Bargaining: 
Axiomatic and Strategic approaches.

We have seen in class that the Nash-Bargaining solution follows from the axiomatic approach (see page 382 in Cahuc and Zylberberg for a discussion of the axioms). The Nash solution reads

\[ S^N = \text{ArgMax}_S [u_1(S) - d_1]^{\beta} [u_2(1 - S) - d_2]^{1-\beta} \] (1)

where \( \beta \) is the bargaining power of player 1.

The main problem with this approach is the foundation of the bargaining parameter. This is why we have considered the strategic approach of the Rubinstein-Stahl model.

When the horizon is finite, the solution of the strategic game is

\[ u_1(S^*_1) - d_1 = \delta_1 (u_1(S^*_1) - d_1) \] (2)

\[ u_2(1 - S^*_1) - d_2 = \delta_2 (u_2(1 - S^*_2) - d_2) \] (3)

where \( S^*_1 \) is the share that goes to player 1 when she makes the first offer, whereas \( S^*_2 \) is the share that goes to player 1 when player 2 makes the first offer. Notice that \( \delta_1 < 1 \) denote the discount factor of player 1. (You can check that this condition is satisfied by the specific solution derived in class where \( d_1 = d_2 = 0, \delta_1 = \delta_2 \) and \( u_1(x) = u_2(x) = x \).

Symmetrically

\[ u_2(1 - S^*_1) - d_2 = \delta_2 (u_2(1 - S^*_2) - d_2) \] (3)

1. Consider that the time length \( \Delta \) between each bargaining interval is very small, so that

\[ \delta_i \simeq 1 - r_i \Delta; \text{ where } i = 1, 2. \]

Then (2) and (3) can be rewritten as follows

\[ u_1(S^*_2(\Delta)) - u_1(S^*_1(\Delta)) \simeq r_1 \Delta (u_1(S^*_1(\Delta)) - d_1) \] (4)

\[ u_2(1 - S^*_1(\Delta)) - u_2(1 - S^*_2(\Delta)) \simeq r_2 \Delta (u_2(1 - S^*_2(\Delta)) - d_2) \] (5)

Let \( \Delta \rightarrow 0 \). Then the Right-Hand sides of (4) and (5) tend to zero, so that

\[ \lim_{\Delta \rightarrow 0} S^*_1(\Delta) = \lim_{\Delta \rightarrow 0} S^*_2(\Delta) = S^* \] (6)

This result is very intuitive. It simply states that when players make offers and counter-offers very quickly, it does not matter who starts the round of talk.

2. Consider the derivative of \( u_i(S) \) evaluated at \( S^* \)

\[ \frac{\partial u_1(S)}{\partial S} \bigg|_{S=S^*} = \lim_{\Delta \rightarrow 0} \frac{u_1(S^*_2(\Delta)) - u_1(S^*_1(\Delta))}{S^*_2(\Delta) - S^*_1(\Delta)} \] (7)

\[ \frac{\partial u_2(1 - S)}{\partial S} \bigg|_{S=S^*} = \lim_{\Delta \rightarrow 0} \frac{u_2(1 - S^*_2(\Delta)) - u_2(1 - S^*_1(\Delta))}{S^*_2(\Delta) - S^*_1(\Delta)} \] (8)
Now take the ratio of (7) to (8) and use relations (4) and (5) to obtain
\[
\frac{u_1'(S^*)}{u_2'(S^*)} = \left( \frac{r_1}{r_2} \right) \left( \frac{u_1(S^*) - d_1}{u_2(1 - S^*) - d_2} \right)
\]  
(9)

The solution of the Nash Bargaining is such that
\[
\frac{u_1'(S^N)}{u_2'(S^N)} = \left( \frac{1 - \beta}{\beta} \right) \left( \frac{u_1(S^N) - d_1}{u_2(1 - S^N) - d_2} \right)
\]  
(10)

Hence when
\[
\beta = \frac{r_2}{r_1 + r_2}
\]  
(11)

the two solutions coincide.

To analyze equation (11) let \(r_1\) go to zero. Then \(\beta\) goes to one. So when player 1 is infinitely patient, he has all the bargaining player. In other terms, the most patient of the two player also has the higher bargaining power.