THE SPATIALLY DOMINANT EQUILIBRIUM OF A GAME

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Abstract. A new framework for equilibrium selection is presented: Playing games recurrently in space and time may render one of the equilibria 'spatially dominant': Prevailing initially on a large enough finite part of the space it will take over on the whole space in the long run. In particular it will drive out the other equilibria along travelling waves. This new dominance concept is compared with the Harsanyi–Selten risk–dominance concept.

The notion of dominance appears in many different fields. In Game Theory, different Nash equilibria have been compared by dominance. As shown by Harsanyi and Selten (1988), the notion of risk–dominance is often more significant and meaningful than what may seem the more natural concept of Pareto (or payoff)–dominance.

Similarly, in Dynamical Systems, there is a notion of dominance, suggested by Fife (1979), which compares the strength of different (stable) equilibrium states, in the context of reaction diffusion equations. Here the direction of travelling wave solutions connecting two equilibria plays a decisive role.

It is the purpose of this paper to point out a fundamental connection between these two notions of dominance. This observation leads to a new approach to the problem of equilibrium selection in normal form games, as initiated in Hofbauer (1997).[1]

The outline of this paper is as follows. In the first four sections, we briefly review these two notions of dominance, and link them in the simplest case, namely

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[1]Other recent approaches to equilibrium selection, based on stochastic dynamics, are due to Kandori, Mailath and Rob (1993) and Young (1993). Somewhat closer to the present approach are the spatial models of Ellison (1993), Blume (1995) and Kosfeld (1996). See also Sugden (1995) and Yegorov (1997) for spatial models in economics.

symmetric 2×2 games, played in space and time. In sections 5–7 we extend this idea to 2×2 bimatrix games and two classes of n–person coordination games, relating the dynamic concept of spatial dominance with the game theoretic concepts of risk–dominance and Nash products. Throughout these sections we implicitly use only the best reply structure of the game. The last section (8) points to some of the difficulties which arise when information beyond the best reply structure enters the game.

1. Games and Risk–dominance

Consider an n person binary choice game: Each of the n players has two pure strategies: A_i and B_i. The frequency of the B_i strategy will be denoted by p_i, hence that of A_i is $1-p_i$. With this notation the pure strategy $\mathbf{B}=(B_i)$ corresponds to $p=\mathbf{1}=(1,\dots,1)$ and $\mathbf{A}=(A_i)$ to $\mathbf{0}=(0,\dots,0)$. The state space, i.e., the set of all mixed strategy profiles, is the n–dimensional cube $[0,1]^n$. Let $\pi^i(p_1,\dots,p_n)=\pi^i(p_i;p_{-i})$ denote the payoff for player i. For a normal form game this is a linear expression in each p_j, for each i. The essential information about the game is contained in the payoff differences

$$d^i(p)=\pi^i(B_i;p_{-i})-\pi^i(A_i;p_{-i}), \tag{1.1}$$

often called the *incentive function* for player i. $\mathbf{A}$ is a Nash equilibrium if $d^i(\mathbf{0})\le 0$ for all i, while $\mathbf{B}$ is a Nash equilibrium if $d^i(\mathbf{1})\ge 0$ for all i. For strict equilibria, all these inequalities are strict. Binary games with both $\mathbf{A}$ and $\mathbf{B}$ being strict equilibria were called *bipolar* games by Selten (1995). A further property of many bipolar games is the monotonicity of the incentive functions,

$$\frac{\partial d^i}{\partial p_j}\ge 0 \quad \text{for all } i\ne j \tag{1.2}$$

which is equivalent to the monotonicity of the best reply correspondence:

$$p\le q \quad \Rightarrow \quad BR(p)\le BR(q).$$

A two–person binary choice game is usually described by a pair of 2×2 payoff matrices

$$\begin{pmatrix} a_1,a_2 & b_1,b_2 \\ c_1,c_2 & d_1,d_2 \end{pmatrix} \tag{1.3}$$

which leads to the incentive functions

$$\begin{aligned} d^1(p)&=(c_1-a_1)(1-p_2)+(d_1-b_1)p_2 \\ d^2(p)&=(b_2-a_2)(1-p_1)+(d_2-c_2)p_1. \end{aligned} \tag{1.4}$$

$\mathbf{A}$ is a strict equilibrium if $d^1(\mathbf{0}) = c_1 - a_1 < 0$ and $d^2(\mathbf{0}) = b_2 - a_2 < 0$, and $\mathbf{B}$ is a strict equilibrium if $d^1(\mathbf{1}) = d_1 - b_1 > 0$ and $d^2(\mathbf{1}) = d_2 - c_2 > 0$. $\mathbf{B}$ is said to *risk–dominate* $\mathbf{A}$ if the *Nash products* satisfy the inequality

$$d^1(\mathbf{1})d^2(\mathbf{1}) > d^1(\mathbf{0})d^2(\mathbf{0}). \tag{1.5}$$

Harsanyi and Selten (1988, p. 87, Theorem 3.9.1) show that the concept of the risk–dominant equilibrium is characterized by the following set of three axioms:

1. Invariance with respect to isomorphisms
2. Best reply invariance
3. Monotonicity

The first simply means that the concept is immune against renamings of strategies and players. The third means that if $\mathbf{A}$ risk–dominates $\mathbf{B}$ then also $\mathbf{A}'$ risk–dominates $\mathbf{B}$ whenever $a_i' \geq a_i$ for $i = 1, 2$. The second axiom requires that the result is the same for two games that have the same best reply structure among mixed strategies. This is a consequence of the rationality postulate underlying classical game theory. In particular, the result depends only on the incentive functions, and moreover is invariant under multiplications of these by (possibly different) positive constants.

Hence, for 2×2 games, the risk–dominance concept depends only on the fractions $\frac{a_1 - c_1}{d_1 - b_1}$ and $\frac{a_2 - b_2}{d_2 - c_2}$. Now it is easy to see that together with the other two axioms this implies (1.5).

In contrast, equilibrium $\mathbf{B}$ is said to be Pareto dominant (or payoff dominant), if $d_1 > a_1$ and $d_2 > a_2$. However, this simpler concept of dominance does not satisfy axiom 2. We will return to this point in section 8.

For $n \geq 3$ players there seems to be no generally convincing and natural extension of the concept of risk–dominance. The obvious generalization of the products of incentives (1.5) has been considered by Güth (1992), more sophisticated extensions can be found in Harsanyi and Selten (1988), Güth and Kalkofen (1989), Selten (1995), etc. The troubles with different equilibrium selection criteria have been illuminated in the simple class of symmetric bipolar n person games by Carlsson and van Damme (1993) and Kim (1996). There are two special classes of bipolar games where the situation looks better: Unanimity games, and games with linear incentives. These classes will be discussed in more detail in sections 6 and 7.

2. Spatio–temporal games

Let us consider now n player populations distributed in space (which is assumed to be the one–dimensional continuum $\mathbb{R}$ for simplicity). Let $\rho_{A_i}(x,t)$ and $\rho_{B_i}(x,t)$ denote the densities of A_i and B_i players, i.e., $\int_a^b \rho_{A_i}(x,t)dx$ denotes the number of A_i players in the interval $[a,b]$ at time t. We assume that these densities will change in time, due to local interaction and migration of players, and satisfy a system of reaction–diffusion equations

$$\frac{\partial \rho_{A_i}}{\partial t} = -F_i(\rho) + d\frac{\partial^2 \rho_{A_i}}{\partial x^2}, \qquad \frac{\partial \rho_{B_i}}{\partial t} = F_i(\rho) + d\frac{\partial^2 \rho_{B_i}}{\partial x^2}. \tag{2.1}$$

The second term on the right hand side of (2.1) models random migration of players at a uniform, strategy– and player–independent rate $d > 0$, see Fife (1979, ch. 1). Then the density of the total i population, $\rho_i(x,t) = \rho_{A_i}(x,t) + \rho_{B_i}(x,t)$, satisfies a diffusion equation

$$\frac{\partial \rho_i}{\partial t} = d\frac{\partial^2 \rho_i}{\partial x^2}, \tag{2.2}$$

and hence converges under mild assumptions to a spatially constant: $\rho_i(x,t) \to \rho_i$, as $t \to \infty$, i.e., each player population will be equally spread over the line $\mathbb{R}$. Let us assume that this equilibrium is reached, i.e., $\rho_i(x,0) = \rho_i(x,t) = \rho_i$ is independent of x. Then the frequencies $p_i(x,t) = \frac{\rho_{B_i}(x,t)}{\rho_i}$ satisfy a reaction diffusion system

$$\frac{\partial p_i}{\partial t} = f_i(p) + d\frac{\partial^2 p_i}{\partial x^2}. \tag{2.3}$$

The first term, the reaction term, $F_i(\rho)$, models the local interaction of players and the resulting adaptations of their strategy according to their local experience. There are at least two sensible choices for this reaction term[2], arising from two different mechanisms of adaptation: 1) best response, and 2) imitation. In 1) we assume that a certain proportion of (randomly chosen) players at each spot x switches to the locally best reply at any time t (see Gilboa and Matsui (1991) and Hofbauer (1994)). For binary games this means for (2.1)

$$F_i(\rho) = \begin{cases} -\rho_{B_i}, & \text{if } d^i(p) < 0 \\ \rho_{A_i}, & \text{if } d^i(p) > 0. \end{cases} \tag{2.4}$$

For frequencies this leads to

$$f_i(p) = \begin{cases} -p_i, & \text{if } d^i(p) < 0 \\ 1 - p_i, & \text{if } d^i(p) > 0. \end{cases} \tag{2.5}$$

in (2.3). In 2) we assume that players change their strategies by imitation, according to the proportional imitation rule, see Schlag (1995). This leads in (2.1) to

$$F_i(\rho) = \rho_{A_i}\rho_{B_i} d^i(p)/\rho_i$$

and in (2.3) to the replicator dynamics, i.e.,

$$f_i(p) = p_i(1 - p_i)d^i(p). \tag{2.6}$$

More general selection, imitation, or myopic adjustment dynamics, satisfying

$$f_i(p) < 0 \Leftrightarrow d^i(p) < 0 \tag{2.7}$$

would also be sensible for the reaction term.

In case of a symmetric binary game, if we assume a single player population, the system (2.1) reduces to a single reaction diffusion equation. For such, there is a well–developed theory (see e.g., Fife, 1979), whose relevant aspects are summarized in the next section.

[2]Note that both of these reaction terms leave the local density $\rho_i(x,t)$ fixed (at ρ_i) since they both involve switching. This may not be the case if the reaction term is to model natural selection, as shown in Cressman and Vickers (1997).

3. Dominance for bistable reaction diffusion equations

Consider a single reaction diffusion equation of the form (2.3), with the diffusion rate normalized to $d = 1$ (which can always be achieved by a scaling of the x axis).

$$p_t = p_{xx} + f(p) \tag{3.1}$$

We assume that the reaction dynamics is *bistable:* There are two stable equilibria at, say $p = 0$ and $p = 1$, separated by an unstable equilibrium at $\hat{p} \in (0,1)$:

$$\begin{aligned} f(p) < 0 \quad &\text{for} \quad 0 < p < \hat{p} \\ f(p) > 0 \quad &\text{for} \quad \hat{p} < p < 1 \end{aligned} \tag{3.2}$$

We restrict attention to solutions $p(x,t)$ of the reaction diffusion equation (3.1) with initial conditions $p(x,0) \in [0,1]$. The maximum principle for parabolic PDEs then implies that a solution of (3.1) exists for all $t \geq 0$ and satisfies again $p(x,t) \in [0,1]$. Some solutions exist also for all negative times; among them are the stationary solutions $p(x,t) = P(x)$, satisfying

$$P'' + f(P) = 0 \tag{3.3}$$

and more generally, the travelling wave solutions $p(x,t) = P(x - ct)$ where c is called the wave speed, and the function P determines the shape of the wave. There are infinitely many such travelling wave solutions, but there is a unique one (up to translation), which satisfies the 'boundary conditions'

$$P(-\infty) = 0, \qquad P(\infty) = 1, \tag{3.4}$$

This particular travelling wave which connects the two stable equilibria is called a *bistable wave*. The sign of c determines the direction of this wave: If $c < 0$, i.e., the wave moves to the left, then $p(x,t) \to 1$ for $t \to \infty$ along the wave, i.e., the equilibrium 1 takes over in the long run along this special solution. The stable equilibrium 1 (i.e. strategy B) dominates the stable equilibrium 0 (i.e. strategy A) in this sense. In the critical case $c = 0$ there is a stationary solution (a standing wave) satisfying (3.3–4).

It is of utmost interest to determine the sign of c, i.e., which of the two equilibria 0,1 dominates the other. For scalar reaction diffusion equations on the line this is done by the following simple criterion.

Theorem 1. (Kanel, 1962, Aronson and Weinberger, 1975, Fife–McLeod, 1977)
For a bistable equation of the form (3.1-2) the following conditions are equivalent.
(i) There exists a unique travelling wave satisfying (3.4), and its speed $c < 0$.
(ii) For any initial condition satisfying $p(x,0) = 0$ *for* $x < a$ *and* $p(x,0) = 1$ *for* $x > b$, *one has* $p(x,t) \to 1$ *as* $t \to \infty$, *for each* $x \in \mathbb{R}$.
(iii) **1** *is asymptotically stable in the compact-open topology, i.e., there exists* $L > 0$

and $\varepsilon > 0$ *such that for each initial function satisfying* $p(x, 0) > 1 - \varepsilon$ *for* $x \in [-L, L]$, $p(x, t) \to 1$ *as* $t \to \infty$, *uniformly for* $x \in C$, *for each compact set* $C \subset \mathbb{R}$.
(iv) There exists a 'standing pulse' solution, i.e., a stationary solution $P(x) > 0$, *satisfying (3.3) and* $\lim_{x\to\pm\infty} P(x) = 0$.
(v) $\int_0^1 f(p)dp > 0$.
(vi) $V(1) > V(0)$, *where* V *is a potential function for the reaction term:* $V'(p) = f(p)$.

For a proof of these and related statements see Fife (1979, ch. 4), or Volpert et al (1994).

The first three conditions describe various strengths of the spatial dominance concept. (i) and (ii) can be visualized as 1 *drives out* 0, along the travelling wave, or along more general solutions. (ii) is a stability property of the wave. Actually it can be shown that the bistable wave is asymptotically stable with respect to the uniform topology on $\mathbb{R}$.

(iii) is an even stronger form: It is sufficient that initially in a large enough, but finite part of the space, 1 is predominant. Then it will be able to invade and take over the whole space in the long run. This happens usually in the form of a two–sided wave front. (iii) also assesses that 1 has an infinitely larger basin of attraction than 0. (Note that both 0 and 1 are asymptotically stable in the uniform topology: E.g., if $0 \le p(x, 0) < \hat{p}$ for all x, then $p(x, t) \to 0$.) Starting from a 'randomly chosen' initial function $p(x, 0) \in [0, 1]$ will lead to 1, since some translate $p(x - c, 0)$ will satisfy the assumption in (iii).

(iv) may be informally expressed as follows: The existence of a finite part of the solution close to one (more precisely: $p > \hat{p}$) is as good as an infinite piece close to zero (more precisely: $p < \hat{p}$).

Finally, (v) and (vi) are explicit conditions for determining the dominant of the two equilibria. The larger of the two areas enclosed by the graph of f below and above the p–axis determines the dominant state. For (vi) note that both 0 and 1 are local maxima of V. It is the *global* maximum of V that corresponds to the dominant equilibrium.

In the marginal case, when there is equality in (v) and in (vi), there still exists a unique bistable wave, but with zero speed, $c = 0$, so that it is in fact a stationary solution, a so–called 'standing wave'. In this case the two equilibria 0 and 1 may be considered equally strong: All initial conditions of the form described in (ii) converge uniformly to (a shift of) this standing wave. None is spatially dominant in the sense of (iii). However, except for this marginal case, Theorem 1 applies either to $f(p)$ or to $-f(1 - p)$ and hence either 0 or 1 is spatially dominant.

It is straightforward to apply Theorem 1 to some simple spatio-temporal games: For a symmetric 2×2 game with payoff matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, and with the replicator dynamics (2.6) as reaction term, $f(p) = p(1 - p)(p - \hat{p})$, it follows easily from (v) or (vi) that $c > 0$ iff $\hat{p} > \frac{1}{2}$ iff $a > b$. The same result holds if we take the BR dynamics as reaction dynamics. Hence, for this simplest class of bipolar games, the spatial dominance concept described by Theorem 1 agrees with the risk–dominance concept described in section 1.

It is instructive to compare the behaviour described by Theorem 1 with that of the stochastic spatio–temporal model of Kosfeld (1996). In his cellular automaton with a local updating rule related to the proportional imitation rule fo Schlag (1994), the asymptotic speed of propagation is $\frac{1}{4}\frac{a-b}{a+b}$ (in dimension 1, conditional on the convergence to B). In the present reaction diffusion model, taking the replicator dynamics (2.6) as reaction dynamics, the wave speed c is $(a-b)(\frac{d}{2(a+b)})^{1/2}$, while for the best response dynamics, it is $(a-b)(\frac{d}{ab})^{1/2}$.

Theorem 1 will be exploited further in section 8.

An analog of Theorem 1 is not known for several space dimensions, nor for systems of reaction diffusion equations. It is conjectured that the first four conditions are still equivalent for bistable systems, if the reaction term enjoys some monotonicity properties. The proof of Theorem 1 exploits the explicit condition (v), which makes no sense for systems. Condition (vi) applies to gradient systems, where partial results are known (see Reineck (1988)). In general, the computation of the direction of the wave is extremely difficult for systems of reaction diffusion equations. For some estimates in mutualistic systems see Mischaikow and Hutson (1993). A technique using singular perturbation theory is developed in Hutson and Mischaikow (1996) for systems with very different diffusion rates. For our case of equal diffusion rates in (2.1), these techniques do not apply.

Fortunately, if we take the best response dynamics as reaction term — keeping in mind that most equilibrium selection theories assume rationality on the part of the players — the system (2.1) remains tractable and to a sufficient extent explicitly solvable. This enables us to extend much of Theorem 1 to determine the spatially dominant equilibrium for more general classes of games.

Definition. A Nash equilibrium is said to be *spatially dominant* if the corresponding spatially constant stationary solution of the system (2.3) satisfies the analog of property (iii) of Theorem 1, i.e., if it is asymptotically stable in the topology of uniform convergence on compact intervals.

Obviously at most one equilibrium can be spatially dominant, hence if existence can be shown, this is a property that provides a way of equilibrium selection. The spatially dominant equilibrium may depend on the reaction dynamics, as is obvious from condition (v) of Theorem 1. Concrete examples will be treated in section 8. If it is not specified otherwise, we agree to use the best response dynamics (2.5).

4. Spatial Dominance for the BR Dynamics

In this section we sketch the proof of Theorem 1 for the best response dynamics, as this will be the key for the new results in this paper, derived in the following sections. As the assertions (i) and (ii) related to travelling waves have been treated in detail in Hofbauer (1997), I focus here on the part (iii) and its relation to (iv).

Let $\psi(x)$ denote a stationary solution of (3.1) such that $\psi(x) < 1$ for all x and $\psi(x) < 0$ for all x with large absolute value. Then $\psi_+(x) = \max(0, \psi(x))$ is a *subsolution with compact support* of (3.1). We will show that the existence of such a subsolution of compact support implies the spatial dominance of **1**. (See Fife (1979), Theorem 4.8). Consider the solution $v(x,t)$ of (3.1) with initial value $v(x,0) = \psi_+(x)$. Then by the maximum principle, $v(x,t)$ is monotonically increasing in t for each fixed $x \in \mathbb{R}$. Hence $\lim_{t\to+\infty} v(x,t) =: w(x)$ exists, satisfies $\psi_+(x) < w(x) \leq 1$ and is a stationary solution of (3.1). By the strict maximum principle we even have $v(x,t) > \psi(x)$ for each $t > 0$. Denoting the shifted subsolution by $\psi_\alpha(x) = \psi_+(x - \alpha)$, we obtain for fixed $t > 0$: $v(x,t) > \psi_\alpha(x)$ for all $x \in \mathbb{R}$ and all small enough α. In the limit $t \to \infty$ this leads to $w(x) \geq w(x - \alpha)$ for α small enough, which can only hold if w is a constant function. Hence w is the smallest spatially constant equilibrium larger than $\psi(x)$, which in our case is $w(x) \equiv 1$. Monotonicity of the flow shows again that for all initial funcions $\psi_+(x) \leq u(x,0) \leq 1$ the corresponding solution $u(x,t) \to 1$ as $t \to \infty$, as claimed. That the convergence is uniform on bounded intervals follows from Dini's theorem.

This proof extends to systems provided that the reaction term generates a monotone flow (see Fife (1979), section 5). For (2.3) with reaction term (2.5) or (2.6) this is the case if (1.2) holds. Furthermore, one may have to impose some conditions to ensure $w = 1$.

For constructing such a subsolution ψ of compact support, it is useful to start with a *standing pulse*. Consider the function

$$\psi_r(x) = \begin{cases} e^{-|x|}\frac{e^r - e^{-r}}{2}, & \text{if } |x| > r \\ 1 - e^{-r}\frac{e^x + e^{-x}}{2}, & \text{if } |x| < r \end{cases} \tag{4.1}$$

for any $r > 0$. It satisfies[3]

$$\psi_r'' + 1_{[-r,r]} - \psi_r = 0 \tag{4.2}$$

and it is the unique C^1 solution of this differential equation which obeys the boundary conditions $\psi_r(\pm\infty) = 0$. Note that

$$\psi_r(r) = \frac{1}{2}(1 - e^{-2r}) < \frac{1}{2} \tag{4.3}$$

Hence the unique (up to translation) standing pulse solution of the scalar equation (3.1) with $f(p) = 1_{[\hat{p},1]}(p) - p$ is given by $\psi_r(x)$ with $\psi_r(r) = \hat{p}$. By (4.3) this exists if and only if $\hat{p} < \frac{1}{2}$ in accordance with Theorem 1 (iv),(v).

From this standing pulse we can easily obtain a subsolution of compact support which implies the spatial dominance of the equilibrium **1** as shown above. We simply consider the perturbation

$$\psi_{r,\varepsilon}(x) = \psi_r(x) - \varepsilon(e^x + e^{-x}) \tag{4.4}$$

[3] 1_I denotes the characteristic function of the set I.

This function satisfies again (4.2). Hence, with $r, \varepsilon > 0$ chosen such that

$$\psi_{r,\varepsilon}(r) = \frac{1}{2}(1 - e^{-2r}) - \varepsilon(e^r + e^{-r}) = \hat{p}, \tag{4.5}$$

$\psi_{r,\varepsilon}(x)$ yields a subsolution of compact support as required above. (4.5) has a solution (choose r large enough, and then ε) whenever $\hat{p} < \frac{1}{2}$. This shows the implication (v) $\Rightarrow$ (iii) in Theorem 1, i.e., the spatial dominance of the risk–dominant equilibrium for symmetric 2×2 games.

5. 2×2 COORDINATION GAMES

Any 2×2 game with two strict equilibria (**A** and **B**) is equivalent (in the sense of having same incentives) to a coordination game or unanimity game

$$\begin{pmatrix} a_1, a_2 & 0, 0 \\ 0, 0 & b_1, b_2 \end{pmatrix} \quad (a_i, b_i > 0). \tag{5.1}$$

The incentive functions (1.4) simplify then to

$$\begin{aligned} d^1(p) &= -a_1(1 - p_2) + b_1 p_2 \\ d^2(p) &= -a_2(1 - p_1) + b_2 p_1 \end{aligned}$$

and satisfy obviously the monotonicity conditions (1.2). The incentives vanish at the interior equilibrium

$$E = (\hat{p}_1, \hat{p}_2) = (\frac{a_2}{a_2 + b_2}, \frac{a_1}{a_1 + b_1}) \tag{5.2}$$

B is *risk dominant* over **A**, if **B** has the higher Nash product,

$$a_1 a_2 < b_1 b_2, \tag{5.3}$$

or equivalently

$$\hat{p}_1 + \hat{p}_2 < 1. \tag{5.4}$$

Theorem 2. *When* **B** *risk–dominates* **A** *then* **B** *is spatially dominant*[4]*. Furthermore, there exists a unique monotone travelling wave for (2.3) that connects the two strict equilibria. Its wave speed c is negative, i.e.,* **B** *drives out* **A**.

Proof. The statements concerning travelling waves have been shown in Hofbauer (1997). Hence we are left to show that risk dominance implies spatial dominance.

Recalling (4.4) we make an Ansatz $P(x) = (P_1(x), P_2(x))$ with

$$P_i(x) = \psi_{r_i,\varepsilon_i}(x) \tag{5.5}$$

[4]Theorems 2,3 and 4 refer to spatial dominance with reaction term given as the best response dynamics.

for a subsolution of compact support. To make (4.2) consistent with a stationary solution of (2.3) for the game (5.1), the conditions

$$\begin{aligned} P_1(r_2) &= \hat{p}_1 \\ P_2(r_1) &= \hat{p}_2 \end{aligned} \tag{5.6}$$

have to be satisfied. For $r_1 < r_2$ this becomes more explicitly

$$\begin{aligned} \frac{1}{2}(e^{r_1} - e^{-r_1})e^{-r_2} - \varepsilon_1(e^{r_2} + e^{-r_2}) &= \hat{p}_1 \\ 1 - \frac{1}{2}(e^{r_1} + e^{-r_1})e^{-r_2} - \varepsilon_2(e^{r_1} + e^{-r_1}) &= \hat{p}_2 \end{aligned} \tag{5.7}$$

Adding these two equations shows $\hat{p}_1 + \hat{p}_2 < \psi_{r_1}(r_2) + \psi_{r_2}(r_1) = 1 - e^{-r_1-r_2} < 1$, hence a solution of (5.7) exists only if the equilibrium $\mathbf{B} = \mathbf{1}$ is risk–dominant, which we assume in the following. Furthermore (5.7) implies the estimates

$$\begin{aligned} \frac{1}{2}e^{r_1-r_2} &> \hat{p}_1 \\ 1 - \frac{1}{2}e^{r_1-r_2} &> \hat{p}_2 \end{aligned} \tag{5.8}$$

Assuming WLOG $\hat{p}_1 \le \hat{p}_2$ (and hence $\hat{p}_1 < \frac{1}{2}$) (otherwise interchange the two players) we can choose $r_1 - r_2 < 0$ in such a way that the inequalities (5.8) are satisfied. Then choose $r_1 + r_2$ large enough and suitable $\varepsilon_1, \varepsilon_2 > 0$ to satisfy (5.7). This yields a subsolution of compact support (under the assumption (5.4)). Since $\psi_{r_i,\varepsilon_i}(0) > \psi_{r_i,\varepsilon_i}(r_j) = \hat{p}_i$ the only constant stationary solution $w = (w_1, w_2) \ge P(x)$ is $w = \mathbf{1}$. Hence $\mathbf{1}$ is spatially dominant by the argument in section 4. $\square$

In a similar way one can show that if $\mathbf{B}$ is risk–dominant then there exists a standing pulse solution: One has to set $\varepsilon_i = 0$ in (5.5) and (5.6). The two equations (5.7) have a unique solution $r_1, r_2 > 0$ if and only if (5.4) holds.

6. n–Person Unanimity Games

A special class of bipolar games are n–person unanimity games (also called pure coordination games) which are the natural generalization of the 2–person game (5.1): If all players unanimously choose A (resp., B) then player i gets payoff $a_i > 0$ (resp., $b_i > 0$), otherwise, if there is no such coordination, the payoff is 0 to everybody. The incentive functions for these games are given by

$$d^i(p) = b_i \prod_{j \neq i} p_j - a_i \prod_{j \neq i}(1 - p_j) \tag{6.1}$$

The classical equilibrium selection principle for this class of games is based on Nash's bargaining theory, see Harsanyi and Selten (1988). If $\mathbf{B}$ has the higher 'Nash product', i.e., if

$$b_1 \cdots b_n > a_1 \cdots a_n \tag{6.2}$$

then strategy **B** is the preferred outcome over **A**. Although this criterion is not as well–founded as in the 2–person case (there seems to be no similarly convincing axiomatic characterization) it is generally accepted, see e.g. Güth (1992), Güth and Kalkofen (1989). The first dynamic model that justifies this criterion was given in Hofbauer (1997) where it was shown that along a bistable travelling wave, the equilibrium with the higher Nash product drives out the other equilibrium. More precisely, the equivalence of Theorem 1 (i) and (ii) with (6.2) was established there. In the following I will complete this by showing spatial dominance, i.e., part (iii) of Theorem 1.

Theorem 3. *In an n–person unanimity game, the strict equilibrium with the higher Nash product is the spatially dominant equilibrium of the game.*

We first prove a general result.

Lemma 1. *For a binary n person game whose incentive functions (1.1) satisfy the monotonicity condition (1.2), $\mathbf{B} = \mathbf{1}$ is spatially dominant if there exist $r_i > 0$ such that for all $i = 1, \ldots, n$*

$$d^i\left(\Phi(r_1 - r_i), \ldots, \Phi(r_n - r_i)\right) > 0. \tag{6.3}$$

Here Φ denotes the function

$$\Phi(z) = \begin{cases} \frac{1}{2}e^z, & \text{for } z \leq 0 \\ 1 - \frac{1}{2}e^{-z}, & \text{for } z > 0, \end{cases} \tag{6.4}$$

which satisfies the symmetry condition

$$\Phi(z) + \Phi(-z) = 1 \tag{6.5}$$

Proof. As in (4.4) and (5.5) we make again an Ansatz $P_i(x) = \psi_{r_i,\varepsilon_i}(x)$ for a subsolution of compact support. If we choose $r_i > 0$ such that

$$d^i(P_1(r_i), \ldots, P_n(r_i)) = 0 \tag{6.6}$$

then

$$d^i(P_1(x), \ldots, P_n(x)) \gtreqless 0 \quad \Leftrightarrow \quad |x| \lesseqgtr r_i \tag{6.7}$$

since the P_i are unimodal (symmetric, strictly decreasing for $x > 0$) and $\frac{\partial d^i}{\partial p_j} \geq 0$, and hence $d^i(P(x))$ strictly decreases as x increases from 0 to ∞.

Now (4.1) implies an upper estimate for $\psi_r(x)$,

$$\psi_r(x) < \Phi(r - |x|) \tag{6.8}$$

from which we obtain $P_i(x) < \Phi(r_i - |x|)$.

Inserting these into (6.6) we obtain (6.3) as a necessary condition for (6.6). Since the estimate (6.8) becomes sharp as $|x| \to \infty$, $\Phi(r_i - r_j) - \psi_{r_i}(r_j)$ becomes arbitrarily small if r_j becomes very large. Hence (6.3) implies the existence of a solution of (6.6) after replacing $r_i \to r_i + C$ (C large enough) and then choosing suitable $\varepsilon_i > 0$.

Since $d^i(P(0)) > 0$ for all i by (6.7), the smallest equilibrium w with $w_i > P_i(x)$ for all x is $w_i = 1$. By the general argument in section 4, $\mathbf{B} = \mathbf{1}$ is spatially dominant. □

Note that for 2×2 games (5.1), the inequality (6.3) boils down to (5.8).

Φ is the shape of the standing (increasing) bistable wave in (3.1) with $f(p) = 1_{[\frac{1}{2},1]}(p)$. If there exist $r_i > 0$ such that the inequalities in (6.3) become equations then we have found a standing wave solution $P(x) = (\Phi(r_i - x))$ connecting in a decreasing way the two equilibria $\mathbf{0}$ and $\mathbf{1}$. Then neither equilibrium is spatially dominant. If all inequalities in (6.3) are reversed then $\mathbf{0}$ is spatially dominant.

Proof of Theorem 3. For (6.1), (6.3) can be rewritten as

$$\prod_{j \neq i} \frac{1 - \Phi(r_j - r_i)}{\Phi(r_j - r_i)} < \frac{b_i}{a_i}, \tag{6.9}$$

or

$$\sum_j f(r_i - r_j) < c_i \tag{6.10}$$

with the notations

$$f(x) = \begin{cases} \log \frac{1-\frac{1}{2}e^{-x}}{\frac{1}{2}e^{-x}} = \log(2e^x - 1), & \text{for } x \geq 0 \\ -f(-x), & \text{for } x < 0 \end{cases} \tag{6.11}$$

and $c_i = \log \frac{b_i}{a_i}$. If $\mathbf{B}$ has the higher Nash product, i.e., (6.2) holds, then $\sum_i c_i > 0$. f is strictly monotonically increasing and $f(x) \approx x$ as $x \to \infty$.

We show that for each sequence $c_1 \leq c_2 \cdots \leq c_n$ with $\sum_i c_i = 0$ there exists a sequence $r_1 \leq r_2 \leq \cdots \leq r_n$ such that

$$f_i(r) = \sum_j f(r_i - r_j) = c_i. \tag{6.12}$$

By changing coordinates, $s_i = r_{i+1} - r_i$, $g_i = f_{i+1} - f_i$, we see that the map $s \mapsto g(s)$ from $\mathbb{R}_+^{n-1}$ to itself, which is conjugate to the map $r \mapsto f(r)$, has the following monotonicity properties: g_i increases strictly monotonically in s_i from 0 to ∞, and $g_i(s) \approx n s_i$, whenever $s_i \to \infty$. Hence, modulo a coordinate transformation $z = \frac{s}{1+s}$, this map is conjugate to a map $z \mapsto h(z)$ of the cube $[0,1]^{n-1}$ into itself which maps each face into itself. Such a continuous map is surjective, by standard theorems from topology. This proves the claim (6.12) (thereby implying the existence of a standing wave in case of equality of the Nash products in (6.2)) and hence the existence of a solution to (6.10) for each sequence c_i with $\sum c_i > 0$. □

7. Potential games with linear incentives

Another important extension of 2 person unanimity games to n person games are bipolar games with linear incentives, as considered by Selten (1995).

The incentive functions take the form

$$d^i(p) = \sum_{j=1}^{n} \alpha_{ij} p_j - s_i \tag{7.1}$$

with $\alpha_{ii} = 0$. I assume further that

$$\alpha_{ij} = \alpha_{ji} > 0. \tag{7.2}$$

Denote $\alpha_i = \sum_{j=1}^{n} \alpha_{ij}$. Note that the special case $n = 2$, $\alpha_{12} = \alpha_{21} = 1$ and $s_1 = \hat{p}_2$, $s_2 = \hat{p}_1$ covers the 2×2 coordination games.

The positivity of the coefficients in (7.2) implies the monotonicity (1.2) of the best reply structure as required in section 4. The symmetry of the 'interaction matrix' α_{ij} implies that the game is a 'potential game' in the sense of Monderer and Shapley (1996). For binary choice games this means that there exists a potential function $V(p)$ such that $\frac{\partial V}{\partial p_i} = d^i(p)$. A specific potential function for (7.1) is given by

$$V(p) = \frac{1}{2} \sum_{i,j} \alpha_{ij} p_i p_j - \sum_i s_i p_i. \tag{7.3}$$

Note that $V(\mathbf{A}) = V(\mathbf{0}) = 0$ and $V(\mathbf{B}) = V(\mathbf{1}) = \frac{1}{2} \sum_{i,j} \alpha_{ij} - \sum_i s_i$, and if these are (strict) equilibria, then they are (strict) local maxima of V. For a potential game, each Nash equilibrium is a critical (extremal) point of V and each local maximum is a Nash equilibrium. A natural way of equilibrium selection for potential games (see Monderer and Shapley (1996), p. 134) is to choose the (usually unique) global maximum of V. Applying this criterion to (7.1), $\mathbf{B}$ is selected over $\mathbf{A}$ iff

$$V(\mathbf{B}) > V(\mathbf{A}), \quad \text{i.e.,} \quad \frac{1}{2} \sum_{i,j} \alpha_{ij} > \sum_i s_i. \tag{7.4}$$

Relating this criterion to the present spatial approach, it was shown in Hofbauer (1997) that along a bistable travelling wave connecting $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{B}$ drives out $\mathbf{A}$ iff (7.4) holds. (Such a wave exists, if there are no other strict equilibria besides $\mathbf{A}$ and $\mathbf{B}$, but not necessarily in general.) In the following Theorem we show that if $\mathbf{B}$ is the global maximizer of V then $\mathbf{B}$ is actually spatially dominant (even if there are other equilibria). Note that $V(\mathbf{1}) > V(p)$ for all $p \in [0,1]^n$ implies in particular — by choosing the corners of the hypercube for p — the inequalities

$$\sum_{i \in I} s_i < \sum_{i \in I} \alpha_i - \frac{1}{2} \sum_{i,j \in I} \alpha_{ij} = \frac{1}{2} \sum_{i,j \in I} \alpha_{ij} + \sum_{i \in I, j \notin I} \alpha_{ij} \quad \forall I \subset \{1, 2, \ldots, n\}. \tag{7.5}$$

Note that (7.5) implies (7.4) for $I = \{1, 2, \ldots, n\}$, and $s_k < \alpha_k$ for $I = \{k\}$, i.e., that $\mathbf{B}$ is a strict equilibrium. Whether $s_i > 0$, i.e., $\mathbf{A}$ is an equilibrium, is irrelevant in the following result. Also conversely, (7.5) implies that $\mathbf{1}$ is the strict global maximum of V, since V is affine linear in each p_i.

Theorem 4. *If a binary n–person game satisfies (7.1), (7.2) and (7.5), then* **B** *is spatially dominant.*

Proof. To apply Lemma 1 we have to find $r_i > 0$ such that[5]

$$\sum_j \alpha_{ij}\Phi(r_j - r_i) > s_i. \tag{7.6}$$

By suitably increasing the s_i it is enough to show that, for fixed α_{ij}, the corresponding system of equations

$$\sum_j \alpha_{ij}\Phi(r_j - r_i) = s_i \tag{7.7}$$

has a solution r_i for all choices of s from the polyhedron

$$\begin{aligned} S =& \{s \in \mathbb{R}^n : \sum_i s_i = \frac{1}{2}\sum_{i,j}\alpha_{ij} \text{ and } \sum_{i\in I} s_i \le \frac{1}{2}\sum_{i,j\in I}\alpha_{ij} + \sum_{i\in I, j\notin I}\alpha_{ij} \quad \forall I \subset \{1,2,\dots,n\}\} \\ =& \{s \in \mathbb{R}^n : \sum_i s_i = \frac{1}{2}\sum_{i,j}\alpha_{ij} \text{ and } \sum_{i\in I} s_i \ge \frac{1}{2}\sum_{i,j\in I}\alpha_{ij} \quad \forall I \subset \{1,2,\dots,n\}\} \end{aligned} \tag{7.8}$$

This is done by induction on n, i.e., the size of the matrix (α). Suppose this claim holds for all smaller sized symmetric nonnegative (α). Consider a partition of $\{1,\dots,n\} = I \cup J$ into two nonempty disjoint subsets and choose r_i such that $r_i >> r_j$ whenever $i \in I$ and $j \in J$. Then $\Phi(r_i - r_j) \to 1$ and $\Phi(r_j - r_i) \to 0$ so that (7.7) becomes

$$\begin{aligned} \sum_{k\in I}\alpha_{ik}\Phi(r_k - r_i) =& s_i \quad (i \in I) \\ \sum_{l\in J}\alpha_{jl}\Phi(r_l - r_j) =& s_j - \sum_{k\in I}\alpha_{jk} \quad (j \in J). \end{aligned} \tag{7.9}$$

Applying the induction hypothesis to the subsets I and J it is easy to check that a solution r_i exists for all $s \in S$ with $\sum_{i\in I} s_i = \frac{1}{2}\sum_{i,j\in I}\alpha_{ij}$, i.e., in the corresponding boundary face of the polyhedron S.

Hence each s in the boundary of S can be obtained by (7.7). Refining this argument by considering partitions into k subsets yields boundary faces of dimension $n-k$. In particular, the $n!$ corners of S can be reached by letting $r_i - r_j \to \infty$ for all $i \neq j$, after choosing a certain ordering of the r_i, say $r_{\sigma_1} < \cdots < r_{\sigma_n}$ for a certain permutation σ, thus giving $s_i = \sum_{j:\sigma_j > \sigma_i}\alpha_{ij}$. A standard argument from topology implies again that all interior points s of S can be reached by a suitable choice of r as well. $\square$

[5] That (7.5) is necessary for (7.6), follows immediately by summing (7.6) for $i \in I$ and applying (6.5).

2 person unanimity games with incomplete information. This is a special class of games considered by Selten (1995) which can be reduced to the above class. Additionally to (7.2) we normalize now the entries $\sum_{i,j}\alpha_{ij}=2$. The game runs as follows: Out of n given players, two are chosen with probabilities α_{ij}. (This is the incomplete information part). The two selected players i,j play a 2×2 unanimity game, with payoffs a_i,b_i, and a_j,b_j as in (5.1). Then the expected payoffs are given by

$$\begin{aligned}\pi(A_i,p_{-i}) &= a_i\sum_j \alpha_{ij}(1-p_j)\\ \pi(B_i,p_{-i}) &= b_i\sum_j \alpha_{ij}p_j,\end{aligned} \tag{7.10}$$

which leads to linear incentive functions

$$d^i(p)=\pi(B_i)-\pi(A_i)=(b_i+a_i)\sum_j \alpha_{ij}p_j - a_i\sum_j \alpha_{ij}$$

Dividing WLOG by b_i+a_i this conforms with (7.1) with $s_i=\frac{\alpha_i a_i}{a_i+b_i}$. Obviously any bipolar potential game with linear incentives (7.1), (7.2) can be interpreted in this way, by choosing $a_i,b_i>0$ suitably. Theorem 4 leads to criteria for the spatial dominance of the strict equilibria **B** resp. **A**. In particular, for spatial dominance of **B** the condition $V(\mathbf{A})<V(\mathbf{B})$ is necessary which reads now

$$\sum_{i=1}^n \alpha_i\frac{a_i}{a_i+b_i}<\sum_{i=1}^n \alpha_i\frac{b_i}{a_i+b_i}. \tag{7.11}$$

As discussed in more detail in Hofbauer (1997), this criterion differs from Selten's (1995) dominance relation based on generalized Nash products: He selects **B** over **A** iff $\prod_i b_i^{\alpha_i}>\prod_i a_i^{\alpha_i}$. It would be interesting to compare these criteria in experiments (Kuon (1994)).

8. Difficulties with Equilibrium Selection beyond Best Responses

The concept of risk–dominance is intimately connected with the best reply structure of the game. This is best seen from the axiomatic description in 2×2 games. However, in some situations, there is more to a game than its best reply structure. For example, in biological applications of evolutionary game theory, payoffs are fitnesses and they have an absolute meaning beyond von Neumann–Morgenstern utilities. In such situations, risk–dominance may not be the relevant equilibrium selection criterion. In fact, there is an increasing number of dynamic models that — already in the simplest case of symmetric 2×2 games — do not necessarily select the risk–dominant equilibrium, but maybe the payoff dominant one, e.g., see Samuelson (1997). In the following I show that the same phenomena and difficulties can arise also within the present framework of spatio–temporal models with reaction–diffusion equations, if we allow other reaction dynamics than the best reply dynamics.

Pareto versus risk dominance.

Consider a symmetric 2×2 game with payoff matrix now taken in the general form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{8.1}$$

Here we take as reaction term an imitation model from Weibull (1995), Hofbauer and Weibull (1996) with a nonlinear increasing transformation g of payoff into fitness (recall that p denotes the frequency of strategy B):

$$\dot{p} = f(p) = p\,(g(\pi(B,p)) - \bar{g}) \tag{8.2}$$

with $\bar{g} = pg(\pi(B,p)) + (1-p)g(\pi(A,p))$. Hence

$$f(p) = p(1-p)\,[g(dp + c(1-p)) - g(bp + a(1-p))] \tag{8.3}$$

Note that unlike the best response dynamics and the replicator dynamics, (8.3) is not invariant under linear payoff transformations.

We assume $a > c$ and $d > b$ so that both pure strategies A and B are strict equilibria. Let $\hat{p} = \frac{a-c}{a+d-b-c}$ denote the frequency of B at the unstable mixed equilibrium. $a - c > d - b$ or $\hat{p} > \frac{1}{2}$ means that A is risk dominant, $a > d$ means that A is payoff dominant.

Theorem. *1. If g is a convex function then the spatially dominant equilibrium of the bistable equation (3.1) with reaction term (8.3) is either the risk–dominant or the Pareto dominant equilibrium of the game.*
2. If g is strictly convex and $\hat{p}$ is sufficiently close to $\frac{1}{2}$, i.e., both strict equilibria are nearly equally risky, then the payoff dominant equilibrium is spatially dominant for (3.1) with (8.3).
3. If the function g is not convex then there are games (8.1) for which A is payoff and risk dominant while B is the spatially dominant equilibrium.

Proof. 1. We have to show that if for a game the payoff dominant and the risk dominant equilibrium coincide then this equilibrium is also spatially dominant. Let A be this equilibrium, i.e., $a > c$, $a > d > b$ and $a - c > d - b$. Convexity of g implies

$$\varepsilon g(bp+a(1-p)) + (1-\varepsilon)g(ap+b(1-p)) \geq g\left((\varepsilon b + (1-\varepsilon)a)p + (\varepsilon a + (1-\varepsilon)b(1-p)\right) \tag{8.4}$$

Choosing $\varepsilon = \frac{a-d}{a-b}$ (which is in $(0,1)$ by assumption) makes the right hand side to

$$g\left(dp + (a+b-d)(1-p)\right) > g(dp + c(1-p)) \tag{8.5}$$

Multiplying this estimate by $p(1-p)$, integrating over the interval $[0,1]$ and using the symmetry $p \leftrightarrow 1-p$ yields for (8.3)

$$\int_0^1 f(p)dp < 0. \tag{8.6}$$

By Theorem 1 (v) this shows the spatial dominance of A.

2. If A and B are equally risky then $a + b = c + d$ which turns the inequality in (8.5) into an equality. For compensation, the strict convexity of g implies that there is strict inequality in (8.4). Hence the conclusion (8.6) remains valid, even after a small perturbation of the payoffs that make B risk dominant.

3. is left to the reader. □

Remark. The statements 1 and 2 of the theorem hold more generally for dynamics that satisfy the convex monotonicity axiom introduced in Hofbauer and Weibull (1996).

Best reply versus replicator dynamics.

Replicator and best reply dynamics, the two most important forms of game dynamics, depend — in contrast to (8.2) — only on the incentive functions d^i of the game.

As seen in section 3 for symmetric 2×2 games, the risk–dominant equilibrium is spatially dominant when either dynamics is taken as reaction term in (3.1). Nevertheless, for more general games this is no longer true. We illustrate this in two cases.

1. For *symmetric bipolar n person games*, with $n \geq 3$, the (single) incentive function $d(p)$ is a polynomial of degree $n - 1$ in p. The BR dynamics leads to a similar criterion as for $n = 2$, namely $\mathbf{A}$ is spatially dominant iff $\hat{p} > \frac{1}{2}$ (where $\hat{p}$ is the frequency of B at the unstable symmetric mixed equilibrium, which we assume to be unique). It is comforting that the *long run equilibrium* of Kandori et al (1993) leads to the same criterion.

On the other hand, the integral condition (v) leads to a different criterion for spatial dominance for the replicator dynamics $f(p) = p(1 - p)d(p)$ which seems to coincide with the criterion of Foster and Young (1990), see also Kim (1996).

Adding further to this troublesome ambiguity, many other approaches to equilibrium selection, in particular the Harsanyi–Selten (1988) definition of risk–dominance based on the tracing procedure, the global games approach of Carlsson and van Damme (1993) and the forward looking dynamic approach of Matsui and Matsuyama (1995), lead to even more different selection criteria for this class of games, see Carlsson and van Damme (1993), Kim (1995) and van Damme (1996).

This divergence of different equilibrium selection methods for this relatively simple class of symmetric bipolar games suggests that a complete, generally accepted theory of equilibrium selection will remain a dream.

2. For *asymmetric* 2×2 *coordination games* it was shown in Hofbauer et al (1997), that with the replicator dynamic as reaction term the notion of spatial dominance is NOT in agreement with risk–dominance like in Theorem 2. The reason is that the replicator dynamics does not depend solely on the best reply structure of the game. It is not invariant under rescaling the payoffs of the two players by different factors. The two player populations may adjust with different speeds, in contrast to the BR dynamics.

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