Wealth distribution and aggregate time-preference: Markov-perfect equilibria in a Ramsey economy

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Abstract: We study Markov-perfect Nash equilibria (MPNE) of a Ramsey-Cass-Koopmans economy in which households are aware of their influence on prices. The Ramsey conjecture fails to hold such that households other than the most patient one own positive wealth in the steady state. This confirms results that have been derived in the same model using an open-loop equilibrium concept. In contrast to the competitive and the open-loop equilibrium, the steady state of the MPNE depends on the utility functions of the households. Since the MPNE cannot be determined analytically, a high-order least squares projection method is employed.

Journal of Economic Literature classification codes: C63, C73, D31, O41

Keywords: Ramsey-Cass-Koopmans model, strategic saving, Markov-perfect Nash equilibrium, aggregate time-preference, projection method.

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1 Introduction

We consider the standard infinite-horizon growth model under the non-standard assumption that households realize their influence on the factor prices. Using a high-order least squares projection method, we solve for Markov-perfect Nash equilibria (MPNE) of that model. The analysis focuses on the wealth distribution and the aggregate time-preference rate in steady states generated by MPNE, but there is also some discussion about the transition dynamics towards the steady state.

A well-known shortcoming of the standard Ramsey-Cass-Koopmans model with price-taking households (see [14, 5, 11]) is its prediction about the long-run distribution of wealth. If all households share the same time-preference rate, then this distribution is indeterminate. If, on the other hand, households differ with respect to their time-preferences, the wealth distribution is degenerate in the sense that only the most patient household owns a positive amount of capital (see Becker [2]). In Sorger [16] it was argued that this feature of the model contains a seed for its own destruction. If the steady state is asymptotically stable (as it is under reasonable assumptions), then it follows that, over time, the capital market becomes a monopoly. This, in turn, renders one of the basic assumptions of the model questionable, namely that of price-taking households. To circumvent this problem, Sorger [16] suggested to assume that the households realize their market power (at least when the economy is close to the steady state). Under this assumption it was shown that the long-run distribution of wealth is not necessarily degenerate and that impatient households may be willing to maintain positive capital holdings forever. These results about the steady states of the strategic Ramsey-Cass-Koopmans model were later generalized by Becker [3] and Sorger [17]. Moreover, Becker and Foias [4] derived sufficient conditions for the asymptotic stability of the steady state in that model.

The studies by Sorger, Becker, and Foias [16, 17, 3, 4] are all based on the concept of an open-loop Nash equilibrium of the strategic Ramsey-Cass-Koopmans model. This equilibrium concept is as close as possible to the usual competitive model while still containing that crucial strategic element that turns out to be responsible for the emergence of a non-degenerate long-run wealth distribution. On the other hand, the open-loop Nash equilibrium concept assumes that all households can credibly commit to all their future consumption and labor supply decisions. This means in particular that the households’ decisions are not state-contingent. As the motivation for Sorger’s [16] suggestion is only compelling if market forces have already steered the economy close to a degenerate wealth distribution, an equilibrium concept based on state-contingent policy functions seems to be more appropriate than one that is based on full commitment. The most obvious choice for such an equilibrium concept is MPNE. The property of Markov-perfection assumes only a limited form of commitment to future consumption and labor decisions and captures an ongoing and, hence, much stronger form of strategic interaction between the players than the open-loop equilibrium. The first contribution of the present paper is therefore to check whether and how the main insights from the above mentioned studies carry over to the concept of MPNE.

We find that in a MPNE there exists a unique steady state, that this steady state is asymptotically stable, and that the long-run wealth distribution is typically non-degenerate. As far as the steady state wealth distribution is concerned, it follows therefore that the main insights
from the literature cited above do carry over to the MPNE in a qualitative sense. Surprisingly, we also find that the quantitative differences between the MPNE steady state and the open-loop steady state are not as big as one might expect. One aspect, however, in which the MPNE differs significantly from the open-loop equilibrium (and from the competitive steady state as well) is that the steady state generated by the MPNE depends on the utility functions of the households. More specifically, we find that a reduction of a household’s intertemporal elasticity of substitution leads, ceteris paribus, to an increase of that household’s steady state capital stock and to a reduction of the other households’ steady state capital holdings. In a competitive equilibrium or an open-loop equilibrium, on the other hand, the location of the steady states is independent of all preference parameters except for the time-preference factors. The intuitive explanation for the influence of the utility functions on the steady state is that, in a MPNE, each household takes into account the reaction of its opponents to changes in the state of the economy. Since this reaction depends obviously on the form of the utility function, it follows that preference parameters like the elasticity of intertemporal substitution affect the location of the steady state. The present paper also emphasizes the strategic substitutability of the capital holdings of different households. If one household reduces its capital stock, the interest rate increases and the wage rate decreases. This creates a substitution effect which induces the other households to increase their own capital stocks. Of course, there is also an income effect associated with the change in factor prices and we show that the relative strength of income and substitution effects differs between rich and poor households.

A second contribution of the present paper is the study of the aggregate time-preference rate in the strategic Ramsey economy. We measure the aggregate time-preference rate by the real interest rate generated in a MPNE. In a recent paper, Gollier and Zeckhauser [8] have shown that, in every Pareto-optimal allocation, the aggregate time-preference rate is a weighted average of the individual time-preference rates with the weights being proportional to the individual tolerances for consumption fluctuations. This result does not automatically carry over to our framework, because households have market power and the equilibrium allocation is therefore not Pareto-optimal. For a realistic calibration of the model, however, we can show that the aggregate time-preference rate differs only slightly, though systematically from the prediction made by Gollier and Zeckhauser [8]. The systematic bias results from two sources. First, the strategic interaction of the households exerts upward pressure on the interest rate for the same reason that prices in an oligopoly exceed competitive prices. Second, proportionality of the weights to the individual tolerances for consumption fluctuations is perturbed by the potentially different degrees of commitment power that the households have. A household with a low elasticity of intertemporal substitution does not react quickly to changes in factor prices such that it can be said to have higher commitment power. Conversely, a household with a high elasticity of intertemporal substitution can easily be influenced by its opponents, because it reacts quickly to any changes in factor prices. Using this interpretation of the elasticity of intertemporal substitution, we argue that the aggregate time-preference rate can be expressed as a weighted average of the individual time-preference rates with more weight given to the households that have higher commitment power.

A further contribution of our paper is about the dynamics generated by MPNE. We show that the transition dynamics are typically non-monotonic, and that both the speed and the direction of adjustment are highly affected by the households’ elasticities of intertemporal substitution.
Finally, the paper contributes to the literature on computational techniques, since it presents an efficient algorithm for the numerical computation of the MPNE in the strategic Ramsey-Cass-Koopmans model. Since the model cannot be solved for a steady state without determining the equilibrium strategies, standard perturbation methods (like local linearization) cannot be used. We therefore solve the model using projection methods as suggested by Judd [10]. More precisely, we implement a least squares projection method with a high degree of approximation. It turns out that, unlike in many growth models, the least squares approach performs very well in our setup. It displays excellent convergence properties and delivers highly accurate solutions.

The assumption that households possess market power is of course quite uncommon in the literature. We would like to emphasize that we do not propose to replace the competitive model by one with strategically interacting households as a standard framework of macroeconomic analysis. Nevertheless, we would like to mention two variants of the present setup in which market power of households is uncontroversial. Firstly, one could interpret the households as different countries in a global economy with integrated markets. If the inhabitants of each single country act in a coordinated way (for example, because they are controlled by a central planner), then it follows that the world economy exhibits the main features of the present model. Secondly, Lane, Tornell, and Velasco [18, 19, 12] have analyzed economies with powerful groups of households (for example clans or ethnic or religious communities) that also act in a coordinated way. Their models also resemble the one in the present paper in the sense that these groups exert market power on the capital market.

The remainder of the paper is organized as follows. The model is formulated in section 2 where we describe the behavior of the economic agents and their interaction in a MPNE. We also briefly discuss alternative equilibrium concepts that have been used in the literature. Section 3 presents the equilibrium conditions. The main results of the numerical analysis are presented in section 4. After explaining the calibration, we discuss how the steady state depends on the time-preference profile of the population as well as on the commitment power of the households (as measured by their elasticities of intertemporal substitution). Representing the steady state interest rate as a weighted average of the individual time-preference rates, we can show that the weights are increasing functions of the commitment power of the households. We also compare this finding to the results from Gollier and Zeckhauser [8]. We conclude section 4 by discussing the transition dynamics to the steady state. The steady state appears to be asymptotically stable in all our numerical experiments, even if the transition dynamics can be non-monotonic. The details of our numerical algorithm are described in section 5 and concluding remarks are contained in section 6.

2 Model formulation

In this section we describe the infinite-horizon economy under consideration. Except for the assumption that households do not take prices as given, this is the standard Ramsey-Cass-Koopmans model with borrowing constraints as in Becker [2]. The departure from price taking behavior requires a game-theoretic equilibrium concept. We will therefore define Markov-perfect Nash equilibria.
2.1 Production

Time is measured in discrete periods $t \in \{0, 1, 2, \ldots\}$. The production sector consists of a continuum of firms, which transform capital and labor into a homogeneous output good. All firms have access to a common technology, which is described by the production function $F : \mathbb{R}_{+}^2 \mapsto \mathbb{R}_{+}$. This function is assumed to be linearly homogeneous and to satisfy standard smoothness and curvature assumptions. The partial derivatives of $F$ will be denoted by $F_K$ and $F_L$, respectively.

There is free entry into the production sector and all firms act as price takers on the output market as well as on both factor markets. Given the aggregate factor supplies $K_t$ and $L_t$, this implies that the rental rate of capital and the real wage in period $t$ are given by

$$r_t = F_K(K_t, L_t) \quad \text{and} \quad w_t = F_L(K_t, L_t),$$

respectively, and that the firms make zero profits.

2.2 Households

The economy is populated by finitely many infinitely-lived households, who own the production factors and derive utility from consumption. Although much of the analysis can be carried out for an arbitrary finite number $H$ of households we restrict the presentation to $H = 2$. This allows us to focus on those issues that form the main contribution of this paper without hiding the mechanics of the model behind cumbersome notation.

Household $i \in \{1, 2\}$ has the utility function $u^i : \mathbb{R}_{+} \mapsto \mathbb{R}$ and the time-preference factor $\beta^i \in (0, 1)$. Without loss of generality, we assume household 1 to be at least as patient as household 2, that is, $\beta^1 \geq \beta^2$. The life-time utility of household $i$ is

$$\sum_{t=0}^{+\infty} (\beta^i)^t u^i(c^i_t),$$

where $c^i_t$ is the household’s consumption rate in period $t$. The utility functions $u^1$ and $u^2$ are assumed to satisfy standard smoothness and curvature assumptions.

We denote by $\ell^i_t$ the fraction of period $t$ that the household spends working and by $k^i_t$ the capital holdings of household $i$ at the beginning of period $t$. The flow budget constraint can then be written as

$$k^i_{t+1} = (1 - \delta + r_t)k^i_t + w_t\ell^i_t - c^i_t.$$

The parameter $\delta \in [0, 1]$ is the rate of capital depreciation and $k^i_0$ is the initial wealth of household $i$. In addition to this flow budget constraint, the household has to satisfy the non-negativity constraints

$$c^i_t \geq 0 \quad \text{and} \quad k^i_{t+1} \geq 0.$$
for all $t \geq 0$.

### 2.3 Markov-perfect Nash equilibrium

If one would assume that the households act as price-takers, then the model described above would coincide with the standard Ramsey-Cass-Koopmans model as studied for example by Becker [2]. We assume, however, that the households are aware of the fact that they can affect the factor prices. In other words, the households take the inverse factor demand functions in (1) as given. In order to solve the model under these assumptions, we employ the concept of Markov-perfect Nash equilibrium (MPNE). This requires that we describe the households’ behavior by strategies that depend on a minimal payoff-relevant state of the economy. In the present case, this is the vector of capital stocks $x_t = (k^1_t, k^2_t) \in \mathbb{R}^2_+$. The behavior of household $i$ is given by two functions $\ell^i_t : \mathbb{R}^2_+ \mapsto [0,1]$ and $c^i_t : \mathbb{R}^2_+ \mapsto \mathbb{R}_+$ with the interpretation that household $i$ supplies $\ell^i_t = \ell^i(x_t)$ units of labor and consumes $c^i_t = c^i(x_t)$ units of output, if the capital holdings of the two households at the outset of period $t$ are given by $x_t = (k^1_t, k^2_t)$.

We assume that the households play a Nash equilibrium. This means that each household takes the strategies of its opponent as given. Moreover, as already mentioned above, both households take the inverse factor demand functions in (1) as given. Denoting the opponent of household $i$ by $j$ (and vice versa), it follows that household $i$ maximizes its objective functional (2) subject to the two flow budget constraints

\[
k^i_{t+1} = R(k^1_t + k^2_t, \ell^i_t + \ell^j_t + \ell^i(x_t)) k^i_t + W(k^1_t + k^2_t, \ell^i_t + \ell^j_t + \ell^i(x_t)) \ell^i_t - c^i_t, \quad (5)
k^j_{t+1} = R(k^1_t + k^2_t, \ell^i_t + \ell^j_t + \ell^i(x_t)) k^j_t + W(k^1_t + k^2_t, \ell^i_t + \ell^j_t + \ell^i(x_t)) \ell^j_t - c^j_t \quad (6)
\]

and the inequality constraints

\[
0 \leq c^i_t \leq R(k^1_t + k^2_t, \ell^i_t + \ell^j_t + \ell^i(x_t)) k^i_t + W(k^1_t + k^2_t, \ell^i_t + \ell^j_t + \ell^i(x_t)) \ell^i_t, \quad (7)
0 \leq \ell^i_t \leq 1, \quad (8)
\]

where $R(K, L) = 1 - \delta + F_K(K, L)$ and $W(K, L) = F_L(K, L)$.

A MPNE consists of four functions $c^1$, $c^2$, $\ell^1$, and $\ell^2$ such that the following is true for all $i \in \{1,2\}$ and all initial states $x_0 = (k^1_0, k^2_0) \in \mathbb{R}^2_+$: given $c^j$ and $\ell^i$, the problem of maximizing (2) subject to (5)-(8) has an optimal solution $(k^i_{t+1}, k^j_{t+1}, c^i_t, \ell^i_t)_{t=0}^{+\infty}$ which satisfies $c^i_t = c^i(x_t)$ and $\ell^i_t = \ell^i(x_t)$.

Given a MPNE $(c^1, c^2, \ell^1, \ell^2)$, an equilibrium path from the initial state $x_0 = (k^1_0, k^2_0)$ is a sequence of capital stocks $(x_t)_{t=0}^{+\infty}$, where $x_t = (k^1_t, k^2_t)$, that satisfies

\[
k^i_{t+1} = R(k^1_t + k^2_t, \ell^1_t + \ell^2_t + \ell^i(x_t)) k^i_t + W(k^1_t + k^2_t, \ell^1_t + \ell^2_t + \ell^i(x_t)) \ell^i_t - c^i(x_t) \quad (5)
\]

for all $i \in \{1,2\}$ and all $t \geq 0$. A steady state is simply a constant equilibrium path, that is, a vector $x = (k^1, k^2) \in \mathbb{R}^2_+$ such that

\[
k^i = R(k^1 + k^2, \ell^1(x) + \ell^2(x)) k^i + W(k^1 + k^2, \ell^1(x) + \ell^2(x)) \ell^i(x) - c^i(x)
\]

holds for all $i \in \{1,2\}$. 
2.4 Alternative equilibrium definitions

As has been mentioned above, the competitive version of the model has been thoroughly investigated. Becker [2], for example, defines a competitive equilibrium in the model described above as a sequence of factor prices \( (r_t, w_t)_{t=0}^{+\infty} \) and allocations \( (c^i_t, \ell^i_t, k^i_{t+1})_{t=0}^{+\infty}, \ i \in \{1,2\} \), such that the following conditions are true: (i) given the sequences of factor prices and the initial state \( x_0 = (k^1_0, k^2_0) \), the allocation \( (c^i_t, \ell^i_t, k^i_{t+1})_{t=0}^{+\infty} \) solves the problem of maximizing (2) subject to (3)-(4) and (ii) the equations in (1) hold for \( K_t = k^1_t + k^2_t \) and \( L_t = \ell^1_t + \ell^2_t \). Becker [2] also shows that there exists a unique initial state \( (k^1_0, k^2_0) \) that allows for a competitive equilibrium consisting of constant sequences (i.e., a steady state). Moreover, the Ramsey conjecture holds, that is \( k_2 = 0 \) whenever \( \beta^1 > \beta^2 \).

The extreme long-run wealth distribution generated by competitive equilibria in the Ramsey-Cass-Koopmans model has concerned many researchers. Sorger [16] has argued that this outcome makes the price-taking assumption implausible because the most patient household 1 has a monopolistic position on the capital market. To address this issue, Sorger [16] has formulated two variants of an equilibrium definition that takes market power properly into account. The first one assumes that the households still behave as price takers on the labor market but that they realize their influence on the interest rate. An equilibrium according to this definition consists of sequences \( (c^i_t, \ell^i_t, k^i_{t+1})_{t=0}^{+\infty} \) solves the problem of maximizing (2) subject to the flow budget constraint

\[
k^i_{t+1} = R(k^1_t + k^2_t, \ell^1_t + \ell^2_t)k^i_t + w_t \ell^i_t - c^i_t
\]

and the non-negativity constraints (4), and such that the second equation in (1) holds for \( K_t = k^1_t + k^2_t \) and \( L_t = \ell^1_t + \ell^2_t \). This equilibrium definition has also been adopted by Becker [3, 4].

The second variant of an open-loop Nash equilibrium defined by Sorger [16] assumes that the households realize their influence on both the interest rate and the wage rate. More specifically, an equilibrium consists of sequences \( (c^i_t, \ell^i_t, k^i_{t+1})_{t=0}^{+\infty}, \ i \in \{1,2\} \), such that, given \( (w_t)_{t=0}^{+\infty}, \ (c^i_t, \ell^i_t, k^i_{t+1})_{t=0}^{+\infty} \), and the initial state \( x_0 = (k^1_0, k^2_0) \), the sequence \( (c^i_t, \ell^i_t, k^i_{t+1})_{t=0}^{+\infty} \) solves the problem of maximizing (2) subject to the flow budget constraint

\[
k^i_{t+1} = R(k^1_t + k^2_t, \ell^1_t + \ell^2_t)k^i_t + W(k^1_t + k^2_t, \ell^1_t + \ell^2_t)\ell^i_t - c^i_t
\]

and the non-negativity constraints (4). Equilibria defined in this way have also been studied in Sorger [17].

No matter which of the two equilibrium definitions proposed by Sorger [16] one adopts, there can exist constant equilibria (steady states) in which the strictly less patient household owns positive wealth forever. In other words, the extreme long-run wealth distribution predicted by the competitive model does not necessarily occur in the strategic model with open-loop strategies. One important purpose of the present paper is to show that this result holds also for MPNE. In this regard it is worth pointing out that our definition of an MPNE assumes that the households realize their control over both the interest rate and the wage rate. We take this approach because, contrary to the open-loop case that has been considered in the literature, it is not possible to consistently define a recursive equilibrium in which households correctly...
anticipate the evolution of the wage rate without realizing their control over it. To see this first note that in a recursive equilibrium the households must know the state \( x_t \) because they make state-contingent decisions. On the other hand, in a recursive equilibrium the households must also know how the wage rate \( w_t \) depends on the (aggregate) state \( x_t \) of the economy. Taking both of these observations together, it follows that the households know how they can affect the wage rate.

### 3 Equilibrium conditions

In this section we summarize the equilibrium conditions for a MPNE. To this end, recall that the state variable is \( x = (k^1, k^2) \). It will be useful to define

\[
\mathcal{I}^i(x, \ell^i, \ell^j) = R(k^1 + k^2, \ell^1 + \ell^2)k^i + W(k^1 + k^2, \ell^1 + \ell^2)\ell^i.
\]

\( \mathcal{I}^i(x, \ell^i, \ell^j) \) is the income of household \( i \) written as a function of both households’ capital stocks and labor supplies. Moreover, let us denote by \( V^i(x) \) the optimal value function of household \( i \)'s optimization problem. This value function satisfies the Bellman equation

\[
V^i(x) = \max_{c^i, c^j} \left\{ u^i(c^i) + \beta^i V^i(\mathcal{I}^i(x, \ell^i, \ell^j(x)) - c^i, \mathcal{I}^j(x, \ell^j(x)) - c^j(x)) \right\}
\]

(9)

In what follows, a prime indicates the evaluation at ‘next period’s’ state \( x' = (k'^1, k'^2) \).

With appropriate Inada-type conditions on the utility functions we can rule out that the left-hand inequality in (7) becomes binding. The first-order optimality condition for the choice of the consumption rate \( c^i \) is therefore

\[
u_{c^i}(c^i(x)) \geq \beta^i V_{k^i}^i(x')
\]

(10)

with equality if \( k'^i > 0 \). The first-order optimality condition for the maximization with respect to \( \ell^i \), on the other hand, is

\[
V_{k^j}^i(x') \mathcal{I}^i_{\ell^i} + V_{k^j}^i(x') \mathcal{I}^j_{\ell^j} \begin{cases} 
\leq 0 & \text{if } \Gamma(x) = 0, \\
= 0 & \text{if } \Gamma(x) \in (0, 1), \\
\geq 0 & \text{if } \Gamma(x) = 1,
\end{cases}
\]

(11)

where here and in what follows the arguments of the functions \( \mathcal{I}^i \) and \( \mathcal{I}^j \) as well as their derivatives are omitted for better readability. Finally, from the envelope theorem it follows that

\[
V_{k^i}^i(x) = \beta^i \left\{ V_{k^j}^i(x')[\mathcal{I}_{k^i} + \mathcal{I}_j \ell^j(x)] + V_{k^j}^i(x')[\mathcal{I}_{k^i} + \mathcal{I}_j \ell^j(x) - c^j_k(x)] \right\}
\]

and

\[
V_{k^j}^i(x) = \beta^i \left\{ V_{k^j}^i(x')[\mathcal{I}_{k^i} + \mathcal{I}_j \ell^j(x)] + V_{k^j}^i(x')[\mathcal{I}_{k^i} + \mathcal{I}_j \ell^j(x) - c^j_k(x)] \right\}.
\]

2These arguments are \( (x, \Gamma(x), \ell^j(x)) \) and \( (x, \Gamma(x), \ell^j(x)) \), respectively.
4 Results

This section presents the main results of the paper. We start by describing the calibration of the model and by discussing the labor supply decisions of the households. Then we analyze the properties of the steady state generated by a MPNE. In particular, we investigate the long-run effects of changes in the time-preference profile of the population and the effects of changes in the households’ elasticities of intertemporal substitution. Finally, we analyze the transition dynamics implied by the MPNE.

We calibrate the model with functional forms and parameter values commonly used in the literature. The production technology is described by a Cobb-Douglas production function of the form

$$F(K, L) = AK^\alpha L^{1-\alpha},$$

where $A$ denotes the technology level and $\alpha$ is the income share of capital. The instantaneous utility function of household $i \in \{1, 2\}$ is

$$u^i(c) = \begin{cases} (c^{1-\sigma^i} - 1)/(1 - \sigma^i) & \text{if } \sigma^i \neq 1, \\ \log(c) & \text{if } \sigma^i = 1, \end{cases}$$

where $\sigma^i$ is the inverse of household $i$’s elasticity of intertemporal substitution. The technology and preference parameters are chosen as follows. The depreciation rate is set to $\delta = 0.025$ and the income share of capital is set to $\alpha = 0.36$. The technology level is arbitrarily (though without loss of generality) set to $A = 1$. The values for $\beta^1, \beta^2, \sigma^1, \text{ and } \sigma^2$ will vary throughout our analysis. We consider time-preference factors $\beta^i$ between 0.98 and 0.995. As for the elasticity of intertemporal substitution parameters $\sigma^i$, we assume values between 0.5 and 5. These values cover the range that is typically considered in the literature when calibrating models to quarterly data.

We solve the calibrated model numerically for a MPNE. Details of the numerical algorithm will be discussed in section 5. In all parameter constellations that we consider, it turns out that both households find it optimal to provide their entire labor endowment (normalized to be equal to 1). Although this is intuitively plausible, it is far from obvious as there are different (and partly ambiguous) effects at work.

Firstly, by reducing its labor supply, a household can increase the wage rate $W(K, L)$, which is a decreasing function of $L = \ell^1 + \ell^2$. If the wage rate responds very strongly to changes of the labor supply, this might induce a household to work little. For the Cobb-Douglas production technology used in our numerical analysis, however, the elasticity of the wage rate with respect to $L$ is smaller than 1 in absolute value so that this effect does not occur. More specifically, under the assumptions of a Cobb-Douglas technology, household $i$’s income $I^i(x, \ell^i, \ell^j)$ is increasing with respect to $\ell^i$.

Secondly, $\ell^i$ affects also the opponent’s income $I^j(x, \ell^i, \ell^j)$ and the direction of this effect depends on whether household $j$ supplies a higher or a lower capital-labor ratio than household $i$. The effect of changing $\ell^i$ on household $j$’s future capital stock (i.e., on the second argument of $V^j$ on the right-hand side of (9)) is therefore ambiguous.

See Sorger [17] for further discussion of this point.
And finally, because the behavior of both households in the future depends on the future states of the economy which, in turn, depend (also) on current labor supply decisions, there may be further strategic reasons for households to supply less than their full time-endowment to the firms. We believe, however, that at least in the case of a Cobb-Douglas technology, it is always optimal to supply the full endowment and our numerical results confirm this conjecture. In what follows, we shall therefore suppose that $f(x) = 1$ holds for all $x \in \mathbb{R}_+^2$ and all $i \in \{1, 2\}$ without further mentioning it.

### 4.1 Steady states

Let us now discuss some properties of the steady states generated by MPNE. We first focus on the role of the time-preference factors in determining the long run equilibrium, and later emphasize the role of the elasticities of intertemporal substitution.

#### 4.1.1 The role of time-preference factors

Table 1 shows the steady state capital stocks, consumption rates, and factor prices for several combinations of the discount factors $\beta^1$ and $\beta^2$. There are several observations that we would like to emphasize regarding these numbers.

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<th>$\beta^2$</th>
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<th>0.99</th>
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<td>(61.90, 15.66, 77.56)</td>
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<td>0.99</td>
<td>0.995</td>
<td>(31.05, 54.09, 85.14)</td>
<td>(37.99, 37.99, 75.98)</td>
<td>(43.31, 25.27, 68.58)</td>
<td>(47.02, 15.39, 62.40)</td>
</tr>
<tr>
<td></td>
<td>0.99</td>
<td>(2.71, 2.88)</td>
<td>(2.75, 2.75)</td>
<td>(2.83, 2.60)</td>
<td>(2.90, 2.44)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.0076, 2.47)</td>
<td>(1.0101, 2.37)</td>
<td>(1.0125, 2.28)</td>
<td>(1.0148, 2.21)</td>
</tr>
<tr>
<td>0.985</td>
<td>0.995</td>
<td>(19.85, 55.99, 75.84)</td>
<td>(25.88, 41.95, 67.83)</td>
<td>(30.70, 30.70, 61.40)</td>
<td>(34.44, 21.61, 56.05)</td>
</tr>
<tr>
<td></td>
<td>0.99</td>
<td>(2.57, 2.94)</td>
<td>(2.61, 2.81)</td>
<td>(2.66, 2.66)</td>
<td>(2.73, 2.51)</td>
</tr>
<tr>
<td></td>
<td></td>
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<td>(1.0127, 2.28)</td>
<td>(1.0152, 2.20)</td>
<td>(1.0176, 2.12)</td>
</tr>
<tr>
<td>0.98</td>
<td>0.995</td>
<td>(12.15, 56.18, 68.34)</td>
<td>(17.60, 43.62, 61.22)</td>
<td>(21.92, 33.61, 55.53)</td>
<td>(25.40, 25.40, 50.81)</td>
</tr>
<tr>
<td></td>
<td>0.99</td>
<td>(2.43, 2.99)</td>
<td>(2.46, 2.86)</td>
<td>(2.51, 2.72)</td>
<td>(2.57, 2.57)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.0126, 2.28)</td>
<td>(1.0153, 2.19)</td>
<td>(1.0179, 2.12)</td>
<td>(1.0204, 2.05)</td>
</tr>
</tbody>
</table>

First line ($k^1, k^2, K$), second line ($c^1, c^2$), third line ($R, W$)

Parameters: $\sigma^1 = 3$, $\sigma^2 = 1$, $\alpha = 0.36$, $\delta = 0.025$, $A = 1$

Note: numbers are rounded, hence $k^1$ and $k^2$ need not exactly sum up to $K$.

First, the Ramsey conjecture does not hold in the MPNE, i.e., the less patient household owns positive wealth in the steady state. The intuition behind this result is exactly the same as in the open-loop case studied by Sorger [16, 17] and Becker [3]. To explain this intuition, suppose that the economy starts in the competitive steady state in which household 1 owns the entire capital stock. In this situation, household 1 benefits from saving less than in the competitive steady state, because doing so increases the interest rate $R(K, L)$ without causing any sacrifice.
of consumption.\textsuperscript{4} If this pushes the interest rate above the time-preference rate of household 2, it creates a substitution effect that induces the latter to save some capital. But even if the interest rate is still below household 2’s time-preference rate, household 2 may already start saving, because an increase of its capital holdings from 0 to some positive value increases also the wage rate $W(K, L)$. This, in turn, raises household 2’s wage income which, after all, is the only source of income for household 2 in the competitive steady state. Of course, saving a positive amount of capital means that the household must sacrifice some present consumption and it depends again on its time-preference whether the household finds this sacrifice worth making.

The above arguments emphasize the strategic substitutability of the two households’ capital stocks. If one household reduces its capital holdings, the interest rate increases and the wage rate decreases. This creates incentives for the other household to increase its own capital stock (substitution effect). Of course, there is also an income effect associated with the change in factor prices. For the rich household 1, a reduction in household 2’s capital stock leads to a higher income, since the increased interest income more than compensates for the reduced wage income. The opposite is true for the poor household, for whom the wage is the dominant source of income. It is also worth mentioning that the elasticities of intertemporal substitution determine to a large extent how quickly the households react to the substitution and income effects. We shall discuss the implications of this observation in more detail further below.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$\beta^2$ & 0.995 & 0.99 & 0.985 & 0.98 \\
\hline
0.995 & (48.49, 48.49, 96.98) & (56.55, 29.70, 86.26) & (61.90, 15.66, 77.56) & (64.70, 5.59, 70.29) \\
& (48.49, 48.49, 96.98) & (53.12, 32.31, 85.43) & (55.17, 20.72, 75.89) & (55.59, 12.32, 67.91) \\
& (48.49, 48.49, 96.98) & (96.98, 0, 96.98) & (96.98, 0, 96.98) & (96.98, 0, 96.98) \\
0.99 & (31.05, 54.09, 85.14) & (37.99, 37.99, 75.98) & (43.31, 25.27, 68.58) & (47.02, 15.39, 62.40) \\
& (32.31, 53.12, 85.43) & (37.99, 37.99, 75.98) & (41.26, 26.79, 68.05) & (42.94, 18.40, 61.34) \\
& (0, 96.98, 96.98) & (37.99, 37.99, 75.98) & (75.98, 0, 75.98) & (75.98, 0, 75.98) \\
0.985 & (19.85, 55.99, 75.84) & (25.88, 41.95, 67.83) & (30.70, 30.70, 61.40) & (34.44, 21.61, 56.05) \\
& (20.72, 55.17, 75.89) & (26.79, 41.26, 68.05) & (30.70, 30.70, 61.40) & (33.11, 22.58, 55.70) \\
& (0, 96.98, 96.98) & (0, 75.98, 75.98) & (30.70, 30.70, 61.40) & (61.40, 0, 61.40) \\
0.98 & (12.15, 56.18, 68.34) & (17.60, 43.62, 61.22) & (21.92, 33.61, 55.53) & (25.40, 25.40, 50.81) \\
& (12.32, 55.59, 67.90) & (18.40, 42.94, 61.34) & (22.58, 33.11, 55.70) & (25.40, 25.40, 50.81) \\
& (0, 96.98, 96.98) & (0, 75.98, 75.98) & (0, 61.40, 61.40) & (25.40, 25.40, 50.81) \\
\hline
\end{tabular}
\caption{Steady state capital stocks implied by different equilibrium concepts}
\end{table}

All entries are of the form $(k^1, k^2, K)$. First line: MPNE; second line: open-loop; third line: competitive.

Parameters: $\sigma^1 = 3$, $\sigma^2 = 1$, $\alpha = 0.36$, $\delta = 0.025$, $A = 1$

Note: numbers are rounded, hence $k^1$ and $k^2$ need not exactly sum up to $K$.

It is also worth noting that the open-loop steady state (as analyzed by Sorger [16, 17] and Becker [3]) typically does not coincide with the steady state generated by the MPNE. In the latter, households do not only realize their direct influence on prices, but they also anticipate their indirect influence via the other household’s reaction to changes in their capital holdings. Because of the strategic substitutability mentioned above, the reaction of the opponent

\textsuperscript{4}This is the usual argument according to which a monopolist uses quantity rationing in order to increase its profit.
mitigates an individual’s effect on prices. As a consequence, the difference between the Markov-
perfect steady state and the competitive one is not as big as the difference between the open-loop
steady state and the competitive one. More specifically, the steady state in the MPNE typi-
cally features more dispersed individual capital holdings than the open-loop steady state. In
most of the cases, the aggregate capital stock in the MPNE steady state is also higher than in
the open-loop steady state. The above arguments are correct as long as both households own
positive amounts of capital in the steady state. If only household 1 has positive wealth (i.e.,
if the Ramsey conjecture holds), the Markov-perfect steady state coincides with the open-loop
steady state. The above findings are illustrated by table 2 which is generated using the same
parameter values as those underlying table 1.

Let us now return to table 1. Our second observation regarding that table is that the ranking of
households according to their steady state capital stocks coincides with the ranking according
to their time-preference rates: more patient households own more wealth in the steady state.
In particular, if the households share the same time-preference rate, their capital holdings in
the Markov-perfect steady state are identical to each other. Furthermore, in this special case
the capital distribution in the steady state generated by the MPNE coincides with the capital
distribution in the (symmetric) competitive steady state and the open-loop steady state. An
analytical proof of this result for the case of identical households is presented in a previous
version of this paper, see Pichler and Sorger [13]. Our numerical analysis underlying table 1
shows that homogeneity of the time-preference rates is all that is needed for the validity of
the result but that households may differ with respect to the parameter values for \( \sigma_1 \) and
\( \sigma_2 \). A possible explanation for the coincidence of steady state capital holdings in competitive,
open-loop, and MPNE equilibria in the case of completely identical households is that neither
net trades nor intertemporal substitution occurs in a steady state equilibrium and that the
households therefore do not have any incentive to change relative prices. Out of a steady
state, however, an intertemporal trade-off between consumption and saving is possible so that,
even in the case of completely identical households, the competitive equilibrium, the open-loop
equilibrium, and the MPNE generate mutually different transition dynamics.

Finally, the results in table 1 demonstrate that, in the case where the households differ with
respect to their time-preference factors, their intertemporal elasticities of substitution para-
eters \( \sigma_1 \) and \( \sigma_2 \) affect the long-run steady state of the economy. For example, the steady
state associated with \( \beta_1 = 0.99, \beta_2 = 0.98, \sigma_1 = 3, \) and \( \sigma_2 = 1 \) features an aggregate capital
stock equal to \( K = 62.40 \) and individual capital holdings equal to \( k_1 = 47.02 \) and \( k_2 = 15.39 \).
The steady state characterized by the same time-preferences factors but different elasticities of
substitution \( \sigma_1 = 1 \) and \( \sigma_2 = 3 \) features a lower aggregate capital stock, \( K = 61.22 \), and less
dispersed individual capital holdings, \( k_1 = 43.62 \) and \( k_2 = 17.60 \). We find this result particu-
larly interesting, since the elasticities of intertemporal substitution do neither affect the steady
states of the competitive equilibrium nor the steady state of the open-loop equilibrium. In the
following section, we therefore elaborate on this observation and try to provide some intuition
behind it.
4.1.2 The role of elasticities of intertemporal substitution

To illustrate the role of the elasticities of intertemporal substitution, we display Markov-perfect steady states for different combinations of $\sigma^1$ and $\sigma^2$ in table 3. We observe that an increase in the parameter $\sigma^i$ leads, ceteris paribus, to an increase in household $i$’s steady state capital stock, whereas it reduces the steady state capital holdings of household $j$. For example, given $\sigma^1 = 3$, an increase of $\sigma^2$ from 0.5 to 5 increases household 2’s steady state capital stock from 14.41 to 17.45, whereas it reduces the steady state capital stock held by household 1 from 48.54 to 43.87. The aggregate capital stock $K$ declines from 62.95 to 61.31. It is not generally true, however, that increases in $\sigma^i$ reduce the aggregate capital stock. In particular, our results show that the effect of an increase in $\sigma^i$ on the aggregate capital stock depends on the time-preference profile in the population. If $\beta^i > \beta^j$, then an increase in $\sigma^i$ increases $K$, whereas if $\beta^i < \beta^j$, the opposite is true.5

### Table 3: Markov-perfect steady states for different elasticities of substitution

<table>
<thead>
<tr>
<th>$\sigma^1$</th>
<th>0.5</th>
<th>1</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>(45.69, 16.36, 62.05)</td>
<td>(44.61, 16.98, 61.60)</td>
<td>(43.04, 17.98, 61.02)</td>
<td>(42.41, 18.42, 60.83)</td>
</tr>
<tr>
<td></td>
<td>(2.89, 2.45)</td>
<td>(2.87, 2.46)</td>
<td>(2.85, 2.47)</td>
<td>(2.85, 2.48)</td>
</tr>
<tr>
<td>1</td>
<td>(46.65, 15.74, 62.40)</td>
<td>(45.42, 16.47, 61.89)</td>
<td>(43.62, 17.60, 61.22)</td>
<td>(42.89, 18.10, 60.99)</td>
</tr>
<tr>
<td></td>
<td>(2.90, 2.44)</td>
<td>(2.88, 2.45)</td>
<td>(2.86, 2.46)</td>
<td>(2.85, 2.47)</td>
</tr>
<tr>
<td>3</td>
<td>(48.54, 14.11, 62.95)</td>
<td>(47.02, 15.39, 62.40)</td>
<td>(44.77, 16.85, 61.61)</td>
<td>(43.87, 17.45, 61.31)</td>
</tr>
<tr>
<td></td>
<td>(2.92, 2.42)</td>
<td>(2.91, 2.44)</td>
<td>(2.88, 2.45)</td>
<td>(2.86, 2.46)</td>
</tr>
<tr>
<td>5</td>
<td>(49.57, 13.63, 63.20)</td>
<td>(47.87, 14.77, 62.64)</td>
<td>(45.38, 16.43, 61.81)</td>
<td>(44.40, 17.10, 61.49)</td>
</tr>
<tr>
<td></td>
<td>(2.94, 2.42)</td>
<td>(2.92, 2.43)</td>
<td>(2.88, 2.45)</td>
<td>(2.87, 2.46)</td>
</tr>
</tbody>
</table>

Parameters: $\beta^1 = 0.99$, $\beta^2 = 0.98$, $\alpha = 0.36$, $\delta = 0.025$, $A = 1$

Note: numbers are rounded, hence $k^1$ and $k^2$ need not exactly sum up to $K$.

To understand these results one must answer the following two questions: why does the steady state depend on the elasticities of intertemporal substitution at all, and why is the aggregate capital stock in the steady state a decreasing (increasing) function of the more patient (less patient) household’s elasticity of intertemporal substitution?

To address the first question, recall that in the competitive model, income of household $i$ is given by $R_t k^i_t + W_t$, where $R_t$ and $W_t$ are factor prices that household $i$ perceives as given. The Euler equation requires that the marginal rate of substitution $u^i_c(c^i_t)/[\beta^i u^i_c(c^i_{t+1})]$ is greater than or equal to the real interest factor $R_{t+1}$ (with equality if positive capital is held in period $t+1$). In a steady state, this marginal rate of substitution is necessarily equal to $1/\beta^i$ so that the steady state condition becomes $1/\beta^i \geq R_{t+1}$. As the factor prices do not depend on the preferences of the households, the steady state itself must also be independent of the form of the utility functions (i.e., independent of $\sigma^1$ and $\sigma^2$). The same argument applies to the open-loop

---

5As already outlined in the previous section, an increase in $\sigma^i$ does not affect the steady state if $\beta^1 = \beta^2$. 

13
steady states analyzed by Sorger [16, 17] as well. In that case the income of household $i$ is 
\[ R(k_{1t} + k_{2t}, 2)k_{1t} + W(k_{1t} + k_{2t}, 2), \]
where the household takes the functions $R$ and $W$ as well as the opponent’s capital stocks $(k^j_{it})_{t=0}^{\infty}$ as given. The Euler equation requires now that the marginal rate of substitution is greater than or equal to 
\[ (d/dk_{1t}+1)[R(k_{1t+1} + k_{2t+1}, 2)k_{1t+1} + W(k_{1t+1} + k_{2t+1}, 2)], \]
which is still independent of the utility functions of the two households.

In contrast to the competitive and the open-loop equilibrium, a household in a MPNE neither takes the factor prices nor the opponent’s capital stocks as given. Instead, household $i$ maximizes its total discounted utility subject to the two flow budget constraints (5)-(6). The latter constraint involves the correctly anticipated policy function $c^j$ of household $j$. The form of this policy function, however, depends obviously on the elasticity of intertemporal substitution $1/\sigma^j$. For example, if $\sigma^j$ is very large (such that household $j$ has a very low elasticity of intertemporal substitution), then $c^j(k^1, k^2)$ is nearly constant. In the opposite case, if $\sigma^j$ is close to 0, the elasticity of intertemporal substitution is very high and the graph of $c^j(k^1, k^2)$ is not as flat. Since the shape of the opponent’s policy function $c^j$ affects one of the two flow budget constraints that household $i$ must take into account, it is clear that the elasticities of intertemporal substitution can (and, in general, do) affect the location of the steady state generated by a MPNE.

To address the second question, namely why the effects of changes in $\sigma^i$ on the aggregate capital stock in the steady state depend on whether household $i$ is more or less patient than its opponent, it is useful to reinterpret the above arguments in terms of commitment power. If $\sigma^i$ is small (high elasticity of intertemporal substitution), then it follows that household $i$ reacts to changes in $k^j$ very quickly. In other words, household $i$ can easily be influenced by household $j$. We can interpret this by saying that household $i$ has only weak commitment power. A large value of $\sigma^i$, on the other hand, implies that household $i$ reacts only slowly to any changes of household $j$’s capital stock, and household $i$ can therefore be said to have relatively high commitment power. To summarize, an increase of $\sigma^i$ implies a corresponding increase of the commitment power of household $i$, and vice versa for a reduction of $\sigma^i$.

Now suppose that $\sigma^2$ increases such that the more patient household 1 loses commitment power relative to household 2. This is likely to shift the steady state interest rate towards the time-preference rate of (the more powerful) household 2. It follows that the steady state interest rate increases and, hence, that the aggregate capital stock in the steady state decreases. In the opposite case, where $\sigma^1$ increases, commitment power is shifted towards the more patient household 1. This implies that the steady state interest rate decreases which, in turn, triggers a corresponding increase of the steady state aggregate capital stock. This is exactly what we observe in table 3.

### 4.1.3 Commitment power and aggregate time-preference

The above argument rests on the intuitively plausible assumption that the steady state interest rate is a weighted average of the time-preference rates of the two households, whereby the household with higher commitment power is given a higher weight. Although we have no general proof of this property, it holds in all our numerical examples. Formally, there exists a
value $\lambda \in (0, 1)$ such that the equation

$$R = \lambda \frac{1}{\beta_1} + (1 - \lambda) \frac{1}{\beta_2}$$

is satisfied. Table 4 below shows the weight $\lambda$ as a function of $\sigma^1$ and $\sigma^2$ for fixed time-preference factors $\beta^1$ and $\beta^2$. It is clearly seen that $\lambda$ (the weight given to household 1) is an increasing function of $\sigma^1$ and a decreasing function of $\sigma^2$. Since, according to our arguments from above, both an increase of $\sigma^1$ and a reduction of $\sigma^2$ correspond to an increase of household 1’s commitment power, these results confirm our conjecture about the relation between individual time-preference rates, commitment power, and the steady state interest rate.

Table 4: The interest rate weight $\lambda$

<table>
<thead>
<tr>
<th>$\sigma^1$</th>
<th>0.5</th>
<th>1</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.5287</td>
<td>0.5103</td>
<td>0.4870</td>
<td>0.4792</td>
</tr>
<tr>
<td>1</td>
<td>0.5425</td>
<td>0.5224</td>
<td>0.4952</td>
<td>0.4856</td>
</tr>
<tr>
<td>3</td>
<td>0.5642</td>
<td>0.5428</td>
<td>0.5111</td>
<td>0.4989</td>
</tr>
<tr>
<td>5</td>
<td>0.5738</td>
<td>0.5520</td>
<td>0.5191</td>
<td>0.5061</td>
</tr>
</tbody>
</table>

Parameters: $\beta^1 = 0.99, \beta^2 = 0.98, \alpha = 0.36, \delta = 0.025, A = 1$

In an important paper, Gollier and Zeckhauser [8] have shown that the aggregate time-preference rate of a heterogeneous population can be expressed as a weighted average of the time-preference rates of the agents, whereby the weights are proportional to the individual tolerances for consumption fluctuations. This result has been derived under the assumption of Pareto-optimal allocations such that it is not applicable to the Nash equilibria analyzed in the present paper. Nevertheless, it is interesting to study to what extent the predictions of Gollier and Zeckhauser [8] fail to hold in the MPNE. In order to do this, we shall therefore identify the aggregate time-preference rate with the steady state interest rate and compare it to the value that is predicted by Gollier and Zeckhauser [8].

In our setting, the tolerance for consumption fluctuations for household $i$ is given by $T^i = c^i/\sigma^i$. The aggregate time-preference according to Gollier and Zeckhauser [8] is therefore given by

$$\hat{R} = \frac{c^1/(\sigma^1 \beta^1) + c^2/(\sigma^2 \beta^2)}{c^1/\sigma^1 + c^2/\sigma^2}.$$  

Table 5 below shows the ratio $(\hat{R} - R)/R$, where $R$ is the actual equilibrium interest rate in the steady state. This ratio measures the relative prediction error of $\hat{R}$.

All the numbers in table 5 are smaller than half a percentage point (in absolute value). This shows that Gollier and Zeckhauser’s aggregate time-preference rate $\hat{R}$ is a rather good predictor of the steady state interest rate. There are, however, two observations that we would like to point out. First, $\hat{R}$ is systematically higher than $R$ in the lower left corner of the table and it is systematically lower than $R$ in the upper right corner. Taking into account the behavior of the interest rates $R$ given in table 3, we conclude that the steady state interest rate $R$ varies

---

6The parameter values used to generate table 5 are the same ones as those in table 4.
Table 5: Comparison with Gollier and Zeckhauser [8]

<table>
<thead>
<tr>
<th>σ^i</th>
<th>σ^2 = 0.5</th>
<th>σ^2 = 1</th>
<th>σ^2 = 3</th>
<th>σ^2 = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>-0.01%</td>
<td>-0.19%</td>
<td>-0.39%</td>
<td>-0.45%</td>
</tr>
<tr>
<td>1</td>
<td>0.17%</td>
<td>-0.02%</td>
<td>-0.29%</td>
<td>-0.37%</td>
</tr>
<tr>
<td>3</td>
<td>0.40%</td>
<td>0.26%</td>
<td>-0.03%</td>
<td>-0.16%</td>
</tr>
<tr>
<td>5</td>
<td>0.47%</td>
<td>0.36%</td>
<td>0.11%</td>
<td>-0.03%</td>
</tr>
</tbody>
</table>

Parameters: β^1 = 0.99, β^2 = 0.98, α = 0.36, δ = 0.025, A = 1

more strongly with the elasticities of intertemporal substitution than Gollier and Zeckhauser’s aggregate time-preference \( \hat{R} \). Second, even when the two households have equal commitment power as measured by the elasticity of intertemporal substitution (this corresponds to the entries on the diagonal of table 5), the value \( \hat{R} \) underestimates the steady state interest rate \( R \). This is just another illustration of the fact that the strategic interaction of the two households exerts upward pressure on the interest rate. To summarize, in the dynamic game of the present paper, the result from Gollier and Zeckhauser [8] fails to hold because of the non-optimality of the Nash equilibrium. This failure is systematic but quantitatively not very important (at least when the model is realistically calibrated). Furthermore, commitment power is one aspect that can explain the systematic bias of \( \hat{R} \).

4.2 Transition dynamics

So far we have focussed our attention on the properties of steady states generated by MPNE. In this section we discuss the transition dynamics associated with such an equilibrium. Figure 1 displays the transition to the steady state starting at eight different capital endowments. Each panel corresponds to a different profile of elasticities of intertemporal substitution.

We observe that, in general, convergence to the steady state does not occur monotonically. This feature is most pronounced in the top right panel, which corresponds to the parameter values \( σ^1 = 5 \) and \( σ^2 = 1 \). Because of the low elasticity of substitution of household 1, household 2 moves its capital stock more quickly than household 1. This can easily lead to an overshooting phenomenon. Starting for example from the capital endowment \( k^1_0 = 54.51, k^2_0 = 19.76 \) in the top right panel, household 2 reduces its capital stock to below 14 before it starts to save again in order to reach its steady state capital stock \( k^2 = 14.77 \). The intuition behind this result is straightforward. Initially, both households own too much capital such that the interest rate is very low. Therefore, both households reduce their capital holdings. When household 2 has reduced its capital stock to the level it would like to hold in the steady state, however, the capital stock of household 1 has not yet adjusted completely and the interest rate is still relatively low. Therefore, household 2 finds it optimal to further decrease its capital holdings. After some time, the interest rate reaches a level at which household 2 finds it optimal to increase its capital holdings again, whereas household 1 continues to reduce its capital stock.

\[^7\]This is in line with analogous findings in capital accumulation games with reversible investment following Fershtman and Muller [6] or Reynolds [15]; see also chapter 13.4.2 in Fudenberg and Tirole [7]. In that literature, however, oligopolistic firms decide on capital accumulation whereas here the households play that role.
Figure 1: Transition dynamics for different fluctuation tolerance profiles

Parameters: $\beta_1 = 0.99$, $\beta_2 = 0.98$, $\alpha = 0.36$, $\delta = 0.025$, $A = 1$, and $\sigma_1 = 1, \sigma_2 = 1$ [top left], $\sigma_1 = 5, \sigma_2 = 1$ [top right], $\sigma_1 = 1, \sigma_2 = 5$ [bottom left], and $\sigma_1 = 5, \sigma_2 = 5$ [bottom right].

Note: to indicate the speed of convergence, every 50th realization is marked with an arrow.

until the economy reaches its steady state. For parameter constellations that are more extreme than those in figure 1, the overshooting can become so strong that one household gives up its entire capital stock before accumulating positive wealth again; see Pichler and Sorger [13].

The bottom panels illustrates the convergence to the steady state associated with $\sigma_1 = 1$ and $\sigma_2 = 5$ (left panel) and $\sigma_1 = \sigma_2 = 5$ (right panel). Although the trajectories in the top left and the bottom right panel are very similar, the speed of adjustment is much slower in the bottom right panel. This can be seen from the more narrowly spaced arrows in the bottom right panel as compared to the top left panel. The interpretation of this observation is also obvious, since both households only moderately adjust their capital holdings when they have a low elasticity of intertemporal substitution.
5 Numerical strategy

In this section we present our strategy to solve for the MPNE numerically. We show how to compute the two value functions $V^1$ and $V^2$ as well as the four policy functions $c^1$, $c^1$, $l^1$, and $P$ so that they (approximately) satisfy the equilibrium conditions stated in section 3. Once the equilibrium value functions and policy functions have been obtained, it is straightforward to compute the corresponding steady state as the fixed point of the system dynamics.

We apply a projection method with Chebyshev polynomials, as introduced into economics by Judd [10]. In a recent paper, Aruoba et al. [1] document that this approach is particularly well suited to compute very precise solutions of dynamic equilibrium economies. This feature makes projection methods especially attractive for our application, as we compute the steady state as the fixed point implied by equilibrium policy functions: to obtain sufficiently precise results for steady states, highly accurate approximations to the policy functions are necessary. In order to pin down the unknown coefficients in the approximating functions, we apply a least squares approach. This is often problematic in the context of economic growth models, since algorithms based on least squares typically fail to converge if the initial parameter guess is not sufficiently close to the true solution. In our application, however, the least squares approach turns out to work extremely well: it displays excellent convergence properties (even for a naive initial guess), and it delivers highly accurate solutions.

The basic idea behind our numerical strategy is to approximate both households’ value functions by parametric functional forms and to compute the value function coefficients by minimizing the sum of the squared Bellman equation errors at a number of pre-determined grid points. As a first task, we thus need to decide upon the functional forms to be used for approximation. We choose a complete basis of Chebyshev polynomials such that the value function of each household $i \in \{1, 2\}$ is approximated by

$$\hat{V}(x; \theta^i) = \sum_{n=0}^{P} \sum_{m=0}^{P-n} \theta^i_{nm} T_n(\xi(k^1))T_m(\xi(k^2)).$$

$T_n$ denotes the univariate Chebyshev polynomial of order $n$, which is defined recursively by $T_0(z) = 1$, $T_1(z) = z$, and $T_n(z) = 2zT_{n-1}(z) - T_{n-2}(z)$ for all $n = 2, 3, \ldots$. Chebyshev polynomials form a basis of the space of continuous functions and constitute a family of orthogonal polynomials in the interval $[-1, 1]$. To benefit from this orthogonality property, we need to transform the original state variables $k^1$ and $k^2$ into the interval $[-1, 1]$. We use a linear mapping

$$\xi(k^i) = \frac{2k^i - k^i - 1}{k^i - k^i}, \quad i \in \{1, 2\},$$

where $k^i$ and $\bar{k}^i$ denote the lower and upper bounds of the capital stock for household $i$. Finally, the matrix $\theta^i$ contains the value function coefficients for household $i \in \{1, 2\}$ and $P$ denotes the order of approximation. To guarantee sufficiently accurate results, we implement a high-order approximation and set $P = 15$. 

\footnote{See Judd [10] and Heer and Maussner [9] on this issue.}

18
For given $\theta = (\theta^1, \theta^2)$, we can invert the nonlinear system of equilibrium conditions (10)-(11) to obtain policy functions $\tilde{c}^1(x; \theta)$, $\tilde{c}^2(x; \theta)$, $\tilde{I}^1(x; \theta)$, and $\tilde{I}^2(x; \theta)$. By construction, these functions together with the approximate value functions $\tilde{V}(x; \theta^0)$ satisfy the households’ first-order conditions exactly. However, they do not generally satisfy the households’ Bellman equations (9). To obtain a good numerical approximation of the MPNE, we seek parameter vectors $\bar{\theta}^1$ and $\bar{\theta}^2$ that make the errors in the Bellman equations very small. We achieve this goal by iteratively minimizing the sum of squared errors at a grid of points in the state space.\footnote{We build our grid following the standard approach in the literature. We select $n > P$ grid points for each capital stock as the zeros of the Chebyshev polynomial of order $n$. We then combine these points to create a bivariate grid of size $n^2$ consisting of the points $x_m = (k_m^1, k_m^2)$, $m = 1, 2, \ldots, n^2$. Our implementation sets $n$ equal to $P + 1$, since we find that increasing $n$ even further has only a negligible effect on the accuracy of the solution while increasing the computational effort substantially.}

Before discussing the steps involved in this procedure in detail, let us emphasize a simplifying feature that we utilize in our solution procedure. Since households derive no disutility of labor and the production function is Cobb-Douglas, we are confident that providing the entire labor endowment is optimal for both households in equilibrium. Throughout our algorithm, we thus postulate that $I^1(x) = I^2(x) = 1$ holds for all $x$, such that we need not explicitly solve for optimal labor supply from the equilibrium condition (11). This simplifies the procedure substantially, since only the two policy functions for consumption have to be obtained from the (nonlinear) system of first-order conditions at every iteration step. After we have solved for the equilibrium, we check that full labor provision is indeed optimal for both households.

Recall that we approximate the MPNE using parametric functions, whereby the parameter vectors $\bar{\theta}^1$ and $\bar{\theta}^2$ minimize the sum of squared Bellman equation errors at the grid points $(k_m^1, k_m^2)$, $m = 1, 2, \ldots, n^2$. Our algorithm to compute $\bar{\theta}^1$ and $\bar{\theta}^2$ proceeds as follows:

1. We start by deriving an initial guess for $\theta_0 = (\theta_0^1, \theta_0^2)$.

2. For every grid-point $m \in \{1, \ldots, n^2\}$, we conjecture that $\tilde{I}^1(x_m; \theta_0) = \tilde{I}^2(x_m; \theta_0) = 1$ and solve the two first-order conditions

$$u^i_{c^i}(c^i) = \beta^i \tilde{V}_{c^i}(I^1(x_m, 1, 1) - c^1, I^2(x_m, 1, 1) - c^2; \theta_0)$$

for $c^1$ and $c^2$. We denote the solution by $c^1 = \tilde{c}^1(x_m; \theta_0)$ and $c^2 = \tilde{c}^2(x_m; \theta_0)$.

3. Then we construct for all $i \in \{1, 2\}$ and all $m \in \{1, 2, \ldots, n^2\}$

$$\nu^i_m = u^i(\tilde{c}^i(x_m; \theta_0)) + \beta^i \tilde{V}(I^1(x_m, 1, 1) - \tilde{c}^1(x_m; \theta_0), I^2(x_m, 1, 1) - \tilde{c}^2(x_m; \theta_0); \theta_0).$$

We then obtain an updated parameter vector $\theta$ as

$$\theta = \arg\min_{\theta = (\theta^1, \theta^2)} \sum_{m=1}^{n^2} \left( \left[ \tilde{V}(x_m; \theta^1) - \nu^1_m \right]^2 + \left[ \tilde{V}(x_m; \theta^2) - \nu^2_m \right]^2 \right).$$

We check whether $|\theta_0 - \theta| < \varepsilon$, where $\varepsilon$ denotes our convergence criterion (set at $\varepsilon = 10^{-6}$). If convergence has not been achieved, we set $\theta_0 = \theta$ and return to the beginning of step 2. If $|\theta_0 - \theta| < \varepsilon$ holds, we set $\bar{\theta} = \theta$ and proceed to the next step.
4. We check the validity of the full labor supply conjecture under which $\bar{\theta}$ has been derived. To this end, we confirm that

$$\tilde{V}_k(I(x_m, 1, 1) - \tilde{c}_1(x_m; \bar{\theta})^2(x_m; \bar{\theta})I_2(x_m, 1, 1)$$

$$= \tilde{V}_k(I(x_m, 1, 1) - \tilde{c}_2(x_m; \bar{\theta})^2(x_m; \bar{\theta})I_2(x_m, 1, 1) \geq 0$$

holds for all $i \in \{1, 2\}$ and all $m \in \{1, \ldots, n^2\}$.

Having derived the equilibrium policy functions, it is straightforward to numerically compute steady states. This boils down to solving a system of two nonlinear equations in the two unknown steady state capital stocks, given by

$$k^1 = R(k^1 + k^2, 2) + W(k^1 + k^2, 2) - \tilde{c}_1(k^1, k^2; \bar{\theta}),$$

$$k^2 = R(k^1 + k^2, 2) + W(k^1 + k^2, 2) - \tilde{c}_2(k^1, k^2; \bar{\theta}).$$

Further details on the numerical solution procedures can be obtained from our MATLAB codes which are available from the authors upon request.

6 Concluding remarks

The work presented here is primarily meant to address the problematic prediction of the standard Ramsey-Cass-Koopmans model concerning the (degenerate) long-run distribution of wealth. Sorger’s [16] suggestion regarding this issue is convincing only if the economy is close to the steady state. By applying an equilibrium concept that assumes that households have state-contingent strategies, we have tried to make this argument more coherent.

We have confirmed the results derived by several authors who used an open-loop equilibrium concept. In particular, we could show that the Ramsey conjecture does not hold. An interesting difference to the open-loop equilibrium (and to the competitive framework as well) is that the steady state of the model is not independent of the utility functions of the households. This results from the fact that the households take their own as well as their opponents’ flow budget constraints into account when they make their optimal consumption/saving decision.

Furthermore, we have compared the aggregate time-preference rate that is reflected in the real interest rate to predictions derived by means of the results from Gollier and Zeckhauser [8]. This shows that the strategic interaction of the households distorts the aggregate time-preference rate systematically but only slightly.

Markov-perfect Nash equilibria in dynamic games are hardly ever analytically tractable. The Ramsey-Cass-Koopmans model with strategically acting households is no exception. In the present paper, we have therefore employed numerical methods to study the equilibrium outcome of this game.

\[10\text{In all our numerical examples, these conditions are satisfied.}\]
References


