Stochastic Processes - SS 2006

Exercises

1. Show that the probability that a branching process becomes extinct precisely in generation n is $P(X_n = 0) - P(X_{n-1} = 0)$.

2. What happens in the long run when $g(z) = 0.3 + 0.5z + 0.2z^3$? Find the mean and variance of the size of generation 100, and hence the mean size conditional on non extinction.

3. Given $g(z) = 0.4 + 0.2z + 0.4z^2$, find the pgfs of the sizes of generations one and two. Find also the chances of dying out

   (a) *by* the second generation;

   (b) *in* the second generation.

4. For what values of λ is $g(z) = 0.2 + (1.2 - \lambda)z + (\lambda - 0.4)z^3$ a probability generating function (pgf). When is the corresponding branching process certain to die out.

5. Let $Y_n = X_n/\mu^n$ so that $\mathbb{E}(Y_n) = 1$. Show that $Var(Y_n) \to \sigma^2/(\mu^2 - \mu)$ when $\mu > 1$. Find the limiting values of $E(Y_n|X_n \neq 0)$ and $Var(Y_n|X_n \neq 0)$.

6. Suppose that, when $n \geq 1$, Y_n immigrants arrive in addition to the offspring of generation $n-1$, where Y_n are i.i.d. random variables with pgf h(z). Let $k_n(z)$ be the pgf of the size of generation n, taking account of the immigrants. Show that $k_{(n+1)}(z) = k_n(g(z))h(z)$, and write down $k_3(z)$ in terms of g, h only.

7. Let $Y_n = X_n|X_n \neq 0$ be the population size conditional on non-extinction. Show that for large n,

$$\mathbb{E}(Y_n) \approx \frac{\mathbb{E}(X_n)}{1-x} \; ; \qquad Var(Y_n) \approx \frac{Var(X_n)}{(1-x)} - \frac{x(\mathbb{E}(X_n))^2}{(1-x)^2}$$

   where x is the probability of extinction, (and is computed as the smallest root of $g(z) = z$).
   (Hint: $\mathbb{E}(X_n^r) = \mathbb{E}(X_n^r|X_n \neq 0)P(X_n \neq 0) + \mathbb{E}(X_n^r|X_n = 0)P(X_n = 0)$)

8. Bets on Red in Las Vegas casino win or lose the amount of the stake with respective probabilities 18/38 and 20/38. Your initial capital is 100 units, you quit when ruined or if you reach 400 units. Compare your chances of ruin if you

   (a) bet one unit every play

   (b) bet 100 units every play

   (c) bet your fortune every play.

9. In the simple random walk, find $\mathbb{E}(S_n)$ and $Var(S_n)$. For p=0.6, use the Central Limit Theorem to estimate the probability that the random walk is more than ten positive steps from its start point after 100 steps.

10. (a) Show for a Markov chain that, for any $n \geq 1$ and subsets $A_0, \ldots, A_{n-1}$ of the state space,

$$P[X_{n+1} = j | X_0 \in A_0, \ldots, X_{n-1} \in A_{n-1}, X_n = i] = p_{ij}.$$

    (b) Verify by giving an example that the following statement is **incorrect**:
    For subsets $A_0, \ldots, A_n$ where A_n is not a singleton, we have

$$P[X_{n+1} = j | X_0 \in A_0, \ldots, X_n \in A_n] = P[X_{n+1} = j | X_n \in A_n].$$

11. Given

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

    Find P^2, the values of $p_{ii}^{(3)}$ and the unique stationary probability vector.

12. Suppose $X_1, X_2 \ldots$ are independent with respective pgfs $g_1, g_2, \ldots$. Then show that $S_n = X_1 + X_2 + \ldots + X_n$ has pgf $g_1 g_2 \ldots g_n$.
Hint: $g(z) = E(z^X)$

13. Suppose $X_1, X_2 \ldots$ are independent, all with same pgf $g(z)$ and that N is a random variable indepndent of these X_i, having pgf h(z). Write

$$S_N = \begin{cases} 0 & \text{if } N = 0; \\ X_1 + X_2 + \ldots + X_n & \text{if } N = n \geq 1 \end{cases}$$

Show that S_N has pgf $h(g(z))$.
Hint Write $\mathbb{E}(z^{S_N}) = \sum_n \mathbb{E}(z^{S_N} | N = n) P(N = n)$ and use the result in 13

14. Let X_n denote the fitness of the nth item produced by a production system with $X_n = 0$ meaning 'good' and $X_n = 1$ meaning 'defective'. Suppose X_n is a Markov chain with transition probability matrix

$$P = \begin{pmatrix} .99 & .01 \\ .12 & 88 \end{pmatrix}$$

What is the probability that the fifth item is defective given that the first item is defective?

15. Let Y_n be the maximum of the first n (independent) throws of a dice . ($Y_n = 1, 2, 3, 4, 5, 6$). Is Y_n a Markov chain? Compute the transition probability matrix.

16. To look *backwards* in time, write $q_{ij} = P(X_n = j | X_{n+1} = i)$. Show that, if the chain begins in a stationary state $\pi > 0$, then $q_{ij} = \pi_j p_{ji} / \pi_i$, and that $Q = (q_{ij})$ is a transition matrix, having the same stationary vector.

17. A recurrent state of a Markov chain is called null recurrent if $p_{ii}^{(n)} \to 0$ as $n \to \infty$ and positive recurrent otherwise. Show that null recurrence and positive recurrence are (equivalence) class properties.
Hint: $i \to j, j \to i \Rightarrow \exists u, v > 0$ such that $p_{ij}^{(u)} > 0, p_{ji}^{(v)} > 0$. Let $\alpha = p_{ij}^{(u)} p_{ji}^{(v)} > 0$. Now show for $n \geq 1$, $p_{ii}^{(u+n+v)} \geq \alpha p_{jj}^{(n)}$. Analogously for $n \geq 1$, $p_{jj}^{(v+n+u)} \geq \alpha p_{ii}^{(n)}$ will hold.

18. Show that an irreducible chain with finitely many states must be **recurrent**.

19. The transition matrix of a Markov chain with state space $\{1, 2, 3, 4, 5\}$ is given by

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}$$

   (a) Determine the communicating classes and identify the ones that are closed.
   (b) Determine which states are transient and the recurrent classes.
   (c) Determine the period of each state.

20. The transition matrix of a Markov chain with state space $\{0, 1, 2\}$ is given by

$$P = \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.2 & 0.5 & 0.3 \\ 0.5 & 0.2 & 0.3 \end{pmatrix}$$

   (a) Compute the stationary distribution of P.
   (b) Find the expected return time of state 2 to state 2

21. Modify the simple random walk so that the steps have size -1,0,1 with respective non-zero probabilities q,r and p, with $q + r + p = 1$. Show that this is an irreducible aperiodic Markov Chain.

22. Consider the Markov chain with state space $\{0, 1, 2\}$ and transition probability matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0 & 1 \end{pmatrix}$$

Let w_i be the probability of absorption into state 0, given that it starts in state i. Determine the probability that the Markov chain ends in state 0 given that it starts in state 1 i.e. w_1.

23. In the last question, determine the expected time to absorption i.e. the expected time starting in state 1 to enter one of the absorbing states $\{0, 2\}$.

24. Given a Poisson process with intensity $\lambda > 0$, alter it by deleting each event independently, with constant probability $p > 0$. Show that the resulting process is Poisson with intensity $(1 - p)\lambda$. If the alteration in the original Poisson process was to delete every alternate event, then would the resulting process still be Poisson?

25. Suppose a random variable Y has a Poisson distribution with parameter $\lambda > 0$. (You know that if Y_1 and Y_2 are Poisson with parameters λ_1 and λ_2 respectively then $Y_1 + Y_2$ is Poisson with $\lambda_1 + \lambda_2$.) A spider climbs up an infinitely high wall. During day i, the spider climbs up Y_i centimetres, where the Y_i are i.i.d. (independent, identically distributed random variables) with Poisson distribution with parameter $\lambda > 0$. During the night, the spider slips back 1 centimetre. Let X_n be the net height gained after n days and nights. Show that

$$Pr(X_n = 0) = \frac{e^{-n\lambda}(n\lambda)^n}{n!}$$

26. Das Auftreten von Erdbeben in der Stadt A werde durch einen Poisson-Prozeß mit $\lambda = 2$ beschrieben. Der Schaden, den ein Erdbeben jeweils anrichtet, sei exponential verteilt mit Parameter $\mu = 10^{-5}$ (Zeiteinheit 1 Jahr, Geldeinheit 1). Berechne den mittleren Schaden pro Jahrzehnt.

27. Die eingebettete Markovkette: Wenn man in einer Markovkette mit stetiger Zeit nur die Sprünge, nicht aber die Verweildauern betrachtet, so entsteht die 'eingebettete Markovkette' mit der Übergangsmatrix

$$S = (s_{ij}) = \begin{cases} \frac{q_{ij}}{q_i} & i \neq j \\ 0 & i = j \end{cases}$$

Sei π die stationäre Verteilung der ursprünglichen Kette und ν jene der eingebetteten Kette, so gilt:

$$\nu_i = c(q_i)\pi_i$$

für eine Konstante c.

28. A pure birth process starting from $Y_0 = 1$ has birth parameters $\lambda_1 = 1, \lambda_2 = 3$ and $\lambda_3 = 2$. By solving the associated differential equation for $P_n(t)$, determine $P_n(t)$ for $n = 1; 2; 3$. ($P_n(t) = P(Y_t = n)$, the probability that there are n members in the population at time t.)

29. In an office telephone messages arrive according to a Poisson process at the mean rate of six per hour and fax messages at three per hour.

    (a) Find the probability that exactly two messages (phone or fax) are received between 9.00 and 9.40.

    (b) Find the probability that the first message after 10.00 occurs before 10.10.

30. Customers arrive at a bank according to a Poisson process at a mean rate of $\lambda = 10$ per minute. A proportion 0.6 wish to draw out money (type A), 0.3 wish to pay in money (type B) and 0.1 wish to do something else (type C).

    (a) If twenty customers arrive in two minutes, what is the probability that just one is of type C?

    (b) How long a time must elapse before there is a probability of 0.9 that at least one customer each of types A and B will have arrived?

    Hint: By similar arguments as in Example 24, one can show that customers of type A, type B and type C arriving are independent Poisson processes with mean rates 0.6λ, 0.3λ and 0.1λ respectively.

31. A population starts at time 0 with a single individual. Let the birth rate be two per week.

(a) What is the probability that after three weeks there are exactly two individuals?

(b) What is the probability that after one week there are between two and four individuals (inclusive)?

32. Let X be a simple birth-death process where individuals have independent $Exp(\mu)$ lifetimes and, during their lifetime give birth at rate λ independently of other individuals. Suppose that $X_0 = 1$. Let the p.g.f. be denoted by G(s,t) i.e.

$$G(s,t) = E(s^{X_t})$$

(a) Construct the Q matrix and deduce the forward equations.

(b) Show that G satisfies

$$\frac{\partial G(s,t)}{\partial t} = (\lambda s - \mu)(s-1)\frac{\partial G(s,t)}{\partial s}$$

Remark: We know that the p.g.f. specifies a discrete distribution completely and uniquely and in this case, it can be shown that

$$G(s,t) = \begin{cases} \frac{\mu(s-1)-(\lambda s-\mu)e^{-(\lambda-\mu)t}}{\lambda(s-1)-(\lambda s-\mu)e^{-(\lambda-\mu)t}} & \mu \neq \lambda \\ \frac{\lambda t(s-1)-s}{\lambda t(s-1)-1} & \mu = \lambda \end{cases}$$

33. Show that if B(t) is a Brownian motion (Wiener Process), then also $B_1(t) = tB(\frac{1}{t})$ is a Brownian motion.