying $\mathcal{L}\bar{u} = f$, $\mathcal{K}\bar{u}_0 = s$. This shows $(\mathcal{L}, \mathcal{K})$ has closed range. ∎

By Definition 6.3, u is a weak solution of

$$\mathcal{L}u = f, \quad \mathcal{K}u_0 = \sigma \ , \tag{8.20}$$

for $f \in L^2(M)$, $\sigma \in PH^{1/2}_*(Y)$, if $u \in L^2_{\mathrm{loc}}(M)$ and

$$\int_M \langle u, \mathcal{L}^\dagger \phi \rangle \, dv_M = \int_M \langle f, \phi \rangle \, dv_M + \oint_Y \langle \sigma, \nu \phi_0 \rangle \, dv_Y \ , \tag{8.21}$$

for all $\phi \in H^1_c(M)$ such that $\mathcal{K}^\dagger \phi_0 = 0$. Similarly, the argument of Theorem 7.3 shows that the weak form of the adjoint problem

$$\mathcal{L}^\dagger u = g, \quad \mathcal{K}^\dagger u_0 = \tau \ , \tag{8.22}$$

for $g \in L^2(M)$, $\tau \in \hat{P} H^{1/2}_*(Y)$, $\hat{P} = 1 - \nu^{-1} P \nu$, is that $u \in L^2_{\mathrm{loc}}(M)$ and

$$\int_M \langle u, \mathcal{L} \phi \rangle \, dv_M = \int_M \langle g, \phi \rangle \, dv_M - \oint_Y \langle \tau, \nu^{-1} \phi_0 \rangle \, dv_Y \tag{8.23}$$

for all $\phi \in H^1_c(M)$ such that $\mathcal{K} \phi_0 = 0$.

We now extend the solvability criterion (Fredholm alternative) of Theorem 7.3 to the non-compact case.

Theorem 8.9 *Under the conditions of Theorem 8.8, suppose the formal adjoint $(\mathcal{L}^\dagger, \mathcal{K}^\dagger)$ also satisfies a Schrödinger-Lichnerowicz estimate (8.7) with the same covariant derivative ∇ and with a curvature term $\hat{\rho}$ satisfying (8.13). Then the system (8.20) with $(f, \sigma) \in L^2(M) \times PH^{1/2}(Y)$ has a solution $u \in \mathbb{H}$ if and only if (f, σ) satisfies*

$$\int_M \langle f, \phi \rangle \, dv_M + \oint_Y \langle \sigma, \nu \phi_0 \rangle \, dv_Y = 0 \ , \tag{8.24}$$

for all $\phi \in \mathbb{H} \cap L^2(M)$ satisfying $\mathcal{L}^\dagger \phi = 0$, $\mathcal{K}^\dagger \phi_0 = 0$. In particular, the system (8.20) is solvable for all $(f, \sigma) \in L^2(M) \times PH^{1/2}(Y)$ if and only if there are no $0 \neq \Psi \in \mathbb{H} \cap L^2(M)$ satisfying $\mathcal{L}^\dagger \Psi = 0$, $\mathcal{K}^\dagger \Psi_0 = 0$.

Remark 8.10 We emphasise that in Theorem 8.9 it is not necessary to impose conditions on $\hat{\rho}$ other than (8.13), and no conditions on the map $\hat{\beta} := \mathcal{L}^\dagger - {}^t a^i \nabla_i$ are needed.

Proof: The necessity of (8.24) follows immediately from the weak form (8.21). To show sufficiency, the argument of Theorem 7.3 applies to reduce to the case $\sigma = 0$, which we now consider.

Let $\mathbb{H}_{\mathcal{K}} = \{u \in \mathbb{H} : \mathcal{K} u_0 = 0\}$. The elliptic estimate (8.7) gives

$$\int_M |\nabla u|^2 \, dv_M \le C \int_M (|\mathcal{L} u|^2 + \rho |u|^2) \, dv_M \ , \quad \forall \ u \in \mathbb{H}_{\mathcal{K}} \ .$$

The arguments used to show (8.16) and (8.17) apply and give

$$\int_M |\nabla u|^2 \, dv_M \le C \int_M |\mathcal{L} u|^2 \, dv_M \quad \forall \ u \in \mathring{\mathbb{H}}_{\mathcal{K}} \ , \tag{8.25}$$

where we define

$$\mathring{\mathbb{H}}_{\mathcal{K}} := \{u \in \mathbb{H}_{\mathcal{K}} : \int_M \langle \nabla u, \nabla \phi \rangle \, dv_M = 0 \ \forall \ \phi \in \ker(\mathcal{L}, \mathcal{K})\} \ . \tag{8.26}$$

Thus the bilinear form $u \mapsto \int_M |\mathcal{L}u|^2 \, dv_M$ is strictly coercive on the Hilbert space $\mathring{\mathbb{H}}_{\mathcal{K}}$, and for each $f \in L^2(M)$ the map $\phi \to \int_M \langle f, \mathcal{L}\phi\rangle \, dv_M$ is bounded on $\mathring{\mathbb{H}}_{\mathcal{K}}$. The Lax-Milgram lemma shows there is $u \in \mathring{\mathbb{H}}_{\mathcal{K}}$ satisfying

$$\int_M \langle \mathcal{L}u, \mathcal{L}\phi\rangle \, dv_M = \int_M \langle f, \mathcal{L}\phi\rangle \, dv_M \quad \forall \, \phi \in \mathring{\mathbb{H}}_{\mathcal{K}} \, .$$

Thus setting $\Psi = \mathcal{L}u - f$ we have

$$\int_M \langle \Psi, \mathcal{L}\phi\rangle \, dv_M = 0 \quad \forall \, \phi \in \mathbb{H}_{\mathcal{K}} \, , \tag{8.27}$$

since $\phi \in \ker(\mathcal{L}, \mathcal{K})$ will also satisfy the relation (8.27). Lemma 8.5 shows that $\Psi \in L^2(M)$ and from Definition 6.3 and (8.27) we see that Ψ is a weak solution of

$$\mathcal{L}^\dagger \Psi = 0, \quad \mathcal{K}^\dagger \Psi = 0 \, .$$

If there are no such non-trivial Ψ then $\mathcal{L}u = f$, and u is the required solution. The arguments of Theorem 7.3 show that $(\mathcal{L}^\dagger, \mathcal{K}^\dagger)$ is elliptic and Theorem 6.4 applies to show $\Psi \in H^1_{\mathrm{loc}}(M)$. Let M_R be the exhaustion of M constructed in Theorem 8.8, with associated cutoff functions $\chi_R \in C^\infty_c(M)$, and let $\Psi_k = \chi_k \Psi \in H^1_c(M) \subset \mathbb{H}$. The assumed Schrödinger-Lichnerowicz estimate (8.7) for $(\mathcal{L}^\dagger, \mathcal{K}^\dagger)$ gives (with $\mathcal{L}^\dagger$ curvature term $\hat\rho$)

$$\int_M |\nabla(\Psi_k - \Psi_l)|^2 \, dv_M \le C \int_M \left(|\mathcal{L}^\dagger(\Psi_k - \Psi_l)|^2 + \hat\rho |\Psi_k - \Psi_l|^2 \right) dv_M \, . \tag{8.28}$$

Since $\mathcal{L}^\dagger \Psi = 0$ we have

$$\int_M |\mathcal{L}^\dagger(\Psi_k - \Psi_l)|^2 \, dv_M \le c \int_M \left(|d\chi_k|^2 |\Psi|^2 + |d\chi_l|^2 |\Psi|^2 \right) dv_M \to 0 \, ,$$

because $\Psi \in L^2(M)$, $|d\chi_k| \le 2$ and $\operatorname{supp} d\chi_k \subset M_{2k} \backslash M_k$. Now

$$\int_M \hat\rho |\Psi_k - \Psi_l|^2 \, dv_M \le \epsilon \int_M |\Psi_k - \Psi_l|^2 \, w \, dv_M$$

by the condition (8.13) on $\hat\rho$, for sufficiently large k, l. By the weighted Poincaré inequality (8.3), this is in turn bounded by ϵ times the left side of (8.28) and may therefore be discarded in (8.28) by choosing ϵ sufficiently small. It follows that Ψ_k is a Cauchy sequence in $\mathbb{H}$, so $\Psi \in \mathbb{H} \cap L^2$ and thus $\mathcal{L}^\dagger \Psi = 0$, $\mathcal{K}^\dagger \Psi_0 = 0$. If there is no such $\Psi \ne 0$ then $\mathcal{L}u = f$, and u is the required solution. More generally we have $\mathcal{L}u = f + \Psi$, $u \in \mathring{\mathbb{H}}_{\mathcal{K}}$, and since $\int_M \langle \mathcal{L}u, \Psi\rangle \, dv_M = 0$ by (8.27), the condition (8.24) (with $\sigma = 0$ and $\phi = \Psi$) shows that $\Psi = 0$ and we have solved $\mathcal{L}u = f$, as required. ■

9 Weighted Poincaré Inequalities

Define the *symmetric part* Γ^S of the connection ∇ by

$$\langle \phi, \Gamma^S(X)\psi\rangle := \tfrac{1}{2}\left(X\langle \phi, \psi\rangle - \langle \phi, \nabla_X \psi\rangle - \langle \nabla_X \phi, \psi\rangle\right) , \tag{9.1}$$

for all smooth sections ϕ, ψ of E and all smooth vector fields X. This gives a linear map $\Gamma^S(X)$ from E to E, symmetric with respect to the scalar product $\langle \cdot, \cdot \rangle$, and linear also in X. Clearly, ∇ is compatible with $\langle \cdot, \cdot \rangle$ if and only if Γ^S vanishes, and for Γ defined by (8.2), $\Gamma^S = \frac{1}{2}(\Gamma + {}^t\Gamma)$. We establish Proposition 8.3 via a special case, based on an argument of Geroch–Perng [11]:

LEMMA 9.1 *Let $\Omega, \tilde\Omega$ be any two relatively compact domains in M, and assume that*

$$\Gamma^S \in L^{n^*}_{\mathrm{loc}}(M)\ . \tag{9.2}$$

There is a constant $\epsilon > 0$ such that for all sections $u \in H^1_{\mathrm{loc}}(M)$ of E we have

$$\epsilon \int_{\tilde\Omega} |u|^2\, dv_M \le \int_{\Omega} |u|^2\, dv_M + \int_M |\nabla u|^2\, dv_M\ . \tag{9.3}$$

PROOF: Let q be any point of $\tilde\Omega$, fix $p \in \Omega$ and let r_p be small enough that the $\mathring{g}$-geodesic ball $B(p, r_p)$ of radius r_p and centred at p, lies within Ω. Let X be a C^∞ compactly supported vector field, such that the associated flow ϕ_t satisfies $\phi_1(B(p, r_p)) \supset B(q, r_q)$ for some $r_q > 0$. (Since M is C^∞ and connected, it is always possible to construct such an X.) Let $\Omega_t = \phi_t(B(p, r_p))$.

By direct calculation and Hölder's inequality we have, for any $u \in H^1_{\mathrm{loc}}(M)$,

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega_t} |u|^2 dv_M &= \int_{\Omega_t} \left(2\langle u, (\nabla_X + \Gamma^S_X)u\rangle + |u|^2 \mathrm{div}_{\mathring{g}} X\right)\, dv_M \\
&\le C\left(\int_{\Omega_t} \left(|u|^2 + |\nabla u|^2\right)\, dv_{\mathring{g}} + \|\Gamma^S\|_{L^{n^*/2}(\Omega_t)} \|u\|^2_{L^{\hat 2}(\Omega_t)}\right)\ ,
\end{aligned}$$

where C depends on $\|X\|_{L^\infty}$, $\|\mathrm{div}_{\mathring{g}} X\|_{L^\infty}$. By the Sobolev inequality in the coordinate ball Ω_t for functions, $\|f\|_{L^{\hat 2}(\Omega_t)} \le C(\|\partial f\|_{L^2(\Omega_t)} + \|f\|_{L^2(\Omega_t)})$. Applying this to $f = |u|$ gives $\|u\|_{L^{\hat 2}(\Omega_t)} \le C(\|Du\|_{L^2(\Omega_t)} + \|u\|_{L^2(\Omega_t)})$, where D is any metric-compatible connection. Since $\Gamma^S \in L^{n^*}$ may be written as $\Gamma_1 + \Gamma_2$, $\Gamma_1 \in L^\infty$, $\|\Gamma_2\|_{L^{n^*}} \le \epsilon$, the Sobolev inequality gives

$$\|u\|_{L^{\hat 2}(\Omega_t)} \le C\left(\|\nabla u\|_{L^2(\Omega_t)} + \|u\|_{L^2(\Omega_t)}\right)\ ,$$

for some constant C depending on Γ. Defining $F(t) = \int_{\Omega_t} |u|^2 dv_{\mathring{g}}$, we have

$$\frac{d}{dt}F(t) \le CF(t) + C\int_M |\nabla u|^2 dv_M\ ,$$

and Gronwall's lemma gives $F(1) \le e^C(F(0) + \int_M |\nabla u|^2 dv_M)$. Thus there is $\epsilon > 0$ such that

$$\epsilon \int_{B(q, r_q)} |u|^2\, dv_M \le \int_\Omega |u|^2\, dv_M + \int_M |\nabla u|^2\, dv_M\ .$$

Since $\tilde\Omega$ has compact closure, it is covered by finitely many such balls $B(q, r_q)$ and (9.3) follows. ∎

Corollary 9.2 *Under condition (9.2), if there is a domain $\Omega \subset M$ and a constant $\epsilon > 0$ such that*

$$\epsilon \int_\Omega |u|^2 \, dv_M \le \int_M |\nabla u|^2 \, dv_M \tag{9.4}$$

for all $u \in C^1_c(M)$, then M admits a weighted Poincaré inequality (8.3).

Proof: By paracompactness and Lemma 9.1, there is a countable locally finite covering of M by domains Ω_k and constants $1 \ge \epsilon_k > 0$, $k \in \mathbb{Z}^+$, such that for each k,

$$\epsilon_k \int_{\Omega_k} |u|^2 \, dv_M \le \int_\Omega |u|^2 \, dv_M + \int_M |\nabla u|^2 \, dv_M \ .$$

This is in turn bounded uniformly by (9.4), so the function

$$w(x) = \sum_{k : x \in \Omega_k} \frac{2^{-k} \epsilon \epsilon_k}{1 + \epsilon} \tag{9.5}$$

is bounded, strictly positive, and satisfies

$$\int_M |u|^2 \, w \, dv_M \le \int_M |\nabla u|^2 \, dv_M \ ,$$

which is the required weighted Poincaré inequality. ∎

This establishes part (i) of Proposition 8.3, and we next turn to the proof of part (ii).

Theorem 9.3 *Suppose that M has a locally finite cover such that*

$$\nabla_i = \partial_i - \Gamma_i \ , \qquad \textit{with } \ \Gamma_i \in L^{n^*}_{\mathrm{loc}} \ . \tag{9.6}$$

If there are no global ∇-parallel sections of the bundle E, then M admits a weighted Poincaré inequality. Equivalently, if M does not admit a weighted Poincaré inequality then M admits a global ∇-parallel section.

Proof: Assume M does not admit a weighted Poincaré inequality, so by Corollary 9.2, for each domain $\Omega \Subset M$ and each constant $\epsilon > 0$, there is $u \in H^1_{\mathrm{loc}}(M)$ such that (9.4) fails. In particular, fixing Ω, for each $k > 0$ there is $u_k \in H^1_{\mathrm{loc}}(M)$ such that

$$\int_\Omega |u_k|^2 \, dv_M = 1, \qquad \int_M |\nabla u_k|^2 \, dv_M \le k^{-1} \ . \tag{9.7}$$

It follows that $\nabla u_k \to 0$ strongly in $L^2(M)$. Under (9.6) Rellich's lemma holds, so there is a subsequence converging strongly to $u \in L^2(\Omega)$. Then $\nabla u = 0$ and $u \neq 0$ in Ω.

Now let M_j, $j = 1, 2, \ldots$ be the exhaustion of M from Theorem 8.8, and let $u_j \in H^1(M_j)$ be the corresponding parallel spinors, constructed in the preceding paragraph. Since $u_j \neq 0$ there is $M'_j \Subset M_j$ such that $\int_{M'_j} |u_j|^2 \neq 0$. Lemma 9.1

applied with M_j replacing M shows there is $\eta_j > 0$ such that for all $v \in H^1_{\mathrm{loc}}(M_j)$,

$$\eta_j \int_{M'_j} |v|^2 \, dv_M \le \int_{M_0} |v|^2 \, dv_M + \int_{M_j} |\nabla v|^2 \, dv_M \ .$$

In particular this implies $\int_{M_1} |u_j|^2 \, dv_M \neq 0$ and we may impose the normalisation $\int_{M_1} |u_j|^2 \, dv_M = 1$. By Rellich's lemma there is $\bar u_1 \in H^1(M_1)$ and a subsequence, also denoted by u_j, such that $u_j \to \bar u_1$ in $H^1(M_1)$ and $\int_{M_1} |\bar u_1|^2 = 1$, $\nabla \bar u_1 = 0$.

Again by Lemma 9.1, for each $k \ge 1$ there is $\epsilon_k > 0$ such that

$$\epsilon_k \int_{M_k} |v|^2 \, dv_M \le \int_{M_1} |v|^2 \, dv_M + \int_{M_{k+1}} |\nabla v|^2 \, dv_M \ , \quad \forall \ v \in H^1_{\mathrm{loc}}(M_{k+1}) \ .$$

Setting $v = u_i - u_j$, $i, j > k$, shows that the sequence u_j is Cauchy in $L^2(M_k)$ and therefore converges strongly in $L^2(M_k)$ for all $k \ge 1$ to some nontrivial $\bar u \in L^2_{\mathrm{loc}}(M)$, and $\nabla \bar u = 0$. ■

Another application of Corollary 9.2 leads to Proposition 8.3 part 3, for asymptotically flat manifolds. In fact the proof works for a much broader class of manifolds:

DEFINITION 9.4 *A* weakly asymptotically flat end $\widetilde M \subset M$ *of a Riemannian manifold* M *with metric* g *is a connected component of* $M \backslash K$ *for some compact set* K*, such that* $\widetilde M \simeq \mathbb{R}^n \backslash B(0,1)$ *and there is a constant* $\eta > 0$ *such that*

$$\eta \, \delta_{ij} \xi^i \xi^j \le g_{ij}(x) \xi^i \xi^j \le \eta^{-1} \delta_{ij} \xi^i \xi^j \ ;$$

for all $x \in \mathbb{R}^n \backslash B(0,1)$ *and all vectors* $\xi \in \mathbb{R}^n$*.*

THEOREM 9.5 *Suppose* (M, g) *is a (connected) Riemannian manifold of dimension* $n \ge 3$*,* $g \in C^0(M)$*, and* M *has a weakly asymptotically flat end* $\widetilde M$*. Suppose also the connection* $\nabla_i = \partial_i - \Gamma_i$ *on* E *satisfies* $\Gamma \in L^{n^*}_{\mathrm{loc}}(M)$ *and the decay conditions*

$$\| r^{-1} \Gamma^S \|_{L^{n/2}(\widetilde M)} + \| \Gamma^S \|_{L^n(\widetilde M)} < \infty \ , \tag{9.8}$$

where Γ^S *is the symmetric, scalar product incompatible, component of* ∇ *defined by Equation (9.1). Then* M *admits a weighted Poincaré inequality.*

REMARK 9.6 The restriction $\dim M \ge 3$ is rather harmless as far as the applications to the positive mass theorems are concerned, since the notion of asymptotic flatness for two dimensional manifolds, relevant to general relativistic applications, has to be defined in a completely different way. An adequate analogue of mass here when $\dim M = 2$ is provided by the Shiohama theorem [27].

REMARK 9.7 The decay condition (9.8) is independent of the choice of flat background metric $\mathring{g}_{ij} = \delta_{ij}$: Equation (9.1) shows that Γ^S is a tensor. By comparison with the g-distance function from any chosen point p, the function r is equivalent to this distance function, which implies the result.

Remark 9.8 The proof below establishes the inequality (9.4) for spinors supported in $\Omega := \mathbb{R}^3 \setminus B(0,R)$ for some R without assuming that $\Gamma \in L^{n^*}_{\mathrm{loc}}(M)$.

Proof: Let $r = (\sum (x^i)^2)^{1/2} \in C^\infty(\widetilde{M})$ and $\chi = \chi(r) \in C^1_c(\widetilde{M})$ satisfy, for some $R_0 > 1$ and $k \geq 10$,

$$\chi(r) = \frac{\log(r/R_0)}{\log k}, \quad 2R_0 \leq r \leq (k-1)R_0$$

and $\chi(r) = 1$ for $r > kR_0$, $\chi(r) = 0$ for $r \leq R_0$. Then $|\chi'(r)| \leq 2/(r \log k)$, so for any section $u \in C^1_c(M)$

$$\int_M |\nabla(\chi u)|^2 \, dv_M \leq 2 \int_M |\nabla u|^2 \, dv_M + \frac{4}{(\log k)^2} \int_{R_0 \leq r \leq kR_0} \frac{1}{r^2} |u|^2 \, dv_M \; . \quad (9.9)$$

Now $\Delta_0(r^{2-n}) = 0$ for $r \geq 1$ in $\mathbb{R}^n$, $n \geq 3$, so for any $v \in C^1_c(\mathbb{R}^n \backslash B(0,R_0))$ we have

$$\begin{aligned} 0 &= -\int_{\mathbb{R}^n} \partial_i(\partial_i(r^{2-n}) \, |v|^2 r^{n-2}) \, dx \\ &= (n-2)^2 \int_{\mathbb{R}^n} r^{-2} |v|^2 \, dx + (n-2) \int_{\mathbb{R}^n} r^{-1} 2 \langle v, (\nabla_r + \Gamma^S_r) v \rangle \, dx \; , \end{aligned}$$

where $\Gamma^S_r = r^{-1} x^i \Gamma^S_i$ and lengths are measured by $\mathring{g}$ and the metric on E. Using Hölder's inequality we obtain

$$\frac{(n-2)^2}{4} \int_{\mathbb{R}^n} r^{-2} |v|^2 \, dx \leq \int_{\mathbb{R}^n} |\nabla v|^2 \, dx + (n-2) \int_{\mathbb{R}^n} r^{-1} |v|^2 |\Gamma^S_r| \, dx \; .$$

The Sobolev inequality in $\mathbb{R}^n$, $n \geq 3$,

$$\left(\int_{\mathbb{R}^n} |v|^{\hat{2}} \, dx \right)^{1-2/n} \leq C_S \int_{\mathbb{R}^n} |Dv|^2 \, dx \; ,$$

where $D = \nabla + \Gamma^S$ is the metric-compatible connection, gives the estimate

$$\begin{aligned} \int_{\mathbb{R}^n} |Dv|^2 \, dx &\leq 2 \int_{\mathbb{R}^n} (|\nabla v|^2 + |v|^2 |\Gamma^S|^2) \, dx \\ &\leq 2 \int_{\mathbb{R}^n} |\nabla v|^2 \, dx + 2 \, C_S \, \|\Gamma^S\|_{L^n(\mathbb{R}^n \backslash B(0,R_0))} \int_{\mathbb{R}^n} |Dv|^2 \, dx \\ &\leq 4 \int_{\mathbb{R}^n} |\nabla v|^2 \; , \end{aligned}$$

provided $2C_S \|\Gamma^S\|_{L^n(\mathbb{R}^n \backslash B(0,R_0))} \leq \frac{1}{2}$. Now (9.8) implies there is $R_0 < \infty$ such that this condition will be satisfied, so for any $v \in C^1_c(\mathbb{R}^n \backslash B(0,R_0))$ we have

$$\begin{aligned} \int_{\mathbb{R}^n} r^{-1} |\Gamma^S_r| \, |v|^2 \, dx &\leq \|r^{-1} \Gamma^S\|_{L^{n/2}(\mathbb{R}^n)} \, C_S \int_{\mathbb{R}^n} |Dv|^2 \, dx \\ &\leq 4 C_S \|r^{-1} \Gamma^S\|_{L^{n/2}(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\nabla v|^2 \, dx \; . \end{aligned}$$

Hence there is $\epsilon > 0$ such that for all $v \in C^1_c(\widetilde{M} \cap \{r > R_0\})$,

$$\epsilon \int_{\widetilde{M}} r^{-2}|v|^2\, dv_M \le \int_{\widetilde{M}} |\nabla v|^2\, dv_M \ . \tag{9.10}$$

Combining (9.10) with $v = \chi u$ and (9.9) gives

$$\begin{aligned}\int_{\{r>kR_0\}} r^{-2}|u|^2\, dv_M &\le \int_{\widetilde{M}} r^{-2}|\chi u|^2\, dv_M \\ &\le C\int_{\widetilde{M}} |\nabla(\chi u)|^2\, dv_M \\ &\le C\int_{\widetilde{M}} |\nabla u|^2\, dv_M + \frac{C}{(\log k)^2}\int_{\widetilde{M}} r^{-2}|u|^2\, dv_M \ ,\end{aligned}$$

where now $|\nabla u|^2 = g^{ij}\langle \nabla_i u, \nabla_j u\rangle$. If k is chosen so that $C/(\log k)^2 \le \frac{1}{2}$ then the last term may be absorbed into the left hand side, giving

$$\int_{r \ge kR_0} r^{-2}|u|^2\, dv_M \le C\int_M |\nabla u|^2\, dv_M \ . \tag{9.11}$$

Lemma 9.1 now applies and gives the required weighted Poincaré inequality, with $w^{-1} = Cr^2$ in the asymptotically flat end. ■

In order to prove part 4. of Proposition 8.3 the following Definition is needed:

Definition 9.9 *A weakly hyperboloidal end $\widetilde{M} \subset M$ is a connected component of $M\backslash K$ for some compact set K, such that $\widetilde{M} \simeq (0, x_0) \times \mathcal{N}$, where $(\mathcal{N}, h)$ is a (boundaryless) compact Riemannian manifold with continuous metric h, with $g|_{\tilde{M}}$ being uniformly equivalent to*

$$\mathring{g} \equiv x^{-2}(dx^2 + h) \ .$$

Here x is the coordinate running along the $(0, x_0)$ factor of $(0, x_0) \times \mathcal{N}$.

We have the following hyperboloidal counterpart of Theorem 9.5:

Theorem 9.10 *Suppose (M, g) is a (connected) Riemannian manifold of dimension $n \ge 2$, $g \in C^0(M)$, and M has a weakly hyperboloidal end $\widetilde{M}$. Suppose also the connection $\nabla_i = \partial_i - \Gamma_i$ on E satisfies $\Gamma \in L^{n^*}_{\mathrm{loc}}(M)$ and the decay condition*

$$\limsup_{x \to 0} |x\Gamma^S_x| < \frac{n-1}{2} \tag{9.12}$$

in $\widetilde{M}$, where Γ^S_x is the symmetric part of ∇_{∂_x}, with norm understood as that of an endomorphism of fibres of E. Then M admits a weighted Poincaré inequality.

Proof: This is essentially McKean's inequality [22]; we follow the proof in [8]. Let, first, f be a function in $C^1([0, x_0] \times \mathcal{N})$ with $f = 0$ at $\{x = 0\}$; we have

$$\begin{aligned} f^2(x, v) &= 2\int_0^x f(s, v)\frac{\partial f(x, v)}{\partial x} ds \\ &\le \frac{n-1}{2}\int_0^x \frac{f^2(s, v)}{s} ds + \frac{2}{n-1}\int_0^x s\left(\frac{\partial f}{\partial x}(s, v)\right)^2 ds \ . \end{aligned} \tag{9.13}$$

Here we use the symbol v to label points in $\mathcal{N}$. Integrating on $[0,x_0]\times\mathcal{N}$, a change of the order of integration in x and s together with some obvious manipulations gives

$$
\begin{aligned}
\int_{[0,x_0]\times\mathcal{N}} f^2\, x^{-n} dx\, d\mu_h &\le \frac{4}{(n-1)^2}\int_{[0,x_0]\times\mathcal{N}} \left(x\frac{\partial f}{\partial x}\right)^2 x^{-n} dx\, d\mu_h \\
&\le \frac{4}{(n-1)^2}\int_{[0,x_0]\times\mathcal{N}} \mathring{g}(df,df)\, x^{-n} dx\, d\mu_h\ . \qquad (9.14)
\end{aligned}
$$

This is the desired inequality on $\tilde{M}$ with metric $\mathring{g}$ for functions, with weight function $w=(n-1)^2/4$. The result for general weakly asymptotically hyperboloidal metrics and for functions follows immediately from the above, using uniform equivalence of g with $\mathring{g}$ on the asymptotic region, and using Lemma 9.1.

Let, finally, v be a smooth compactly supported section of a Riemannian bundle with not-necessarily-compatible connection ∇. Let ϕ be any smooth compactly supported function equal to 1 on the support of v, set

$$
f_\epsilon = \phi\sqrt{\epsilon+\langle v,v\rangle}\ .
$$

We have

$$
\begin{aligned}
\left|\frac{\partial f_\epsilon}{\partial x}\right|^2 &= |d_x\phi|^2(\epsilon+\langle v,v\rangle)+\phi^2\frac{\langle v,(\nabla_x+\Gamma^S_x)v\rangle\langle v,(\nabla_x+\Gamma^S_x)v\rangle}{\epsilon+\langle v,v\rangle} \\
&\le \epsilon|d\phi|^2+\phi^2|(\nabla_x+\Gamma^S_x)v|^2\ .
\end{aligned}
$$

The first line of (9.14) yields

$$
\begin{aligned}
\int_M f_\epsilon^2\, x^{-n}dx\, d\mu_h &= \int_M \phi^2(\epsilon+\langle v,v\rangle)\, x^{-n}dx\, d\mu_h \\
&\le \frac{4}{(n-1)^2}\int_M x^2\left(\epsilon|d\phi|^2+\phi^2|(\nabla_x+\Gamma^S_x)v|^2\right)\, x^{-n}dx\, d\mu_h\ .
\end{aligned}
$$

Passing with ϵ to zero gives

$$
\begin{aligned}
\int_M \langle v,v\rangle\, x^{-n}dx\, d\mu_h &\le \frac{4}{(n-1)^2}\int_M x^2|(\nabla_x+\Gamma^S_x)v|^2\, x^{-n}dx\, d\mu_h \\
&\le \frac{4}{(n-1)^2}\int_M \left((1+\frac{1}{\delta})|\nabla v|^2_{\mathring{g}}+x^2(1+\delta)|\Gamma^S_x v|^2 dx\right)\, x^{-n}dx\, d\mu_h\ ,
\end{aligned}
$$

for any $\delta>0$, and if condition (9.12) holds the last term can be carried over to the left hand side, leading to

$$
C^{-1}\int_M \langle v,v\rangle dv_M \quad \le \quad \int_M |\nabla v|^2_g dv_M\ .
$$

Lemma 9.1 gives then the desired inequality, with a weight function w equal to $1/C$ in the asymptotic region. ■

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