

Recent results in mathematical relativity

Piotr T. Chruściel

GR17, Dublin, July 19, 2004

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Global properties of solutions of the
vacuum Einstein equations

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the slide show for this talk can be downloaded from URL www.phys.univ-tours.fr/~piotr

Shi and Tam: Positivity of Brown-York mass

- Let (M, g) be a three dimensional compact Riemannian manifold with smooth boundary Σ and with $R = g^{ij} R_{ij} \geq 0$

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$$m_{\text{BY}} = \frac{1}{8\pi} \int_{\Sigma} (H_0 - H) d^2\mu$$

H : trace of the extrinsic curvature of Σ in M

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Theorem [Shi and Tam, math.DG/0301047]:

$$m_{\text{BY}} \geq 0$$

with equality if and only if (M, g) is a subset of Euclidean \mathbb{R}^3

proof uses a Bartnik-type extension of g ;

Yau: convexity not needed by embedding into a hyperboloid

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O'Murchadha, Szabados and Tod, gr-qc/0311006: $m_K \neq 0$ for some surfaces in Minkowski

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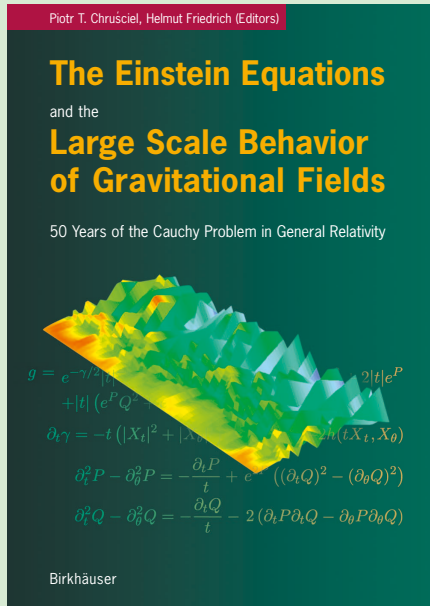
- Eardley-Moncrief version of Penrose's strong cosmic censorship hypothesis:

generic initial data (asymptotically flat, or spatially compact)
do **not** lead to Cauchy horizons

For a recent review of SCC, see L. Andersson's contribution in:

(to appear within a few weeks)

further contributions there on SCC by
Y. Choquet-Bruhat and V. Moncrief



7130_05_ug_chrusciel-friedrich

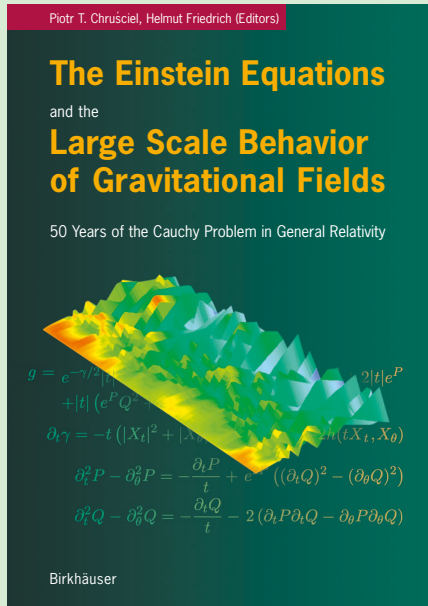
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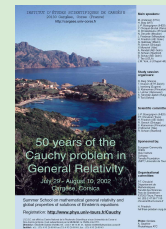
M. Anderson, T. Barbot, R. Bartnik,
H. Bray, PTC, H. Friedrich,
J. Friedman, G. Galloway, R. Geroch,
J. Isenberg, L. Lehner, O. Reula,
A. Rendall, A. Zeghib



7130_05 Ug_chrusciel-Friedrich

image: snapshot of a numerical Gowdy evolution by B. Berger

with a DVD with videos of lectures from the 2002 Cargèse Summer School



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Theorem: [H.Ringström CQG **17** (2000), 713–731]

Curvature blows up in the contracting direction in **almost all** Bianchi models

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Theorem [Ringström]:

SCC holds in this class of metrics

SCC in space-times with two Killing vectors

Previous key contributions by V. Moncrief, A. Rendall and S. Kitchenassamy, A. Rendall and M. Weaver and further related results by J. Isenberg and V. Moncrief, PTC, PTC and M. Chae

Open questions:

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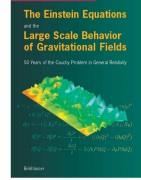
Open questions:

- What about the remaining topologies consistent with two Killing vectors? $(S^3, L(p, q), S^1 \times S^2)$
- What about 2KV models on \mathbb{T}^3 with non-vanishing twist constants? (Gowdy metrics are “zero measure” in the set of all $U(1) \times U(1)$ symmetric metrics on \mathbb{T}^3)

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- Stability in the expanding direction CMC time:

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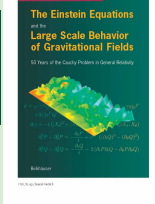
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Theorem: [Choquet-Bruhat (2003), building upon Choquet-Bruhat and Moncrief Ann. Henri Poincaré 2 (2001) 1007]

Spacetimes satisfying the vacuum Einstein equations on a manifold of the form $\Sigma \times S^1 \times \mathbb{R}$, where Σ is a compact surface of **genus $G > 1$** and where the Cauchy data is invariant with respect to $U(1)$ and sufficiently **small** exist for an **infinite proper time** in the expanding direction

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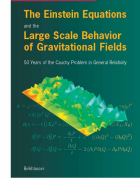
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- Nothing is known in the contracting direction
- The proof of a similar future stability theorem of Andersson and Moncrief of a class of hyperbolic models **without any symmetries** to be found in the same volume.

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Theorem: [Lindblad and Rodnianski, in preparation]

Nonlinear stability holds for the **Einstein-scalar field** equations with initial data with asymptotics

$$g = (1 + 2M/r)\delta + O(r^{-1-\alpha}), \quad K = O(r^{-2-\alpha}),$$

$$\psi_0 = O(r^{-1-\alpha}), \quad \psi_1 = O(r^{-2-\alpha}),$$

with $\alpha > 0$.

Initial data: constraint equations

Initial data (g, K) satisfy the *vacuum constraints equations*:

$$\begin{pmatrix} J \\ \rho \end{pmatrix} (K, g) := \begin{pmatrix} 2(-\nabla^j K_{ij} + \nabla_i \operatorname{tr} K) \\ R(g) - K^{ij} K_{ij} + (\operatorname{tr} K)^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

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- trapped surfaces?
- apparent horizons?

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Theorem [Corvino, Schoen, gr-qc/0301071]:

Asymptotically flat initial data with well defined mass, momentum, center of mass, and angular momentum, can be deformed in the asymptotic region to a Kerr metric, with arbitrarily small change in global charges.

(compare PTC and Delay, gr-qc/0301073)

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Theorem [PTC, Delay, CQG 19 (2002) L71-L79], gr-qc/0203053:

There exist non-trivial asymptotically simple **vacuum** space-times.

Construction of asymptotically simple vacuum space-times:

- Consider a parity-symmetric initial data set $(\mathbb{R}^3, \mathring{g}, \mathring{K} = 0)$, with ADM mass m_0

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Why a parity condition?

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to obtain $\vec{c}' = 0$, one needs to translate by

$$-\frac{\vec{c}}{m}$$

(uncontrollable if m can be very small)

Initial data engineering

- Can one glue together two vacuum initial data sets with a localised deformation near the gluing region?

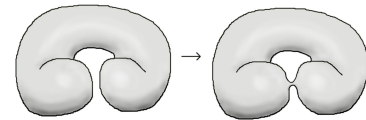


Figure 1

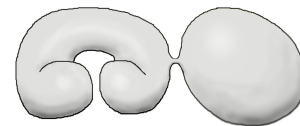
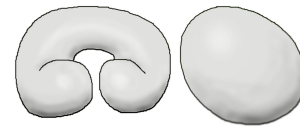


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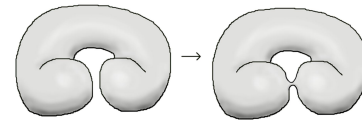


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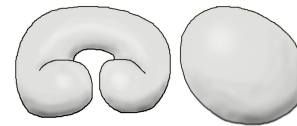


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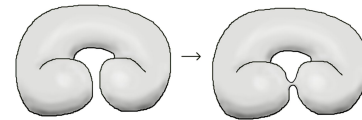


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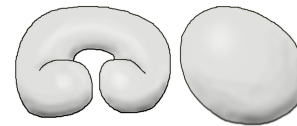


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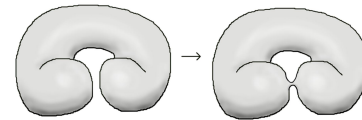


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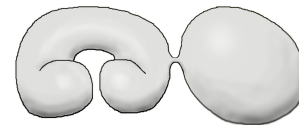
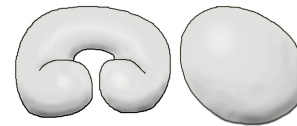


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- Application:

\exists spatially compact **vacuum** space-times without any CMC surfaces

more in

J.Isenberg's talk

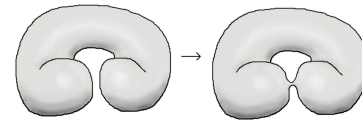


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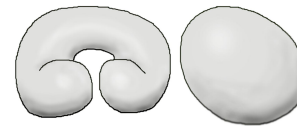


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- The Christodoulou-Klainerman theorem gives a \mathcal{I}^+ with very low differentiability — not enough for peeling theorems (does one care? enough e.g. for the Trautman-Bondi mass-loss formula)

Asymptotic structure

Theorem: [Klainerman and Nicolò CQG 20 (2003) 3215]

Peeling holds for initial data with asymptotics

$$\mathbf{g} = \left(\mathbf{1} + \frac{\mathbf{M}}{2\mathbf{r}} \right)^4 \delta + \mathbf{O}(\mathbf{r}^{-3-\alpha}), \quad \mathbf{K} = \mathbf{O}(\mathbf{r}^{-4-\alpha}), \quad \alpha > 0.$$

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Theorem [Valiente-Kroon gr-qc/0310048]:

Bowen-York and **Brill-Lindquist** initial data do **not** lead to a smooth \mathcal{I}^+ , if any

Asymptotic structure

In Friedrich's Cargèse lectures and in

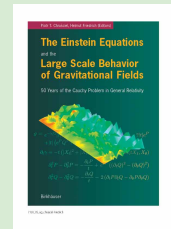
Valiente Kroon CMP 244 (2004) 133, gr-qc/0211024 the following conjecture is made:

Conjecture:

Time symmetric ($K_{ij} \equiv 0$)
initial data lead to a \mathcal{I} of differentiability \mathbf{C}^k

if and only if

the data are **static** up to terms $O(r^{-N(k)})$
for some $N(k)$



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Standard theory [Fischer, Kato, Marsden]: local existence of so-

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Let $\epsilon > 0$, the existence time of the solution can be bounded from below by a function that depends only upon $\|(g, K)\|_{H_{2+\epsilon} \times H_{1+\epsilon}}$

Related results on nonlinear wave equations by Bahouri and Chemin, Smith and Tataru

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L^2 conjecture [Klainerman]:

L^2 regularity of the Riemann tensor should suffice for existence

some partial results towards the conjecture in Klainerman and Rodnianski math.AP/0308123

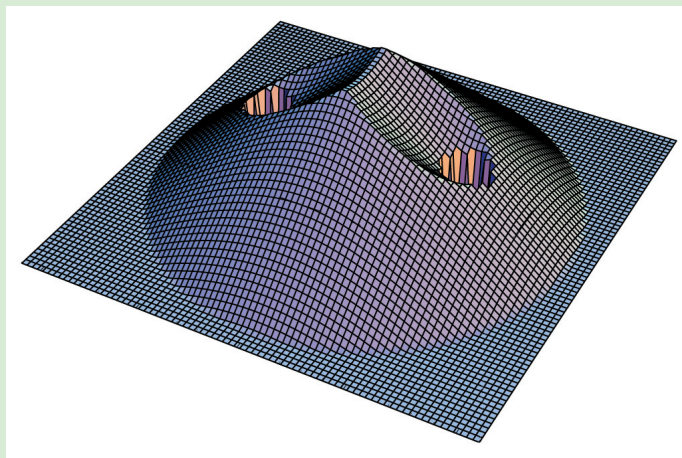
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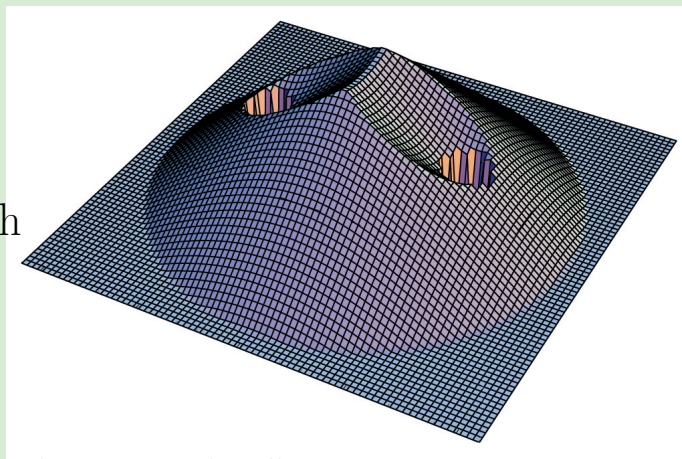
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- Recall Brill-Lindquist :

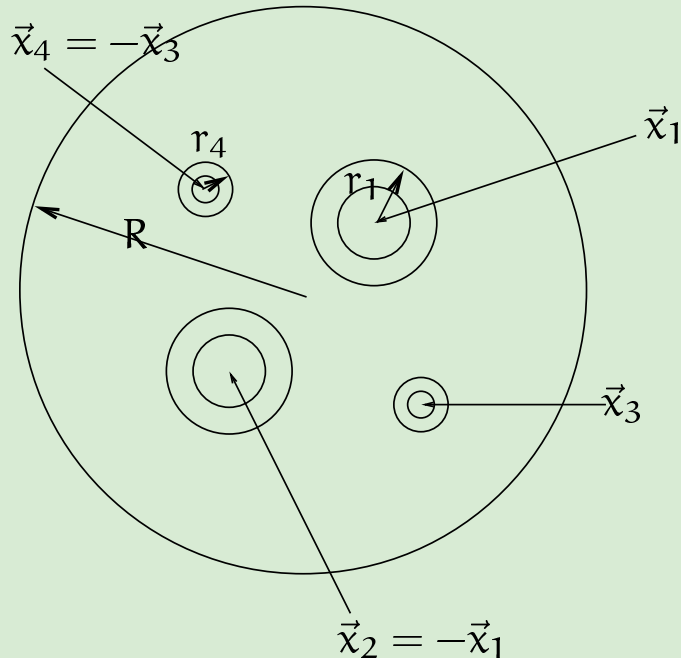
$g = \psi^4(dx^2 + dy^2 + dz^2)$, with

$$\psi = 1 + \sum_{i=1}^I \frac{m_i}{2 |\vec{x} - \vec{x}_i|} .$$

problems with smoothness of Scri? cf. J.Valiente Kroon's talk



The “many Schwarzschild metrics” (PTC, E.Delay, CQG, gr-qc/0203053) a solution exists when $0 \leq m_i \leq \delta$, δ small enough



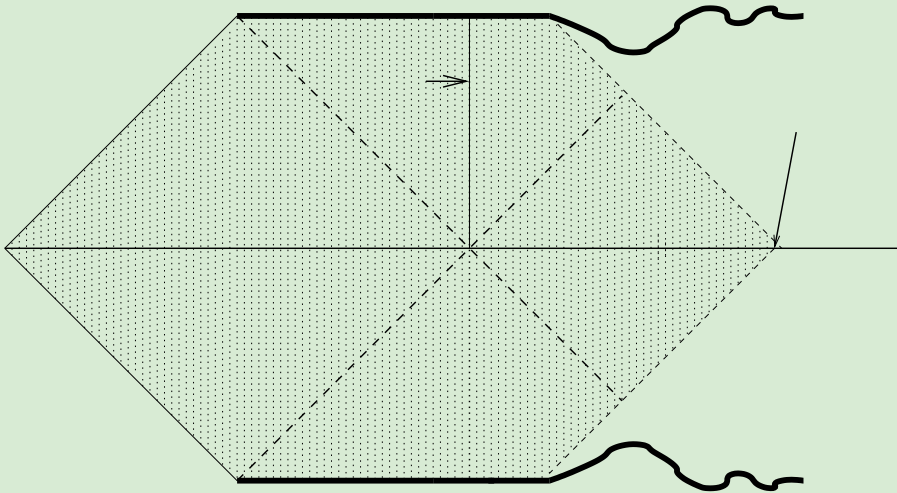
Free parameters: R and (\vec{x}_1, r_1, m_1) , (\vec{x}_3, r_3, m_3)

$$m_2 = m_1, \quad r_2 = r_1$$

$$m_4 = m_3, \quad r_4 = r_3$$

inner small circles are Schwarzschild horizons; metric *exactly* *Schwarzschild* within the outer small circles and outside the big circle

What about event horizons? The resulting space-times are “small”:



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Theorem (Schoen, unpublished): yes, with H **outermost, smooth**

proof using Jang’s equations; previous proof for $K_{ij} \equiv 0$ by Huisken and Ilmanen, and in a support sense (no regularity) by Fu and Howard (unpublished)