Multi-Window Weaving Frames

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Abstract—In this work we deal with the recently introduced concept of weaving frames. We extend the concept to include multi-window frames and present the first sufficient criteria for a family of multi-window Gabor frames to be woven. We give a Hilbert space norm criterion and a pointwise criterion in phase space. The key ingredient are localization operators in a Hilbert space norm criterion and a pointwise criterion in the concept of weaving frames. We extend the concept to include statements can be generalized verbatim to higher dimensions, this work does not gain any deeper insights by using a more general notation.

A. Gabor Frames and Weaving Frames

We denote by $\gamma = (x, \omega) \in \mathbb{R} \times \mathbb{R}$ a point in the time-frequency plane, also called phase space, and use the following notation for a time-frequency shift by $\gamma$:

$$\pi(\gamma)\phi(t) = M_\omega T_x \phi(t) = e^{2\pi i \omega t} \phi(t-x),$$

for $x, \omega, t \in \mathbb{R}$, where $T_x \phi(t) = \phi(t-x)$ and $M_\omega \phi(t) = e^{2\pi i \omega t} \phi(t)$ are the translation and the modulation operator, respectively.

A Gabor system for $L^2(\mathbb{R})$ is generated by a window function $\phi \in L^2(\mathbb{R})$ and an index set $\Gamma \subset \mathbb{R}^2$. It is denoted by

$$\mathcal{G}(\phi, \Gamma) = \{\pi(\gamma)\phi | \gamma \in \Gamma\}.$$

$\mathcal{G}(\phi, \Gamma)$ is called a frame if for all $f \in L^2(\mathbb{R})$

$$\text{A} \|f\|^2_2 \leq \sum_{\gamma \in \Gamma} |\langle f, \pi(\gamma)\phi \rangle|^2 \leq B \|f\|^2_2,$$

with $0 < A \leq B < \infty$ called frame bounds.

Definition I.1. We call a finite family of frames, $\{\mathcal{G}(\phi_j, \Gamma_j)\}_{j=1}^M$, woven if for every partition $\{\sigma_j\}_{j=1}^M$ of $\Gamma$, the family $\{\mathcal{G}(\phi_j, \sigma_j(\Gamma))\}_{j=1}^M$ is a frame for $L^2(\mathbb{R})$.

We will sometimes use the notation $f \approx g$ if there exist constants $0 < A \leq B < \infty$ such that $Ag \leq f \leqBg$.

II. Localization Operators

Definition II.1. For a window $\phi \in L^2(\mathbb{R})$, the short-time Fourier transform (STFT) at $\gamma = (x, \omega) \in \mathbb{R} \times \mathbb{R}$ of a function $f \in L^2(\mathbb{R})$ is defined as $V_\phi f(\gamma) = \langle f, \pi(\gamma)\phi \rangle$ and for $m \in L^\infty(\mathbb{R}^2)$ the time-frequency localization operator with symbol $m$ for $f \in L^2(\mathbb{R})$ is given by

$$H_{m,\phi} f(t) = \int_{\mathbb{R}^2} V_\phi f(\gamma) m(\gamma) \pi(\gamma)\phi(t) d\gamma.$$

Formally we have $H_{m,\phi} f = V_\phi^* m V_\phi f$.

We will be interested in families of localization operators, which cover the entire time-frequency domain, such that we can derive local windows from their eigenfunctions. We require the following property.

Definition II.2. A family of symbols $\{\eta_\gamma : \mathbb{R}^2 \to \mathbb{R} | \gamma \in \Gamma, \eta_\gamma \in L^1(\mathbb{R}^2)\}$ is called well-spread if $\Gamma \subset \mathbb{R}^2$ is a
discrete set without accumulation points and there exists a continuous function \(g \in L^2(\mathbb{R}^2)\) with polynomial decay, such that \(|\eta_\gamma(\xi)| \leq g(\xi - \gamma)\) for \(\xi \in \mathbb{R}^2, \gamma \in \Gamma\).

Note that for localization operators as defined above, well-spread symbol families always lead to well-spread operator families in the sense of [4], i.e., for all \(z \in \mathbb{R}, \gamma \in \Gamma\) we have the point-wise bound
\[
|V_\phi H_{\eta, \phi} f(z)| \leq |T_z g \cdot V_\phi f| \ast |V_\phi \phi|(z).
\]

We will now have a closer look at the time-frequency localization operator \(H_{\eta, \phi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})\). Assuming that the family of symbols \(\{\eta_\gamma | \gamma \in \Gamma\}\) is non-negative and well-spread we can use the results in [4] which state that, under these conditions, the localization operator \(H_{\eta, \phi}\) is positive and trace class and, hence, can be diagonalized. Therefore, we have
\[
H_{\eta, \phi} f = \sum_{k \geq 1} \lambda_k^\gamma \langle f, \phi_k^\gamma \rangle \phi_k^\gamma, \quad f \in L^2(\mathbb{R}),
\]
where \(\{\phi_k^\gamma | k \in \mathbb{N}\}\) is an orthonormal subset of \(L^2(\mathbb{R})\) consisting of eigenfunctions of \(H_{\eta, \phi}\). The sequence of eigenvalues \((\lambda_k^\gamma)_{k=1}^\infty\) is non-increasing sequence with non-negative real numbers. We define a related operator by putting a threshold on the eigenvalues. For \(\varepsilon > 0\) we define
\[
H_{\eta, \phi}^\varepsilon f = \sum_{k: \lambda_k^\gamma \geq \varepsilon} \lambda_k^\gamma \langle f, \phi_k^\gamma \rangle \phi_k^\gamma, \quad f \in L^2(\mathbb{R}).
\]

Considering the derived first \(N^*\) eigenfunctions and observing that they are maximally concentrated within the support of \(\eta_\gamma\) in the sense that among all orthonormal sets of functions, they maximize the quantity \(\sum_{j=1}^{N^*} \int \eta_\gamma(z)|V_\phi \phi_j(z)|^2 dz\), motivates an approach that uses these windows as basic windows generating a multi-window Gabor frame, [6]. A multi-window Gabor system is, by obvious generalization, defined as the set of functions \(\{\pi(\gamma)\phi_k | \gamma \in \Gamma, k \in \mathcal{I}\}\) and denoted by \(\mathbb{G}(\mathcal{E}, \mathcal{I})\).

**Definition III.1.** A family of multi-window Gabor frames \(\mathbb{G}(\mathcal{E}, \mathcal{I})\) is woven if for any partition \(\{\sigma_j\}_{j=1}^M\) of \(\Gamma\), the family \(\mathbb{G}(\{\phi_k^\gamma | k \in \mathcal{I}, \sigma_j(\Gamma)\})_{j=1}^M\) is a frame for \(L^2(\mathbb{R})\).

**III. VARYING THE LOCALIZATION WINDOW**

In a series of papers, cf. [4] for references, it has been shown, that, if a family of localization operators \(H_{\eta, \phi}\) is well-spread and \(\sum H_{\eta, \phi}\) is invertible, one has the norm equivalence \(\|f\|^2 \approx \sum_{\gamma \in \mathbb{R}} \|H_{\eta_\gamma, \phi} f\|^2\). Thereby, however, the window \(\phi\) defining the localization operator remains the same for all \(\gamma\). In the next sections, we show how the window can be varied. The rationale behind this approach is to generate different suitable families of eigenfunctions. We will then see, that, by picking finite sets from the operators’ eigenfunctions, we obtain multi-window Gabor frames, and eventually, due to the different analysis windows, the resulting different multi-window frames will be woven. We proceed to give two conditions on varying analysis windows, for which the overall family of localization operators remains invertible.

**A. Norm-Conditions**

**Definition III.2.** The time-frequency concentration \(\mathcal{E}_{\Omega, \phi} f\) of \(f \in L^2(\mathbb{R})\) w.r.t. \(\phi \in L^2(\mathbb{R})\) is given by
\[
\mathcal{E}_{\Omega, \phi} f = \int_{\Omega} |V_{\phi} f|^2 d\lambda = \langle H_{\Omega, \phi} f, f \rangle.
\]

**Lemma III.2.** Let \(\Omega \subset \mathbb{R}^2\) and \(\phi_1, \phi_2\) be two window functions. Then, for every \(f \in L^2(\mathbb{R})\) we get the estimate
\[
|\mathcal{E}_{\Omega, \phi_1} f - \mathcal{E}_{\Omega, \phi_2} f| \\
\leq (\|\phi_1\|_2 + \|\phi_2\|_2) \|\phi_1 - \phi_2\|_2 \|f\|_2^2.
\]
**Proof:** First, we note that we can write \(H_{\Omega, \phi} = V_{\phi}^* \chi_{\Omega} V_{\phi}\). We get
\[
H_{\Omega, \phi_1} - H_{\Omega, \phi_2} = V_{\phi_1}^* \chi_{\Omega} V_{\phi_1} - V_{\phi_2}^* \chi_{\Omega} V_{\phi_2} \\
= V_{\phi_1}^* \chi_{\Omega} V_{\phi_1} - V_{\phi_2}^* \chi_{\Omega} V_{\phi_2} + V_{\phi_2}^* \chi_{\Omega} V_{\phi_1} - V_{\phi_2}^* \chi_{\Omega} V_{\phi_2} \\
= V_{\phi_1}^* \chi_{\Omega} V_{\phi_1} - V_{\phi_2}^* \chi_{\Omega} V_{\phi_1} \\
= V_{\phi_1}^* \chi_{\Omega} V_{\phi_1} + V_{\phi_2}^* \chi_{\Omega} V_{\phi_2}.
\]
Now we compute
\[
(\|H_{\Omega, \phi_1} f, f\| - \|H_{\Omega, \phi_2} f, f\|) \\
\leq \|\langle V_{\phi_1}^* \chi_{\Omega} V_{\phi_1} f, f \rangle + \langle V_{\phi_2}^* \chi_{\Omega} V_{\phi_2} f, f \rangle\| \\
\leq (\|\phi_1\|_2 + \|\phi_2\|_2) \|\phi_1 - \phi_2\|_2 \|f\|_2^2.
\]

**Proposition III.3.** Let \(\phi_1, \phi_2 \in L^2(\mathbb{R})\) with \(\|\phi_1\|_2 = 1\). Let \(\{\eta_\gamma | \gamma \in \Gamma_1\} \cup \{\eta_\gamma | \gamma \in \Gamma_2\}\) be a well-spread family of non-negative symbols on \(\mathbb{R}^2\) with \(\sum_{\gamma \in \mathbb{R}} \eta_\gamma \approx 1\). If \(\|\phi_1 - \phi_2\|_2 < \frac{1}{2}\), then the following holds.
\[
A \|f\|_2^2 \leq \sum_{\gamma \in \mathbb{R}} \|H_{\eta_\gamma, \phi} f\|_2^2 \leq B \|f\|_2^2.
\]
**Proof:** We define \(m_i = \sum_{\gamma \in \mathbb{R}} \eta_\gamma\), and supp(m_i) = \(\Omega_i\). We get
\[
\sum_{i=1}^2 \left( \sum_{\gamma \in \mathbb{R}} H_{\eta_\gamma, \phi} f, f \right) = \sum_{i=1}^2 \left( H_{m_i, \phi} f, f \right) \\
= \sum_{i=1}^2 \left( \langle V_{\phi_1}^* m_i V_{\phi_1} f, f \rangle + \langle V_{\phi_2}^* m_i V_{\phi_2} f, f \rangle \right) \\
= \sum_{i=1}^2 \int_{\Omega_i} m_i |V_{\phi_1} f|^2 d\lambda \geq A (|\mathcal{E}_{\Omega_1, \phi_1} f| + |\mathcal{E}_{\Omega_2, \phi_2} f|)
\]
with \(A > 0\) as \(\sum_{\gamma \in \mathbb{R}} \sum_{\gamma \in \mathbb{R}} \eta_\gamma \approx 1\). From this assumption we also conclude that \(\Omega_1 \cup \Omega_2 = \mathbb{R}^2\) and we have
\[
\|f\|_2^2 \leq |\mathcal{E}_{\Omega_1, \phi_1} f| + |\mathcal{E}_{\Omega_2, \phi_2} f| \\
\leq |\mathcal{E}_{\Omega_1, \phi_1} f| + |\mathcal{E}_{\Omega_2, \phi_2} f| + |\mathcal{E}_{\Omega_1, \phi_1} f - \mathcal{E}_{\Omega_2, \phi_2} f|.
\]
Therefore, it follows from Lemma III.2 that
\[
\left( 1 + 2\|\phi_1 - \phi_2\|_2 \right) \|f\|_2^2 \leq |\mathcal{E}_{\Omega_1, \phi_1} f| + |\mathcal{E}_{\Omega_2, \phi_2} f|.
\]
\( C_{\phi_1, \phi_2} > 0 \) if \( \| \phi_1 - \phi_2 \|_2 < \frac{1}{2} \). Hence, we get that
\[
\sum_{i=1}^{2} \left( \sum_{\gamma \in \Gamma_i} H_{\eta_i, \phi_i} f, f \right) \geq AC_{\phi_1, \phi_2} \| f \|_2^2.
\]
It is immediate that the upper bound is finite.

B. Phase-Space Conditions

We will now establish sufficient conditions by pointwise estimates in phase space. Let
\[ \varphi_0(t) = 2^{1/4} e^{-\pi t^2} \]
be the standard Gaussian of \( L^2 \)-unit norm. In this section, we consider a collection of windows \( \Phi = \{ \phi_i, i \in \mathcal{I} \} \) such that
\[
| V_{\varphi_0} \phi_i | \ast | V_{\varphi_0} \phi_i |(z) \leq C(1 + |z|^2)^{-s}
\]
for all \( i \in \mathcal{I}, s > 1 \) and a constant \( C \). Also, we let the symbols \( \{ \eta_\gamma : \mathbb{R}^2 \to \mathbb{R} \} \) \( \gamma \in \Gamma, \eta_\gamma \in L^1(\mathbb{R}^2) \) be well-spread with \( \sum_{\gamma \in \Gamma} \| \eta_\gamma \|_2 \approx 1 \) and a window \( \phi_0 \) from \( \Phi \) be chosen for every index \( \gamma \in \Gamma \). We consider the family of localization operators \( H_{\eta_\gamma, \phi_i} \), i.e., at each \( \gamma \) we allow the window to be picked from the collection \( \Phi = \{ \phi_i, i \in \mathcal{I} \} \). We will need the following lemma.

**Lemma III.4.** The family of localization operators \( H_{\eta_\gamma, \phi_i} \), is well-spread, i.e., \( \forall z \in \mathbb{R}^2 \)
\[ |V_0 H_{\eta_\gamma, \phi_i} f(z) | \leq | T_\gamma g \ast V_0 f | \ast | V_{\varphi_0} \varphi_0 |(z). \]

**Proposition III.5.** Consider a collection of windows \( \Phi = \{ \phi_i, i \in \mathcal{I} \} \) with (1) and such that \( | V_{\varphi_0} \varphi_0(z) - V_{\varphi_0} \phi_i(z) | < C_0 | V_{\varphi_0} \varphi_0(z) | \) for all \( i \in \mathcal{I} \) and \( 2 C_0 < \| V_{\varphi_0} \varphi_0 \|_1 \). Furthermore consider a well-spread family of symbols \( \{ \eta_\gamma \} \gamma \in \Gamma \). Then, the following inequalities hold for some positive constants \( A, B \) and all \( f \in L^2(\mathbb{R}^2) \):
\[
\left\| \sum_{\gamma \in \Gamma} H_{\eta_\gamma, \phi_i} f \right\|_2^2 \leq \sum_{\gamma \in \Gamma} | H_{\eta_\gamma, \phi_i} f \|_2^2 \leq B \| f \|_2^2. \tag{2}
\]

**Proof:** We start by recalling from [4], that for (2) to hold, the family of operators need to fulfill two conditions. First, they must be well-spread and second, the sum of operators must be invertible, i.e., we require, for some \( A > 0 \), that
\[
\left\langle \sum_{\gamma \in \Gamma} H_{\eta_\gamma, \phi_i} f, f \right\rangle \geq A \| f \|_2^2. \tag{3}
\]
Well-spreadness is stated in Lemma III.4. We proceed to prove equation (3). First note that we trivially have
\[
\left\langle \sum_{\gamma \in \Gamma} H_{\eta_\gamma, \phi_i} f, f \right\rangle = \sum_{\gamma \in \Gamma} \int_{\mathcal{I}} \left| V_{\varphi_0} f(z) \right|^2 dz = \| f \|_2^2. \]

For \( \tilde{f}, \tilde{g} \in L^1(\mathbb{R}^2) \), the twisted convolution \( \tilde{f} \tilde{g} \) is given by
\[
\tilde{f} \tilde{g}(x, \omega) = \int_{\mathbb{R}^2} \tilde{f}(x', \omega') \tilde{g}(x-x', \omega-\omega') e^{-2\pi i x' \omega} dx' d\omega'.
\]
We observe that
\[
\langle H_{\eta_\gamma, \phi_i} f, f \rangle = \langle \chi_{\eta_\gamma, \phi_i} f, \chi_{\eta_\gamma, \phi_i} f \rangle = \int_{\mathcal{I}} | V_{\varphi_0} f \| V_{\varphi_0} f \| \langle \chi_{\eta_\gamma, \phi_i} f, \chi_{\eta_\gamma, \phi_i} f \rangle dz = \int_{\mathcal{I}} | V_{\varphi_0} f \| V_{\varphi_0} f \| \langle \chi_{\eta_\gamma, \phi_i} f, \chi_{\eta_\gamma, \phi_i} f \rangle dz.
\]
By the reverse triangle inequality we have
\[
| V_{\varphi_0} f \| V_{\varphi_0} f \| \langle \chi_{\eta_\gamma, \phi_i} f, \chi_{\eta_\gamma, \phi_i} f \rangle \| \leq | V_{\varphi_0} f \| V_{\varphi_0} f \| \langle \chi_{\eta_\gamma, \phi_i} f, \chi_{\eta_\gamma, \phi_i} f \rangle \|.
\]
Since
\[
| V_{\varphi_0} f - V_{\varphi_0} f \| V_{\varphi_0} f \| \langle \chi_{\eta_\gamma, \phi_i} f, \chi_{\eta_\gamma, \phi_i} f \rangle \| \leq | V_{\varphi_0} f \| V_{\varphi_0} f \| \langle \chi_{\eta_\gamma, \phi_i} f, \chi_{\eta_\gamma, \phi_i} f \rangle \|
\]
pointwise, we get
\[
| V_{\varphi_0} f \| V_{\varphi_0} f \| \langle \chi_{\eta_\gamma, \phi_i} f, \chi_{\eta_\gamma, \phi_i} f \rangle \| \leq | V_{\varphi_0} f \| V_{\varphi_0} f \| \langle \chi_{\eta_\gamma, \phi_i} f, \chi_{\eta_\gamma, \phi_i} f \rangle \|
\]
Finally, since
\[
\int_{\mathbb{R}^2} | V_{\varphi_0} f \| V_{\varphi_0} f \| \langle \chi_{\eta_\gamma, \phi_i} f, \chi_{\eta_\gamma, \phi_i} f \rangle \| dz \leq \| V_{\varphi_0} f \|_2 \| V_{\varphi_0} f \|_2 \| V_{\varphi_0} f \|_1
\]
we obtain
\[
\sum_{\gamma \in \Gamma} \langle H_{\eta_\gamma, \phi_i} f, f \rangle \geq (1 - 2 C_0 \| V_{\varphi_0} f \|_1) \cdot \| f \|_2^2
\]
which proves the invertibility of the operator sum as desired, whenever \( 2 C_0 < \| V_{\varphi_0} f \|_1 \).

C. Multi-window Gabor frames

In [4] it was shown that the norm equivalence of the sum of localized functions as stated in Prop. III.3 and Prop.III.5 is maintained if the full spectral representation of the operators is replaced by truncated versions. Here, we use a variant of this idea to construct families of local windows which eventually yield multi-window weaving frames. We need the following proposition.

**Proposition III.6.** Let windows \( \phi_i \) and symbols be chosen as in Prop. III.3 or Prop. III.5, respectively. Then we also have that \( \sum_{\gamma \in \Gamma} \| H_{\eta_\gamma, \phi_i} f \|_2^2 \approx \| f \|_2^2 \). As a consequence, there exist constants \( A, B \) such that
\[
\| f \|_2^2 \leq \sum_{\gamma \in \Gamma} \| H_{\eta_\gamma, \phi_i} f \|_2^2 \leq \sum_{\gamma \in \Gamma} \| H_{\eta_\gamma, \phi_i} f \|_2^2 \leq B \| f \|_2^2.
\]
\[
(5)
\]
If we choose \( \varepsilon < \frac{A}{4} \), then the family of collections of all eigenfunctions \( \hat{\phi}_k^\varepsilon \) of \( H_{\gamma,\phi}, \gamma \in \Gamma \), corresponding to eigenvalues bigger than \( \varepsilon \) generates a frame, i.e.

\[
\sum_{\gamma \in \Gamma} \| H_{\gamma,\phi}^\varepsilon \hat{f} \|_2^2 \geq \sum_{\gamma \in \Gamma} \sum_{k: \lambda_k^\varepsilon > \varepsilon} |(f, \hat{\phi}_k^\varepsilon)^2| \geq \| f \|_2^2.
\]

**Proof:** Since \( \| H_{\gamma,\phi}^\varepsilon \hat{f} \|_2 \leq \| H_{\gamma,\phi}^\varepsilon \hat{f} \|_2 + \varepsilon \| \hat{f} \|_2 \) we obtain

\[
\| H_{\gamma,\phi}^\varepsilon \hat{f} \|_2 \leq \| H_{\gamma,\phi}^\varepsilon \hat{f} \|_2 + \varepsilon \| H_{\gamma,\phi} \hat{f} \|_2
\]

and, since \( H_{\gamma,\phi} \) and \( H_{\gamma,\phi}^\varepsilon \) commute, this yields

\[
\| H_{\gamma,\phi}^\varepsilon \hat{f} \|_2 \leq \| H_{\gamma,\phi} \hat{f} \|_2 + \varepsilon \| H_{\gamma,\phi} \hat{f} \|_2
\]

Taking sums and using the norm-equivalences in (5), we obtain

\[
A \| f \|_2^2 \leq \sum_{\gamma \in \Gamma} \| H_{\gamma,\phi}^\varepsilon \hat{f} \|_2^2 + B \varepsilon \| f \|_2^2,
\]

which implies the claim.

The above proposition leads to the following result.

**Theorem III.7.** Let the windows \( \phi_i \) be chosen as in Prop. III.3 or Prop. III.5 and consider a well-spread family of symbols \( \{\eta_{\gamma}\} \in \Gamma \), such that \( \sum_{\gamma \in \Gamma} \eta_{\gamma}(z) \approx 1 \). Then, there exists an \( \varepsilon > 0 \), such that for each fixed \( \phi_i, \gamma \in \Gamma \), the family of collections of all eigenfunctions \( \hat{\phi}_k^\gamma \) of \( H_{\gamma,\phi}, \gamma \in \Gamma \), corresponding to eigenvalues bigger than \( \varepsilon \) generates a (multi-window Gabor) frame and all these frames are woven.

**IV. GAUSSIAN WINDOWS AND ELLIPTIC DOMAINS**

In this section we will have a look at localization operators on elliptic domains with an appropriate dilated Gaussian as carried out by Daubechies [3]. We denote the dilated standard Gaussian by \( \varphi_{0,L}(t) = 2^{1/4} \sqrt{L} e^{-\pi(tL)^2} \). The dilated standard Gaussian is essentially concentrated in an ellipse, which is best seen by computing

\[
\mathcal{V}_{\varphi_{0,L}} \varphi_{0,L}(x,\omega) = e^{-\frac{x^2}{2} L^2} (L^2 x^2 + \omega^2) \cdot \frac{1}{L^2}.
\]

Therefore, the ellipse

\[
E_{L,R} = \left\{ (x,\omega) \in \mathbb{R}^2 : L^2 x^2 + \omega^2 \leq R \right\}
\]

is the appropriate domain to be used for the localization operator. The eigenfunctions of the localization operator \( H_{E_{L,R},\varphi_{0,L}} \) are the dilated Hermite functions [3]. The eigenvalues are given by

\[
\lambda_{L,k}(R) = \lambda_k(R) = 1 - e^{-\pi R^2} \sum_{j=0}^{k} \frac{1}{j!} (\pi R^2)^j.
\]

We note that the eigenvalues depend on the size, but not the shape of the ellipse, whereas the eigenfunctions depend on the shape, but not the size of the ellipse. As a next step we compute how far two dilated Gaussians differ from each other in the \( L^2 \)-norm. For \( L_1, L_2 > 0 \) we have

\[
\| g_{L_1} - g_{L_2} \|_2^2 = 2 - 2 \frac{\sqrt{2L_1 L_2}}{\sqrt{L_1^2 + L_2^2}}.
\]

If we want \( \| g_{L_1} - g_{L_2} \|_2 < \frac{1}{2} \), then

\[
\frac{64 - \sqrt{1695}}{49} < \frac{L_2}{L_1} < \frac{64 + \sqrt{1695}}{49}.
\]

Numerically, this means that for \( 0.47 < \frac{L_2}{L_1} < 2.14 \) we get woven multi-window Gabor frames consisting of sufficiently many Hermite functions. The results can be extended to chirped or rotated Gaussians. A chirped, dilated Gaussian is of the form

\[
\varphi_{c,L}(t) = 2^{1/4} e^{\pi ic t^2} \sqrt{L} e^{-\pi(tL)^2}, \quad c, L > 0.
\]

We compute

\[
\| \mathcal{V}_{\varphi_{c,L}} \varphi_{c,L}(x,\omega) \| = e^{-\frac{x^2}{2} L^2} \left( L^2 x^2 + \omega^2 \right) \cdot \frac{1}{L^2}.
\]

Therefore, a chirped Gaussian is essentially concentrated in a rotated ellipse described by the quadratic form in the exponent in (7). For more details on Gaussians and their concentration in phase space see [5].

**V. CONCLUSION**

We have established sufficient criteria for multi-window Gabor frames consisting of eigenfunctions of a localization operator to be woven. In particular we found out that two finite families of Hermite functions can constitute woven multi-window frames. However, there seems to be a gap between the necessary condition we know from the Balian-Low theorem, which is already sufficient for Gaussians, and the number of Hermite functions we need in our phase-space approach. Also, the problem posed in [1] asks whether any families of rotated Gaussians yield woven Gabor frames. We have seen that if the difference of the Gaussians in the \( L^2(\mathbb{R}) \)-norm is less than \( 1/2 \), we get weaving frames by taking a finite number of generalized Hermite functions derived from the original Gaussians. As a next step, it would be interesting to show that the finite number is \( 1 \) and we only need to take the Gaussians if the index set is sufficiently dense in phase-space, in particular in the case of a lattice with density greater than 1.

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