Finanzdienstleistungen (Anwendungen)

Square root diffusion process
Square-Root Diffusions

Definition
A class of process that includes the square-root diffusion
\[ dr(t) = a(b - r(t))dt + \sigma \sqrt{r(t)}dW(t) \]
with \( W \) a standard one-dimension Brownian motion.
We consider the case in which:
\[ a > 0, b > 0 \]
If \( r(0) > 0 \) then \( r(t) \geq 0 \) for all \( t \)
If \( 2ab \geq \sigma^2 \) then \( r(t) > 0 \) for all \( t \)
All of the coefficients could in principle be time-dependent
i.e. with \( b = b(t) \)
\[ dr(t) = a(b(t) - r(t))dt + \sigma \sqrt{r(t)}dW(t) \]
Square-Root Diffusions

CIR Model
The square-root diffusion process
\[ dr(t) = a(b - r(t))dt + \sigma \sqrt{r(t)}dW(t) \]
can be used as a model of the short rate.
This model was done to illustrate the workings of a general equilibrium model and was proposed by Cox-Ingersoll-Ross (CIR) (1985) as an extension of the Vasicek model.
Square-Root Diffusions

CIR Model
Considering a square-root diffusion process
\[ dr(t) = a(b - r(t))dt + \sigma \sqrt{r(t)}dW(t) \]
Under the no-arbitrage assumption, a bond may be priced using the CIR Model.
The bond price is exponential affine in the interest rate:
\[ P(t,T) = A(t,T)e^{-B(t,T)r(t)} \]
with
\[ A(t,T) = \left( \frac{2he^{0.5(a+h)(T-t)} - 2h}{2h + (a + h)(e^{(T-t)h} - 1)} \right)^{2ab/\sigma^2} \]
Square-Root Diffusion

CIR Model

and

\[ B(t, T) = \left( \frac{2(e^{(T-t)h} - 1)}{2h + (a + h)(e^{(T-t)h} - 1)} \right) ^ {2ab/\sigma^2} \]

\[ h = \sqrt{a^2 + 2\sigma^2} \]
Square-Root Diffusions

Heston Model
Heston proposed a stochastic volatility model in which the price of an asset $S$ is governed by:

$$\frac{dS(t)}{S(t)} = \mu dt + \sqrt{V(t)}dW_1(t)$$

the squared volatility $V(t)$ follows a square-root diffusion

$$dV(t) = a(b - V(t))dt + \sigma \sqrt{V(t)}dW_1(t)$$

And

$$(W_1, W_2), \text{is a two-dimensional Brownian motion with}$$

$$\text{corr}(dW_1, dW_2) = \rho dt$$
Square-Root Diffusions

Heston Model

Description

The first equation gives the dynamic of a stock price

The parameters represent:

- $S(t)$ the stock price at time $t$
- $\mu$ the risk neutral drift
- $\sqrt{V(t)}$ the volatility
Square-Root Diffusions

Heston Model

Description
The second equation gives the evolution of the variance which follows the square-root process.

The parameters represent:

- $b$ the long vol, or long run average price volatility: as $t$ tends to infinity, the expected value of $V(t)$ tends to $b$.
- $\alpha$ the mean reversion parameter; rate at which $V(t)$ reverts to $b$.
- $\sigma$ the vol of vol, or volatility of the volatility; this determines the variance of $V(t)$.
Square-Root Diffusions

**Heston Model**

Simulation

Simulate \( W(t) = (W_1(t), W_2(t)) \) (e.g. with random walk construction) by setting

\[
W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i} B Z_i
\]

with

\[
B^t B = \Sigma,
\]

\[
\Sigma = \begin{pmatrix}
1 & \rho \\
\rho & 1
\end{pmatrix}
\]

and independent standard random normal vectors \( Z_1, Z_2, \ldots \).
Square-Root Diffusions

**Heston Model**

Simulation

Simulate $V(t)$ (e.g. with Euler discretization) at times $t_1, t_2, \ldots, t_n$ by setting

$$V(t_{i+1}) = V(t_i) + a(b - V(t_i))(t_{i+1} - t_i) + \sigma \sqrt{V(t)}(W_2(t_{i+1}) - W_2(t_i))$$

Simulate $S(t)$ at times $t_1, t_2, \ldots, t_n$ by setting

$$S(t_{i+1}) = S(t_i) + \mu S(t_i)(t_{i+1} - t_i) + \sqrt{V(t)} S(t_i)(W_1(t_{i+1}) - W_1(t_i))$$