

# Fully Packed Loop Configurations and Littlewood Richardson coefficients

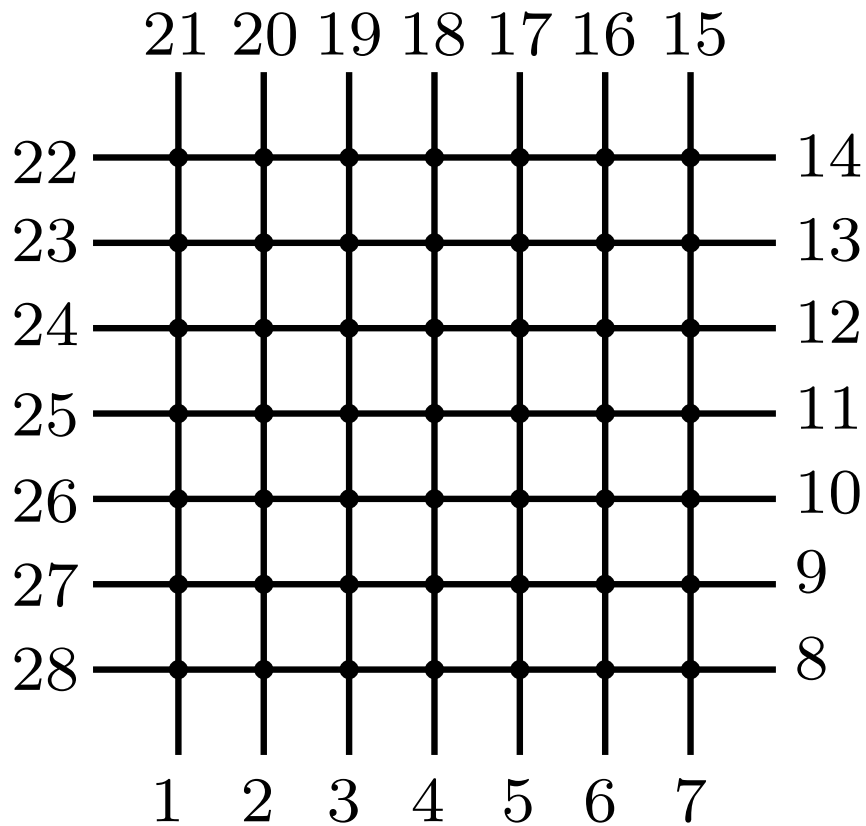
Philippe Nadeau

Faculty of Mathematics, University of Vienna

MIT Combinatorics seminar, February 24th, 2010

# FPL configurations : Definition

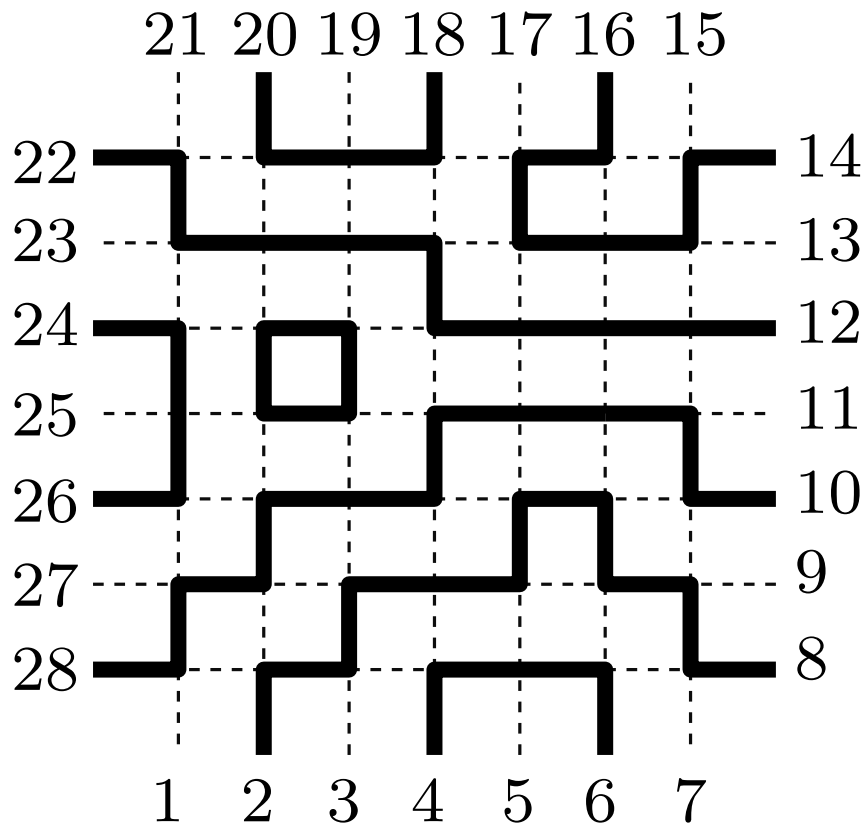
Start with the **square grid**  $G_n$  with  $n^2$  vertices and  $4n$  external edges. In the example, we have  $n = 7$ .





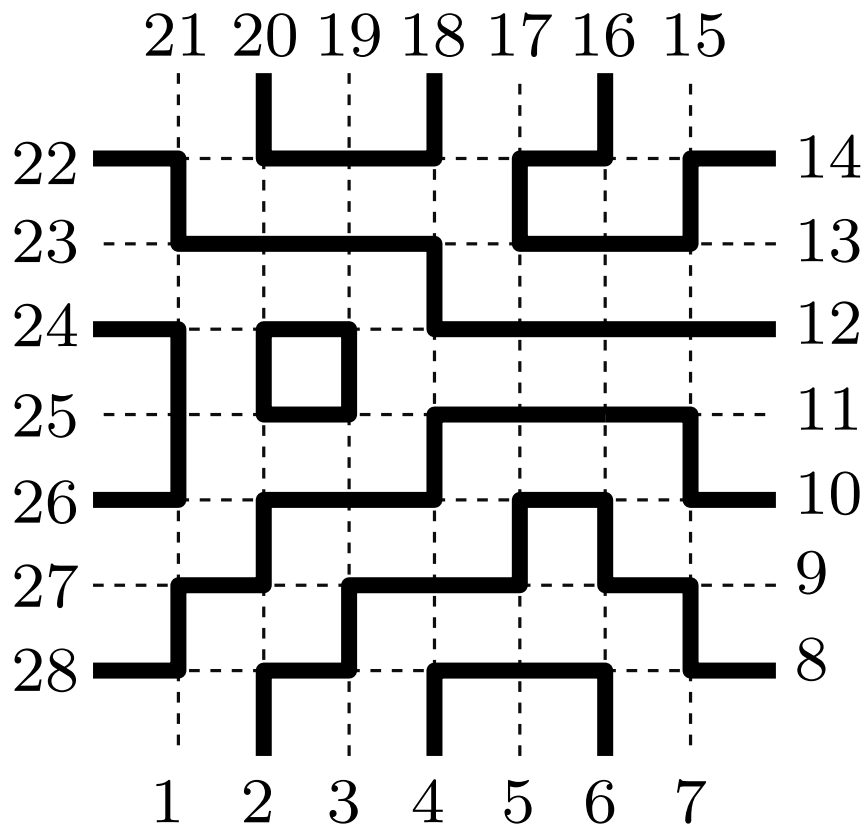
# FPL configurations : Enumeration

Such FPL configurations are in simple bijection with numerous objects : alternating sign matrices, height matrices, configurations of the six vertex model, Gog triangles,...



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FPL of size  $n$  with even boundary

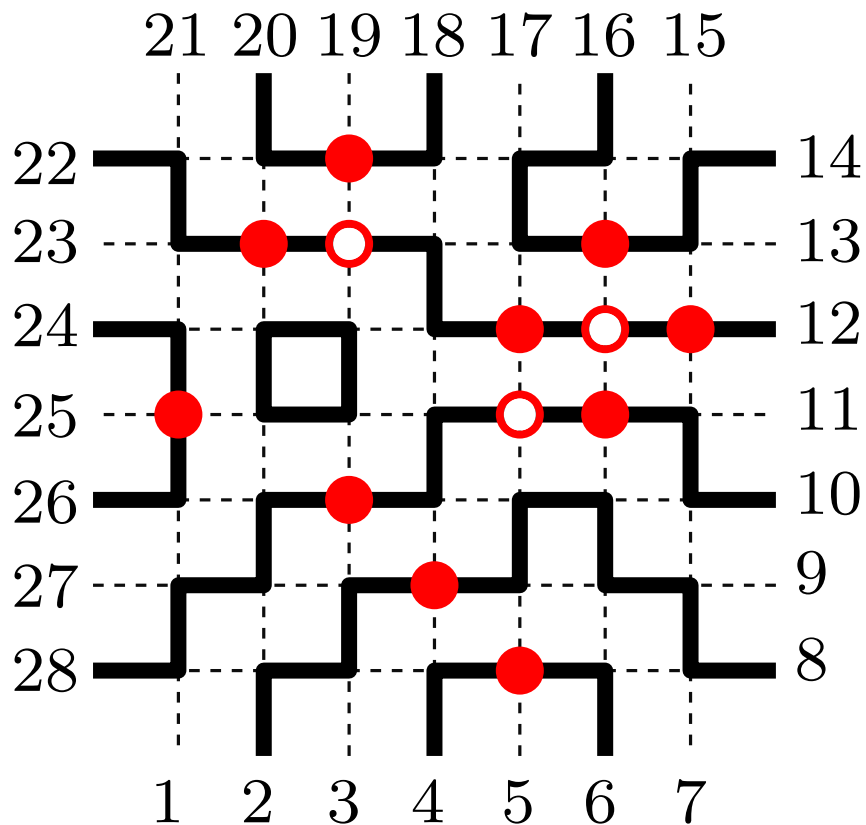
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An ASM is a square matrix with coefficients in  $\{1, 0, -1\}$  such that on each row or column **1 and  $-1$  alternate**, and the **sum is 1**.

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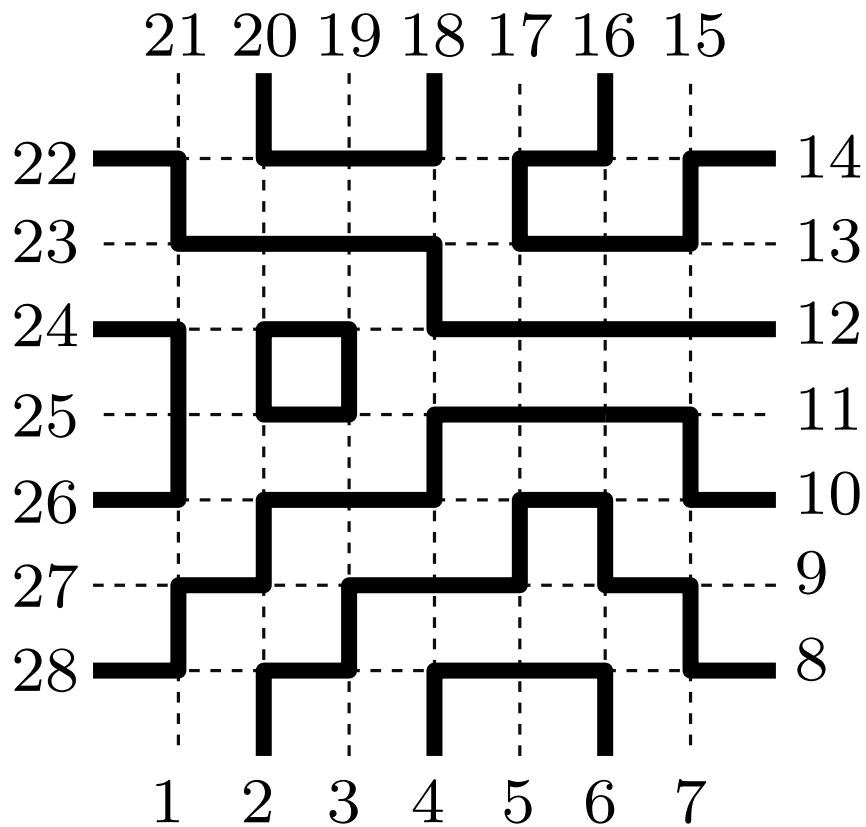
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Here  $1 \rightarrow \bullet$  and  $-1 \rightarrow \circ$

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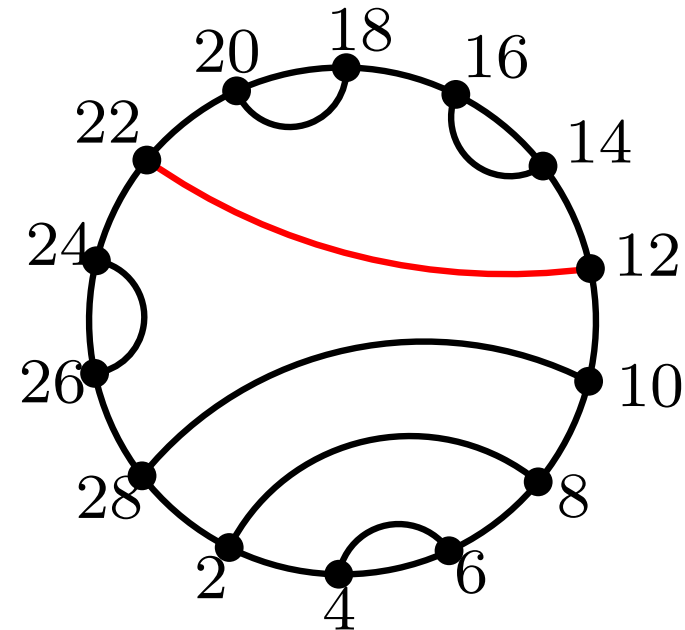
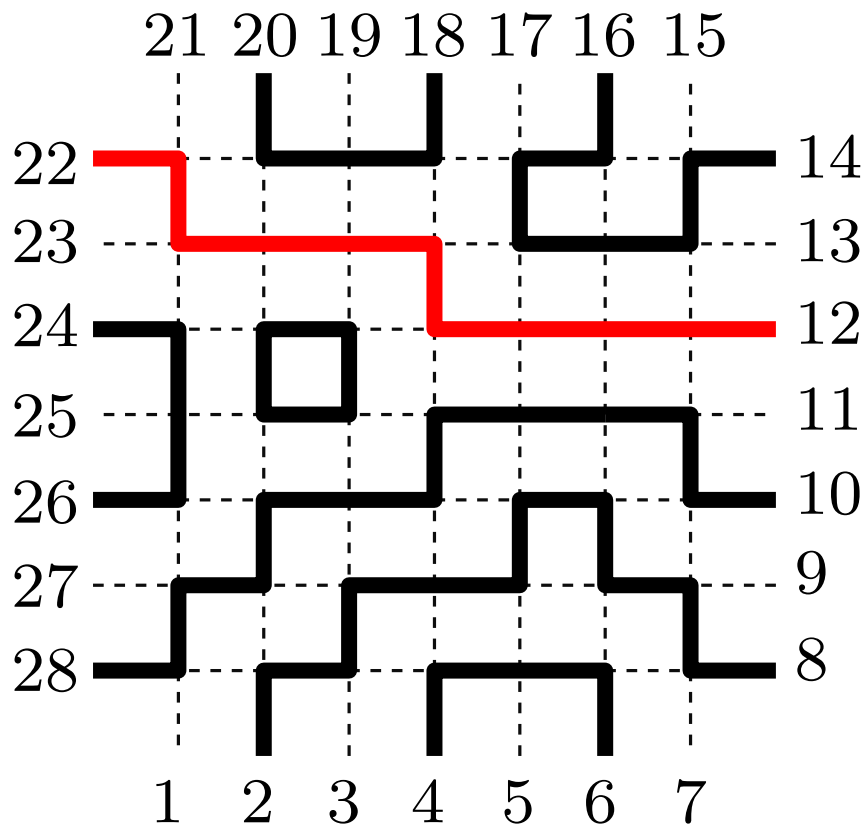
Alternating sign matrices of size  $n$

$$|FPL_n| = A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

[Zeilberger '96, Kuperberg '96]

# FPL configurations : Refined enumeration

Every FPL configuration determines a **link pattern** on the odd or even external edges of the grid  $G_n$ .



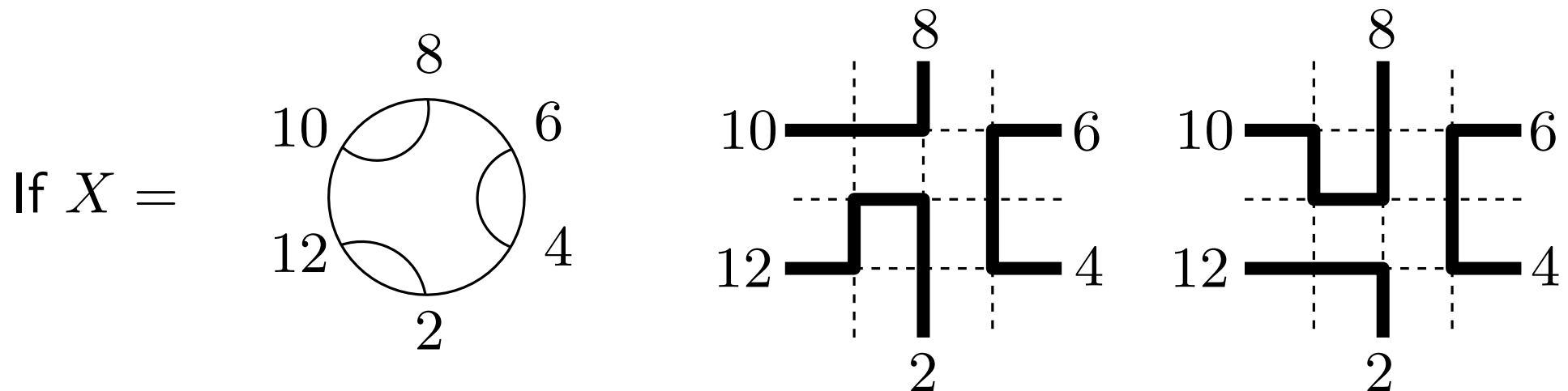
Link pattern = set of  $n$  noncrossing chords between  $2n$  points on a disk

$$|LP_n| = C_n := \frac{1}{n+1} \binom{2n}{n}$$

# FPL configurations : Refined enumeration

Now if we are given a pairing  $X$  of external edges, our main question will be : **how many FPL configurations induce the link pattern  $X$  ?**

**Definition** We note  $A_X$  this number.



For this link pattern we have  $A_X = 2$ .

# FPL configurations : Refined enumeration

Now given a link pattern  $X$ , let  $X'$  be defined by

$$(i, j) \in X' \Leftrightarrow (i - 1, j - 1) \in X$$

**Theorem [Wieland '00]**

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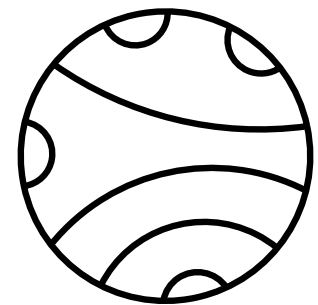
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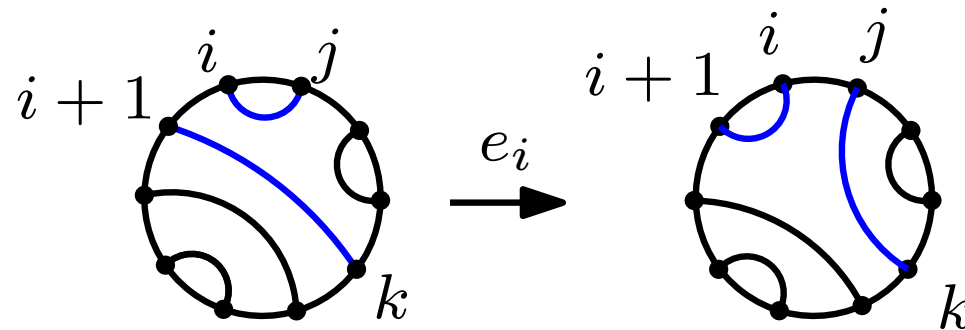
This means that “rotating the link pattern” does not change the number of FPL configurations attached to it.

For enumeration purposes, we can then use **unlabeled** link patterns :



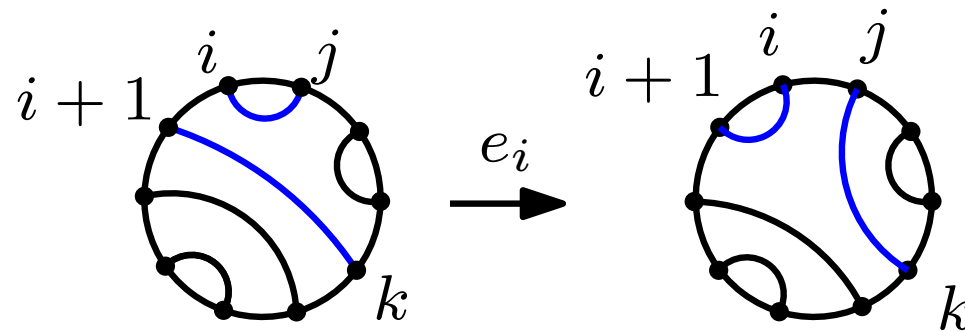
# Motivation : the Razumov-Stroganov conjecture.

**Definition** : We define operators  $e_i$  on link patterns for  $i = 1 \dots 2n$  by  $\{i, j\}, \{i + 1, k\} \in X \rightarrow \{i, i + 1\}, \{j, k\} \in e_i(X)$ .



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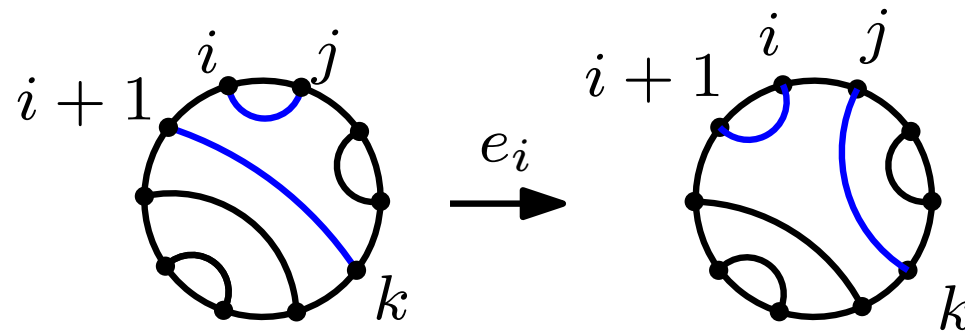


## Markov chain $\mathcal{M}$

- **States** =  $LP_n$ ;
- **Transition probabilities** :  $P(X \rightarrow Y) = \frac{k}{2n}$  where  $k$  is the number of  $i \in \{1, \dots, 2n\}$  such that  $e_i(X) = Y$ .

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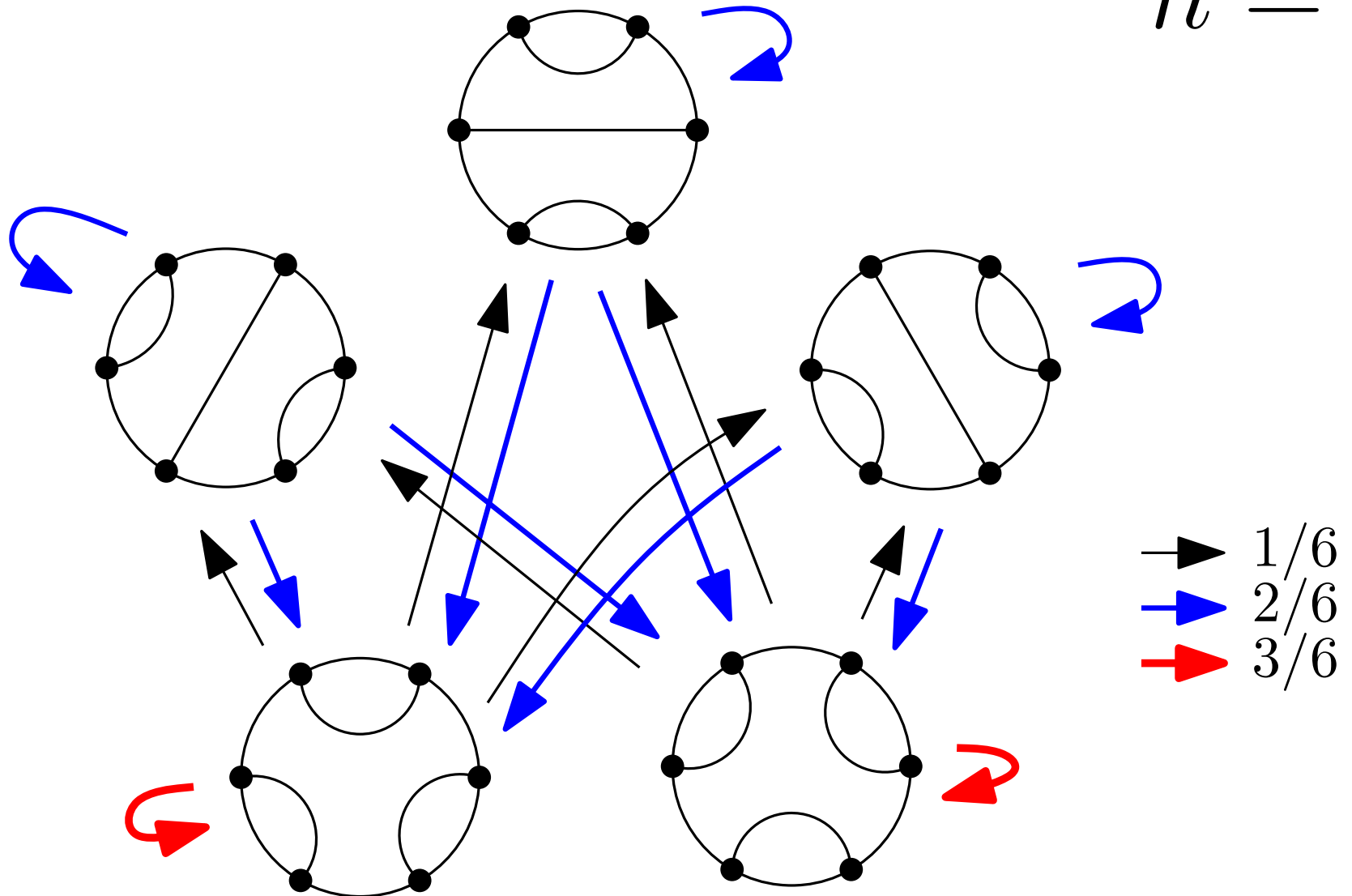
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## Stationary distribution $(\psi_X)$ of $\mathcal{M}$

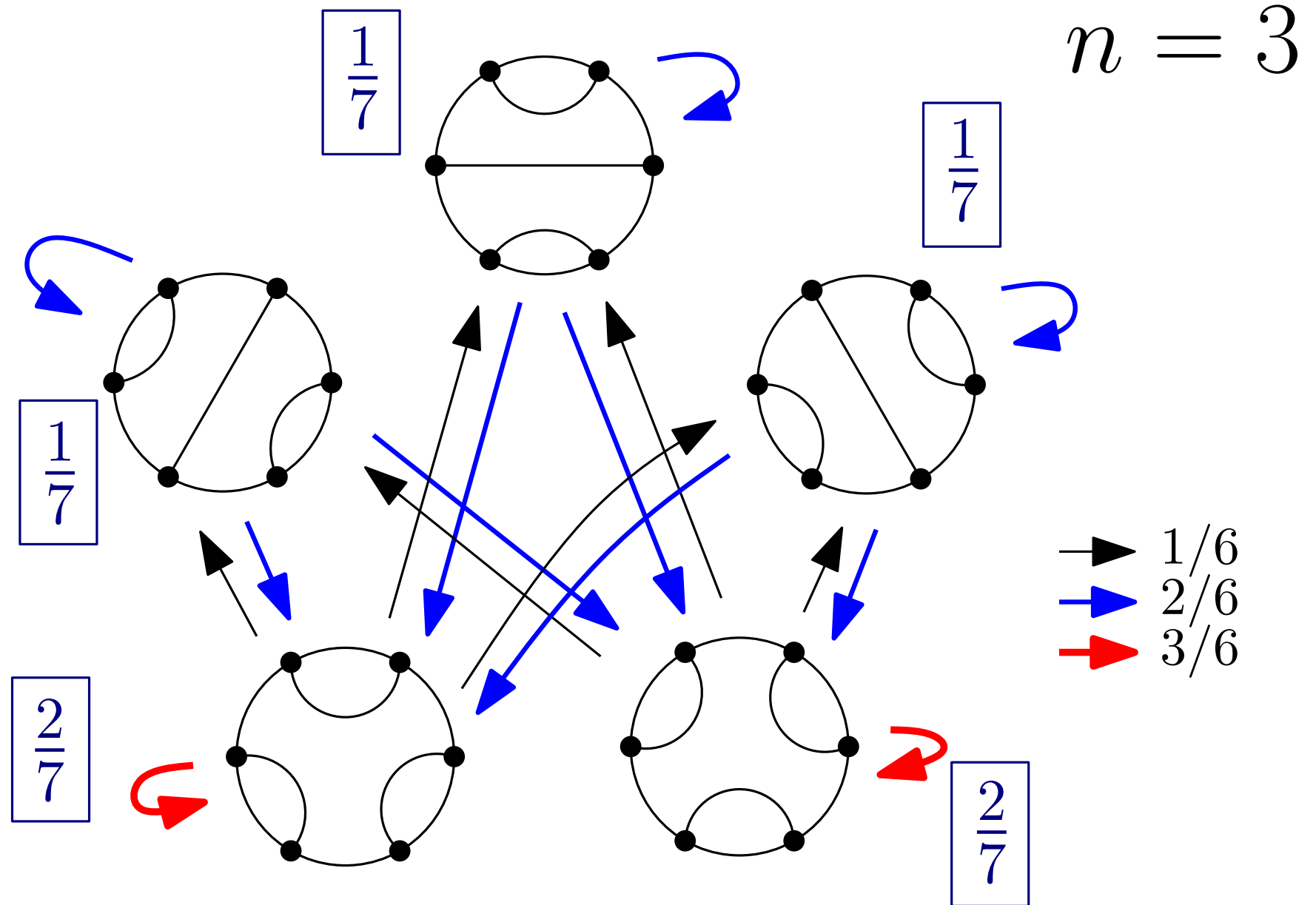
Let  $P$  be the matrix defined by  $P_{XY} = P(X \rightarrow Y)$  where  $X, Y \in LP_n$ . Then there is a unique probability distribution  $(\psi)_X$  on  $LP_n$  such that  $P\psi = \psi$ .

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$$n = 3$$



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**RS conjecture** : The stationary distribution  $(\psi_X)_{X \in LP_n}$  is given by

$$\psi_X = \frac{A_X}{A_n}$$

Another formulation is :  $\forall X, \quad 2nA_X = \sum_{(i,Y), e_i(Y)=X} A_Y$

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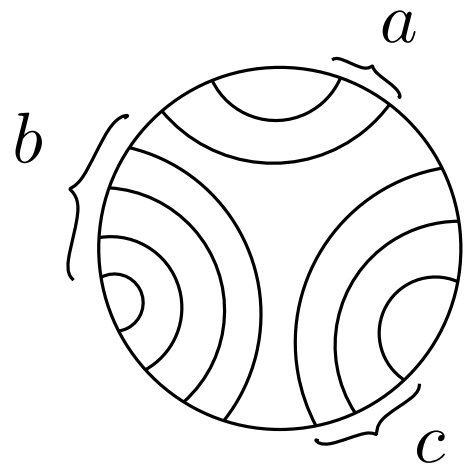
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The numbers  $\psi_X$  were studied in detail by Di Francesco and Zinn-Justin

→ integral expressions (up to a change of basis), multivariate versions, computation in special cases.

For the numbers  $A_X$ , very little is known in contrast.

## Special cases for $A_X$



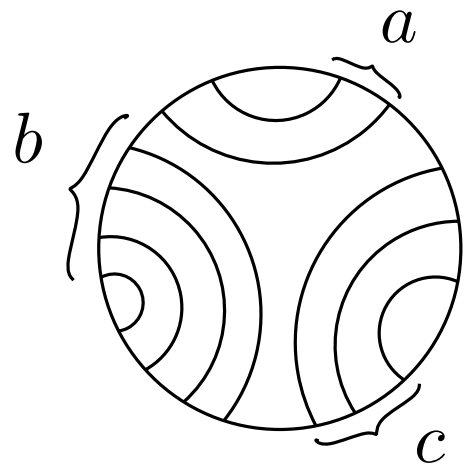
$$= \prod_{i=1}^a \prod_{j=1}^b \prod_{k=1}^c \frac{i + j + k - 1}{i + j + k - 2}$$

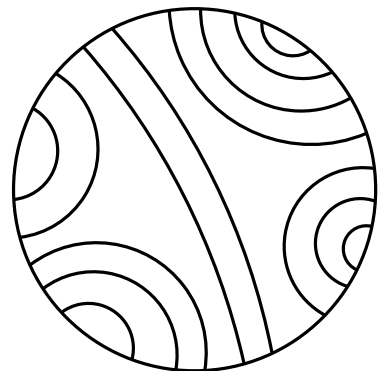


$$= \text{Complicated determinant formulas}$$

[Zinn-Justin, Zuber, Di Francesco, Caselli, Krattenthaler]

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In this talk we will describe a possible approach for the computation of  $A_X$ .

# Outline of the talk

## (0) Long introduction

Why do we want to count FPLs with a given link pattern?

## (1) From the square to the triangle

We will explain a formula expressing numbers  $A_X$  in terms of FPL configurations in a certain triangle (TFPL), which uses *link patterns with nested arches*.

## (2) FPL configurations in a triangle

We will collect various formulas and relations for FPL configurations in the triangle.

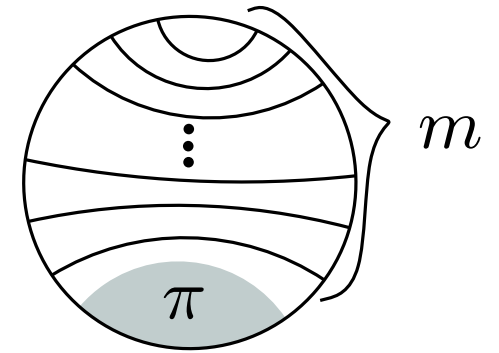
## (3) Extremal TFPL configurations

In a special case, we will show that TFPL configurations are enumerated by the famous **Littlewood-Richardson coefficients**.

(1) From the square to the triangle

# Link patterns with nested arches

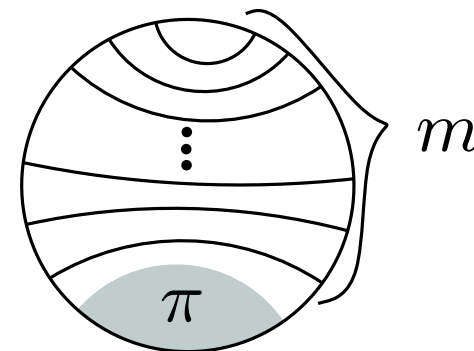
We consider now integers  $n, m \geq 0$ , and link patterns with  $m$  nested arches, and  $\pi$  is a **noncrossing matching** with  $n$  arches.



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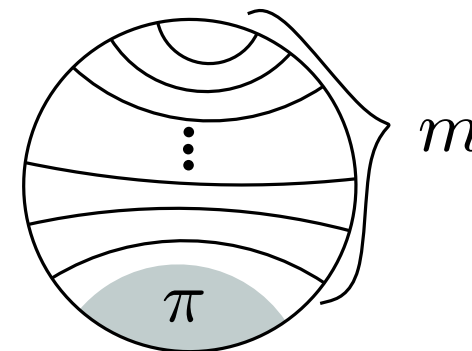


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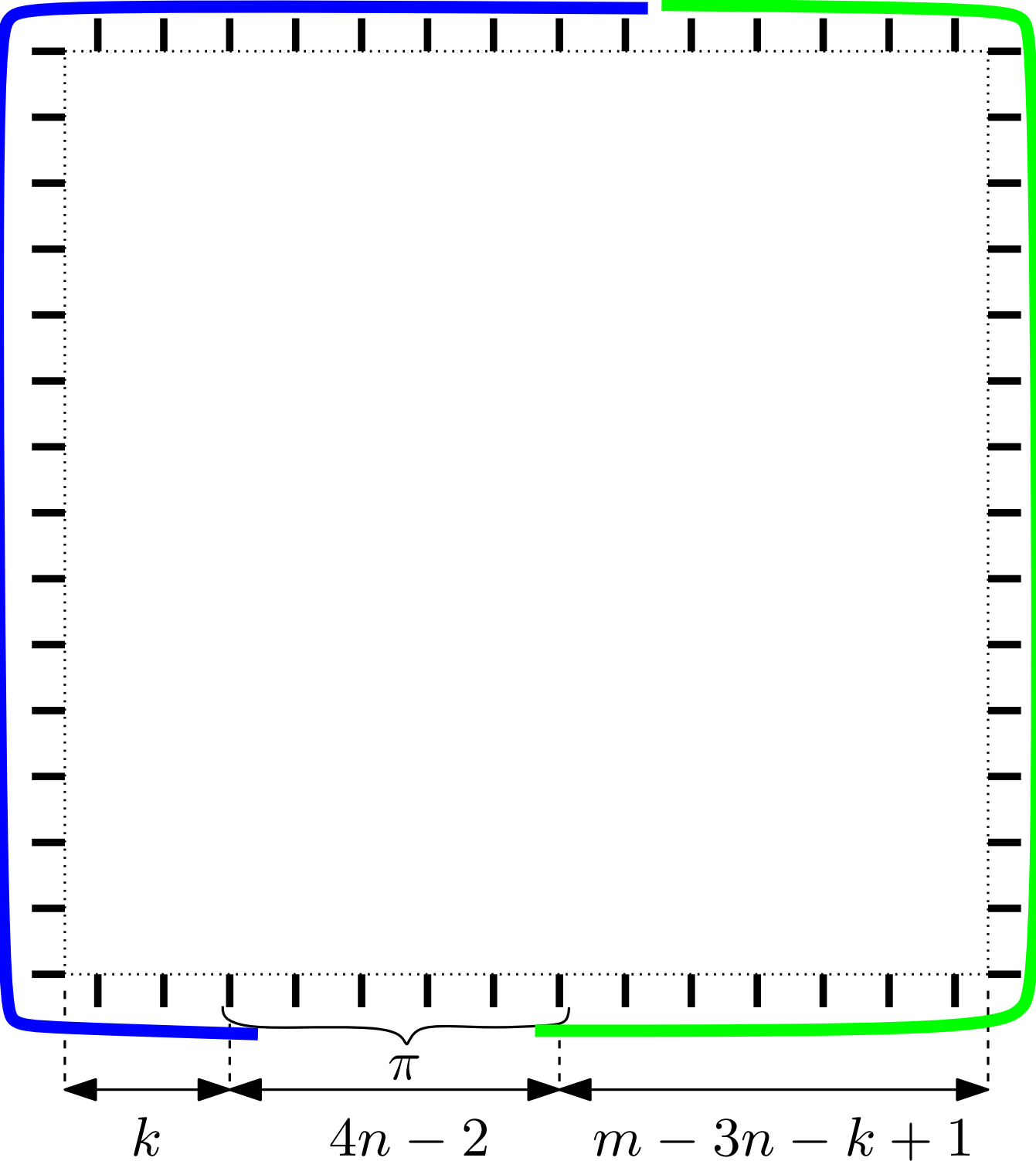
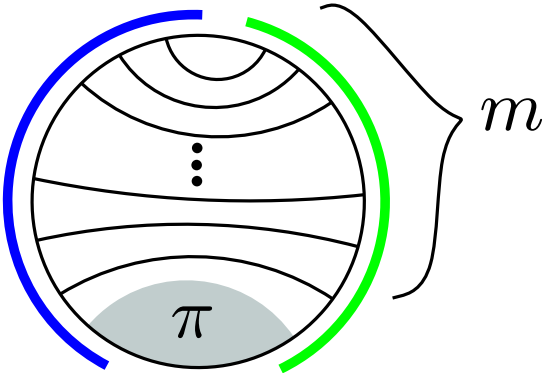


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**Idea** : for  $m$  large enough, we derive an expression for  $A_{\pi}(m)$  based on a certain combinatorial decomposition. It turns out that the expression is actually valid for all  $m \geq 0$ .

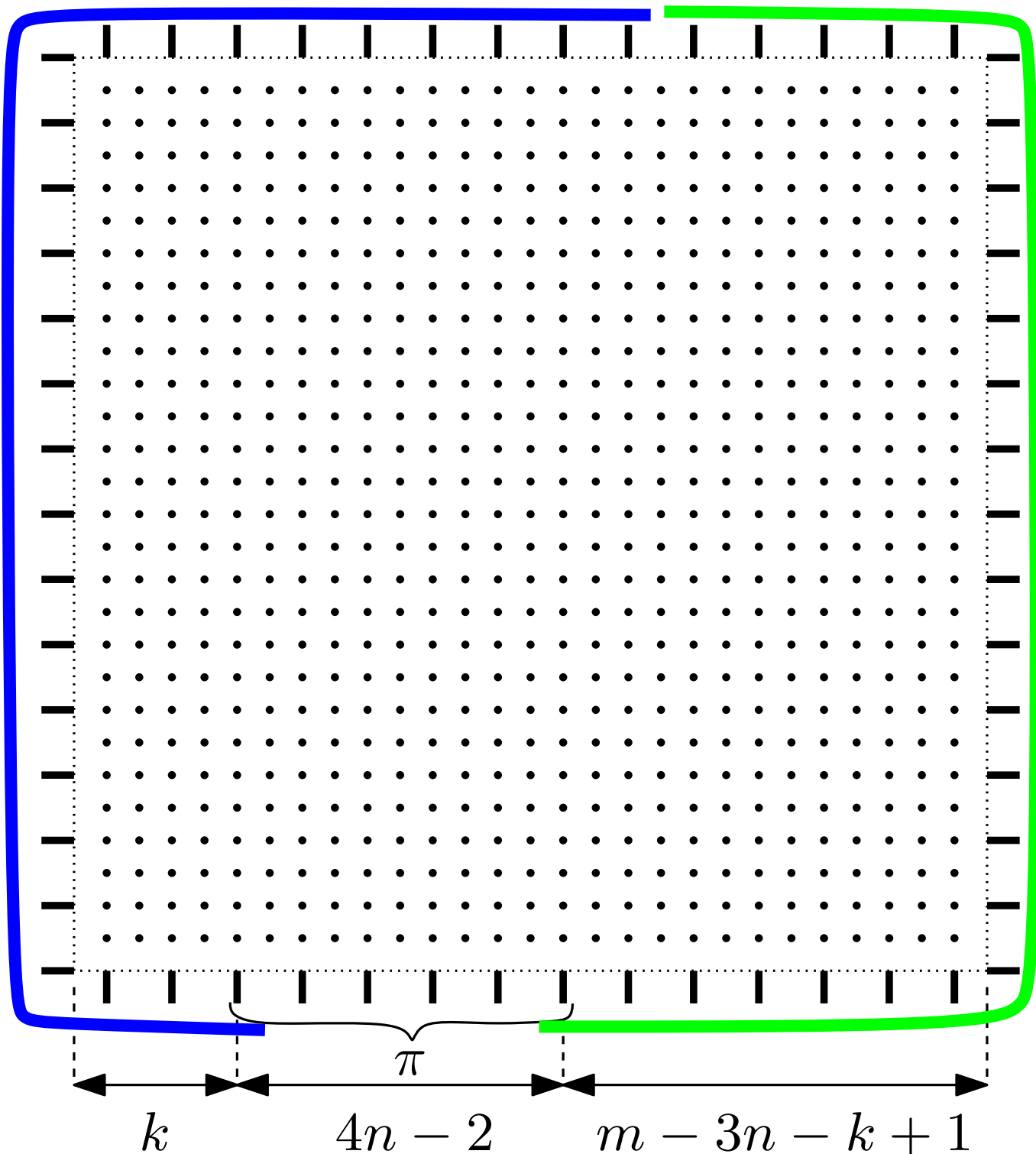
We suppose  $m \geq 3n - 1$ ,  
 and choose  $k$  such that  
 $0 \leq k \leq m - (3n - 1)$ .



Many edges of the grid belong to every FPL configuration respecting the link pattern.

⇒ "Fixed edges"

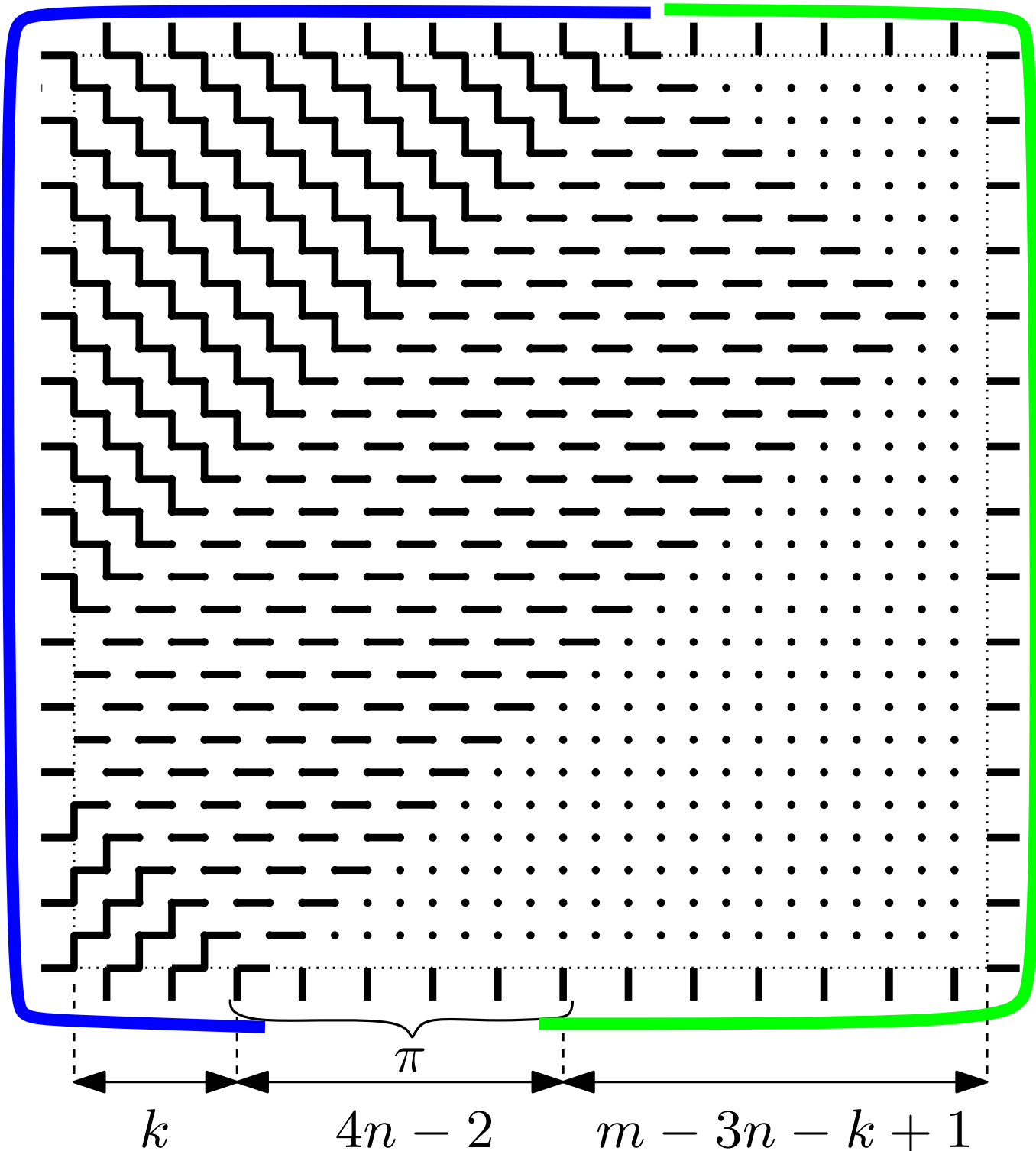
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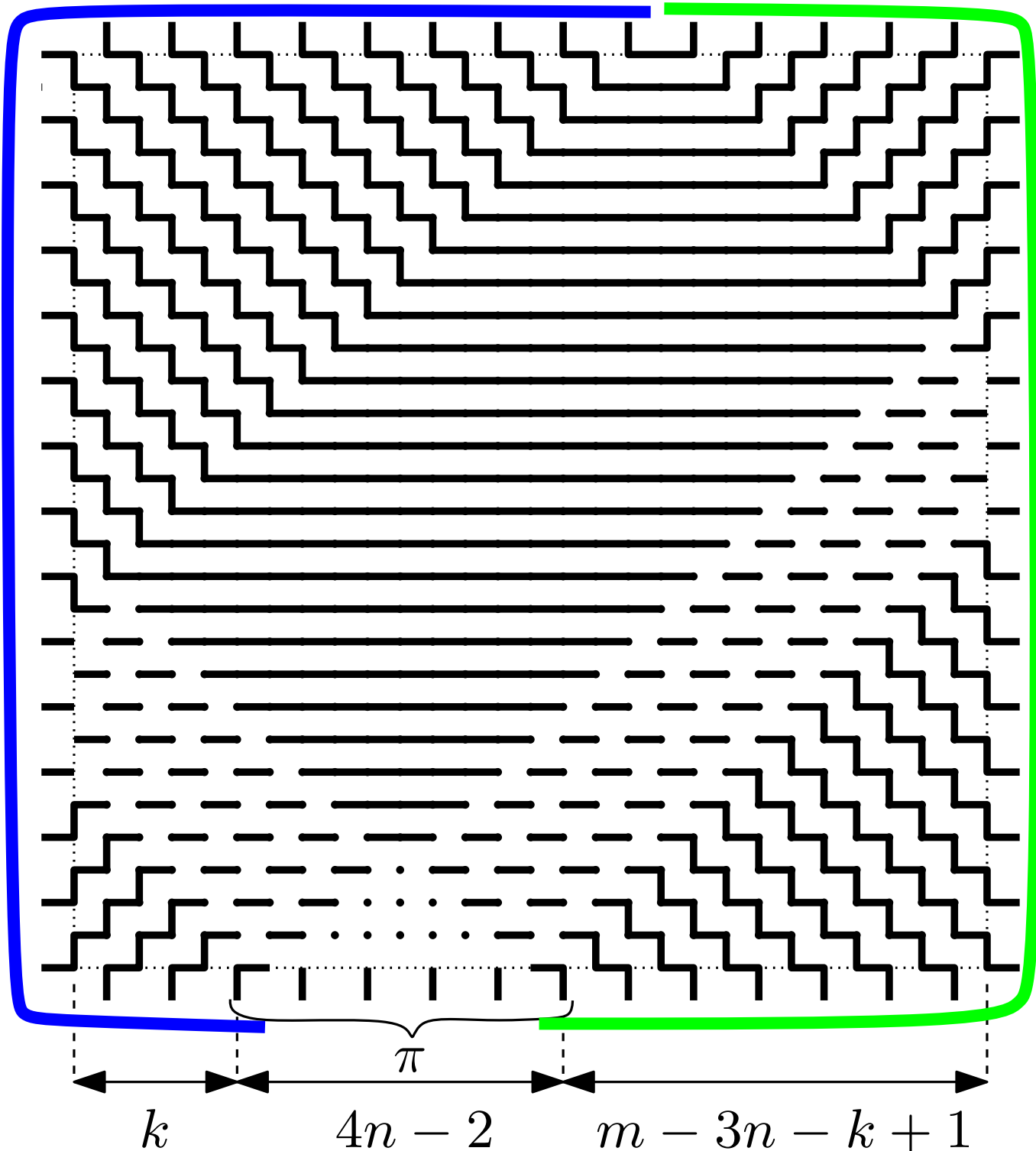
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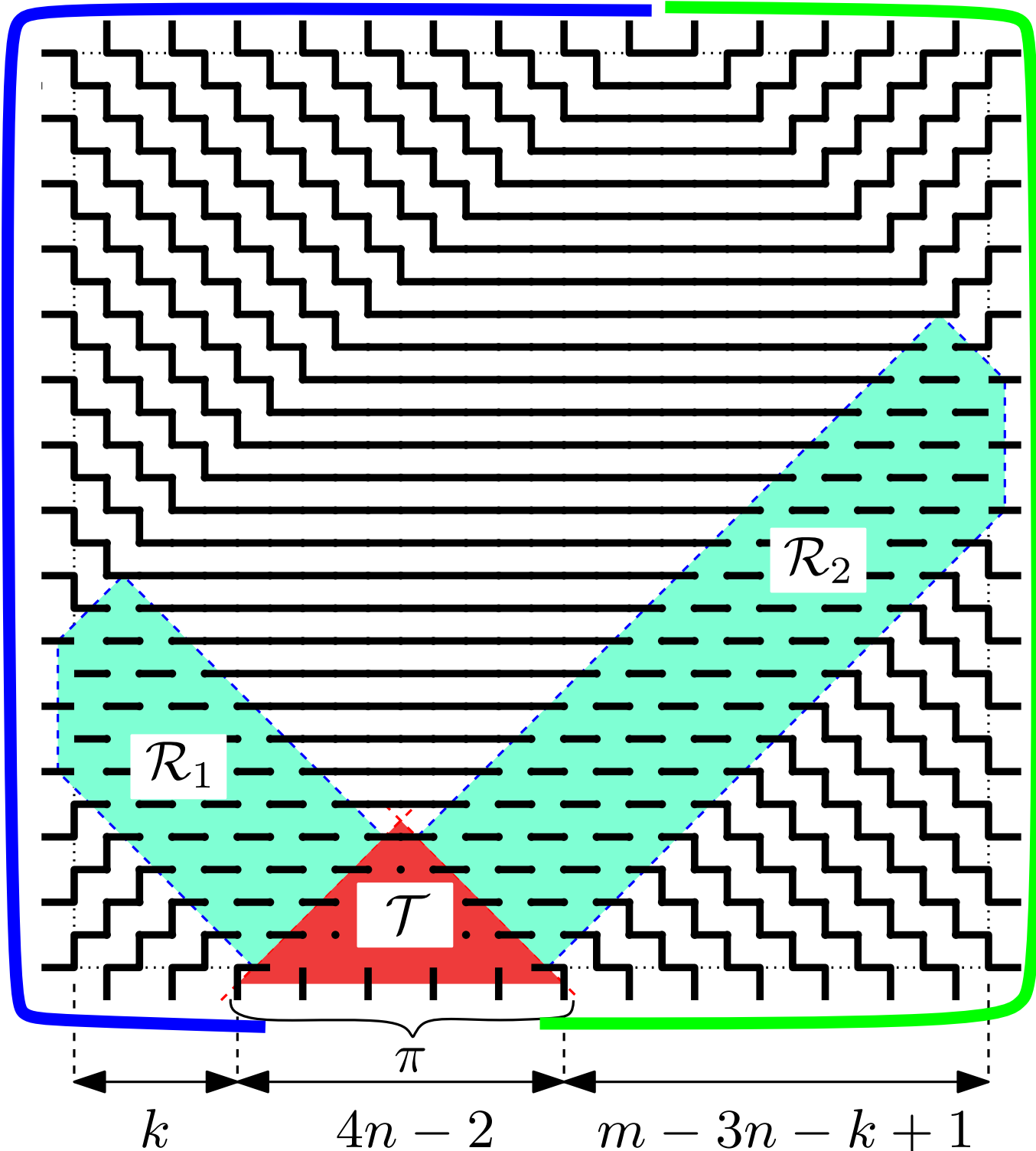
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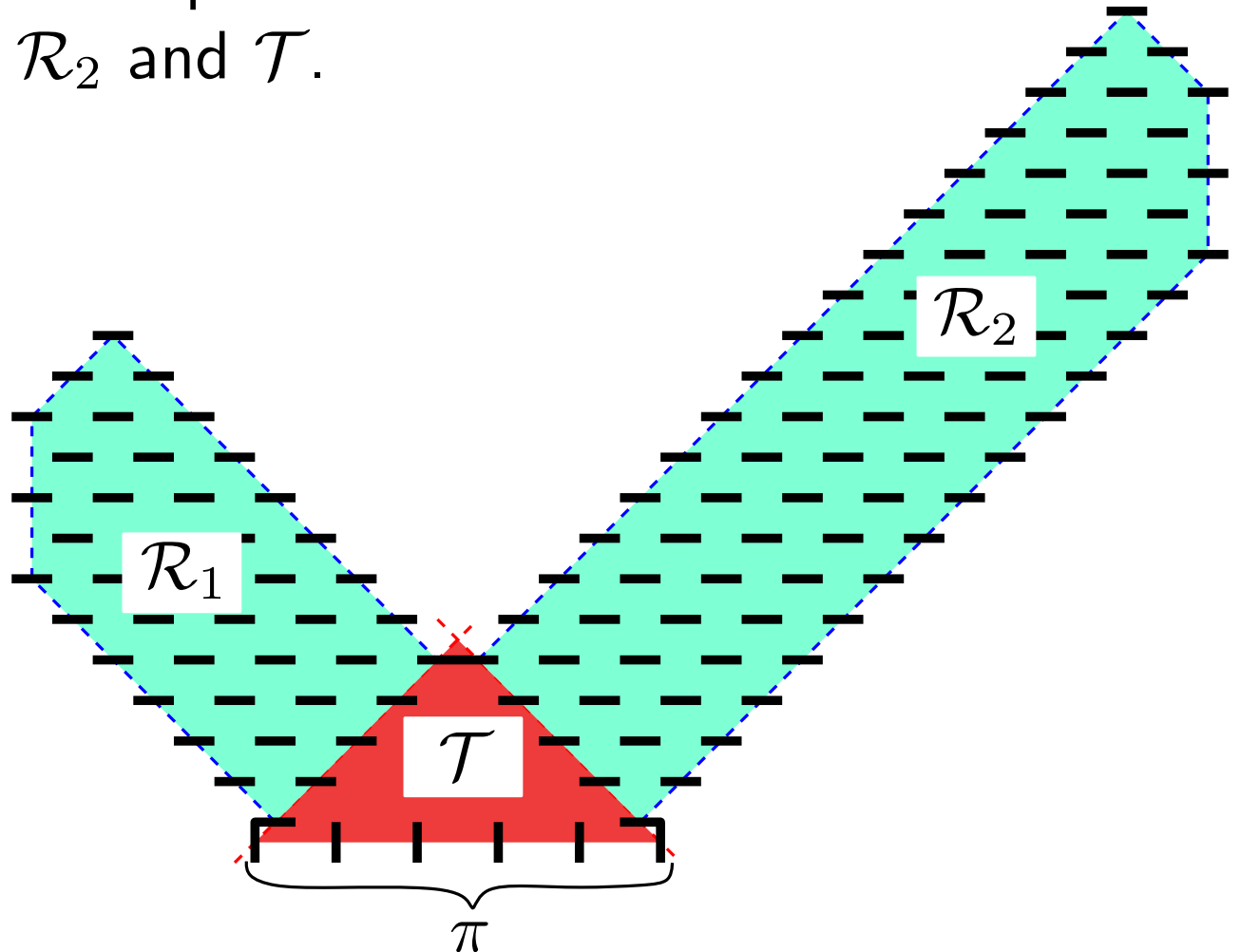
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To compute the numbers  $A_\pi(m)$ , we will count FPL configurations separately in  $\mathcal{R}_1, \mathcal{R}_2, \mathcal{T}$ .

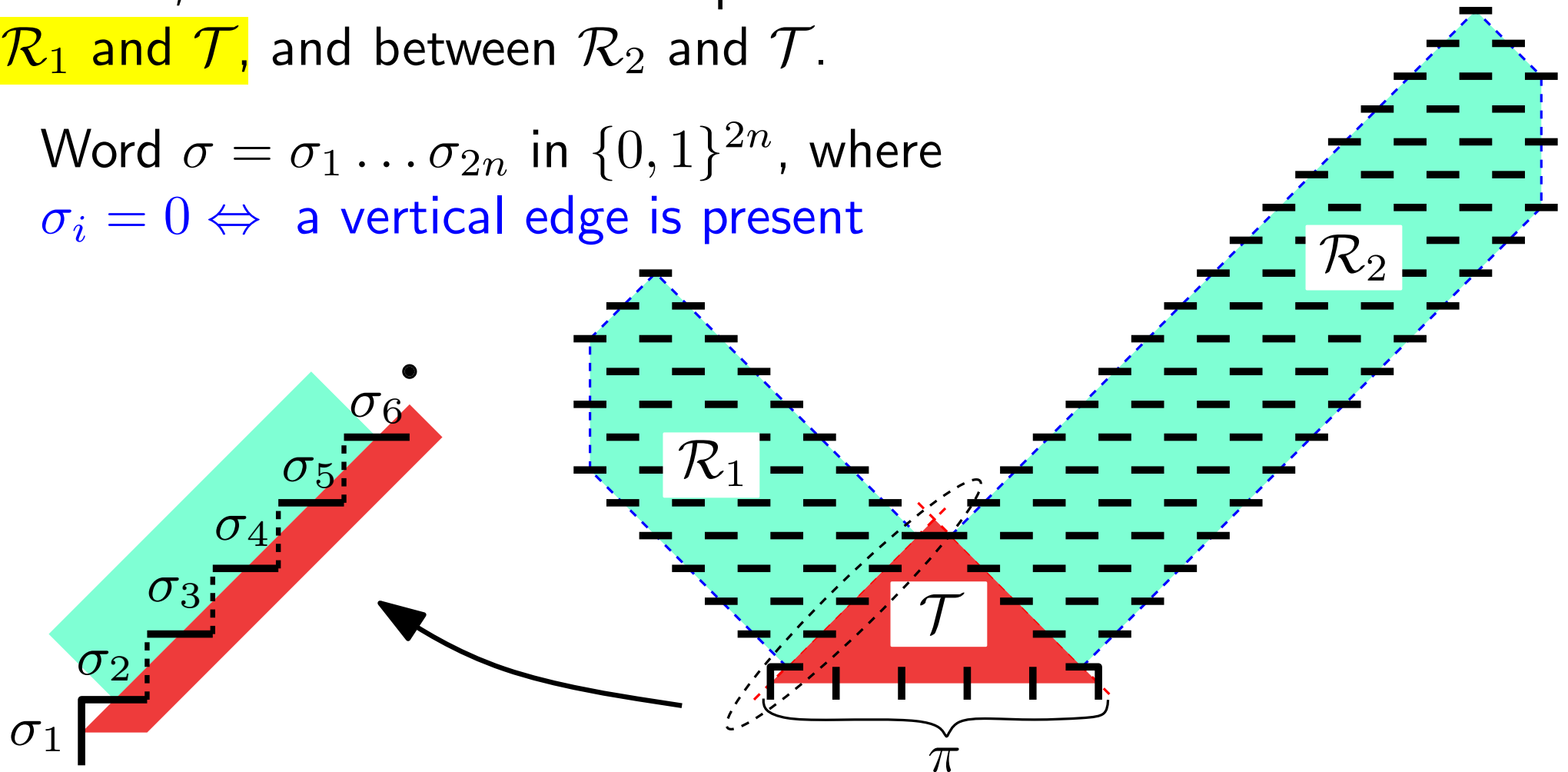
For this, we need to encode the possible boundaries between  $\mathcal{R}_1$  and  $\mathcal{T}$ , and between  $\mathcal{R}_2$  and  $\mathcal{T}$ .



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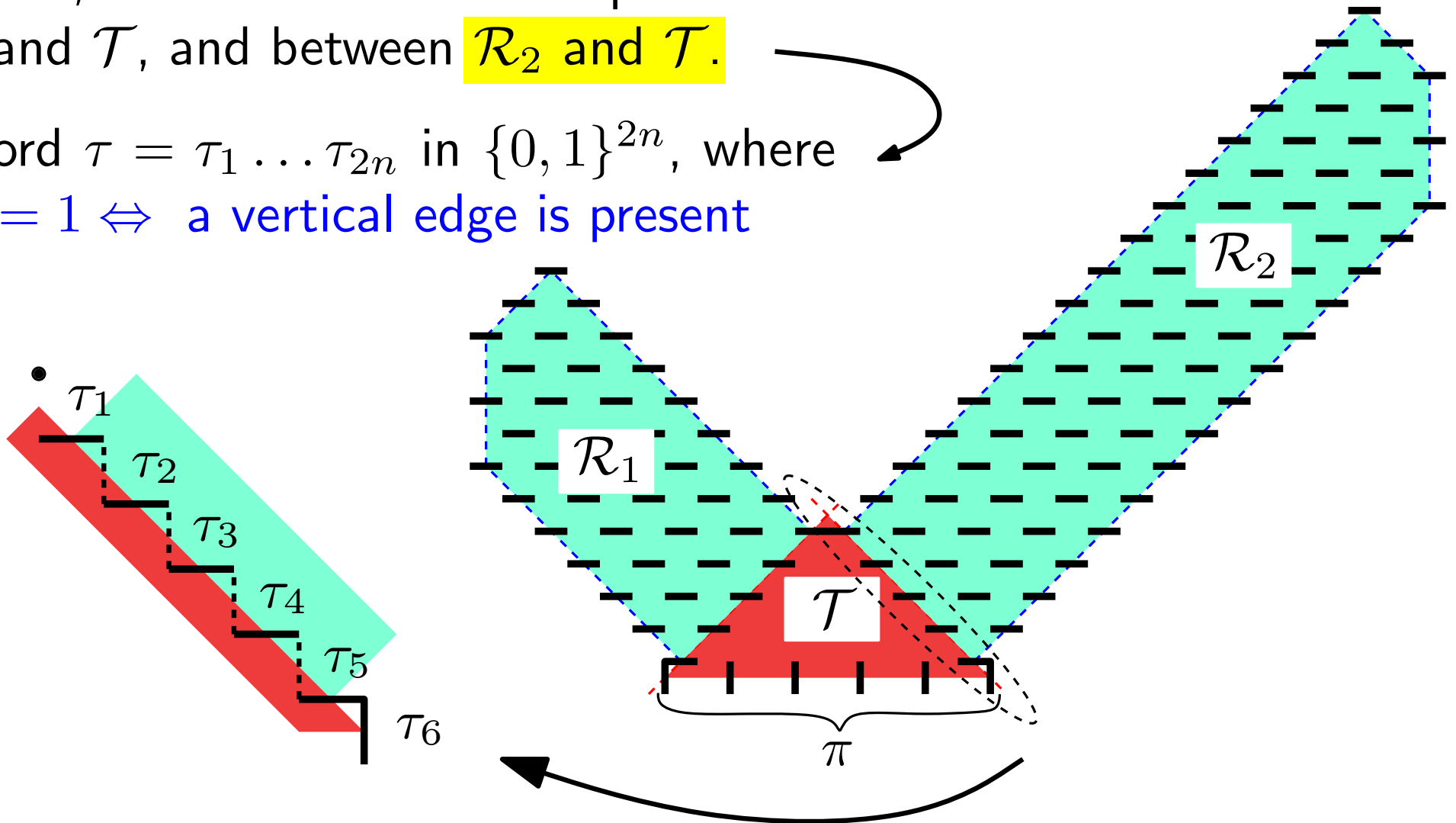
Word  $\sigma = \sigma_1 \dots \sigma_{2n}$  in  $\{0, 1\}^{2n}$ , where  
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Word  $\tau = \tau_1 \dots \tau_{2n}$  in  $\{0, 1\}^{2n}$ , where  
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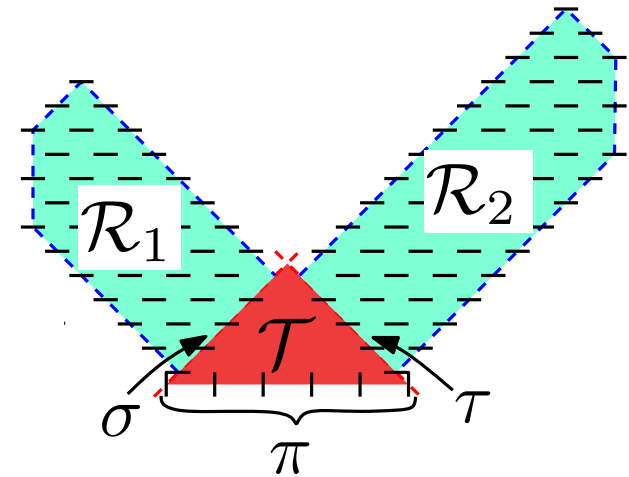
# Putting things together

We can then write, for  $m \geq 3n - 1$  and  $0 \leq k \leq m - (3n - 1)$

$$A_\pi(m) = \sum_{\sigma, \tau} |\mathcal{R}_1(\sigma, k)| \times t_{\sigma, \tau}^\pi \times |\mathcal{R}_2(\tau, m - 3n - k + 1)|$$

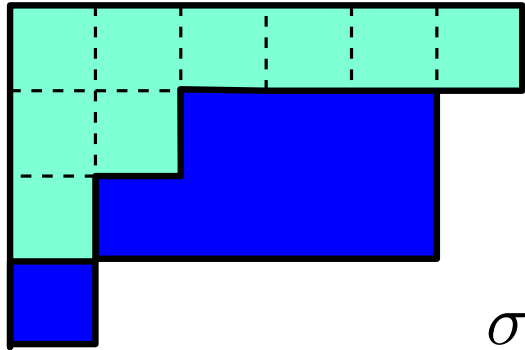
where

- $\sigma, \tau$  are words of length  $2n$  on  $\{0, 1\}$  ;
- $\mathcal{R}_1(\sigma, \cdot), \mathcal{R}_2(\tau, \cdot)$  are the sets of FPL configurations in the regions  $\mathcal{R}_1$  and  $\mathcal{R}_2$  with boundaries  $\sigma, \tau$  respectively ;
- $t_{\sigma, \tau}^\pi$  is the number of FPL configurations in the triangle  $\mathcal{T}$  with boundary data  $\{\sigma, \pi, \tau\}$ .

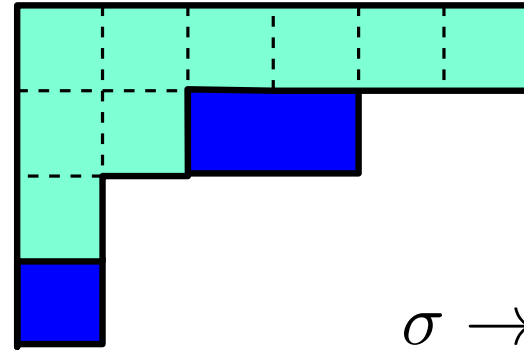






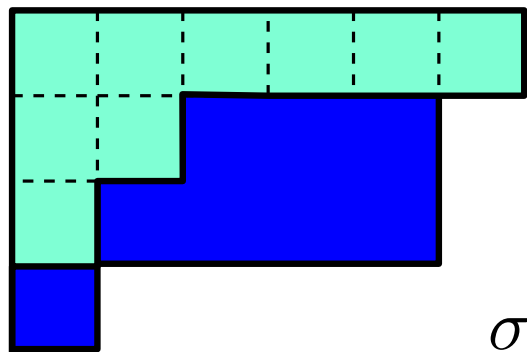


$$\sigma \leq \sigma'$$

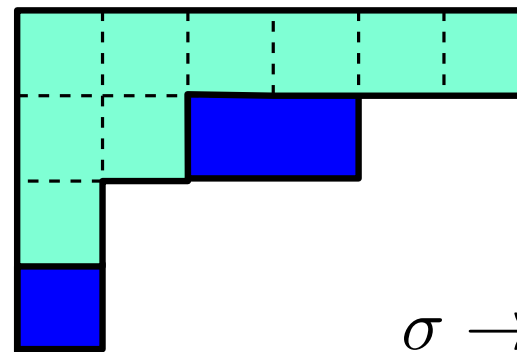


$$\sigma \rightarrow \sigma'$$

At most one more box per column



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## Definition

A **semi standard Young tableau** of shape  $\sigma$  and entries bounded by  $N$  is a filling of the shape  $\sigma$  by integers in  $\{1, \dots, N\}$  such that entries are strictly increasing in columns and weakly increasing in rows.

The number of such tableaux is given by  $SSYT(\sigma, N)$ , a **polynomial in  $N$**  with leading term  $\frac{1}{H(\sigma)} N^{d(\sigma)}$ .

(Here  $H(\sigma)$  is the product of *hook lengths* of the shape  $\sigma$ .)

# Regions $\mathcal{R}_1$ and $\mathcal{R}_2$

**Proposition [Caselli, Krattenthaler, Lass, N. '05]**

Let  $\sigma$  be a word of length  $2n$ , and  $k \in \mathbb{N}$ . There is a bijection between FPLs in  $\mathcal{R}_1(\sigma, k)$  and semistandard Young tableaux of shape  $\sigma$  and length  $n + k$ .

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So for  $m \geq 3n - 1$  (and  $k = 0$ ) we obtain :

$$\begin{aligned} A_\pi(m) &= \sum_{\sigma, \tau} |\mathcal{R}_1(\sigma, 0)| \cdot t_{\sigma, \tau}^\pi \cdot |\mathcal{R}_2(\tau, m - 3n + 1)| \\ &= \sum_{\sigma, \tau} SSYT(\sigma, n) \cdot t_{\sigma, \tau}^\pi \cdot SSYT(\tau^*, m - 2n + 1) \end{aligned}$$

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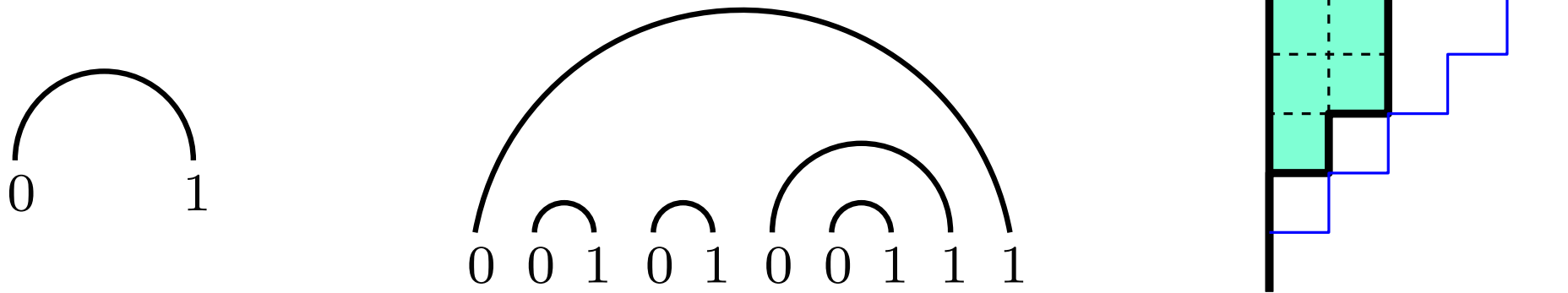
## Theorem [CKLN '05]

$A_\pi(m)$  is a polynomial function of  $m$  for  $m \geq 0$



# Some more definitions

Given a noncrossing matching  $\pi$  of size  $n$ , we can associate to it a word, and thus a Ferrers shape :



The words obtained from matchings are the famous Dyck words :

**Definition** We note  $\mathcal{D}_n$  the words  $w$  such that  $|w|_0 = |w|_1 = n$  and which are smaller than  $(01)^n$ .

We write  $\mathbf{0}_n := 0^n 1^n$ , and  $\mathbf{1}_n := (01)^n$ . Then  $(\mathcal{D}_n, \leq)$  forms a poset with minimum  $\mathbf{0}_n$  and maximum  $\mathbf{1}_n$ .

# The final expression for $A_\pi(m)$

## Theorem [CKLN '04]

For all  $\sigma, \tau, \pi$ , we have  $t_{\sigma, \tau}^\pi \neq 0$  implies  $\sigma \leq \pi$ .

Moreover,  $t_{\pi, \mathbf{0}_n}^\pi = 1$  and  $t_{\pi \tau}^\pi = 0$  if  $\tau \neq \mathbf{0}_n$ .

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As a consequence, the expression for  $A_\pi(m)$  can be restricted to words  $\sigma, \tau \in \mathcal{D}_n$  : for any  $m \geq 0$

$$A_\pi(m) = \sum_{\sigma, \tau \in \mathcal{D}_n} SSYT(\sigma, n) \cdot t_{\sigma, \tau}^\pi \cdot SSYT(\tau^*, m - 2n + 1)$$

One can show then that  $A_\pi(m)$  has leading term  $\frac{1}{H(\pi)} m^{d(\pi)}$ .

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One can show then that  $A_\pi(m)$  has leading term  $\frac{1}{H(\pi)} m^{d(\pi)}$ .

Our goal is to obtain a formula for  $A_\pi(m)$ , so the problem is now to evaluate the numbers  $t_{\sigma, \tau}^\pi$ , i.e. the number of FPLs in a triangle.

## (2) FPL configurations in a triangle



# First properties

A vertical symmetry gives immediately

$$t_{\sigma, \tau}^{\pi} = t_{\tau^*, \sigma^*}^{\pi^*}.$$

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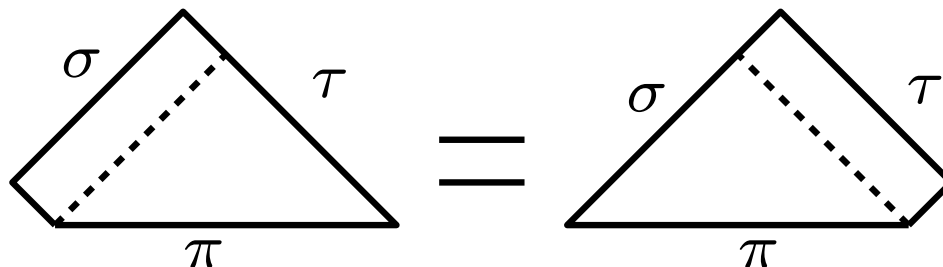
A vertical symmetry gives immediately

$$t_{\sigma, \tau}^{\pi} = t_{\tau^*, \sigma^*}^{\pi^*}.$$

There holds also the following identity, the proof of which is based on Wieland's rotation :

## Théorème [N '09]

$$\sum_{\substack{\sigma_1 \in \mathcal{D}_n \\ \sigma \rightarrow \sigma_1}} t_{\sigma_1, \tau}^{\pi} = \sum_{\substack{\tau_1 \in \mathcal{D}_n \\ \tau^* \rightarrow \tau_1^*}} t_{\sigma, \tau_1}^{\pi}.$$



## Theorem [CKLN '04, N]

$t_{\sigma, \tau}^{\pi} \neq 0$  implies  $\sigma \leq \pi$ .

*Proof (sketch)* the idea is to attach to any TFPL  $f$  certain integers  $N_i(f) \geq 0$  such that if  $f$  has boundaries  $\sigma, \pi, \tau$ , then

$$\pi_i - \sigma_i = N_i(f) - N_{i-1}(f)$$

for all  $i \geq 1$ , and  $N_0(f) = 0$ . These integers  $N_i(f)$  actually *count* certain edges in the configuration  $f$ .

One obtains then :

$$\forall j, \sum_{i \leq j} (\pi_i - \sigma_i) = N_j(f) \geq 0,$$

which is equivalent to  $\sigma \leq \pi$ .

# Common prefixes and suffixes

For  $\sigma = \pi$ , there is just one possible TFPL, which verifies  $\tau = \mathbf{0}_n$ .

What happens when  $\sigma$  is “close” to  $\pi$ ?

A partial answer :  $\sigma$  and  $\pi$  share a common prefix and/or suffix.

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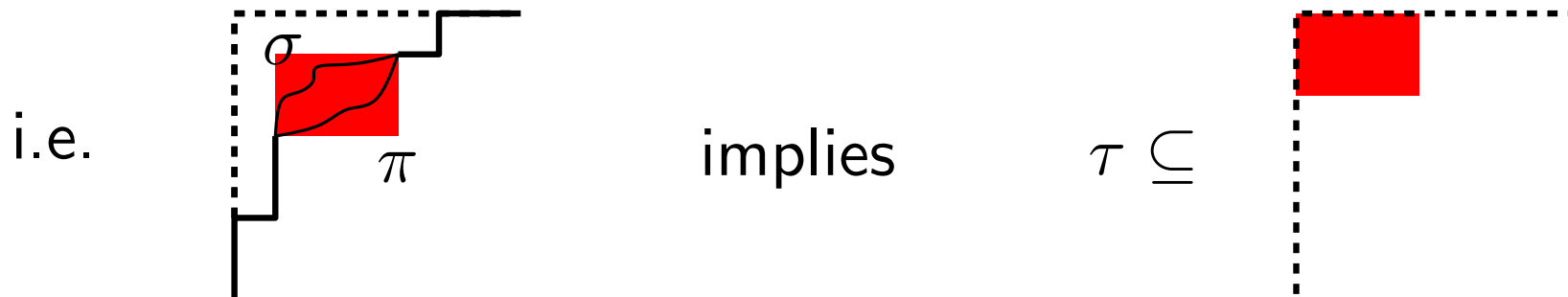
## Proposition [N]

Let  $\pi, \sigma, \tau \in \mathcal{D}_n$ . Let also  $u, v, \sigma', \pi', v$  be such that

$$\sigma = u\sigma'v \quad \text{and} \quad \pi = u\pi'v.$$

Write  $a := |u|_0 + |v|_0$  and  $b := |u|_1 + |v|_1$ .

Then  $t_{\sigma, \tau}^{\pi} \neq 0$  implies  $\tau = 0^a \tau' 1^b$  for a certain  $\tau'$ .



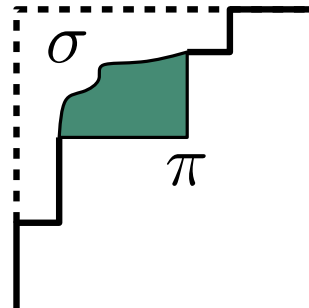
# Common prefixes and suffixes

In a special case we can actually evaluate the coefficient  $t_{\sigma, \tau}^{\pi}$ .

## Proposition

If  $\pi' = 1^{n-b}0^{n-a}$ , then  $t_{\sigma, \tau}^{\pi}$  can be written as a determinant of size  $\min(n-a, n-b)$ , the entries of which are certain binomial coefficients.

This corresponds to the case where the skew shape  $\pi/\sigma$  is a “rotated diagram”.

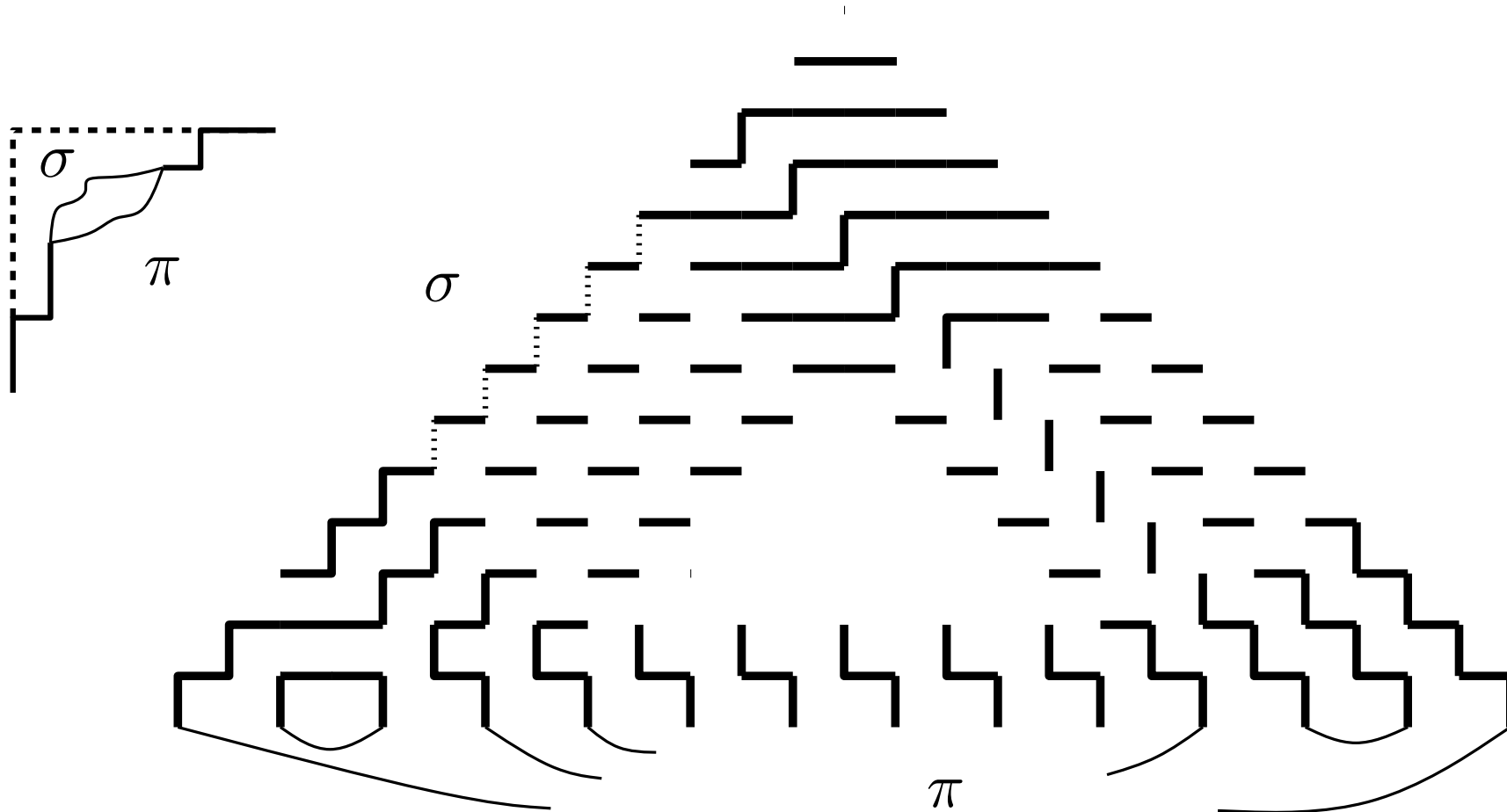


# Common prefixes and suffixes

Idea of Proof : there are many fixed edges.

$$\sigma = 00100\sigma'1011$$

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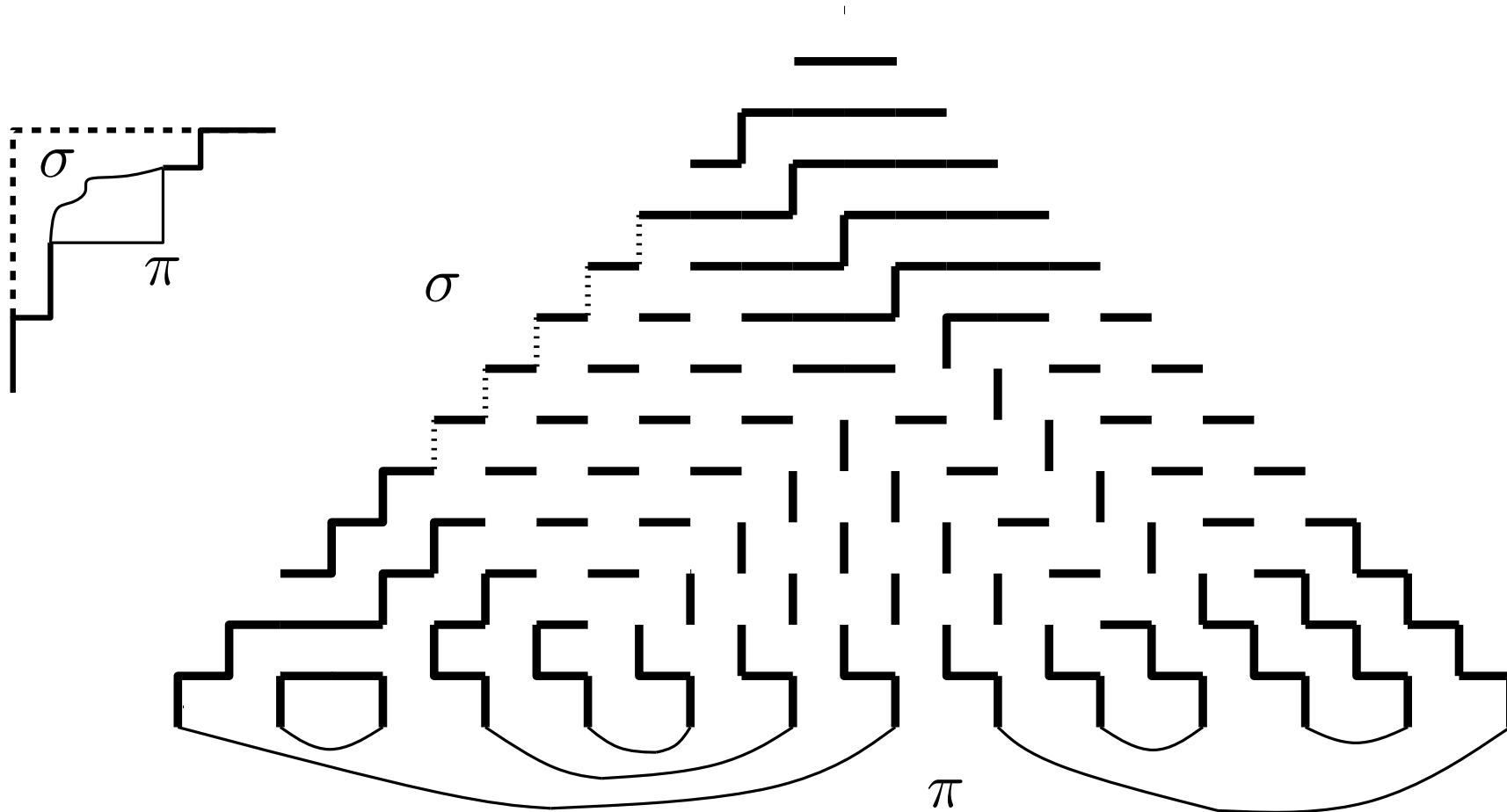


# Common prefixes and suffixes

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(3) Extremal TFPL  
and  
Littlewood-Richardson coefficients.

# Extremal configurations

We previously saw the “non vanishing” constraint  $\sigma \leq \pi$ .  
Thapper proved another important such constraint :

|  $t_{\sigma,\tau}^{\pi} \neq 0$  implies  $d(\sigma) + d(\tau) \leq d(\pi)$ .

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$$t_{\sigma,\tau}^{\pi} \neq 0 \text{ implies } d(\sigma) + d(\tau) \leq d(\pi).$$

Following his idea, one obtains a certain identity in the case  $d(\sigma) + d(\tau) = d(\pi)$  :

**Proposition** For any  $\pi \in \mathcal{D}_n$ ,

$$\frac{1}{H(\pi)} = \sum_{\substack{\sigma, \tau \in \mathcal{D}_n \\ d(\sigma) + d(\tau) = d(\pi)}} t_{\sigma,\tau}^{\pi} \cdot \frac{1}{2^{d(\sigma)} H(\sigma)} \cdot \frac{1}{2^{d(\tau)} H(\tau)}$$

**Definition** : We name **extremal** the TFPL with boundaries  $\{\sigma, \pi, \tau\}$  verifying  $d(\sigma) + d(\tau) = d(\pi)$ .

# Sketch of proof

(a)  $t_{\sigma,\tau}^{\pi} \neq 0$  implies  $d(\sigma) + d(\tau) \leq d(\pi)$ .

$$(b) \frac{1}{H(\pi)} = \sum_{\substack{\sigma, \tau \in \mathcal{D}_n \\ d(\sigma) + d(\tau) = d(\pi)}} t_{\sigma,\tau}^{\pi} \cdot \frac{1}{2^{d(\sigma)} H(\sigma)} \cdot \frac{1}{2^{d(\tau)} H(\tau)}$$

# Sketch of proof

(a)  $t_{\sigma,\tau}^\pi \neq 0$  implies  $d(\sigma) + d(\tau) \leq d(\pi)$ .

(b) 
$$\frac{1}{H(\pi)} = \sum_{\substack{\sigma,\tau \in \mathcal{D}_n \\ d(\sigma)+d(\tau)=d(\pi)}} t_{\sigma,\tau}^\pi \cdot \frac{1}{2^{d(\sigma)} H(\sigma)} \cdot \frac{1}{2^{d(\tau)} H(\tau)}$$

Let us recall that  $A_\pi(m)$  is a polynomial of degree  $d(\pi)$  whose leading coefficient is  $1/H(\pi)$ , and that

$$A_\pi(m) = \sum_{\sigma,\tau} t_{\sigma,\tau}^\pi \cdot SSYT(\sigma, n+k) \cdot SSYT(\tau^*, m+1-k-2n).$$

for  $k$  between 0 and  $m - (3n - 1)$ . We choose then  $k = m/2$  for  $m$  even and large enough. Then we obtain

(a) by comparing coefficients in degree  $> d(\pi)$  and

(b) by comparing them in degree  $= d(\pi)$ .

# Littlewood Richardson coefficients

Let  $\lambda, \mu, \nu$  be partitions, and  $\Lambda(x)$  be the ring of symmetric functions of the variables  $x_1, x_2, \dots$ . The **Schur functions**  $s_\lambda(x)$  can be defined as

$$s_\lambda(x) = \sum_T \prod_i x_i^{T_i},$$

where  $T$  goes through all semistandard Young tableaux of shape  $\lambda$ , and  $T_i$  is the number of cells labeled  $i$ .

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Schur functions form a **basis** of  $\Lambda(x)$ . We can expand  $s_\mu(x)s_\nu(x)$  on this basis, where the coefficients  $c_{\mu,\nu}^\lambda$  are often called the **Littlewood-Richardson (LR) coefficients**.

$$s_\mu(x)s_\nu(x) = \sum_\lambda c_{\mu,\nu}^\lambda s_\lambda(x)$$

# Littlewood Richardson coefficients

By homogeneity of Schur functions, we have

$$c_{\mu, \nu}^{\lambda} \neq 0 \text{ implies } d(\lambda) = d(\mu) + d(\nu).$$

We have also, if  $s_{\lambda}(x, y)$  is the symmetric function  $s_{\lambda}$  in the variables  $x_1, x_2, \dots, y_1, y_2, \dots$

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If we evaluate this at  $x_i = y_i = 1$  for  $i = 1, \dots, m/2$ ,  $x_i = y_i = 0$  for  $i > m/2$ , we obtain polynomials in  $m$  which give the following identity in top degree  $d(\lambda)$  :

$$\frac{1}{H(\lambda)} = \sum_{\mu,\nu} c_{\mu,\nu}^{\lambda} \cdot \frac{1}{2^{d(\mu)} H(\mu)} \cdot \frac{1}{2^{d(\nu)} H(\nu)}$$

# Littlewood Richardson coefficients

As a consequence, there exist  $a_{\sigma\tau} > 0$  such that, for any  $\pi \in \mathcal{D}_n$ ,

$$\sum_{\sigma, \tau} a_{\sigma\tau} c_{\sigma, \tau}^{\pi} = \sum_{\sigma, \tau} a_{\sigma\tau} t_{\sigma, \tau}^{\pi} \quad (E)$$

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## Theorem [N. '09]

For all words  $\pi, \sigma, \tau \in \mathcal{D}_n$  verifying  $d(\sigma) + d(\tau) = d(\pi)$ , we have

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Thanks to equation (E), we need only prove that  $c_{\sigma,\tau}^{\pi} \leq t_{\sigma,\tau}^{\pi}$  for all  $\sigma, \tau, \pi$  such that  $d(\sigma) + d(\tau) = d(\pi)$ .

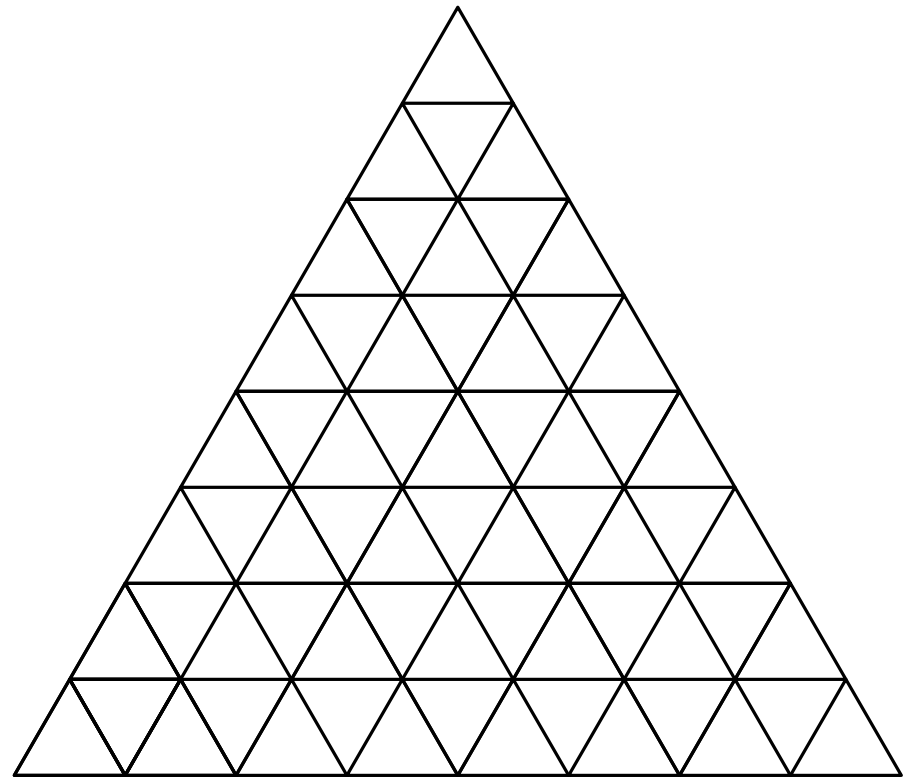
# Computing LR coefficients

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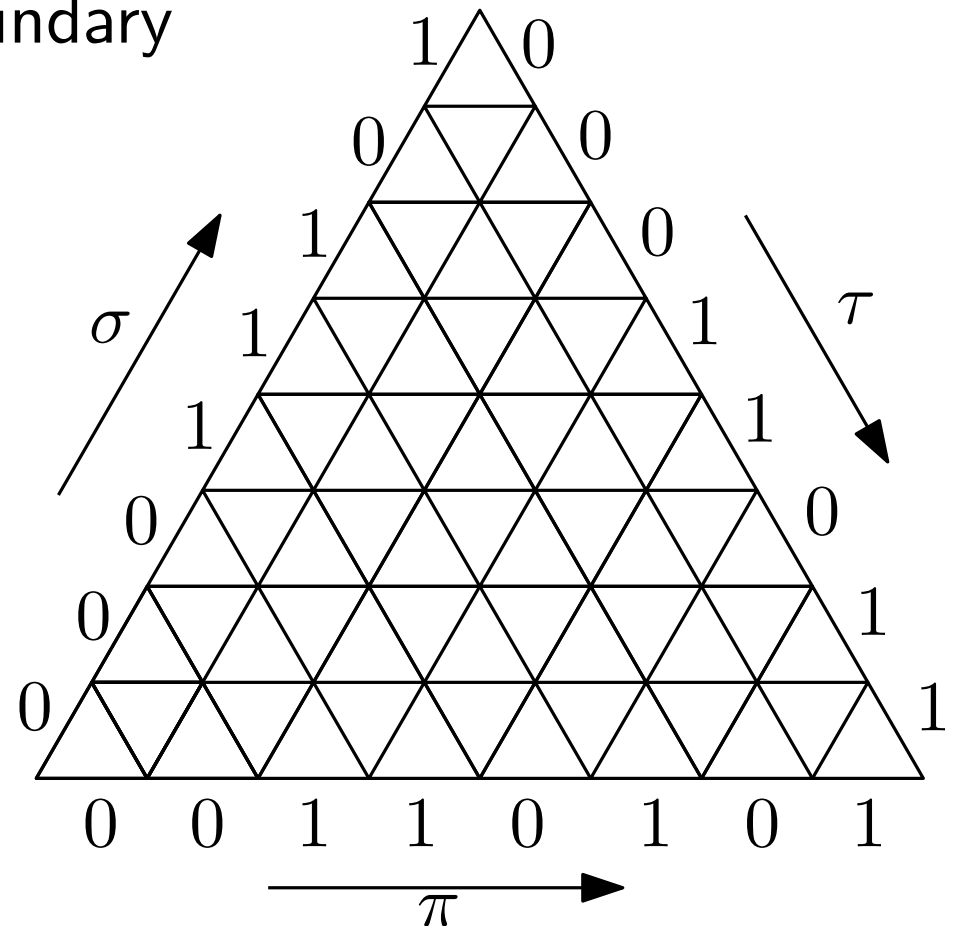
Consider a triangle of size  $2n$  on the triangular lattice.

Fix  $\sigma, \pi, \tau \in \mathcal{D}_n$ , and label the boundary edges of the triangle.

$$\pi = 00110101$$

$$\sigma = 00011011$$

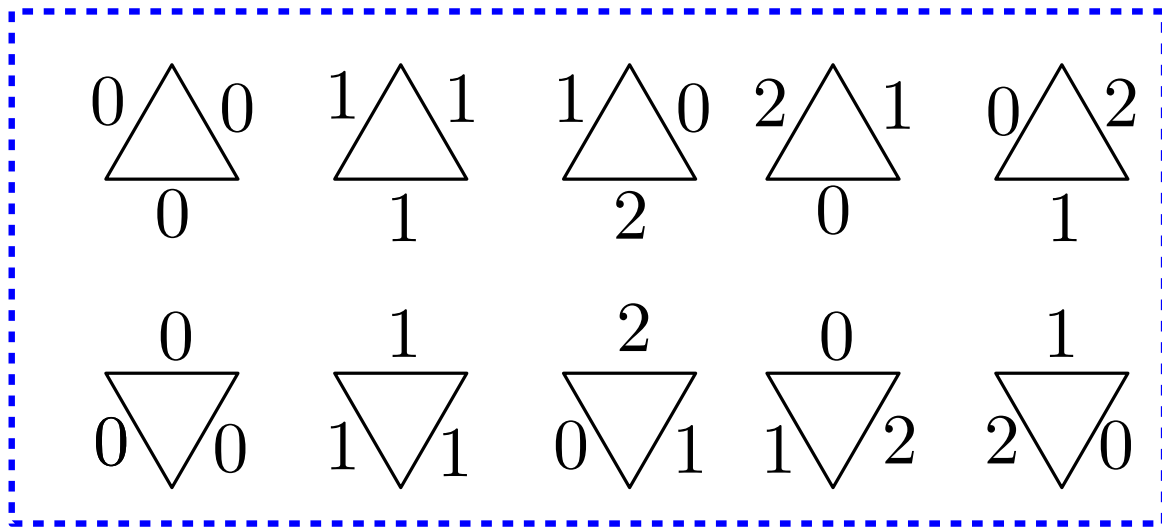
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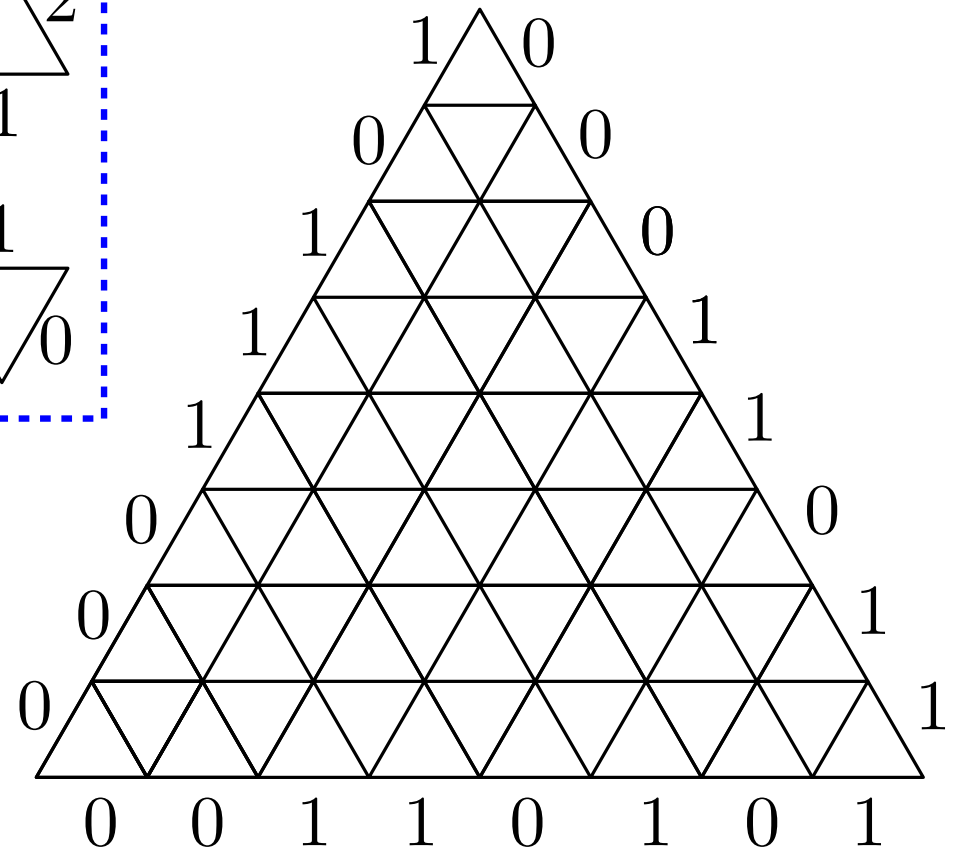
# Definition

A Knutson-Tao puzzle with boundary data  $\sigma, \pi, \tau$  is a labeling of each edge of the triangle by 0, 1 or 2, such that :

- the labels on the boundary are given by  $\sigma, \pi, \tau$  ;
- on each unit triangle, the induced labeling must be among :



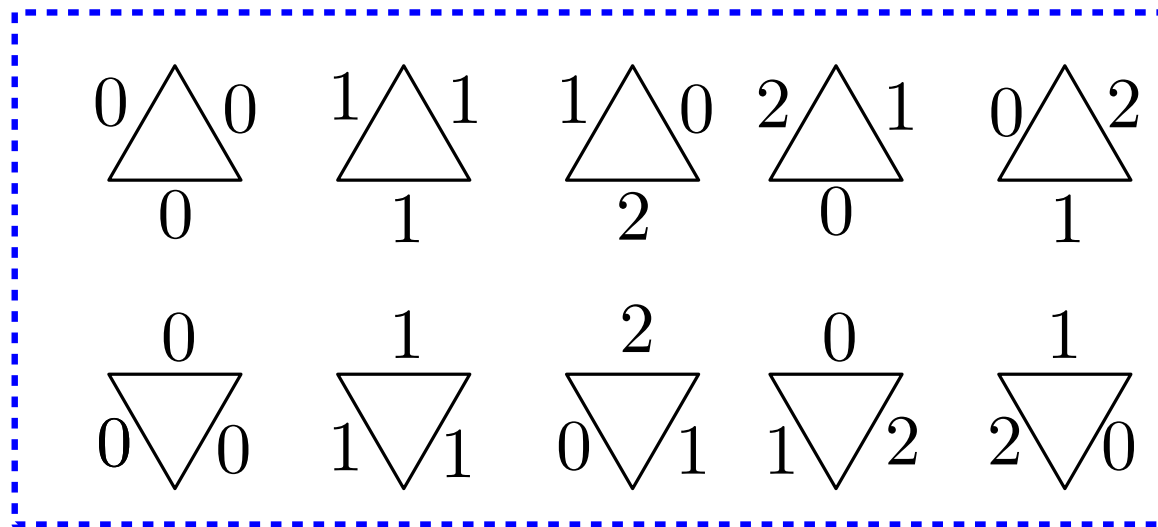
“Only 0s, only 1s, or 0, 1, 2 counterclockwise”



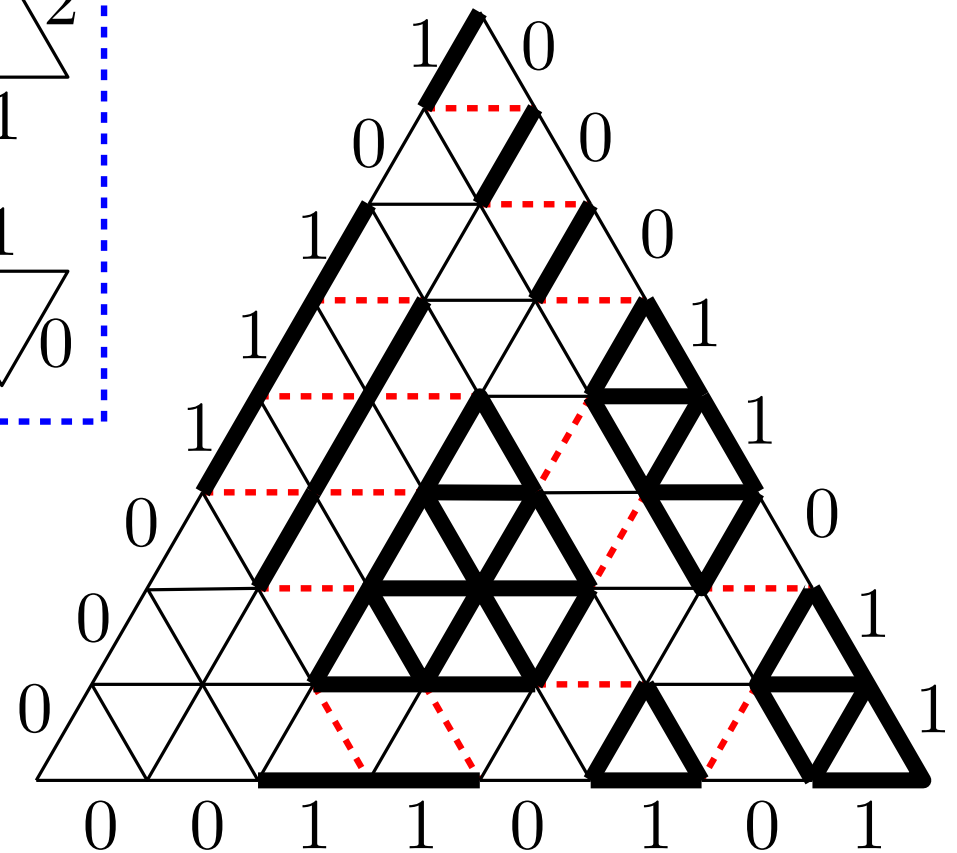
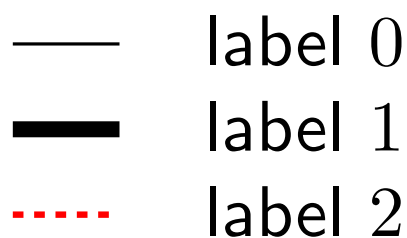
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We will picture the labeling of edges as follows :



## Theorem [Knutson, Tao '03][K., T. and Woodward '03]

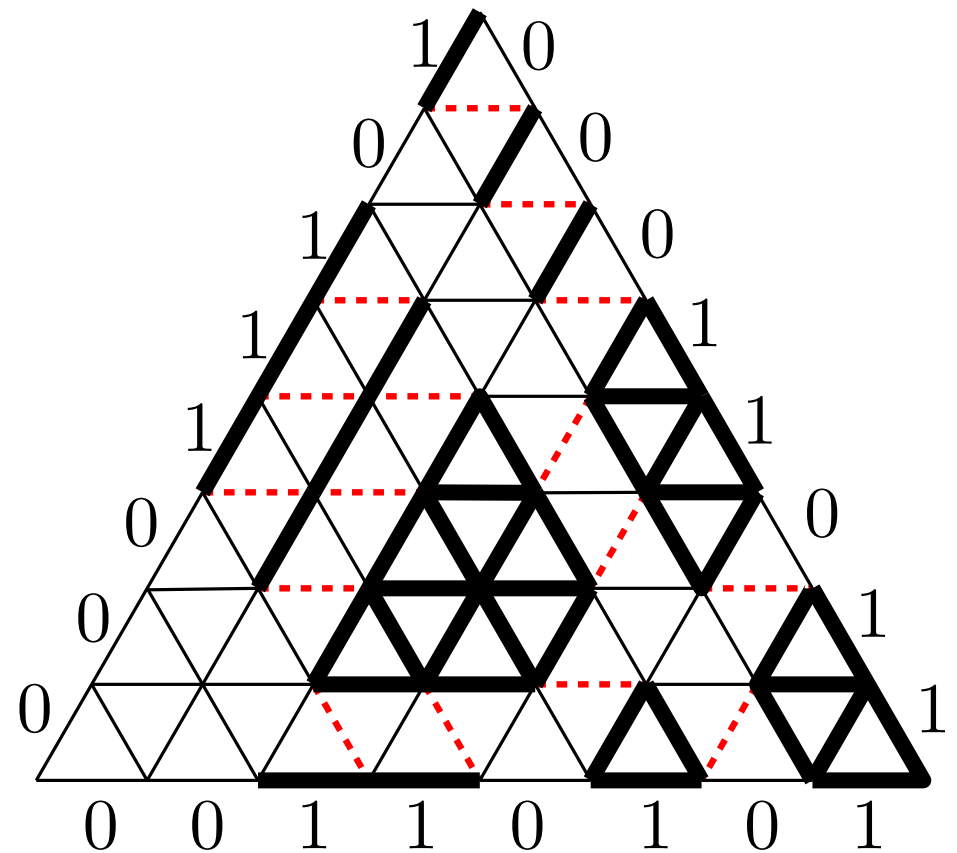
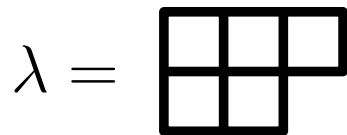
Let  $\sigma, \tau, \pi \in \mathcal{D}_n$ . Then the number of KT-puzzles with boundary data  $\sigma, \pi, \tau$  is equal to the LR coefficient  $c_{\sigma, \tau}^{\pi}$ .

# Theorem [Knutson, Tao '03][K., T. and Woodward '03]

Let  $\sigma, \tau, \pi \in \mathcal{D}_n$ . Then the number of KT-puzzles with boundary data  $\sigma, \pi, \tau$  is equal to the LR coefficient  $c_{\sigma, \tau}^{\pi}$ .

For example, it is easy to see that there is only one puzzle with the boundary data of the example.

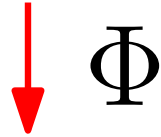
so  $c_{\mu, \nu}^{\lambda} = 1$  where



# From KT puzzles to TFPL configurations.

We fix  $\sigma, \pi, \tau \in \mathcal{D}_n$ , such that  $d(\sigma) + d(\tau) = d(\pi)$ . We will define a map  $\Phi$ .

KT puzzles with boundary data  $\sigma, \pi, \tau$

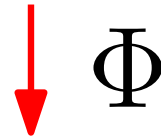


TFPL configurations with boundaries  $\sigma, \pi, \tau$

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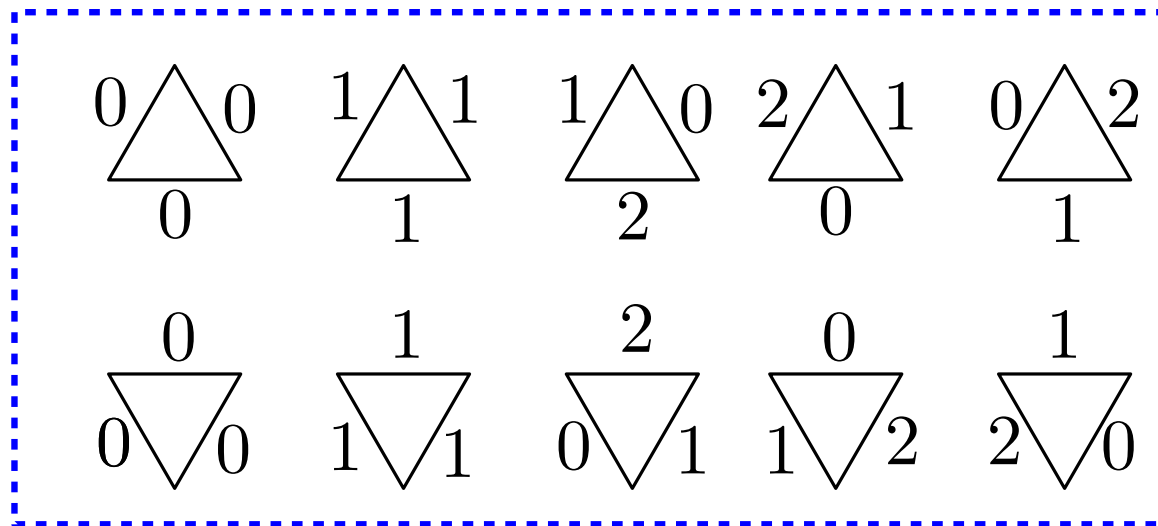
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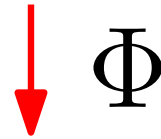
The map is **local** : it changes every small labeled triangle of the puzzle to a piece of a path of a TFPL configuration.



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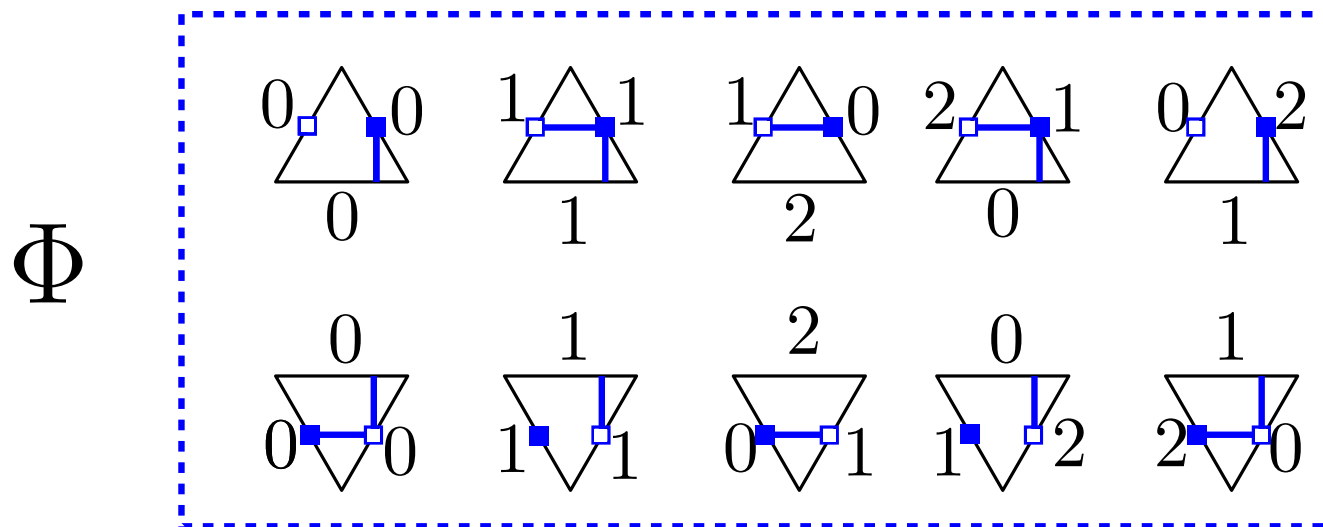
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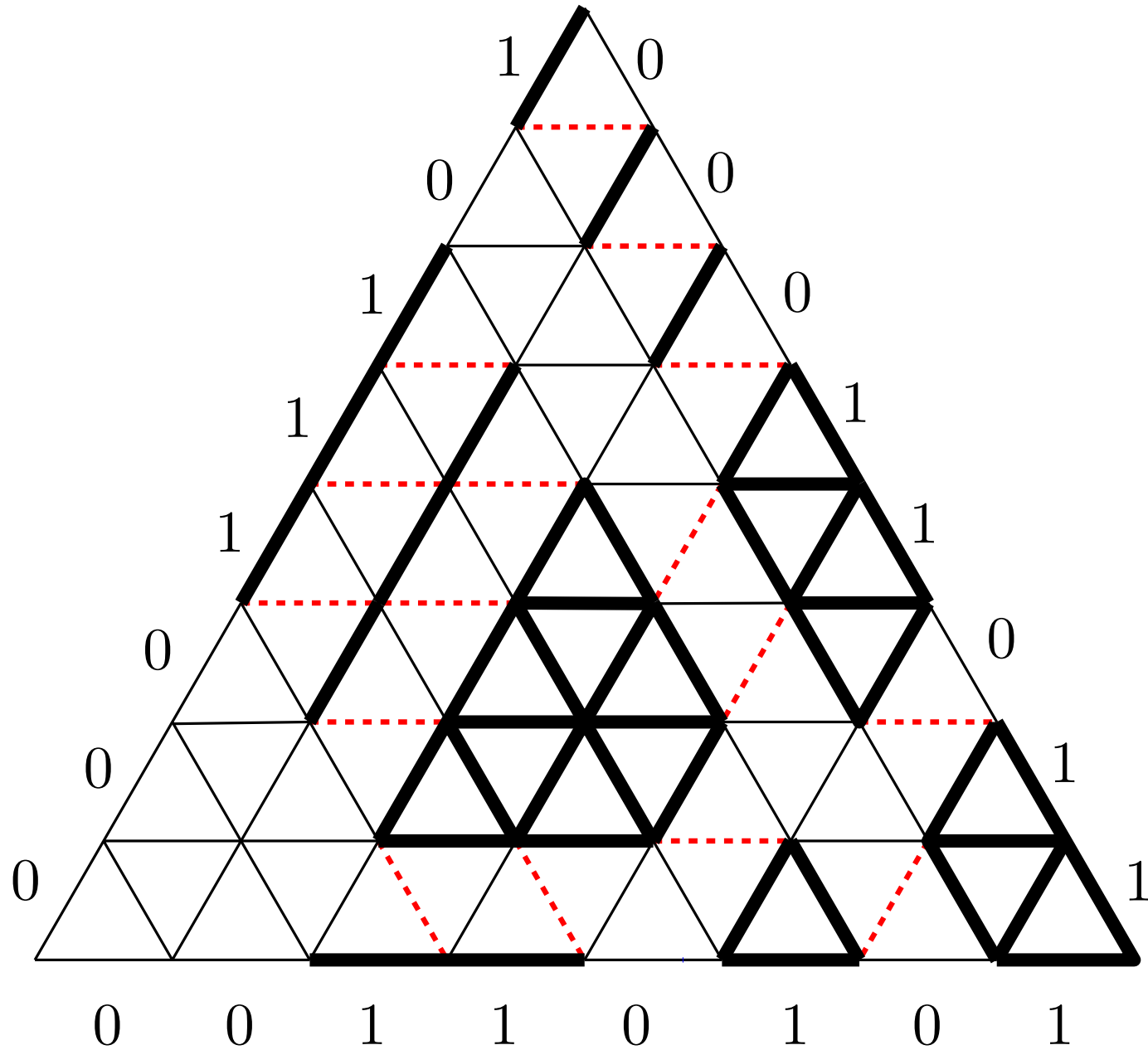


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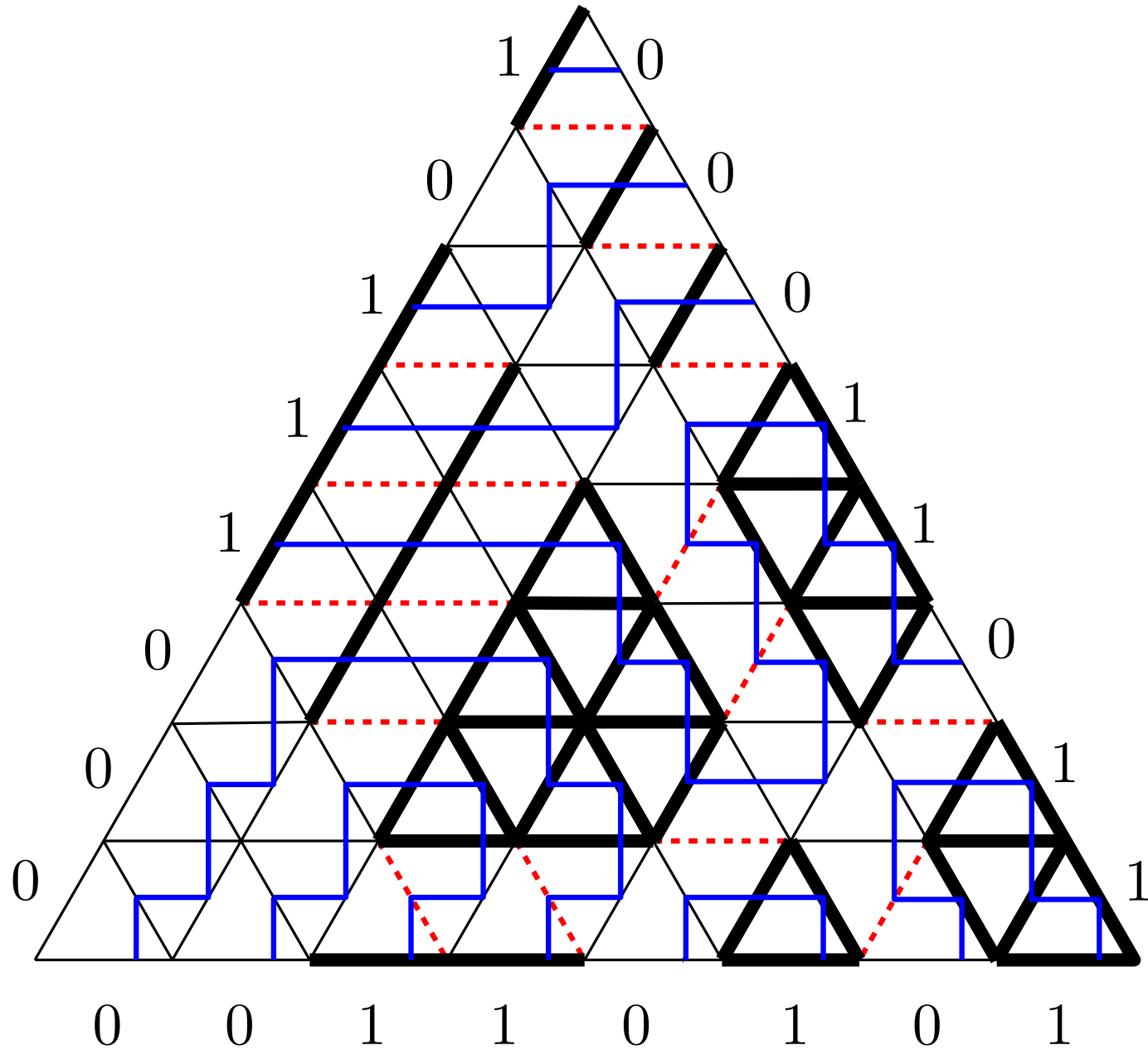
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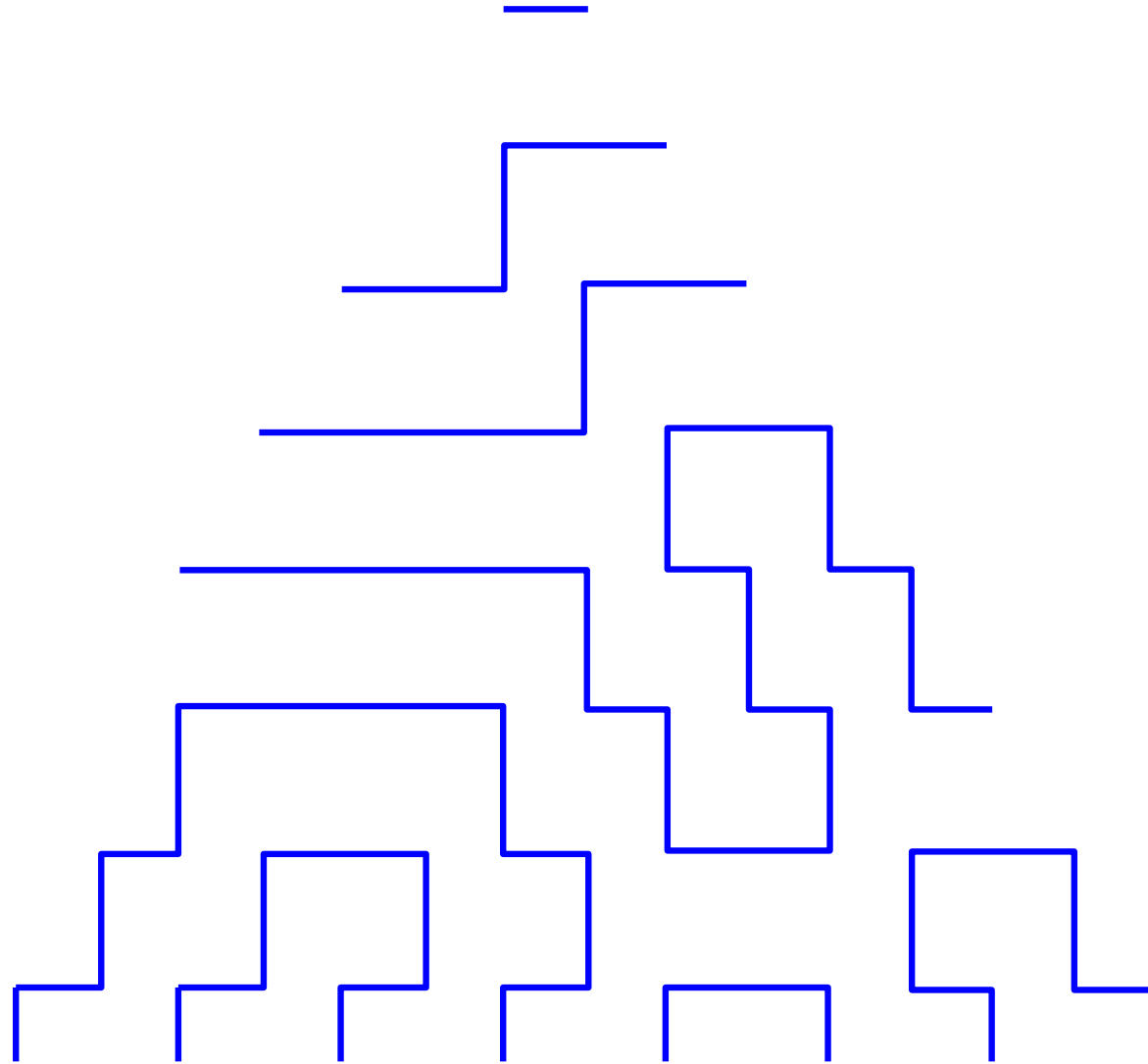
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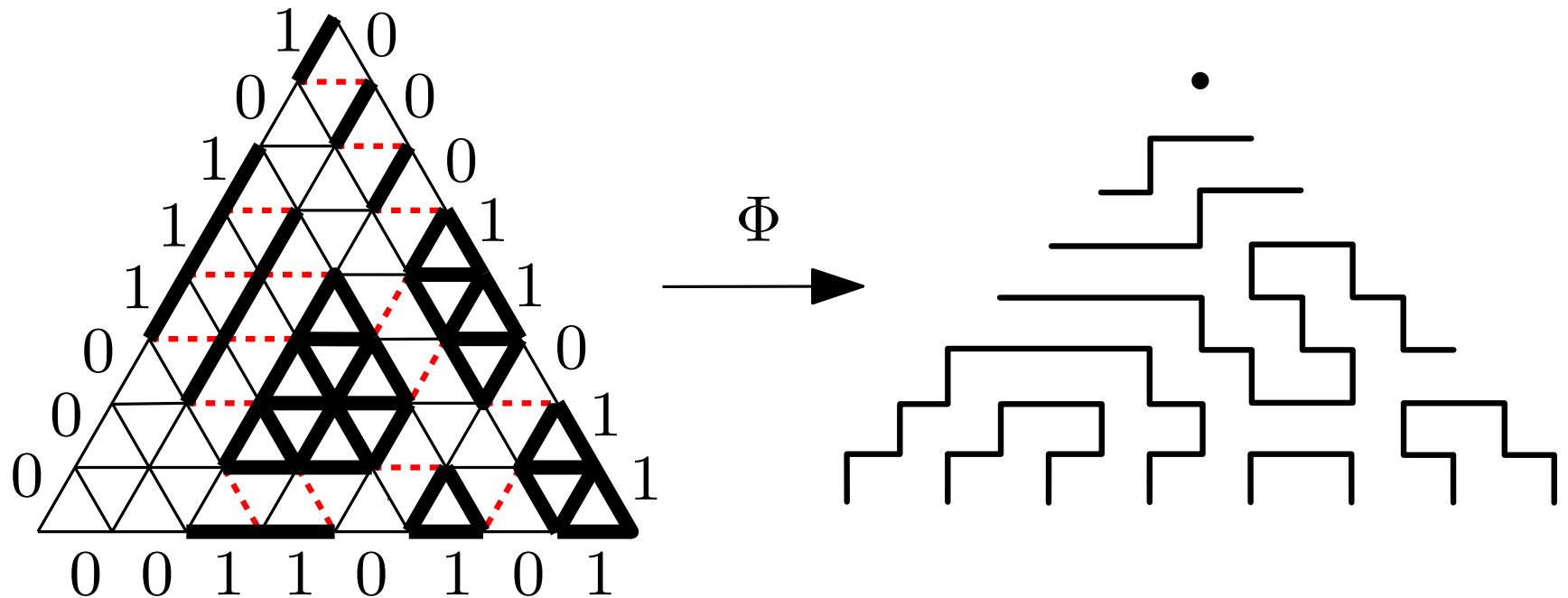
# $\Phi$ is the wanted bijection

One has to prove that  $\Phi$  is :

1. **well defined** :

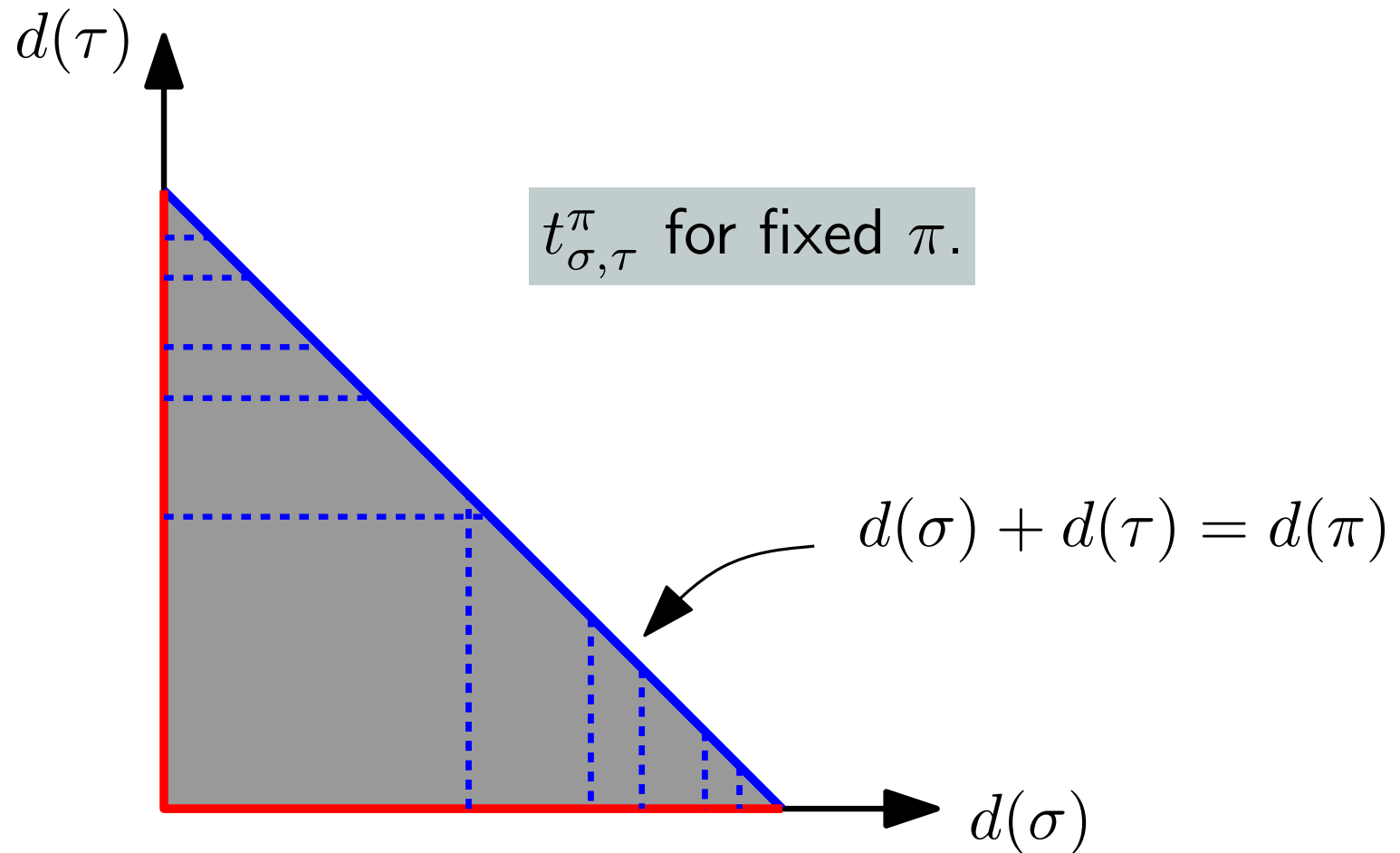
- the vertices of  $\Phi(\text{puzzle})$  are of degree 2 ,
- $\Phi(\text{puzzle})$  verifies the boundary conditions  $\sigma, \tau$ .
- the connectivity of external edges given by  $\pi$  is respected.

2. **injective**.



# Conclusion

We have obtained enumerative results for certain numbers  $t_{\sigma, \tau}^{\pi}$  (in blue). In red are the coefficients  $t_{\sigma, \mathbf{0}_n}^{\pi}$ .



# Conclusion

- To compute  $A_X$ , one needs all coeffs  $t_{\sigma, \tau}^{\pi}$ , and not only the extremal ones. A natural parameter to partition these numbers is  $exc(\pi, \sigma, \tau) := d(\pi) - d(\sigma) - d(\tau) \geq 0$ .  
The LR coefficients form the base case  $exc(\pi, \sigma, \tau) = 0$ ;  
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- Other direction (based on [Thapper '07]). The polynomials  $A_\pi(m)$  verify linear recurrences

$$A_\pi(m) = \sum_{\alpha \leq \pi \in \mathcal{D}_n} c_{\alpha\pi} A_\alpha(m-1),$$

where  $c_{\alpha\pi}$  are integers, defined in terms of the coefficients  $t_{\sigma\mathbf{0}_n}^\pi$ . What are these coefficients  $c_{\alpha\pi}$  ?

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- Related work : joint with T. Fonseca, nice conjectures about the polynomials  $A_\pi(m)$  pointing to **combinatorial reciprocity** for them ; cf arXiv.CO two days ago.

Vielen Dank für Ihre Aufmerksamkeit !