Vector valued Fourier transforms and Fourier type operators

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Outline

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Banach space farm

All Banach spaces are equal but some Banach spaces are more equal than others.

Motivation and Introduction

Problem

Scalar-valued results $\overset{\text{EXTENSION ?}}{\rightsquigarrow}$ Vector-valued results

Possible answers

- Results remain true for any Banach space,
- Only "trivial" extensions remain true,
- Extension depends on the structure and geometry of Banach space.

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Hausdorff-Young inequality

Fourier transform

For a function $f \in L_1(\mathbb{R})$ the Fourier transform $\mathcal{F}_{\mathbb{R}} f$ is given by

$$(\mathcal{F}_{\mathbb{R}} f)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-ist} \, dt$$

Hausdorff-Young inequality

If $1 \le p \le 2$ then we have

$$\|\mathcal{F}_{\mathbb{R}} f\|_{L_{p'}(\mathbb{R})} \le c \, \|f\|_{L_p(\mathbb{R})} \quad \text{for all} \quad f \in L_p(\mathbb{R}).$$

We study Hausdorff-Young inequalities for vector-valued functions.

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Abstract harmonic analysis

- We work in the framework of a *locally compact abelian group* G (shortly: lca) which comes equipped with its *Haar measure* μ_G.
- A *character* γ on G is a continuous homomorphism from G into the torus $\mathbb{T}$. The collection of characters on G is an abelian group under pointwise multiplication and carries a natural locally compact topology. The resulting lca group is the *dual group* G' of G.
- For a function $f \in L_1(G)$, the *Fourier transform* $\mathcal{F}_G f$ is defined by

$$(\mathcal{F}_G f)(\gamma) = \int_G f(t)\overline{\gamma(t)}\, d\mu_G(t) \quad \text{for} \quad \gamma \in G'.$$

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Examples

Integers $\mathbb{Z}$

The characters on $\mathbb{Z}$ are given by $\gamma(k) = z^k$ for some $z \in \mathbb{T}$. It turns out that $\mathbb{Z}' \cong \mathbb{T}$ and the Fourier transform is given by

$$(\mathcal{F}_{\mathbb{Z}} f)(e^{it}) = \sum_{n \in \mathbb{Z}} f(n) e^{-int} \quad \text{for} \quad e^{it} \in \mathbb{T}.$$

Torus $\mathbb{T}$

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Real line $\mathbb{R}$

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$$(\mathcal{F}_{\mathbb{R}} f)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixy}\, dx \quad \text{for} \quad y \in \mathbb{R}.$$

Cantor group $\mathbb{D} = \mathbb{Z}_2^{\infty} = \{0,1\}^{\mathbb{N}}$

For $n \in \mathbb{N}_0$ let $n = \sum_{k=0}^{\infty} n_k 2^k$ with $n_k \in \{0,1\}$. The characters on $\mathbb{D}$ are given by $\psi_n(x) = (-1)^{\langle n,x \rangle}$ with $\langle n,x \rangle = n_0 x_0 + n_1 x_1 + \ldots\ (mod 2)$ for $n \in \mathbb{N}_0$ and $x \in \mathbb{D}$. It turns out that $\mathbb{D}' \cong (\mathbb{N}_0, \oplus)$ and the Fourier transform is given by

$$(\mathcal{F}_{\mathbb{D}} f)(n) = \int_{\mathbb{D}} f \psi_n \, d\mu, \quad \text{for} \quad n \in \mathbb{N}_0.$$

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Fourier type of Banach spaces

The Bochner-Lebegue space

$$L_p^X(G) = \{f : G \to X : \int_G \|f(t)\|_X^p \, d\mu_G(t) < \infty\}$$

Definition (J. Peetre 1969 $G = \mathbb{R}$, M. Milman 1984 general case)

A Banach space X has a *Fourier type* p ($1 \leq p \leq 2$) with respect to G if the operator $\mathcal{F}_G$ originally defined on $L_p(G) \otimes X$ by

$$\mathcal{F}_G \left(\sum_{i=1}^n \varphi_i x_i\right)(\gamma) = \sum_{i=1}^n \left(\mathcal{F}_G \varphi_i\right)(\gamma) x_i, \quad \varphi_i \in L_p(G), x_i \in X$$

can be extended to a bounded operator $\mathcal{F} : L_p^X(G) \to L_{p'}^X(G')$. In other words,

$$\|\mathcal{F}_G f\|_{L_{p'}^X(G')} \leq c \, \|f\|_{L_p^X(G)}.$$

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Fourier type operators

Definition

An operator $T \in \mathcal{L}(X, Y)$ is said to be of *Fourier type* p $(1 \leq p \leq 2)$ with respect to G if the operator

$$\mathcal{F}_G \otimes T : L_p(G) \otimes X \to L_{p'}(G') \otimes Y$$

extends to a bounded linear operator from $L_p^X(G)$ to $L_{p'}^Y(G')$. In other words

$$\|(\mathcal{F}_G \otimes T)f\|_{L_{p'}^Y(G')} \leq c\, \|f\|_{L_p^X(G)}.$$

The class of all operators of Fourier type p equipped with the operator norm of the extended operator (denoted by $\| \cdot | \mathcal{FT}_p^G \|$) is a Banach operator ideal -$\mathcal{FT}_p^G$.

Transference principles

Theorem (M. Milman (1984))

Let $1 < p_1 < p_2 < 2$.

$$\mathcal{FT}_2^G \subset \mathcal{FT}_{p_2}^G \subset \mathcal{FT}_{p_1}^G \subset \mathcal{FT}_1^G = \mathcal{L}.$$

Problem

Do the ideals $\mathcal{FT}_p^G$ depend at all on the infinite lca group G?

More precisely

Let G_1, G_2 be infinite lca groups and $p \in (1, 2)$.

- Inclusion: $\mathcal{FT}_p^{G_1} \subset \mathcal{FT}_p^{G_2}$?
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Known result

Theorem (H. König (1991), M. E. Andersson (1994), J. Garcia-Cuerva et.al. (1998))

$$\mathcal{FT}_p^{\mathbb{R}} = \mathcal{FT}_p^{\mathbb{Z}} = \mathcal{FT}_p^{\mathbb{T}} = \mathcal{FT}_p^{\mathbb{R}^n} = \mathcal{FT}_p^{\mathbb{Z}^n} = \mathcal{FT}_p^{\mathbb{T}^n}$$

Cantor group

$$\mathbb{D} = \{x = (x_n)_{n\in\mathbb{N}} : \quad x_n \in \{0,1\}\}$$

Its continual analogue

$$\mathbb{F} = \{x = (x_n)_{n\in\mathbb{Z}} : \quad x_n \in \{0,1\} \quad \text{and } x_n \to 0 \quad \text{for } n \to -\infty\}$$

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Theorem

Let $1 < p < 2$. For an operator $T \in \mathcal{L}(X, Y)$ the following statements are equivalent

- *T has Fourier type p with respect to group $\mathbb{D}$.*
- *T has Fourier type p with respect to group $\mathbb{D}^m$ for all $m \in \mathbb{N}$.*
- *T has Fourier type p with respect to group $\mathbb{F}$.*
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Moreover, in this case all norms coincide.

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Rademacher type and B-convex spaces

Definition (Rademacher type)

A Banach space X has the Rademacher type p $(1 \leq p \leq 2)$, if there is a constant $c > 0$ such that for any $x_1, \ldots, x_n \in X$

$$\left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\|_{L_2^X} \leq c \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{1/p} .$$

Theorem (G. Pisier, B. Maurey):

A Banach space X is B-convex if and only if it has some nontrivial Rademacher type if and only if it does not contain the spaces ℓ_1^n uniformly.

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Bourgain's Hausdorff-Young inequality for cyclic groups

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A Banach space X is B-convex if, and only if, it has some nontrivial Fourier type with respect to the classical groups or the Cantor group.

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Kwapień's result and factorization though a Hilbert space

Theorem (S. Kwapień (1972):)

A Banach space X has Fourier type 2 with respect to some infinite lca group if and only if it is isomorphic to a Hilbert space.

Open question

Let $\mathcal{H}$ denote the class of all operators T factoring through a Hilbert space. If G is infinite group, is it true that

$$\mathcal{FT}_2^G = \mathcal{H}?$$

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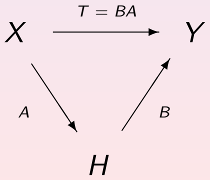

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Transference principle for Fourier type 2 operators

Theorem (A. Hinrichs, M.P.:)

Any operator of Fourier type 2 with respect to the classical groups has Fourier type 2 with respect to all lca groups. More precisely,

$$\mathcal{FT}_2^{\mathbb{T}} \subseteq \mathcal{FT}_2^{G} \quad \text{and} \quad \| T \mid \mathcal{FT}_2^{G} \| \leq \| T \mid \mathcal{FT}_2^{\mathbb{T}} \|$$

holds for all lca groups G and all $T \in \mathcal{FT}_2^{\mathbb{T}}$.

Outline

A la recherche de connections to NuHAG research!?!

Research questions

1. Vector–valued extensions of classical results in time–frequency analysis, vector–valued modulation spaces
2. Vector–valued Fourier multiplier theorems, pseudodifferential operators with operator valued symbols

THANK YOU FOR YOUR ATTENTION